Symmetry Theories, Wigner's Function, Compactification, and Holography

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Motivation

- Intrinsically Relative Theories (e.g., 6D SCFTs with $|\mathbb{D}^{(2)}| \neq N^2$)

[Witten; 1998], ..., [Gukov, Hsin, Pei; 2020]

Not always possible: Relative QFT $\rightarrow$ Absolute QFT

Partition Vector $\rightarrow$ Partition Function

- SymTFT Framework for Mixed States [Luo, Wang, Bi; 2025], [Schäfer-Nameki, Tiwari, Warman, Zhang; 2025], [Qi, Sohal, Chen, Stephen, Prem; 2025]

a) String Compactifications: Mixed State Boundary conditions
b) Holographic Systems: Ensamble of Theories

Outline

- Wigner's Quasi-Probabilistic Function
  - a) Continuum Case
  - b) Discrete Case
  - c) Wavefunction Interpretation
- Wigner's Quasi-Probabilistic Function and SymTFTs for Mixed States
  - a) Folding and Unfolding
  - b) Examples
- Compactification (and Holography)

Wigner (1932)

Classical Statistical Mechanics: Gibbs-Boltzmann formula

$$P(x,p)dxdp = e^{-\beta\epsilon}dxdp\,, \quad \epsilon = \frac{p^2}{2m} + V(x)\,, \quad \beta = \frac{1}{k_B T}$$

Quantum Mechanics: von Neumann formula

$$\langle Q \rangle = \mathrm{Tr}\, Q e^{-\beta H}$$

Drawback: challenging computationally.

Wigner (1932): Continuum Case

Wigner's Alternative: Start with phase space density

$$W(x,p) = \frac{1}{\pi} \int dy\, \psi(x-y) e^{2ipy} \psi(x+y)^*$$

with QM Wavefunction $\psi(x) = \langle x|\psi\rangle$ (pure state). [Wigner, 1932]

Integration over Lagrangian submanifolds in phase space

$$\int dp\, W(x,p) = |\psi(x)|^2 \quad \text{and} \quad \int dx\, W(x,p) = \left| \int dx\, \psi(x) e^{-ipx} \right|^2 = |\widetilde{\psi}(p)|^2$$

Key Equation: Expectation values as phase space integrals

$$\int dxdp\, W(x,p) g(x,p) = \langle \widehat{G} \rangle$$

Wigner (1932): Continuum Case

Generalization to Mixed States:

$$W_\rho(x,p) = \frac{1}{\pi}\int dq\, \langle x-q|\,\widehat{\rho}\,|x+q\rangle e^{2ipq}\,, \qquad \widehat{\rho} = \sum p_i|\psi_i\rangle\langle\psi_i|$$

Integration over Lagrangian Submanifolds in Phase Space:

$$\int dp\, W_\rho(x,p) = \langle x|\,\widehat{\rho}\,|x\rangle\,, \qquad \int dx\, W_\rho(x,p) = \langle p|\,\widehat{\rho}\,|p\rangle$$

Expectation Values as Phase Space Integrals:

$$\mathrm{Tr}\,\widehat{\rho}\widehat{G} = \int dxdp\, W_\rho(x,p) g(x,p)$$

Wigner (1932): Continuum Case

Comments:

- $W_\rho(x,p)$ takes negative values $\rightarrow$ "Quasi-Probabilistic Function"

- Non-Uniqueness: Function of N variables $\mapsto$ Function of $2N$ variables

$$W_\rho^{[r,s]}(x,p) = \frac{r+s}{2\pi} \int dq \, \langle x - rq | \, \widehat{\rho} \, | x + sq \rangle e^{(r+s)ipq}$$

- Computationally favorable: Explicit Quantum Corrections $f_k^{[r,s]}$

$$W_{\rho_t}^{[r,s]}(x,p) = e^{-\beta\epsilon} + \hbar f_1^{[r,s]} + \hbar^2 f_2^{[r,s]} + \dots$$

Discrete Case

Phase space $(x, p) \in \mathbb{Z}_N \times \mathbb{Z}_N$, then [Wootters; 1987], [Bouzouina, Bìevre; 1996], [Bianucci, Miquel, Paz, Saraceno; 2001], ..., [Heckman, Hübner, Murdia; 2025]

$$W_\rho^{[r,s]}(x,p) = \frac{1}{2\pi} \sum_q \langle x - rq | \widehat{\rho} | x + sq \rangle e^{(r+s)ipq} \,, \quad \gcd(r+s, N) = 1$$

with normalization factor change due to

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{\frac{2\pi i}{N} gn(n' - n'')} = \delta_{n',n''}$$

for all units of $g \in \mathbb{Z}_N$, as characterized by $\gcd(g, N) = 1$.

Wavefunction Interpretation

Introduce displacement, dilation and conjugation operators

$$\begin{aligned} U_x &= \exp(-ix\widehat{p}) \quad \text{with} \quad U_x\,|x'\rangle = |x+x'\rangle \\ V_p &= \exp(+ip\widehat{x}) \quad \text{with} \quad V_p\,|p'\rangle = |p+p'\rangle \\ \Delta_r & \qquad\qquad\quad\;\; \text{with} \quad \Delta_r|x\rangle = |r\,x\rangle \\ \mathcal{C} & \qquad\qquad\quad\;\; \text{with} \quad \mathcal{C}\,|x\rangle = |-x\rangle\,,\ \mathcal{C}\,|p\rangle = |-p\rangle \end{aligned}$$

and the doubled operator

$$\begin{aligned} \mathbb{U}_{x,p} &\equiv \exp(-ix\widehat{p} + ip\widehat{x}) \\ \mathbb{U}^{[r]}_{x,p} &\equiv \mathbb{U}_{x,p}\Delta_r \end{aligned}$$

Then, with phase cancellation due to Baker-Campbell-Hausdorff,

$$\begin{aligned} W^{[r,s]}_{\rho}(x,p) &= \frac{1}{2\pi}\sum_q \langle x - rq|\,\widehat{\rho}\,|x+sq\rangle e^{(r+s)ipq} \\ &= \frac{1}{2\pi}\mathrm{Tr}(\widehat{\rho}\,\mathbb{U}^{[s]}_{x,p}\mathcal{C}\,\mathbb{U}^{[r]\dagger}_{x,p}) = \frac{1}{2\pi}\sum_i p_i\langle Z_i|\,\mathbb{U}^{[s]}_{x,p}\mathcal{C}\,\mathbb{U}^{[r]\dagger}_{x,p}|Z_i\rangle \end{aligned}$$

Wavefunction Interpretation

Given the N-dimensional Hilbert space $\mathcal{H}$ consider the double

$$\mathcal{H} \otimes \overline{\mathcal{H}}$$

and map the mixed state $\widehat{\rho}$ to the vector (operator-state mapping)

$$|\rho\rangle\rangle \equiv \sum_i p_i \,|\,i\,\rangle \otimes |\,\bar{i}\,\rangle$$

and, with respect to the point $\Phi = (x, p)$ in phase space, define

$$\langle\langle \Phi^{[r,s]}| \equiv \sum_e \langle e|\mathcal{C}\mathbb{U}_\Phi^{[r]\,\dagger} \otimes \langle e|\mathbb{U}_\Phi^{[s]\,\dagger}$$

which derives from the Choi state [Choi; 1975]

$$|\mathrm{Choi}\rangle \propto \sum_e |e\rangle \otimes |\bar{e}\rangle$$

Wavefunction Interpretation

Previous definitions are such that

$$\langle\langle \Phi^{[r,s]} | \rho \rangle\rangle = W_\rho^{[r,s]}(\Phi)$$

with phase space point $\Phi = (x, p)$. Further, given a classical function $f(\Phi)$ on phase space define

$$\langle\langle f^{[r,s]} | \equiv \sum_{\Phi} f(\Phi) \langle\langle \Phi^{[r,s]} |$$

then one has the overlap

$$\langle\langle f^{[r,s]} | \rho \rangle\rangle = \sum_{\Phi} f(\Phi) \langle\langle \Phi^{[r,s]} | \rho \rangle\rangle = \sum_{\Phi} f(\Phi) W_\rho^{[r,s]}(\Phi) \,.$$

SymTFT and Wavefunction Interpretation

Consider a relative QFT with partition vector $|Z\rangle$ and a choice of absolute form (assuming existence) as specified by the Lagrangian

$$\Lambda \subset \mathbb{D}$$

specifying electric and magnetic bases to the SymTFT Hilbert space $\mathcal{H}$. Then, the QFT partition function with some background $e \in \Lambda$ is

$$Z(\Lambda, e) \equiv \langle \Lambda, e | Z \rangle \,.$$

[Reshetikhin, Turaev; 1991], [Turaev,Viro;1992], [Barrett, Westbury; 1996], [Witten; 1998], [Fuchs, Runkel, Schweigert; 2002], [Kapustin, Saulina; 2010], [Kitaev, Kong; 2011], [Freed, Teleman; 2014], [Gaiotto, Kapustin, Seiberg, Willett; 2014], [Gaiotto and J. Kulp; 2021], [Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schafer-Nameki; 2021], [Freed, Moore, Teleman; 2022], [Kaidi, Ohmori, Zheng; 2022], ...

SymTFT and Wavefunction Interpretation

Independent of a choice of polarization we can associate Wigner's function to a pure state

$$\widehat{\rho}_Z = |Z\rangle\langle Z| \;\mapsto\; W_Z^{[r,s]}(e,m)\,,$$

and from it one extracts the partition functions squared

$$\sum_{m\in\overline{\Lambda}} W_Z^{[r,s]}(e,m) = |Z(\Lambda,e)|^2\,, \qquad \sum_{e\in\Lambda} W_Z^{[r,s]}(e,m) = |Z(\overline{\Lambda},m)|^2\,,$$

summing over Lagrangian submanifolds in phase space.

Mixed States and SymTFTs

More generally consider the mixed states

$$\widehat{\rho}_Z = \sum_i z_i |Z_i\rangle\langle Z_i| \,, \qquad \widehat{\rho}_B = \sum_i b_i |B_i\rangle\langle B_i|$$

and define on phase space the classical function $p_B^{[r,s]}(\Phi)$ by

$$\mathrm{Tr}\,(\widehat{\rho}_B \widehat{\rho}_Z) \equiv \sum_\Phi p_B^{[r,s]}(\Phi) W_{\rho_Z}^{[r,s]}(\Phi) \,.$$

Then define with respect to the doubled SymTFT Hilbert space $\mathcal{H} \otimes \overline{\mathcal{H}}$

$$\langle\!\langle \rho_B | \equiv \sum_{\mathbb{B}} p_B^{[r,s]}(\Phi) \langle\!\langle \Phi^{[r,s]} |$$

which is such that

$$\langle\!\langle \rho_B | \rho_Z \rangle\!\rangle = \mathrm{Tr}\,(\widehat{\rho}_B \widehat{\rho}_Z)$$

Mixed States and SymTFTs

Overall: SymTFT with Hilbert space $\mathcal{H} \otimes \overline{\mathcal{H}}$ and boundary conditions

$$\langle\langle \rho_B | \quad \text{and} \quad | \rho_Z \rangle\rangle$$

derived from mixed states of the original SymTFT with Hilbert space $\mathcal{H}$.

The topological boundary condition $\langle\langle \rho_B |$ is defined with respect to any function on phase space, independent of the existence of a polarization Λ.

A basis to the space of boundary conditions, for any fixed $[r, s]$, are

$$\langle\langle \Phi^{[r,s]} | = \sum_e \langle e | \mathcal{C} \mathbb{U}_{\Phi}^{[r]\,\dagger} \otimes \langle e | \mathbb{U}_{\Phi}^{[s]\,\dagger}$$

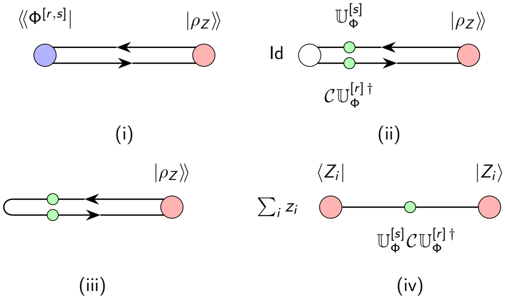
labelled by points Φ in the phase space of the original SymTFT.

Doubled SymTFT Sandwich: $W_{\rho_Z}^{[r,s]}(\Phi)$ in Pictures

Evaluating the trace:

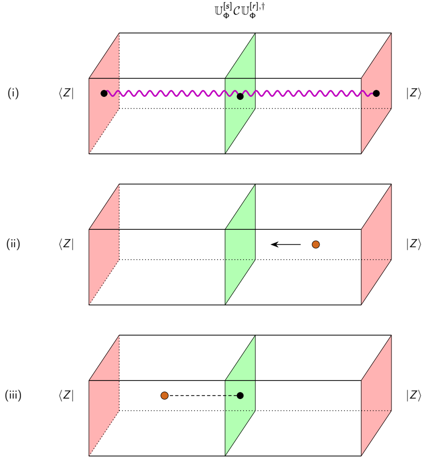
$$W_{\rho_Z}^{[r,s]}(\Phi) = \langle\langle \Phi^{[r,s]} | \rho_Z \rangle\rangle = \sum_i z_i \langle Z_i | \mathbb{U}_\Phi^{[s]} \mathcal{C} \mathbb{U}_\Phi^{[r]\,\dagger} | Z_i \rangle$$

Pictorial Presentation:



Defect and Symmetry Operators: Standard Story

Pure state $\widehat{\rho} = |Z\rangle\langle Z|$:



Example: 6D SCFTs

3-form potential C^I and level matrix K_{IJ} (ADE Cartan Matrix):

$$\mathcal{S}_{7D} = \frac{K_{IJ}}{4\pi} \int_{7D} C^I dC^J \,,$$

Group of Surface Defects: $\mathbb{D} \cong \text{Coker}(K_{IJ})$.

$$|\mathbb{D}| \neq N^2 \;\rightarrow\; \text{Intrinsically Relative}$$

Topological bulk operators $\mathcal{T}_\nu(\Sigma) = \exp\left(i \int_\Sigma \nu_I \widehat{C}^I\right)$ and linking

$$\mathcal{T}_\nu(\Sigma)\mathcal{T}_{\nu'}(\Sigma') = \exp\left(2\pi i \times \nu_I K^{IJ} \nu'_J \times \mathrm{lnk}(\Sigma, \Sigma')\right) \mathcal{T}_{\nu'}(\Sigma')\mathcal{T}_\nu(\Sigma)$$

[Monnier; 2018], [Heckman, Tizzano; 2018], [Gukov, Pei, Putrov, Vafa; 2018], [Hsieh, Tachikawa, Yonekura; 2020], [Gukov, Hsin, Pei; 2020], ...

Example: 6D SCFTs

Defect group $\mathbb{D}$ is the "phase space" and

$$\mathbb{U}_\nu = \exp\left(-iK_{IJ}\int_{6D} C^I \widehat{C}^J\right)$$

with label $[\nu] \in \mathbb{D}$ where $\nu_J = -\int_{3\text{D}} K_{IJ} C^I$.

E.g., for the pure state $\widehat{\rho}_Z = |Z\rangle\langle Z|$ we have Wigner's function

$$W_Z(\nu) = \langle\langle \nu | \rho_Z \rangle\rangle = \langle Z | \mathbb{U}_\nu \mathcal{C} \mathbb{U}_\nu^\dagger | Z \rangle \,,$$

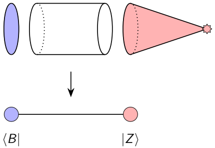
where $r = s = 1$. This is the partition function of a doubled system of 6D SCFTs (which is absolute) with 2-form symmetry group

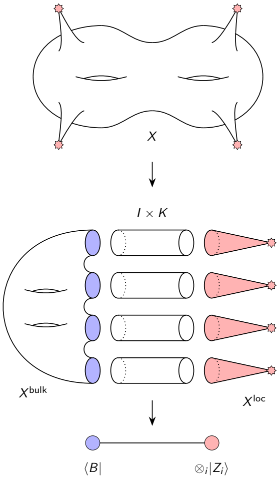
$$\Gamma^{(2\text{-form})} \cong \mathbb{D}^\vee \,.$$

Compactification

Geometric Engineering:

$$X \xrightarrow[\text{\& Local Limit}]{\text{IIA/IIB/M/F}} \mathcal{T}_X$$





[Cvetič, Heckman, Hübner, Torres; 2022], [Gould, Lin, Sabag; 2023], [Cvetič, Dierigl, Lin, Torres, Zhang; 2024]

Compactification: The Quiche

Decomposition of X with cross section K:

$$X = X^\circ \cup (I \times K) \cup X^{\text{loc}}$$

where $X^{\text{loc}} = \sqcup_i X_i^{\text{loc}}$ is a disjoint union of local models centered on all loci supporting localized degrees of freedom.

Symmetry theory derives from the cross section:

$$K \mapsto \mathcal{S} = \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_{\pi_0(K)}$$

Similarly, we have pure state physical boundary conditions:

$$X^{\text{loc}} \mapsto |Z\rangle\!\rangle = |Z_1\rangle \otimes \cdots \otimes |Z_{\pi_0(X^{\text{loc}})}\rangle$$

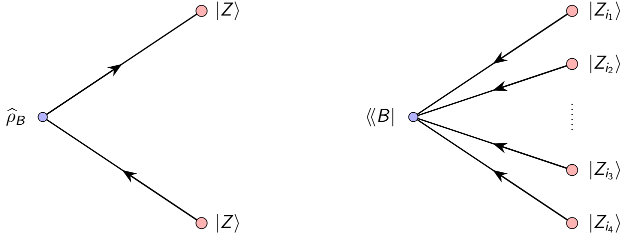
[Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schafer-Nameki; 2021], [Heckman, MH, Torres, Turner, Yu; 2023]

Compactification: "Mixed State Boundary Conditions"

In contrast, boundary conditions do not decompose into tensor products:

$$X^{\circ} \mapsto |B\rangle\rangle \neq \otimes_i |B_i\rangle$$

Compare to Wigner's function (left) approach:



Compactification: Mixed State Boundary Conditions

Decompose (assuming existence of the B_i)

$$|B\rangle\!\rangle = \sum_{i_1,\ldots,i_m} |B_{i_1,\ldots,i_m}\rangle\!\rangle\langle\!\langle B_{i_1,\ldots,i_m}|B\rangle\!\rangle \equiv \sum_{i_1,\ldots,i_m} b^*_{i_1,\ldots,i_m}|B_{i_1,\ldots,i_m}\rangle\!\rangle\,,$$

where $m = \pi_0(K)$ is the number of connected components of K and

$$|B_{i_1,\ldots,i_m}\rangle\!\rangle = |B_{i_1}\rangle \otimes \cdots \otimes |B_{i_m}\rangle\,.$$

Coefficients $b_{i_1,\ldots,i_m} = \langle\!\langle B_{i_1,\ldots,i_m}|B\rangle\!\rangle$ are functions of the background fields.

The overall partition function is

$$\langle\!\langle B|Z\rangle\!\rangle = \sum_{i_1,\ldots,i_m} b_{i_1,\ldots,i_m}\langle B_{i_1}|Z_{i_1}\rangle \ldots \langle B_{i_m}|Z_{i_m}\rangle\,,$$

which is a collection of direct products of absolute theories with partition functions $\langle B_{i_k}|Z_{i_k}\rangle = Z_{i_k}(B_{i_k})$ weighted by $b_{i_1,\ldots,i_m}$.

Example: M-theory on $T^6/\mathbb{Z}_3$

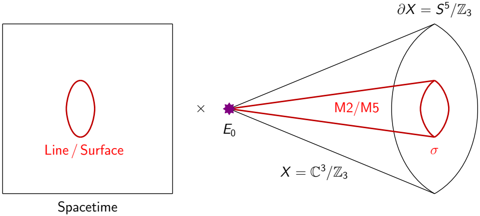
$X = T^6/\mathbb{Z}_3$ contains 27 singularities modelled on $\mathbb{C}^3/\mathbb{Z}_3$

$\Rightarrow$ M-theory on $X = T^6/\mathbb{Z}_3$ contains 27 Seiberg SCFTs $E_0^{\oplus 27}$

Each Seiberg SCFT E_0 has defect lines and surfaces

$$\mathbb{D}^{(1)} \cong \mathbb{Z}_3^{(\mathrm{M2})}\,, \qquad \mathbb{D}^{(2)} \cong \mathbb{Z}_3^{(\mathrm{M5})}$$

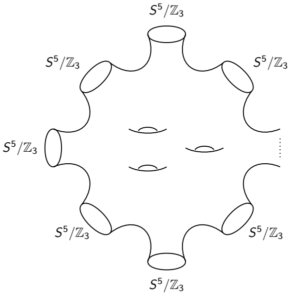
from branes on cones over $H_1(S^5/\mathbb{Z}_3) \cong \mathbb{Z}_3$ and $H_3(S^5/\mathbb{Z}_3) \cong \mathbb{Z}_3$.



[Del Zotto, Heckman, Park, Rudelius; 2015], [Morrison, Schäfer-Nameki, Willett; 2020], [Cvetič, Heckman, Hübner,

Example: M-theory on $T^6/\mathbb{Z}_3$

The 27 copies of the 5D SCFT E_0 interact across the bulk X°:



$$H_n(T^6/\mathbb{Z}_3) = \begin{cases} \mathbb{Z} & n=6 \\ 0 & n=5 \\ \mathbb{Z}^9 \oplus \mathbb{Z}_3^4 & n=4 \\ \mathbb{Z}^2 & n=3 \\ \mathbb{Z}^9 \oplus \mathbb{Z}_3^{17} & n=2 \\ 0 & n=1 \\ \mathbb{Z} & n=0 \end{cases}$$

Boundary conditions $\langle\!\langle B|$ compute from $H_n(T^6/\mathbb{Z}_3)$. For example, an $\mathfrak{u}(1)^{\oplus 9}$ gauge theory (abelian bulk modes) is associated to $\langle\!\langle B|$ and determines the coefficients $b_{i_1,\ldots,i_m}$. [Cvetič, Heckman, Hübner, Torres; 2022]

Computational Details for $\langle\langle B|$

Notation:

$X = T^6/\mathbb{Z}_3$, $X^{\text{loc}} = \sqcup_{i=1}^{27}(\mathbb{C}^3/\mathbb{Z}_3)_i$, $\partial X^{\text{loc}} = \sqcup_{i=1}^{27}(S^5/\mathbb{Z}_3)_i$, $X^\circ = X \setminus X^{\text{loc}}$

Sequences setting boundary conditions:

$$0 \to H_4(X^\circ) \xrightarrow{\jmath_4} H_4(X) \xrightarrow{\partial_4} H_3(\partial X^{\text{loc}}) \xrightarrow{\imath_3} \text{Tor}\, H_3(X^\circ) \to 0$$

$$0 \to \mathbb{Z}^9 \xrightarrow{\jmath_4} \mathbb{Z}^9 \oplus \mathbb{Z}_3^4 \xrightarrow{\partial_4} \mathbb{Z}_3^{27} \xrightarrow{\imath_3} \mathbb{Z}_3^{17} \to 0$$

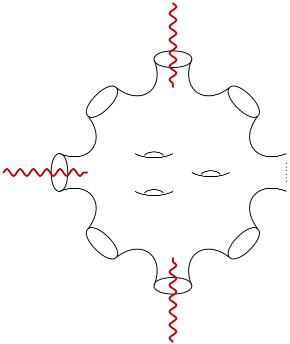
$$0 \to H_2(X^\circ) \xrightarrow{\jmath_2} H_2(X) \xrightarrow{\partial_2} H_1(\partial X^{\text{loc}}) \xrightarrow{\imath_1} \text{Tor}\, H_1(X^\circ) \to 0$$

$$0 \to \mathbb{Z}^9 \xrightarrow{\jmath_2} \mathbb{Z}^9 \oplus \mathbb{Z}_3^{17} \xrightarrow{\partial_2} \mathbb{Z}_3^{27} \xrightarrow{\imath_1} \mathbb{Z}_3^{4} \to 0\,.$$

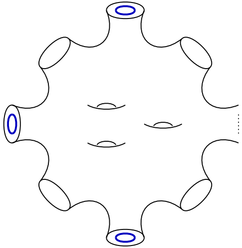
Boundary conditions are such that the overall p-form gauge groups $G^{(p)}$ are

$$G^{(0)} = \left(\mathbb{Z}_3^{23} \times U(1)^9\right)/\mathbb{Z}_3^6\,,$$
$$G^{(1)} = \left(\mathbb{Z}_3^{10} \times U(1)^9\right)/\mathbb{Z}_3^6\,,$$
$$G^{(2)} = U(1)^2\,.$$

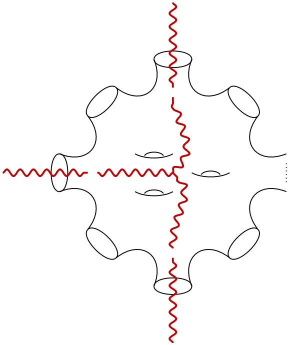
Defect Operators



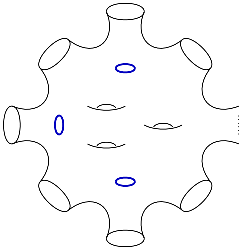
Symmetry Operators



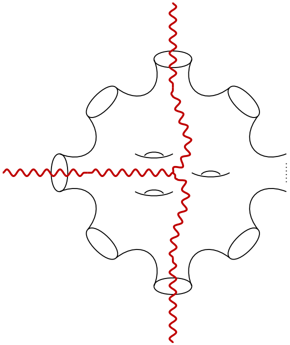
Defect Operators



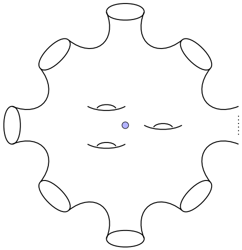
Symmetry Operators



Defect Operators



Symmetry Operators

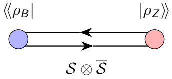


Conclusion and Outlook

- Wigner's Quasi-Probabilistic Function applied to SymTFTs:

  a) Pure states: Computable for intrinsically relative theories,
  Independent of Lagrangian $\Lambda \subset \mathbb{D}$

  b) Mixed states: SymTFT Sandwich



Wave- / Partition function interpretation: $W_{\rho_Z}^{[r,s]}(\Phi) = \langle\langle \Phi^{[r,s]} | \rho_Z \rangle\rangle$

- Applications: String Compactifications

$$\text{Bulk } X^\circ \mapsto \langle\langle X^\circ |$$

- Outlook: Holographic Ensambles