June. 31, 2025

Beijing

45 min.

Higher Structures on Boundary Conformal Manifolds

Shuhei Ohyama (Wien)

arXiv:2304.05356 with Shinsei Ryu,
arXiv:2408.15960 with Kansei Inamura,
arXiv:2505.12525 with Yichul Choi, Hyunsoo Ha,
Dongyeob Kim, Yuya Kusuki, Shinsei Ryu

Apology

This talk is related to the generalized symmetry,
but I won't directly talk about symmetry itself.

Introduction

Parametrized family of Hamiltonians/Moduli space of theories:

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- moduli space of gapped systems

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Q. Geometric Phase in Many-body system/Quantum Field Theory?

Plan of this talk

1. Multi-wavefunction overlap
2. Matrix Product Representation
3. Boundary Conformal Field Theory

Idea: Generalization of the inner prod.

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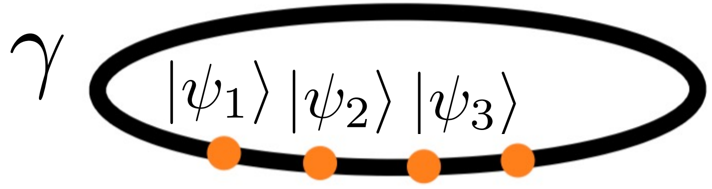

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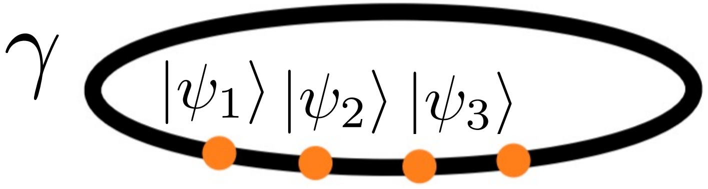

Berry phase $\fallingdotseq \langle\psi_1|\psi_2\rangle \cdot \langle\psi_2|\psi_3\rangle \cdots \langle\psi_L|\psi_1\rangle$

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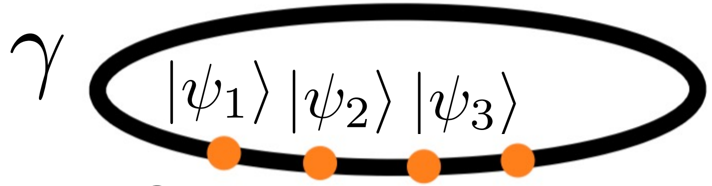

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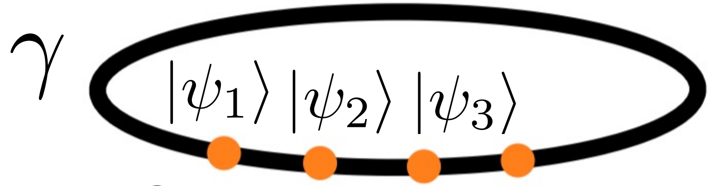

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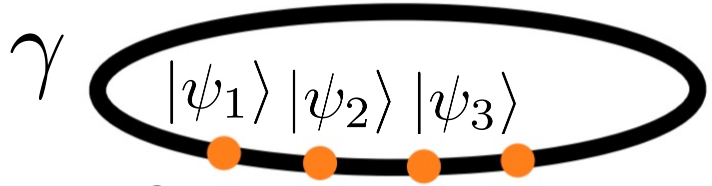

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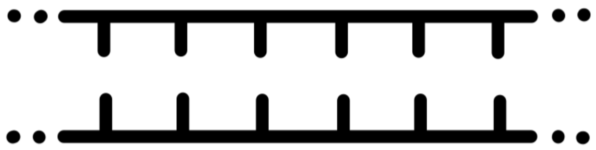

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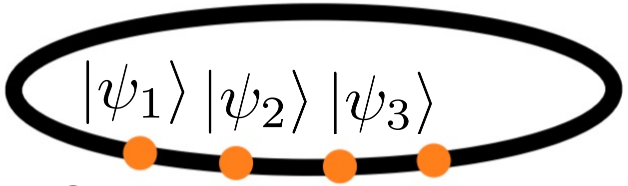

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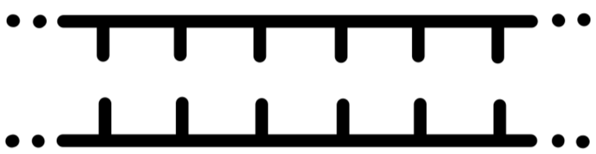

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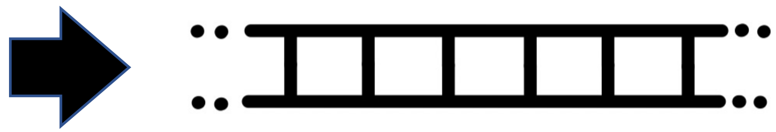

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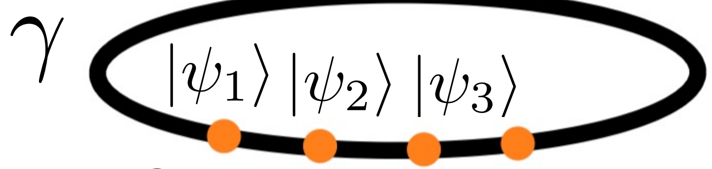

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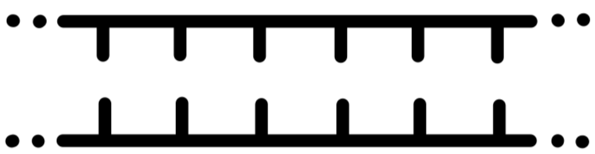

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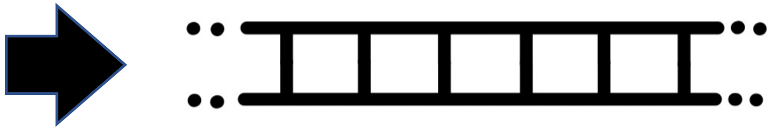

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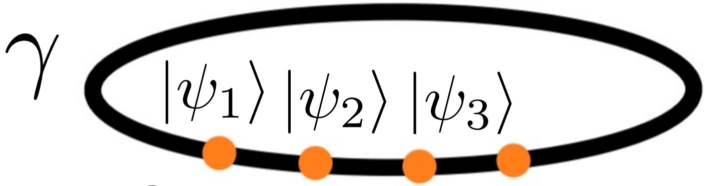

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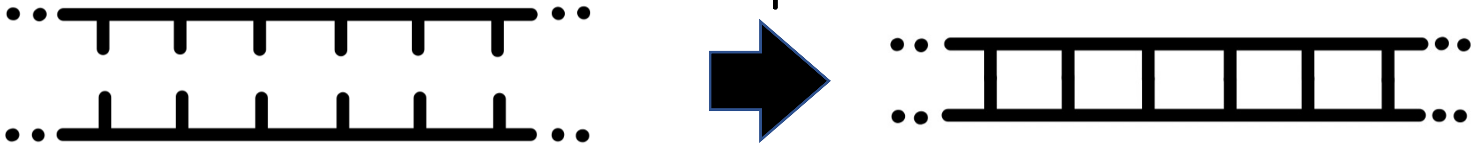

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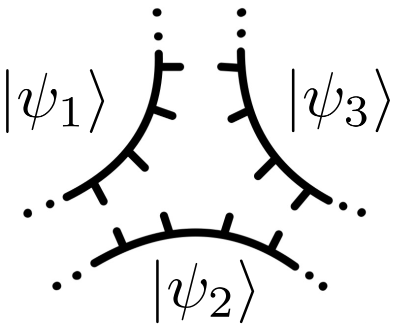

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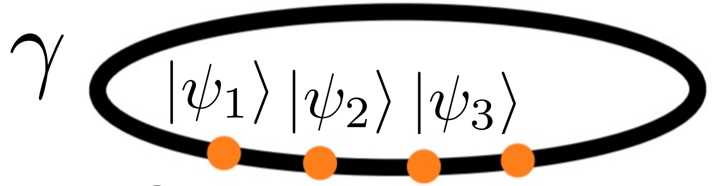

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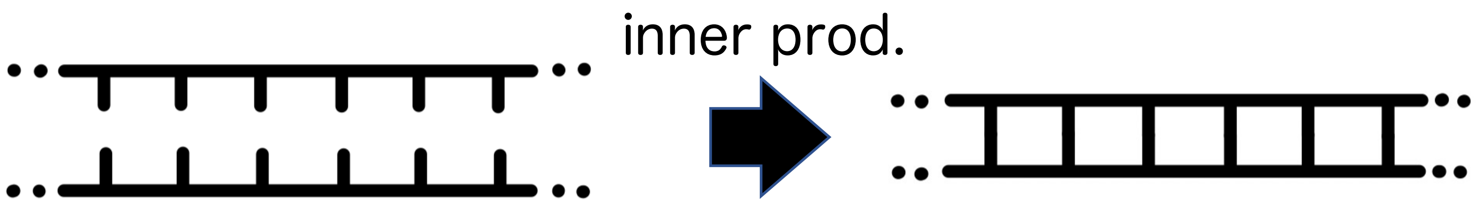

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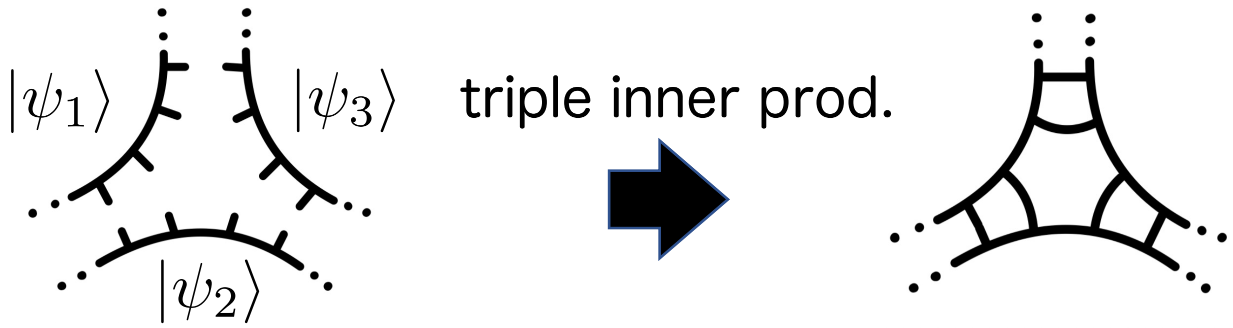

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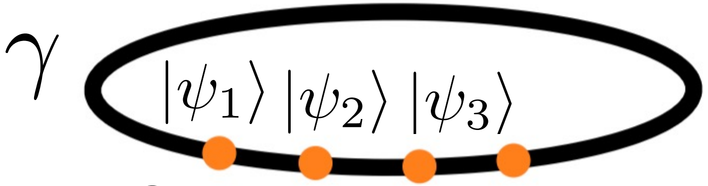

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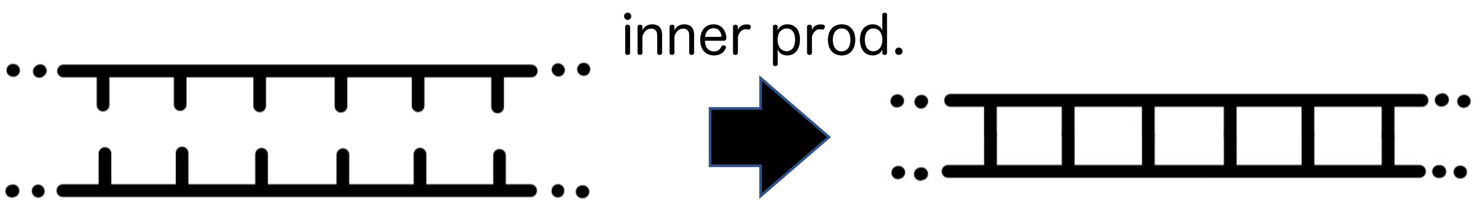

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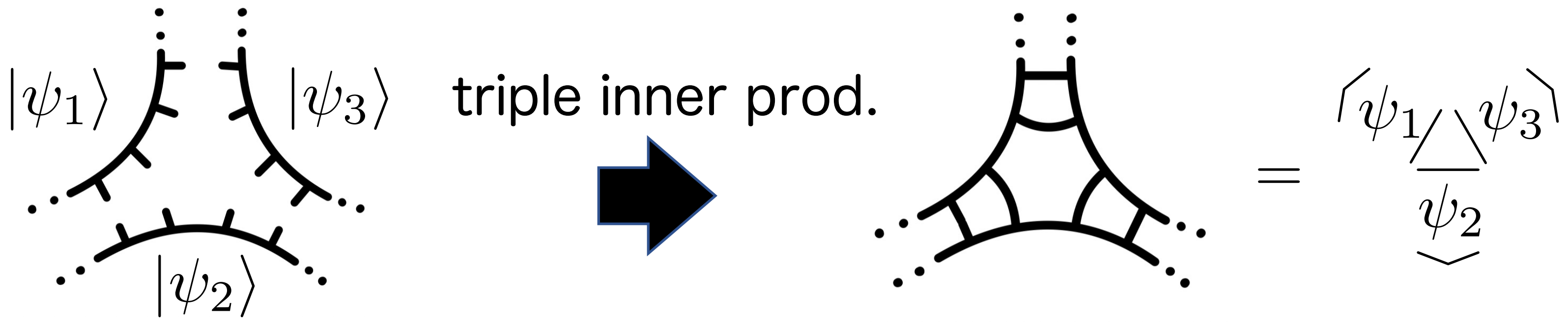

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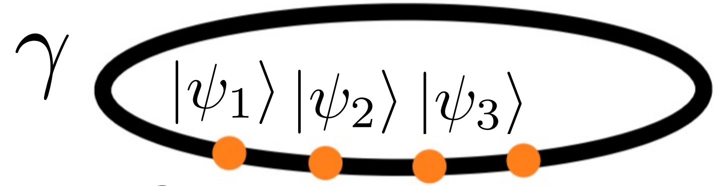

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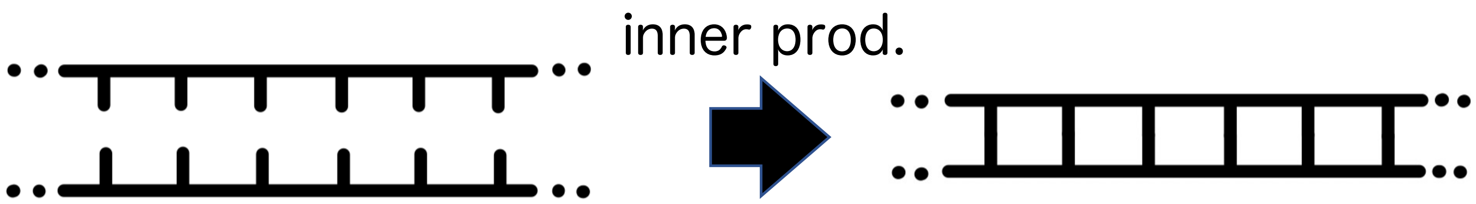

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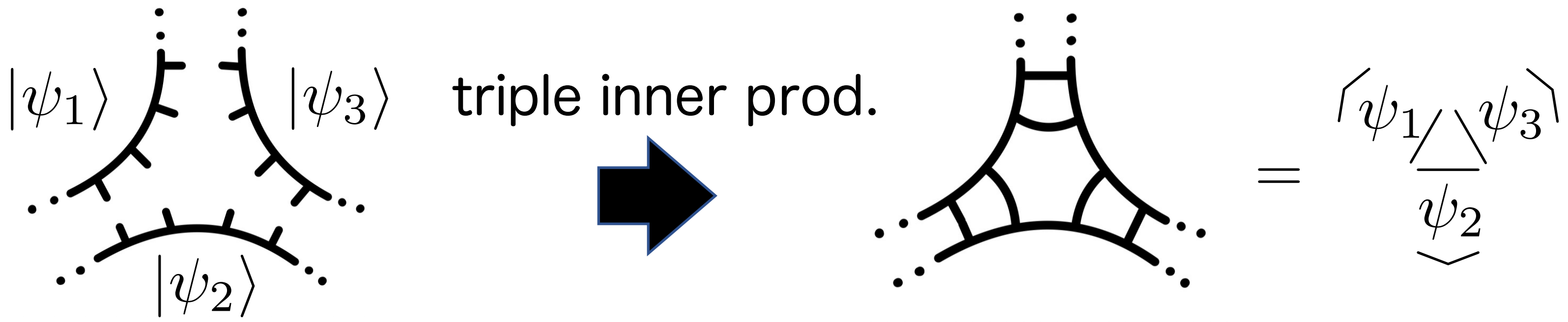

Two implementations: (a) tensor network approach arXiv:2304.05356

(b) boundary conformal field theory arXiv:2507.12525

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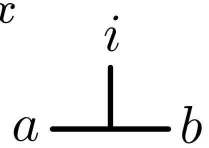
$M_{ab}^i =$ 

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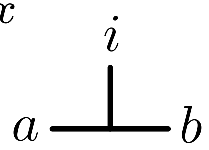
$M^i_{ab} =$ 

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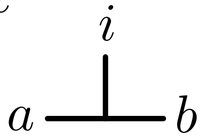
$$M^i_{ab} =$$  $$\longrightarrow \quad |M\rangle_L := \sum_{i_1,\ldots,i_L} \mathrm{tr}(M^{i_1}\cdots M^{i_L})\,|i_1,\ldots,i_L\rangle$$

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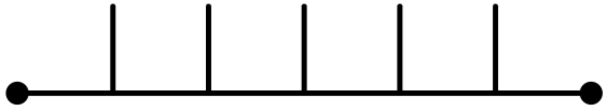
$=$ 

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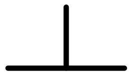

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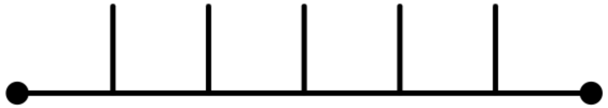

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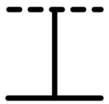

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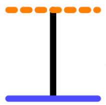

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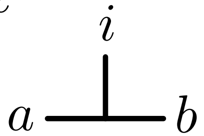
$$M^i_{ab} =$$ 

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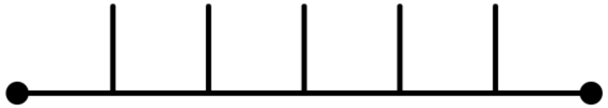
$$=$$ 

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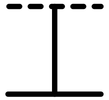
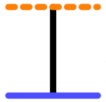
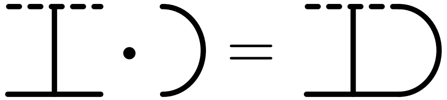
$$T_M = \sum_i M^{i*} \otimes M^i =$$ 

$$T_{MN} = \sum_i M^{i*} \otimes N^i =$$ 

Transfer matrix as an operator:

$$T_M \cdot \Lambda := \sum_i M^i \Lambda M^{i\dagger} \Longleftrightarrow$$ 

Triple inner prod. of MPSs.

[S.O., S. Ryu, Physical Review B 109 (2024) 115152]

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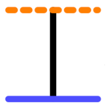
$$T_{MN} = \sum_i M^{i*} \otimes N^i =$$  for two MPS matrices M and N.

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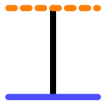
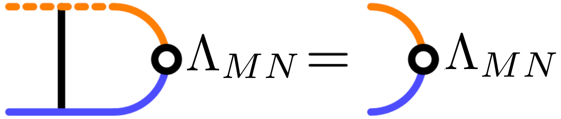
$$T_{MN} = \sum_i M^{i*} \otimes N^i =$$  for two MPS matrices M and N.

Let V_{MN} be a vector space of fixed points of T_{MN} : 

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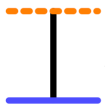
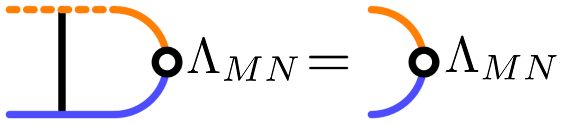
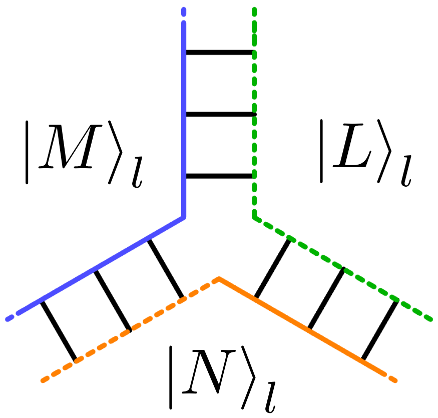
$$T_{MN} = \sum_i M^{i*} \otimes N^i =$$  for two MPS matrices M and N.

Let V_{MN} be a vector space of fixed points of T_{MN} : 



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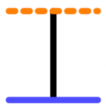
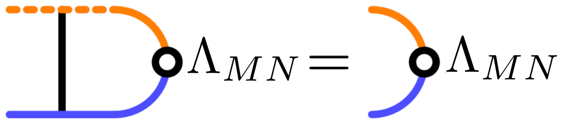
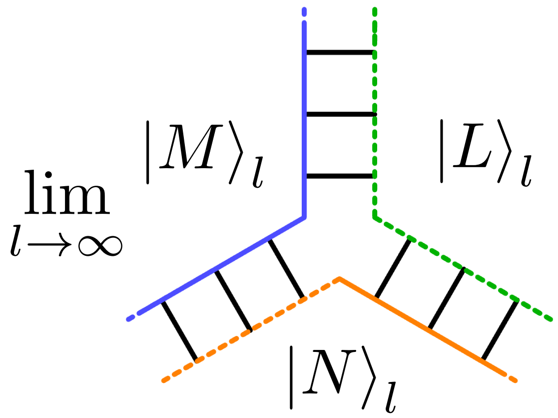
$$T_{MN} = \sum_i M^{i*} \otimes N^i =$$  for two MPS matrices M and N.

Let V_{MN} be a vector space of fixed points of T_{MN} : 



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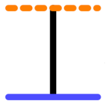
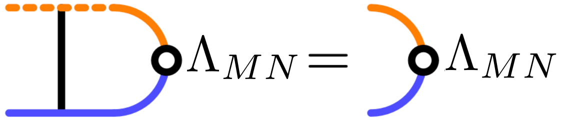
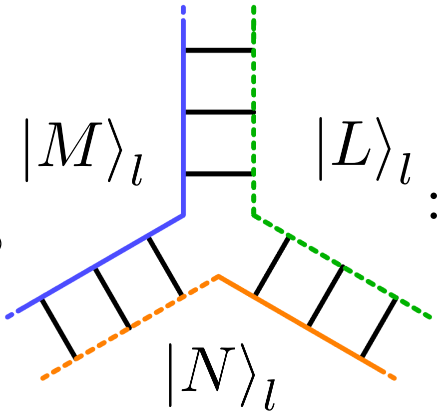
$$T_{MN} = \sum_i M^{i*} \otimes N^i =$$  for two MPS matrices M and N.

Let V_{MN} be a vector space of fixed points of T_{MN} : 

$\lim_{l\to\infty}$ 

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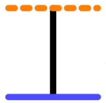
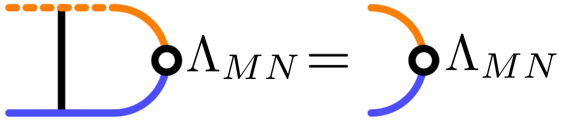
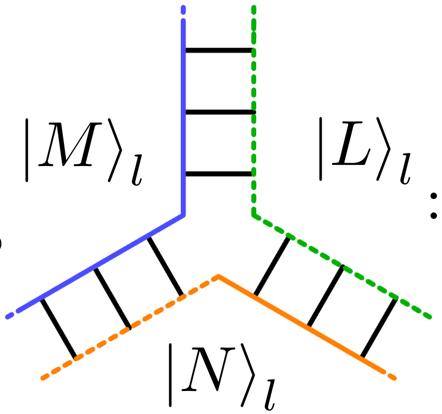
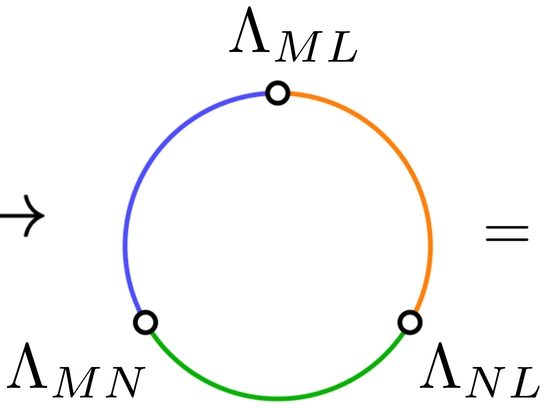
$$T_{MN} = \sum_i M^{i*} \otimes N^i = $$  for two MPS matrices M and N.

Let V_{MN} be a vector space of fixed points of T_{MN} : 

$\lim_{l\to\infty}$ 

$: V_{MN} \otimes V_{NL} \to V_{ML} \longleftrightarrow$ 

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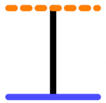
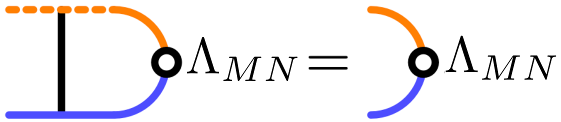
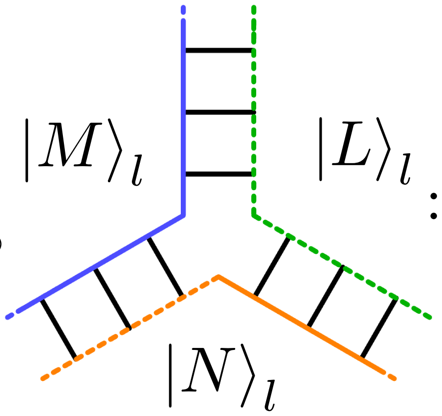
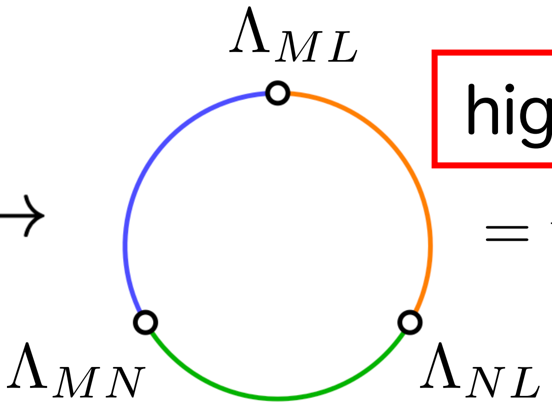
$$T_{MN} = \sum_i M^{i*} \otimes N^i =$$  for two MPS matrices M and N.

Let V_{MN} be a vector space of fixed points of T_{MN} : 

$\lim_{l\to\infty}$ 

$: V_{MN} \otimes V_{NL} \to V_{ML} \longleftrightarrow$ 

higher Berry phase

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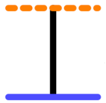
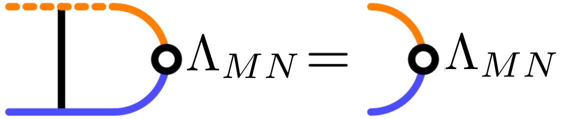
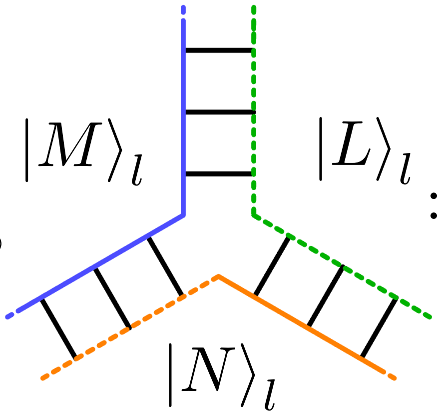
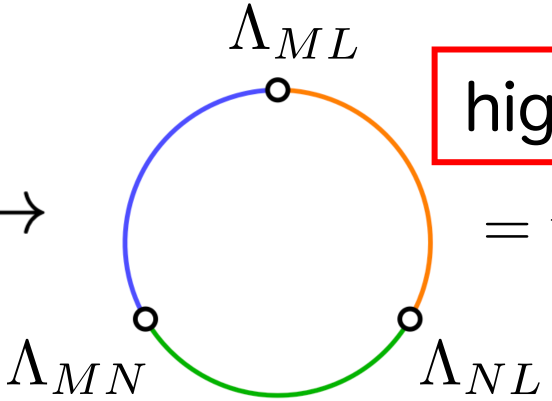
$$T_{MN} = \sum_i M^{i*} \otimes N^i =$$  for two MPS matrices M and N.

Let V_{MN} be a vector space of fixed points of T_{MN} : 

$\lim_{l\to\infty}$ 

$: V_{MN} \otimes V_{NL} \to V_{ML} \longleftrightarrow$ 

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Two applications: 1. gerbe structure over the parameter space X.
2. Fiber functor of C-symmetric SPT phases.

Application 1: Gerbe str.

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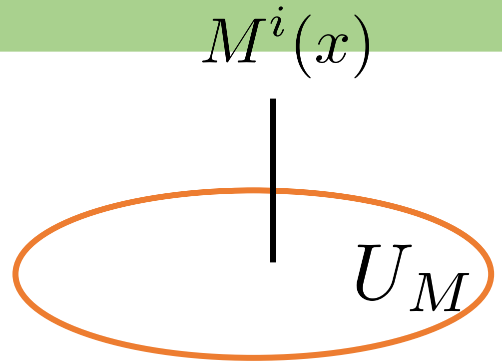

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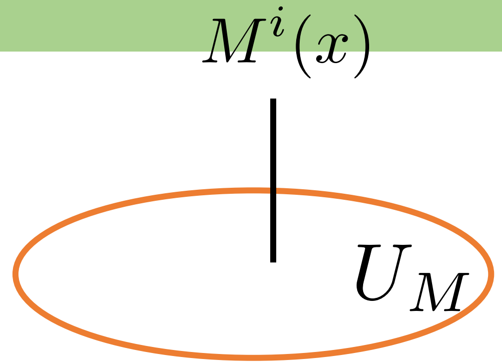

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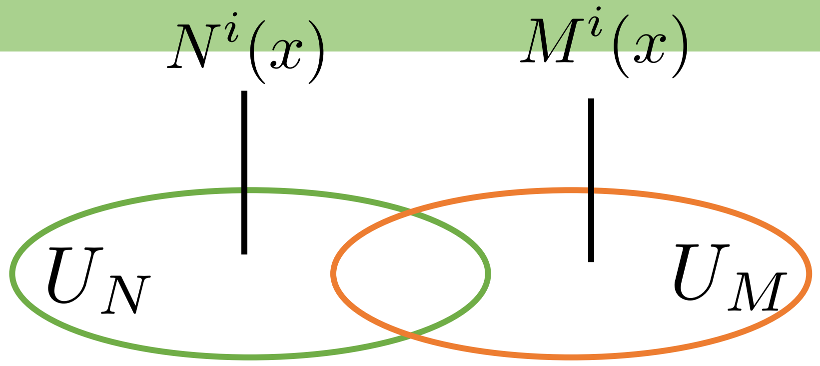

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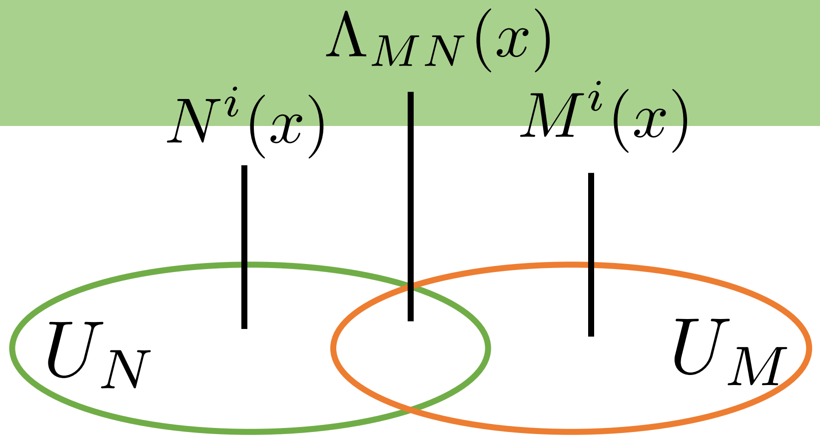

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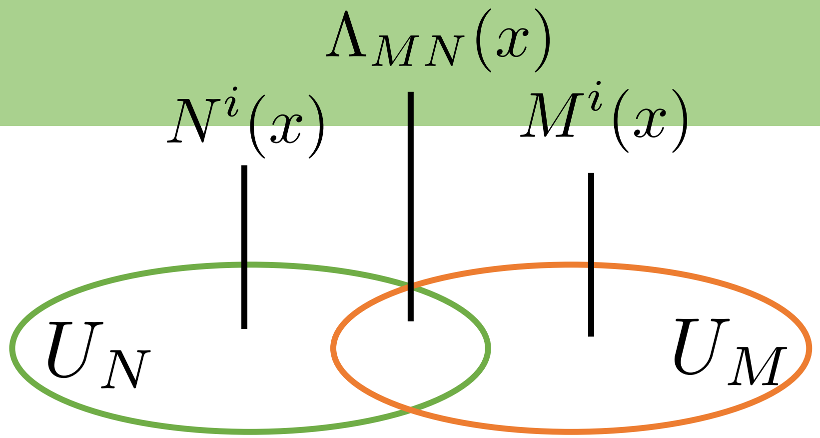

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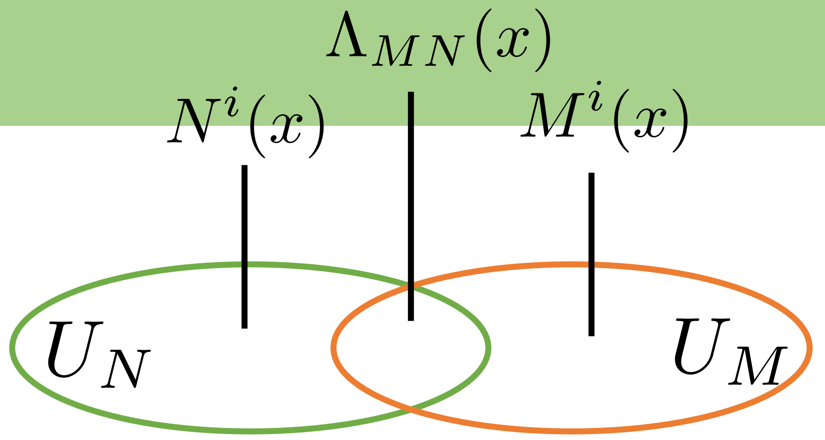

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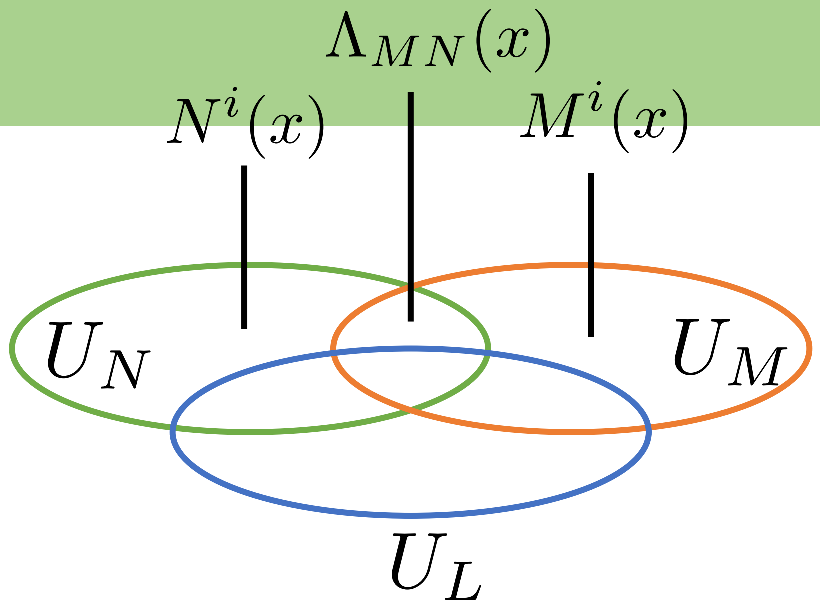

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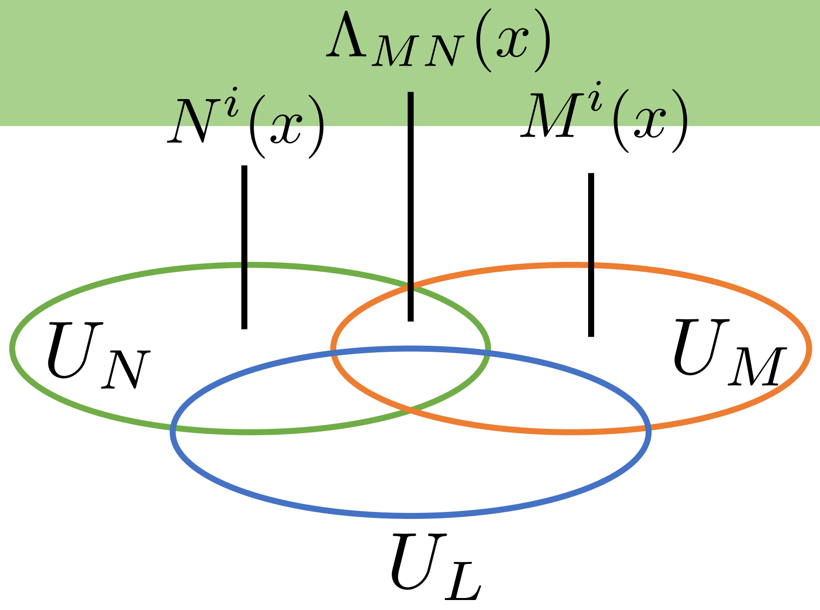

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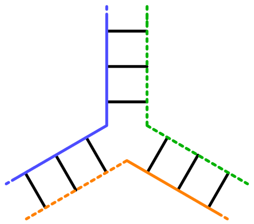
$\lim_{L\to\infty}$  $: V_{MN} \otimes V_{NL} \to V_{ML}$

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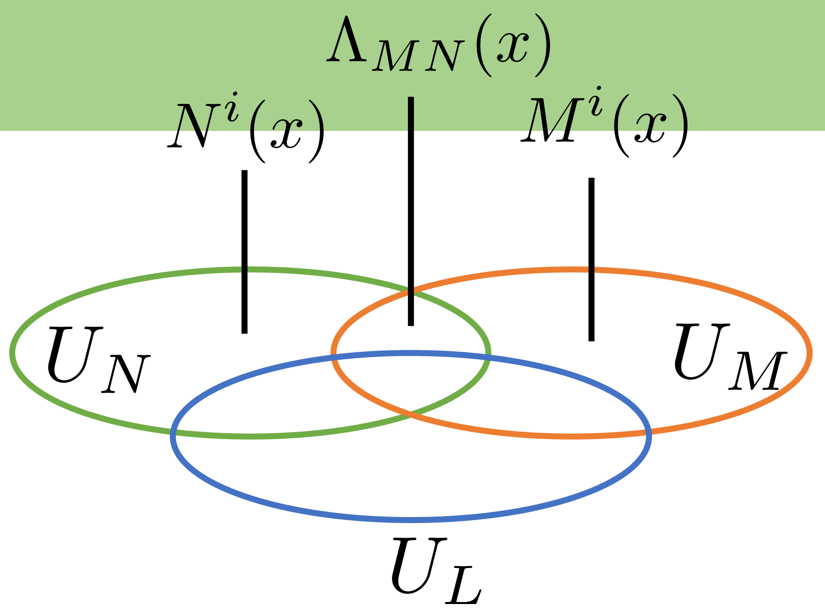

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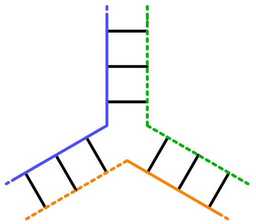
$\lim_{L\to\infty}$  $: V_{MN}\otimes V_{NL}\to V_{ML}\quad\Rightarrow$ gerbe structure over X

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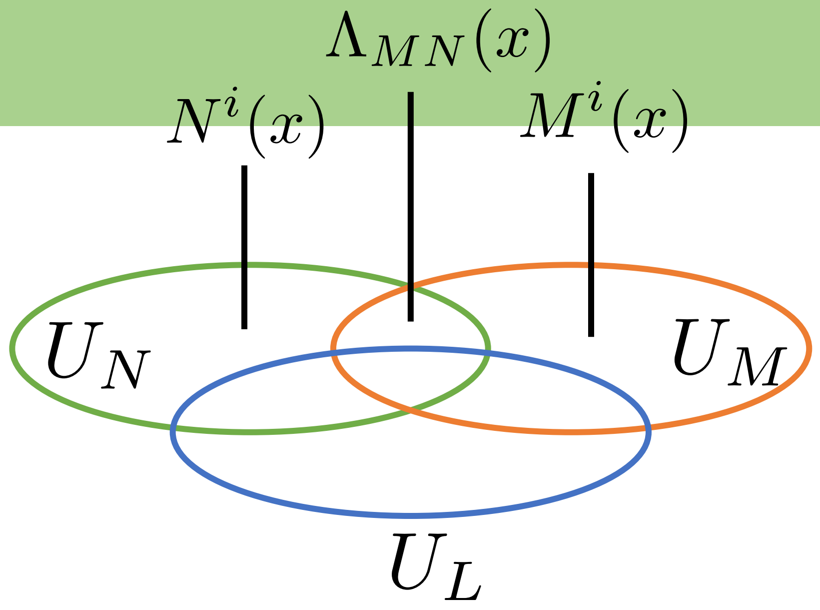

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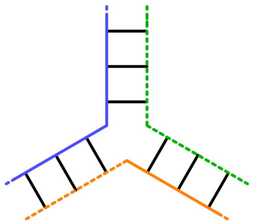
$\lim_{L\to\infty}$  $: V_{MN} \otimes V_{NL} \to V_{ML} \quad \Rightarrow$ gerbe structure over X

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The higher Berry phase is the Diximer-Douady class of the gerbe.

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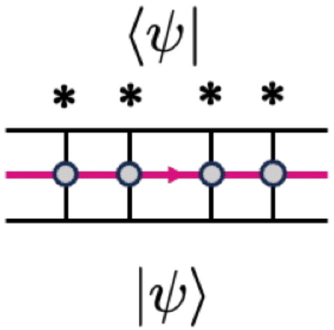

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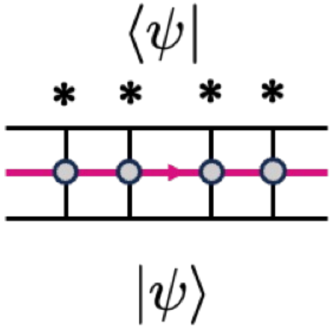

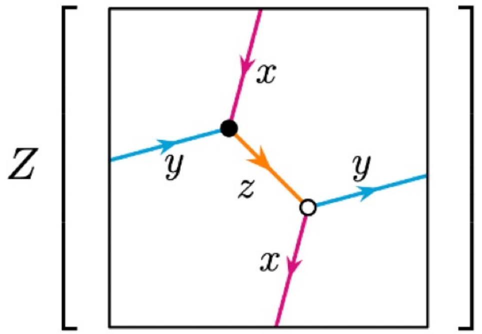

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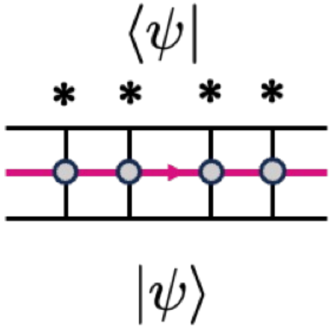

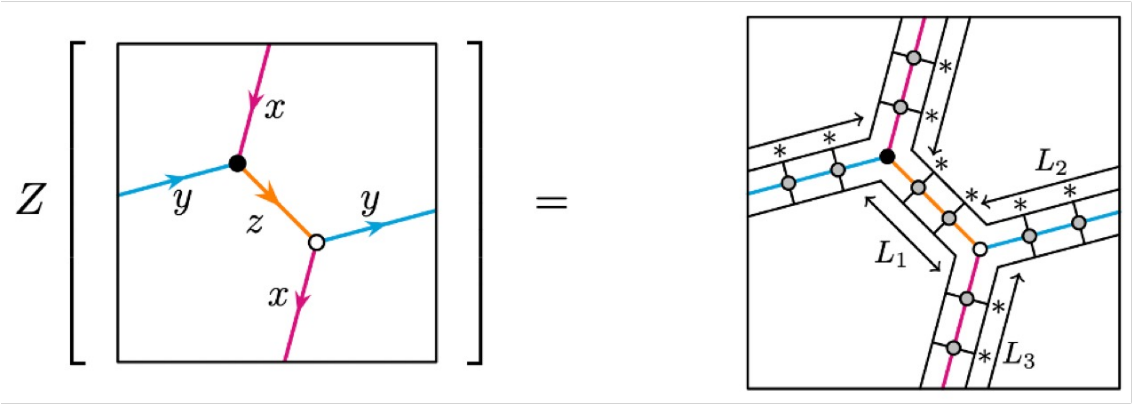

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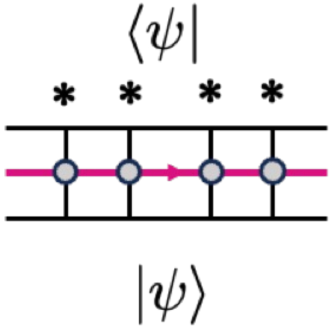

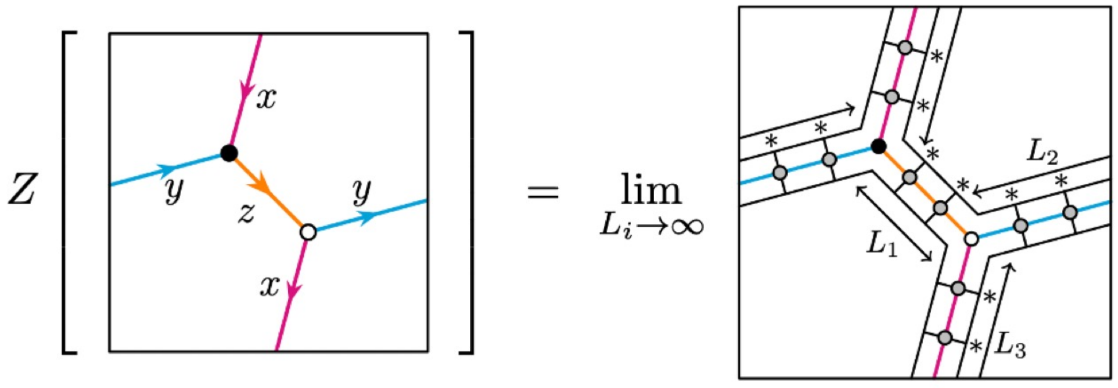

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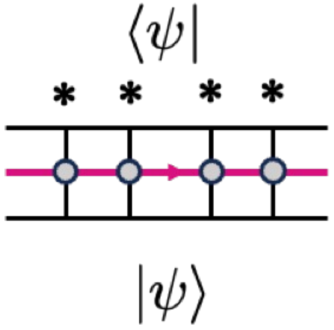

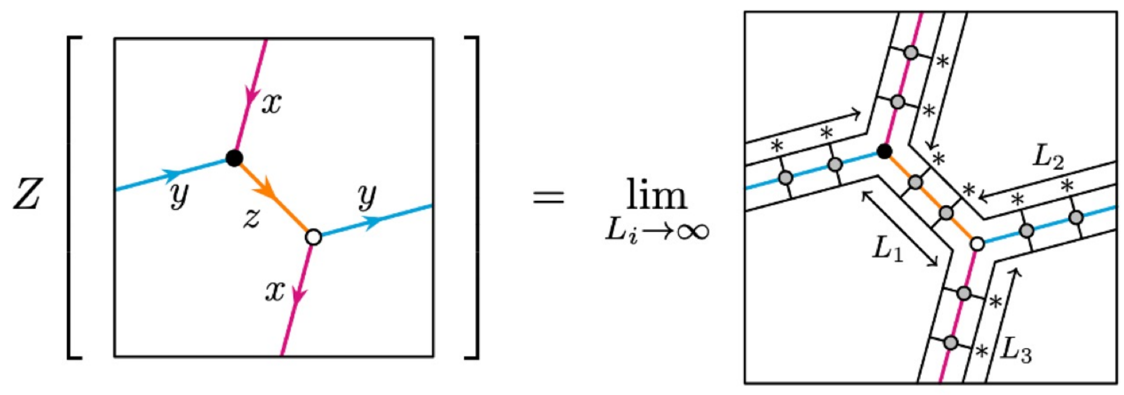

In particular, a fiber functor is computed as follows:

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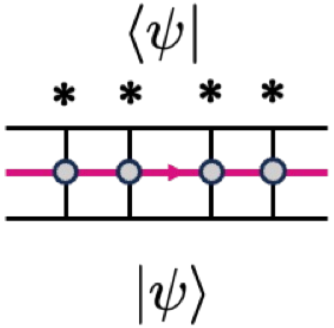

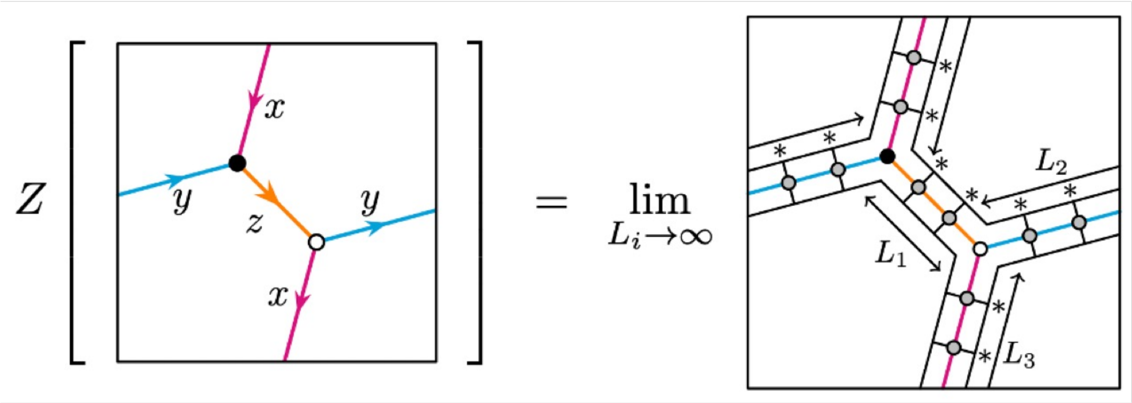

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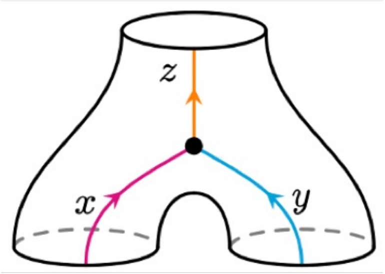

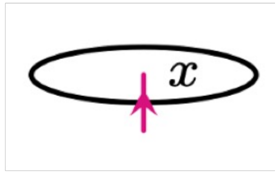

[R. Thorngren et. al., JHEP 04 (2024) 132]

Application 2: Fiber functor

Assume an MPO rep of C-sym:

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Let's consider a $\mathcal{C}$-symmetric SPT phases.

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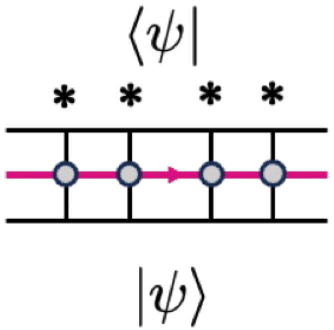

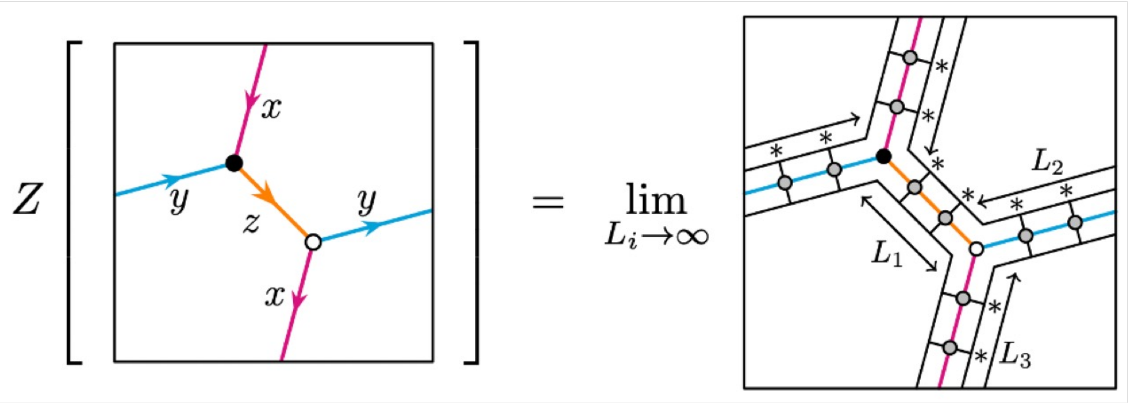

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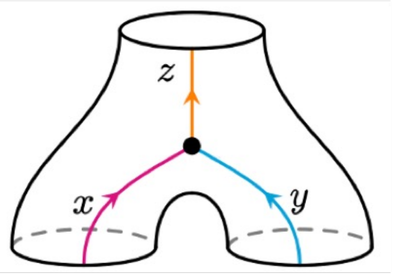

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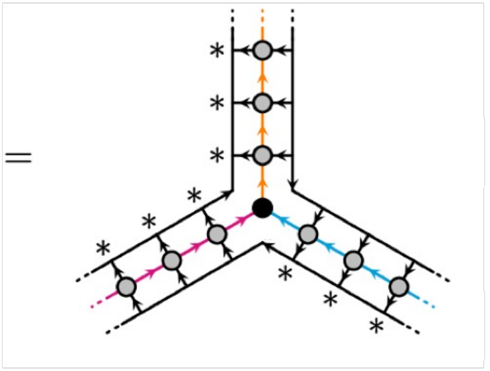
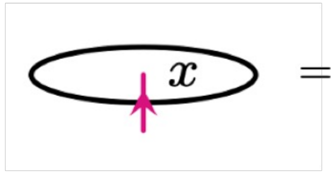
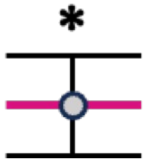




= fixed points of 

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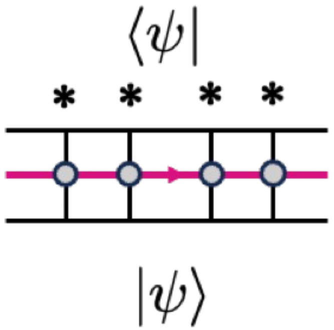

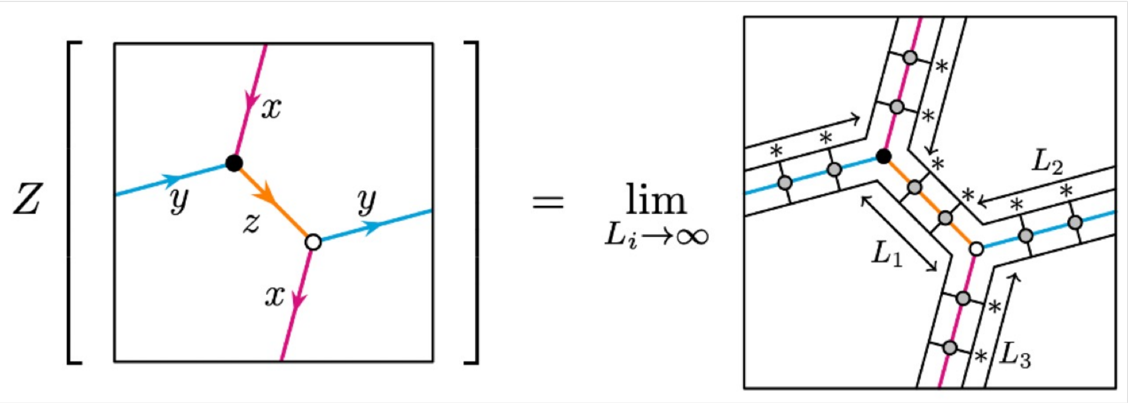

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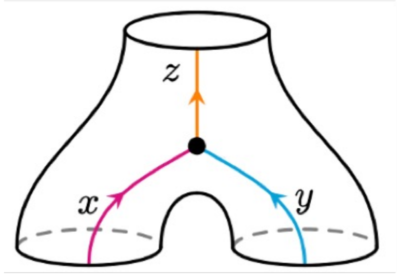

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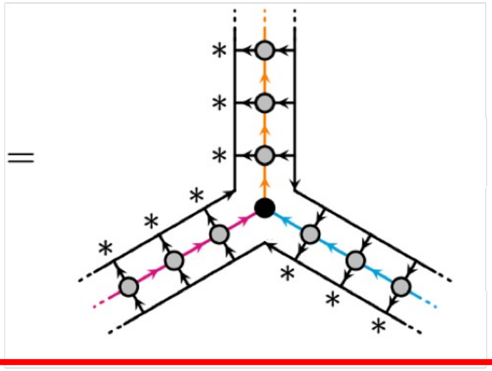
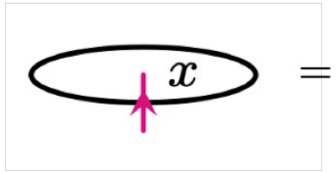
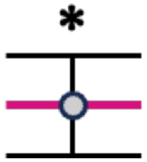




= fixed points of 

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We (partially) justified this method in [Inamura-Ohyama 24].

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Conformal field theory

B_α : boundary condition

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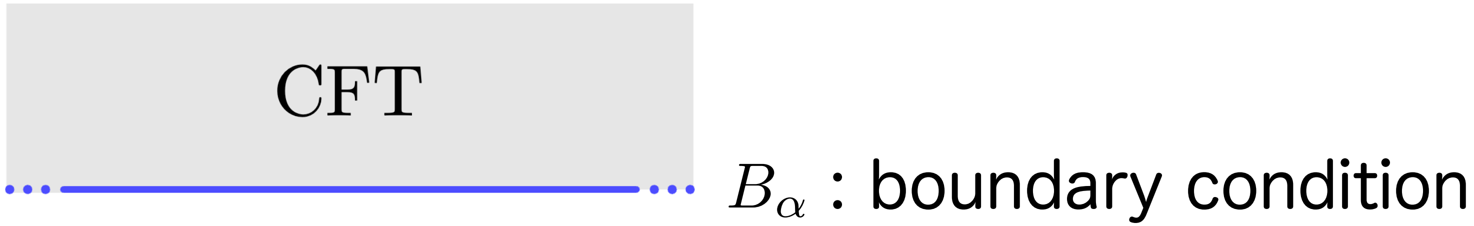
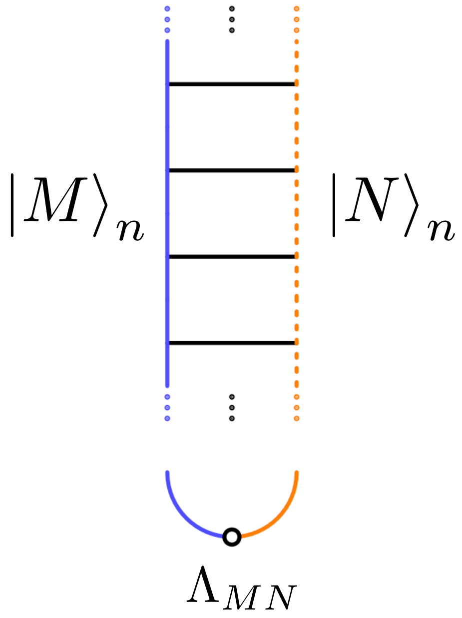


Analogy



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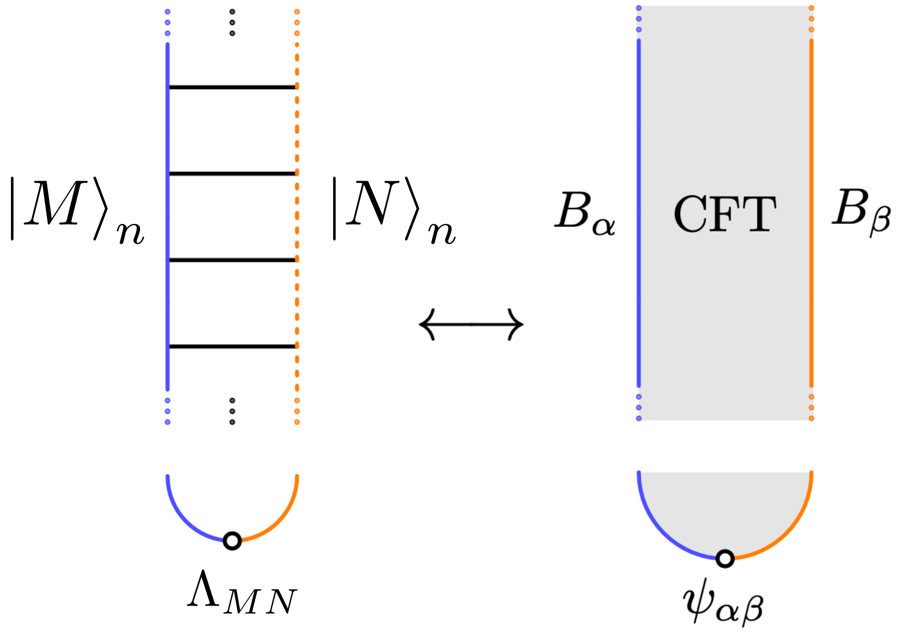

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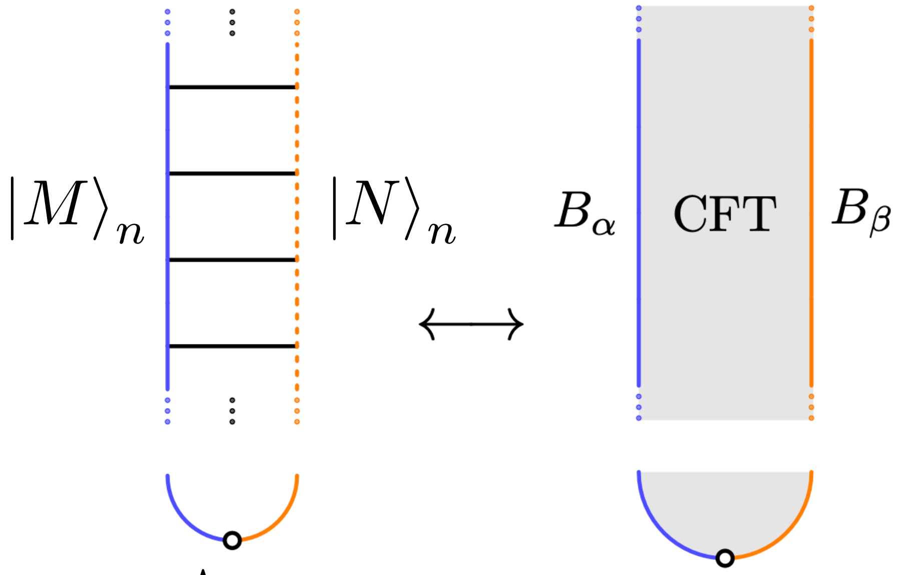

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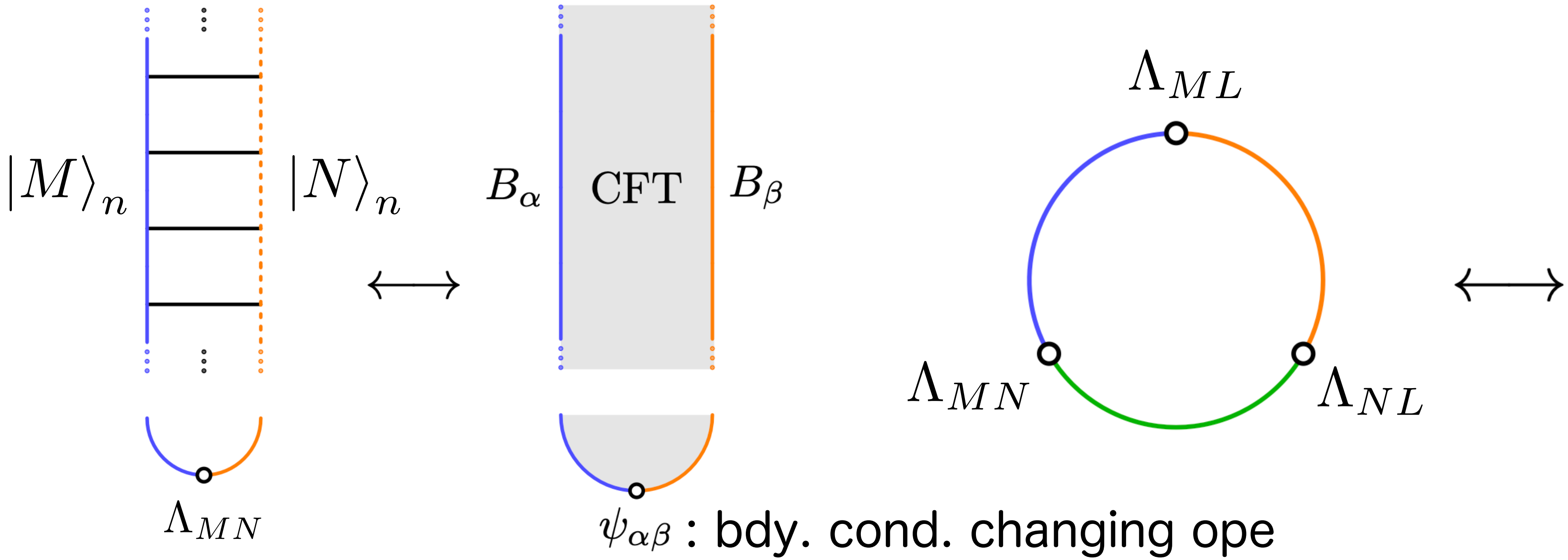

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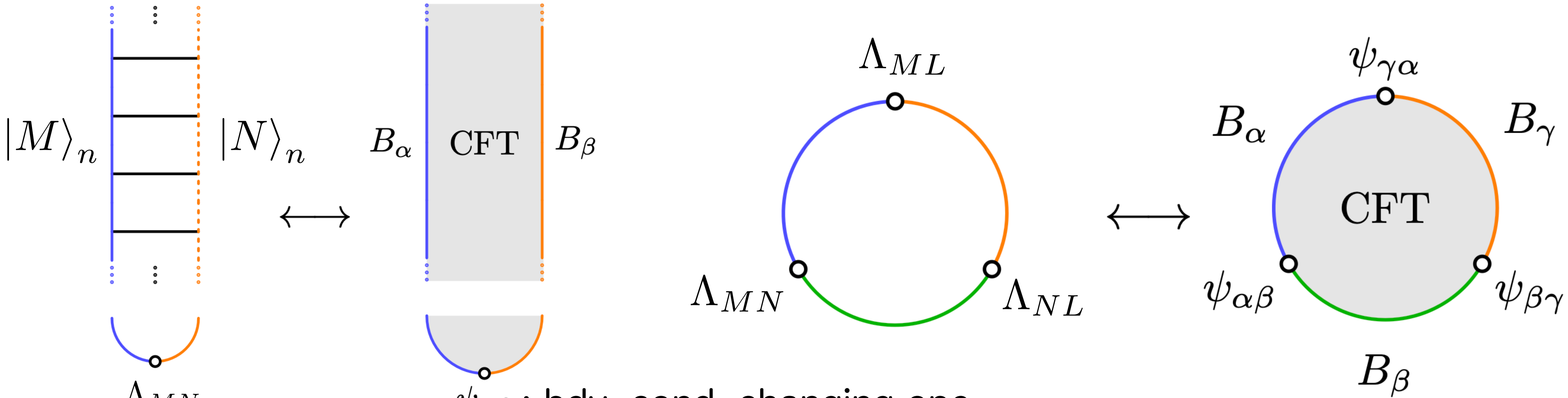

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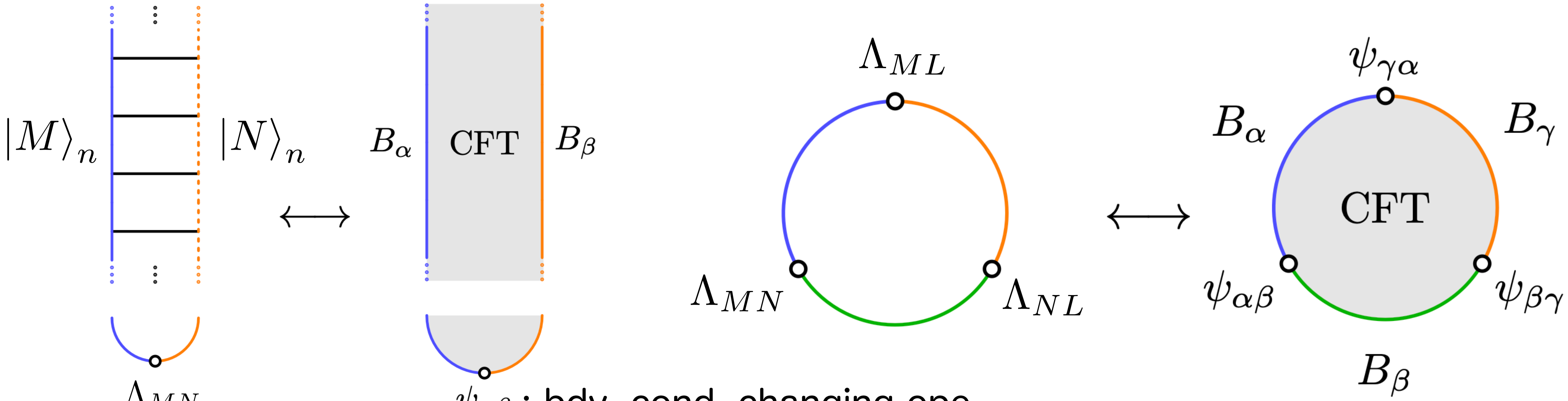

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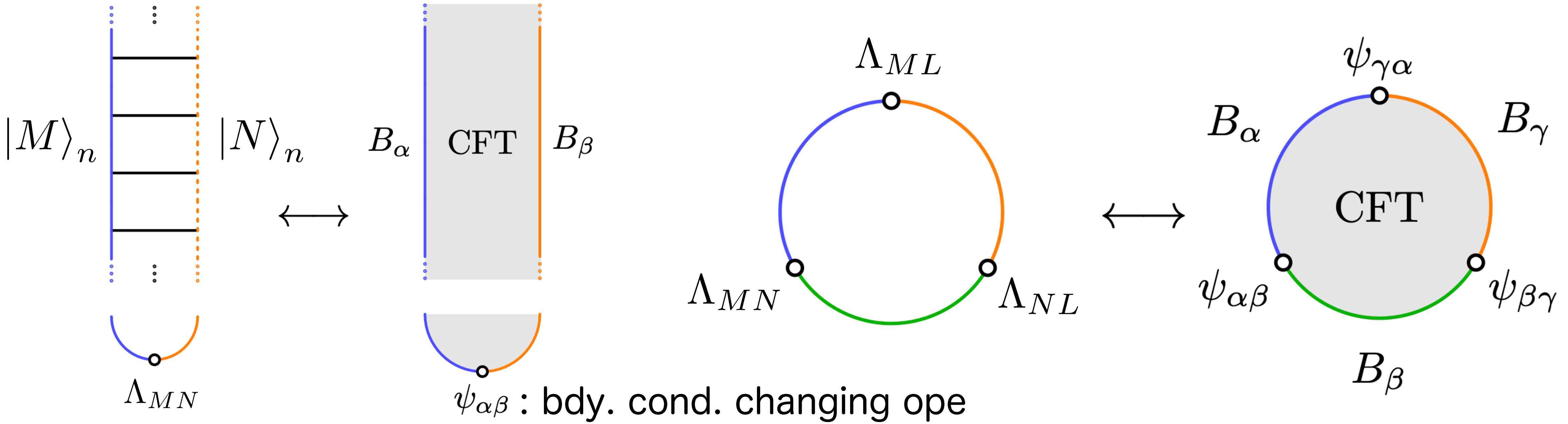

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This defines a “gerbe” str. over boundary conformal manifolds.

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[CHKKOR 25]
[Xueda Wen, 2507.12546]
[Christian Copetti, 2507.15466]

We can compute the higher Berry curvature perturbatively:

$$\mathcal{B} := -i \langle B_\alpha, dB_\alpha, \wedge dB_\alpha \rangle$$

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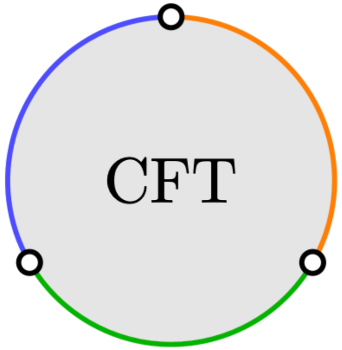

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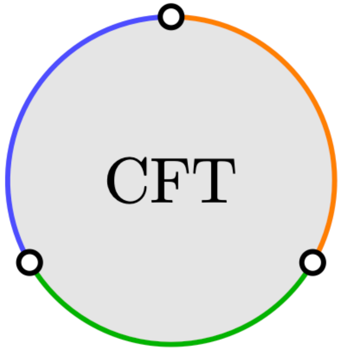

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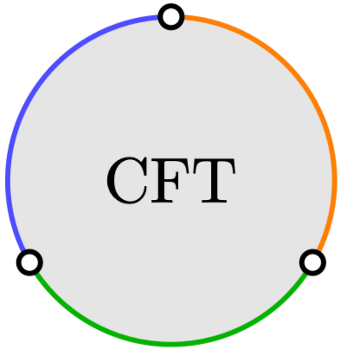

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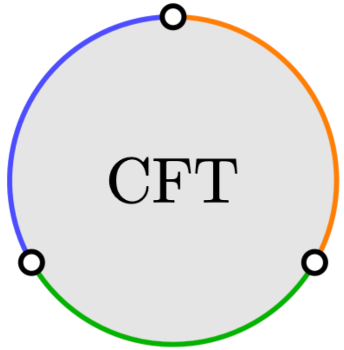

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$\mathcal{B}$ is the 2nd order contribution and $\mathcal{H} = \mathcal{H}_{\mathrm{WZ}}$ $\quad \frac{1}{2\pi}\int_{S^3} \mathcal{H} = k$

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Rem. The orbifold cohomology is computed by Kawasaki: $\mathrm{H}^3_{\mathrm{orb.}}(\mathrm{SU}(2)/\mathbb{Z}_N;\mathbb{Z}) \simeq \mathbb{Z}$

[T. Kawasaki, Mathematische Annalen 206 (1973) 243.]

Loop space formalism

Slogan: Gerbe over $X \longleftrightarrow$ Line bundle over LX

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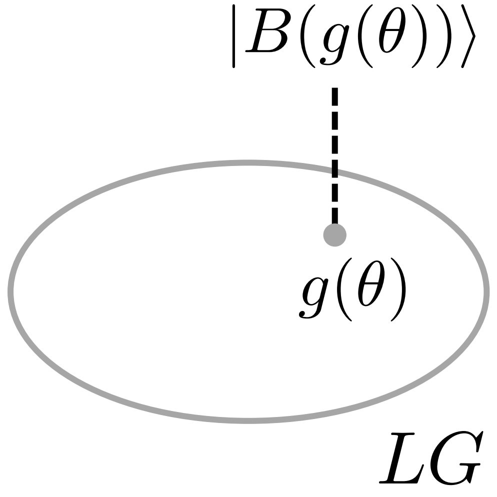

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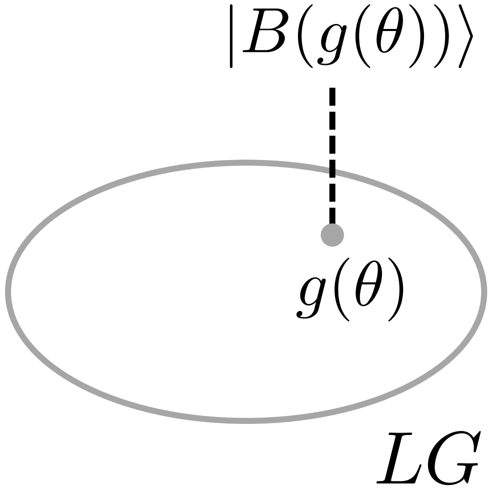

We can compute the ordinary Berry phase for $\mathcal{L}$.

$$\langle B^\epsilon(g(\theta))|B^\epsilon(g(\theta)+\delta g(\theta))\rangle = 1 + iS^k_{\mathrm{WZW}} + \cdots$$

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Loop space formalism

Slogan: Gerbe over $X \longleftrightarrow$ Line bundle over LX

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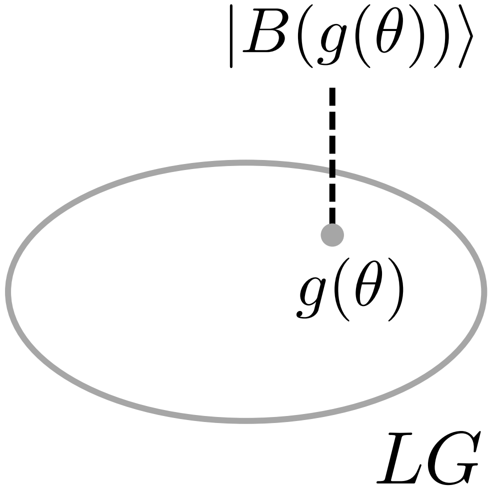

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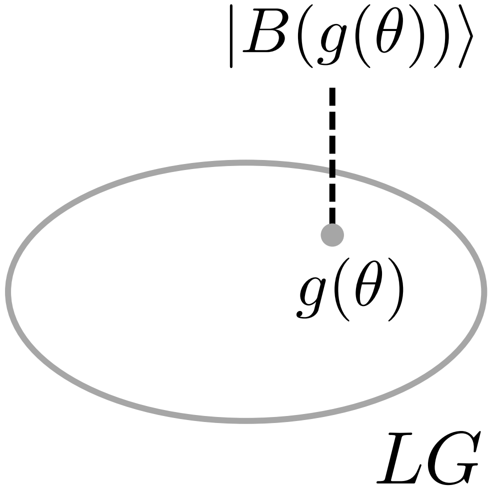

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This is an anomaly of boundary coupling constants.

Summary

Summary

- triple inner product of matrix product states
- triple inner product of boundary states
- line bundle over the loop space

Future work

- Higher structure on conformal manifold
- Spelling out full higher structures on boundary conformal manifolds

Thank you!

Back-up

Injective MPS and Key properties

[D. Perez-Garcia, et.al., Quant. Inf. Comput. 7 (2007) 401]
[D. Pérez-García, et.al., Physical Review Letters 100 (2008)]

Injective MPS matrices:

M^i is injective if the following map is injective as a linear map:

$$\begin{array}{ccc} \mathrm{Mat}_n(\mathbb{C}) & \to & \mathcal{L} \\ X & \mapsto & \sum_{i_1,\ldots,i_L} \mathrm{tr}(XM^{i_1}\cdots M^{i_L})\,|i_1,\ldots,i_L\rangle \end{array}$$

Three important theorems:

(1) An injective MPS matrices generates an invertible state.

(2) Two injective MPS matrices M^i, N^i generate the same state

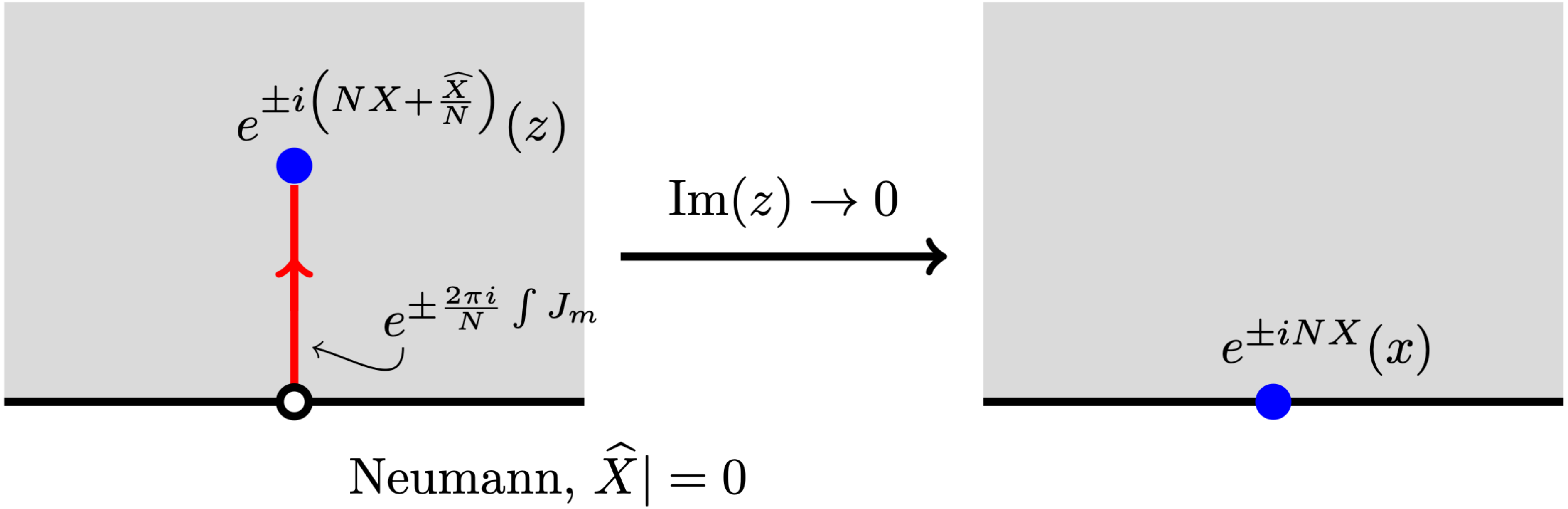
$$\iff \exists ! U \in \mathrm{PU}(n) \text{ s.t. } M^i = UN^iU^\dagger$$

(3) If two injective MPS matrices M^i, N^i generate the same state, the mixed transfer matrix has unique fixed point Λ_{MN}.

Non-chiral deformation

Compact boson

$$J^3(z) = \sqrt{2} i N \partial X(z), \quad J^{\pm}(z) = e^{\pm(iNX + i\frac{\hat{X}}{N})}$$



$$\lim_{\mathrm{Im}(z)\to 0^+}\left[V_{\pm N, \pm\frac{1}{N}}(z) \times \exp\left(\pm\frac{2\pi i}{N}\int_z J_m\right)\right]|B_0\rangle = e^{\pm iNX(x)}|B_0\rangle$$

Perturbative computation

$$
\begin{aligned}
&{}_L\langle 0|\exp\left[i\int_0^{2\pi} d\theta(\omega^a+\delta\omega^a(\theta))J^a(\theta)\right]|0\rangle_L \\
&= {}_L\langle 0|\exp\left[i\left(\omega^a J_0^a+\sum_{m\in\mathbb{Z}}\delta\omega_m^a J_m^a\right)\right]|0\rangle_L \\
&= \sum_{r=0}^{\infty}\frac{i^r}{r!}\,{}_L\langle 0|\left(\omega^{a_1}J_0^{a_1}+\sum_{m_1\in\mathbb{Z}}\delta\omega_{m_1}^{a_1}J_{m_1}^{a_1}\right)\times\cdots\times\left(\omega^{a_r}J_0^{a_r}+\sum_{m_r\in\mathbb{Z}}\delta\omega_{m_r}^{a_r}J_{m_r}^{a_r}\right)|0\rangle_L\,.
\end{aligned}
$$

Discussion: general dimension

	Underlying Geometry	Classification	tensor network
0+1d system	complex line bundle	$\mathrm{H}^2(X;\mathbb{Z})$	—
1+1d system	algebra bundle	$\mathrm{H}^3(X;\mathbb{Z})$	MPS
2+1d system	?? bundle	$\mathrm{H}^4(X;\mathbb{Z})$	PEPS

Conjecture: $\mathrm{K}(\mathbb{Z};d+2)=\mathrm{BAut}(\mathcal{A}_{d+1})$. $\mathcal{A}_{d+1}$: "higher" algebra of a d+1-d TN.

e.g. d=0 $\mathrm{K}(\mathbb{Z};2)=\mathrm{BAut}(\mathbb{C})$, $\mathbb{C}$: cpx. line spand by the ground state.

e.g. d=1 $\mathrm{K}(\mathbb{Z};3)=\mathrm{BAut}(\mathbb{B}(\mathcal{H}))$, $\mathbb{B}(\mathcal{H})$: generated by the MPS matrices.

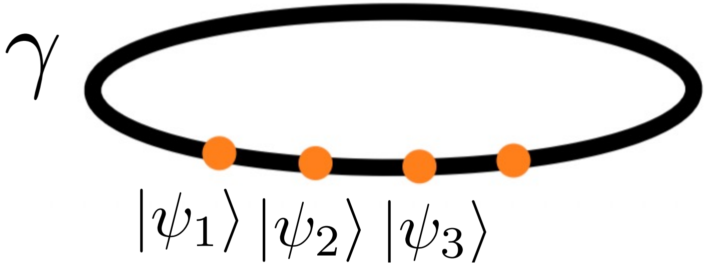
e.g. d=2 $\mathrm{K}(\mathbb{Z};4)\overset{?}{=}\mathrm{BAut}(\mathrm{BiMod}(\mathcal{R}))$, $\mathrm{BiMod}(\mathcal{R})$: cat. of bimodule over $\mathcal{R}$.
($\mathcal{R}$: type-III_1 v.N. alg.)

Can we construct $\mathrm{BiMod}(\mathcal{R})$ from (2+1)-d PEPS?

Remark: Numerical Comp.

The inner product is important to compute the Berry phase.

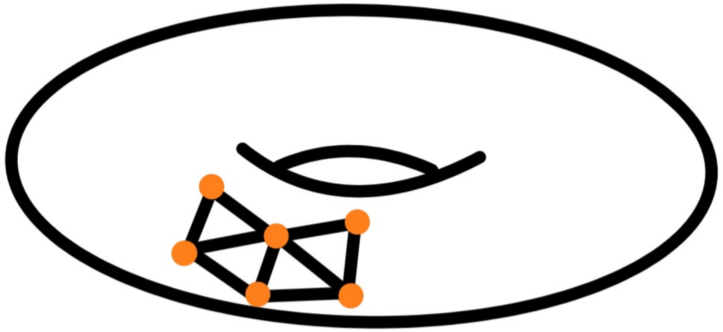
[Fukui, et. al., *J. Phys. Soc. Jap.* 74 (2005).]



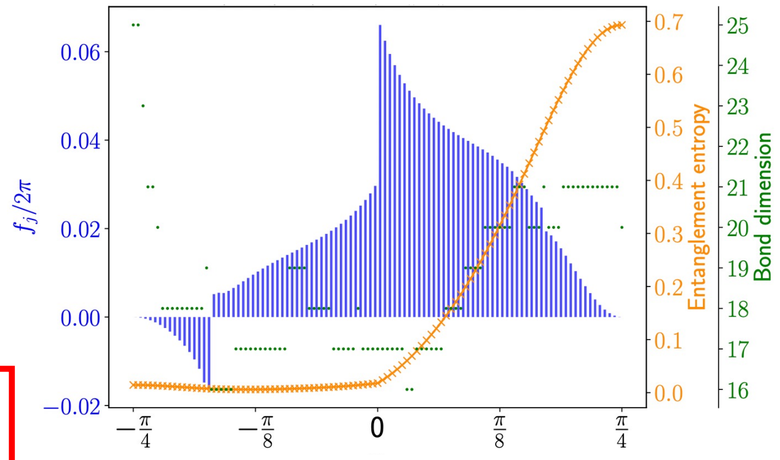
Berry phase ≒ $\langle\psi_1|\psi_2\rangle\cdot\langle\psi_2|\psi_3\rangle\cdots\langle\psi_L|\psi_1\rangle$

We can compute the Berry phase numerically.

We generalized this method to the higher Berry phase.



Higher Berry phase ≒ $\prod_{\triangle}$ (Triple Inner Prod. of $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$)



Higher Berry phase= 1.00029

[K. Shiozaki, N. Heinsdorf, S. O., Phys.Rev.B 112 (2025) 3, 035154]

The total amount ends up with being quantized to 1.