# Symmetry constraints on defect RG flows

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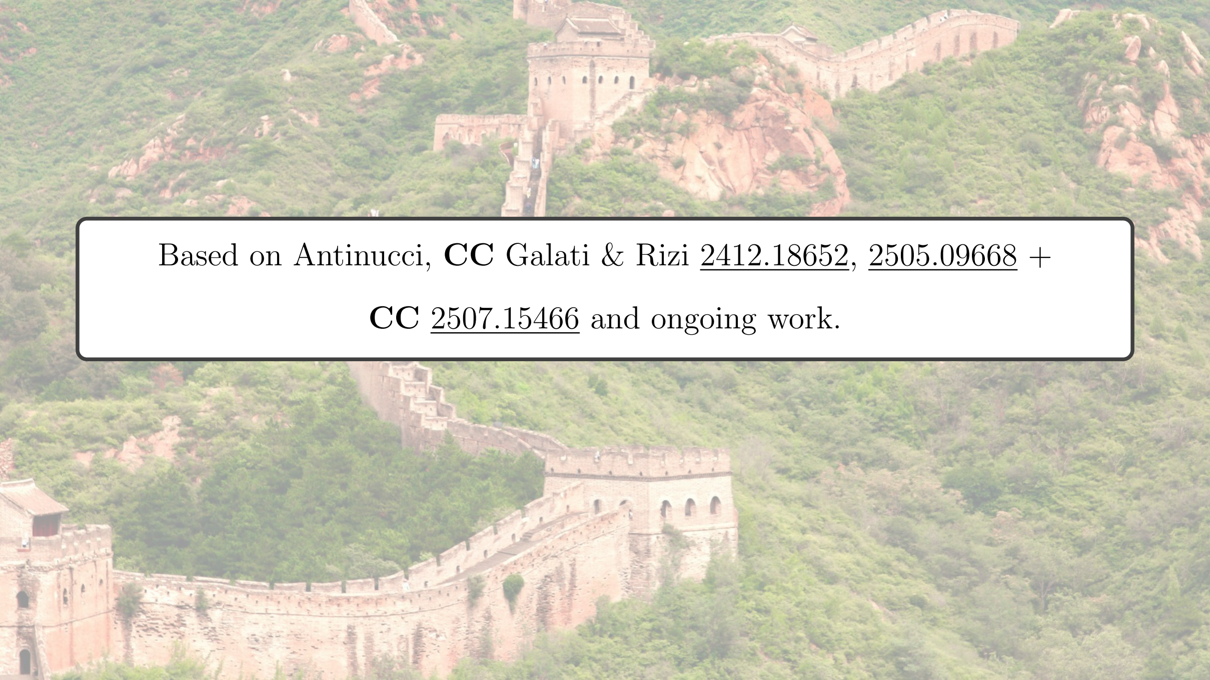
Generalized symmetries in HEP and CMP
**PKU**

Science and Technology Facilities Council



Based on Antinucci, **CC** Galati & Rizi 2412.18652, 2505.09668 +

**CC** 2507.15466 and ongoing work.

# Why Defects?

Defects and their RG flows are ubiquitous in Physics:

HEP-TH
- Wilson lines and 't Hooft operators in gauge theories[Polchinski,Sully '11] [Aharony,Cuomo,Komargodski,Mezei,Raviv-Moshe'23] .
- Pinning field defects in $O(N)$ CFT[Cuomo,Komargodski,Mezei '21 + Raviv-Moshe '22] [Raviv-Moshe, Zhong ´23] [Giombi,Liu '23] .
- Domain walls in SSB scenarios.
- Monodromy defects for free theories[Bianchi,Chalabi,Prochazka,Robinson,Sisti '21] [Giombi,Helfenberger,Ji,Khanchandani '21] [Herzog,Shresta '22] .

COND-MAT
- Lattice impurities (Kondo problem)[Anderson'70,Wilson'75,Affleck,Ludwig'90...]
- Disclocations and Disclination [Barkeshli,Fechisin,Komargodski,Zhong '25] .
- Pinning defects in ferromagnets [Assaad,Herbut '13] [Parisen,Assaad,Wessell '16] .

GEN-SYM
- Topological defects describe Generalized Symmetries [Gaiotto,Kapustin,Seiberg,Willet'14] ...

The List goes on...

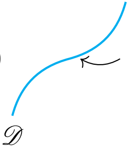
Window into strongly coupled dynamics (e.g. confinement).

Bulk-defect systems are inherently strongly coupled → few analytic results.

# How defects are defined

**[Electric]:**

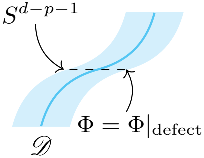
$$S_{\text{bulk}} = \int d^d\mathbf{x}\, \mathscr{L}_{\text{bulk}}(\Phi)$$




$$\int d^p x \,(\, \mathscr{L}(\varphi) + F(\Phi, \varphi))$$

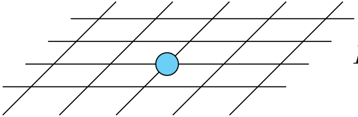
**Ex.** Wilson lines $\mathscr{D} = P \exp\left(i \int A\right)$, $O(N)$ defect $\mathscr{D} = \exp\left(n_i \int \phi^i\right)$ ...

**[Magnetic]:**




**Ex.** 't Hooft (disorder) operators $\frac{1}{2\pi}\int_{S^2} F = 1$.

**[Impurities]:**



$$H = H_0 + H_{\text{impurity}}$$

**Ex.** Kondo problem.

# Defect RG flows

We will focus on the IR fate $\mathscr{D}_{IR}$ of a UV defect/impurity.

The following are common scenarios:

**Screening** Bulk and defect decouple completely $\mathscr{D}_{IR} = \mathbb{1}_p$.

**Conformal** $\mathscr{D}_{IR}$ preserves $SO(2,p) \times SO(d-p)$ conformal group with a single vacuum. [Billó,Gonçalves,Lauria,Meineri '16]

**Topological** $\mathscr{D}_{IR}$ is a nontrivial topological defect in the theory.

**This Talk:** If the bulk has a symmetry $\mathcal{C}$, does it constrain $\mathscr{D}_{IR}$?

⚠ Common setup: bulk CFT fixed → **Defect RG flow**. Our results hold regardless of this assumption, provided we assume that $\mathcal{C}$ acts **faithfully** along the RG.

# Symmetry & Defects I: Symmetric defects

Consider a bulk system with symmetry $G$. For concreteness $G = U(1)$.

In the presence of a defect $\mathscr{D}$ the Ward identities for the $G$ current are modified: [Padayasi,Krishnan,Metlitski,Gruzberg,Meineri '21] [Drukker,Kong,Sakkas '22] [Herzog,Schaub '23] [CC,DiPietro,Ji,Komatsu '23] [Cuomo,Zhang '23] :

$$\partial_\mu J^\mu(\mathbf{x}) = t(x)\,\delta(\Sigma_{\mathscr{D}})\,,$$

a nontrivial **tilt operator** $t(x)$ signals symmetry breaking by the defect.

In order for $\mathscr{D}$ to preserve the symmetry we will need the tilt to trivialize:

$$t(x) = \partial_a\, j^a(x)\,, \quad \text{for some defect current } j^a\,.$$

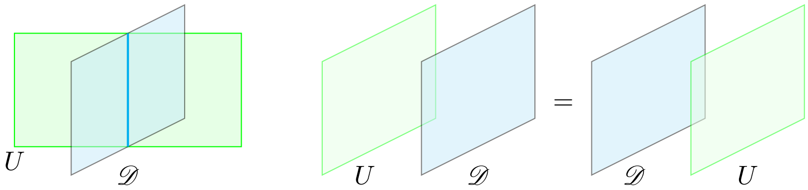
In this case we'll say that $\mathscr{D}$ is **symmetric** wrt $G$.

A symmetric defects allows for an improvement of the symmetry generator $U_\alpha(Y) = \exp\left(i\alpha \int_Y \star J\right)$:

$$U_\alpha(Y) \longrightarrow U_\alpha(Y)\ \exp\left(-i\alpha \int_{Y\cap\Sigma_p} \star j\right)$$

Such that $U_\alpha(Y)$ remains topological in the presence of $\mathscr{D}$. In other words:

$$U_\alpha\ \mathscr{D} = \mathscr{D}\ U_\alpha$$




We can then carry out many of the familiar hep-th procedures, such as turning on gauge fields for the defect symmetry $G$.

A defect $\mathscr{D}$ being **symmetric** does not itself give strong constraints on defect RG.

This follows from the fact that the identity $\mathbb{1}_p$ itself is a symmetric defect:

$$\mathbb{1}_p \, U = U \, \mathbb{1}_p \, .$$

# Symmetry & Defects II: Symmetry-Reflecting defects

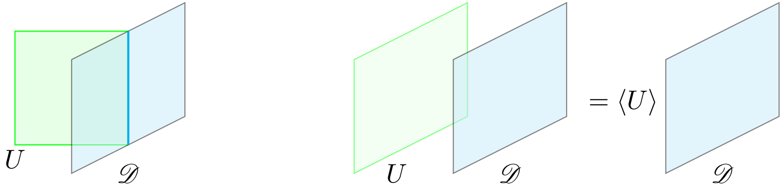
A natural generalization of this concept is what we call **symmetry reflecting defects**:[1]

$$U\,\mathscr{D} = \mathscr{D}\,U = \langle U \rangle\,\mathscr{D}\,.$$

The symmetry defects are **absorbed** by $\mathscr{D}$. For concreteness we focus on $p = d-1$. In terms of the current $J$ it means that, on the defect's worldvolume:

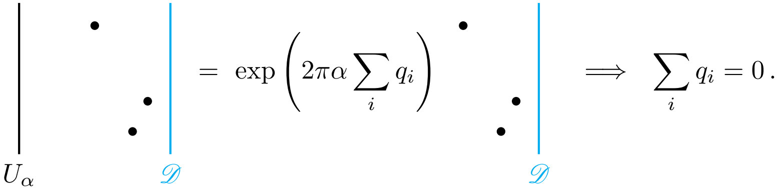
$$J_\perp(x) = \partial_a\,\eta^a(x)\,.$$

Alternatively, the topological operator $U$ can terminate **topologically** on $\mathscr{D}$.

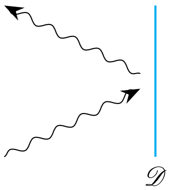



[1] A similar concept for boundary conditions appeared in [Choi,Rayhaun,Sanghavi,Shao '23]

A symmetry reflecting interface preserves the $G$ symmetry **independently** on the two sides. The total symmetry in this case is **at least** $G_L \times G_R$.



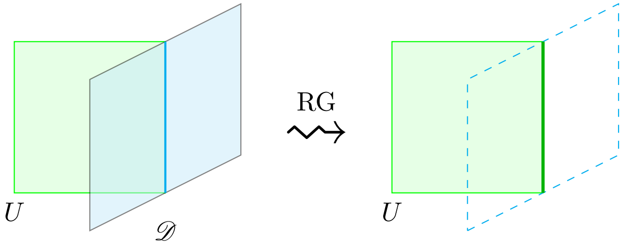
All of the symmetry charge scattering on $\mathscr{D}$ is thus reflected back. $\mathscr{D}$ acts as an **hard wall** for charged objects.




Similar ideas can be formulated using the Defect OPE of charged bulk fields.

# Consequences

A symmetry reflecting defect cannot be screened in the IR.




The IR fixed point can either be:

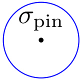
- A nontrivial symmetry reflecting conformal defect.
- A nontrivial (and non-invertible) topological defect.
- A theory of Defect Goldstone modes.

## Example: Deforming Topological defects

A wide class of conformal defects are obtained by the "pinning field" construction:

$$\mathscr{D} =$$  $$+ \lambda \int d^p x \; \sigma_{\text{pin}} \, , \qquad \Delta(\sigma_{\text{pin}}) < p \, .$$

$\mathbb{1}_p$

These defects are symmetric if $U$  $= \overset{\sigma_{\text{pin}}}{\bullet}$ .

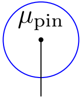


A symmetry reflecting defect can be constructed in a similar manner by deforming a topological defect $\mathscr{N}$ (related ideas [Kormos,Runkel,Watts '09] [Makabe, Watts '17] ):

$$\mathcal{L}\,\mathscr{N} = \mathscr{N}\,\mathcal{L} = d_{\mathcal{L}}\,\mathscr{N}$$

$$\mathscr{D} =$$  $$+ \lambda \int d^p x \, \mu_{\text{pin}} \, ,$$

$\mathscr{N}$

$\mathcal{L}$  $=$  . $\qquad \forall \; \mathcal{L} \in \mathcal{C} \, .$



Interestingly, $\mu_{\text{pin}}$ can be a **nonlocal (twisted)** operator living at the end of an $\mathcal{L}$ line.

**Example**:

$$\text{(1+1)d Ising CFT}\,,\;\; G=\mathbb{Z}_2: \qquad \mathscr{D} = \underset{\color{blue}{1+\eta}}{\color{blue}{\Bigg|}} \; + \lambda \int dx\, \mu_{\frac{1}{16},\frac{1}{16}}\,.$$

The flow can be "bootstrapped" exactly:

$$\underset{\text{KW duality}}{\mathcal{N}} \times \mathscr{D} = \left( \underset{\mathbb{1}_p}{\Bigg\vdots} \; + \lambda \int dx\; \sigma_{\frac{1}{16},\frac{1}{16}} \;\oplus\; \underset{\mathbb{1}_p}{\Bigg\vdots} \; - \lambda \int dx\; \sigma_{\frac{1}{16},\frac{1}{16}} \right) \times \mathcal{N}\,.$$

The term in () brackets flows to $|+\rangle\langle+| \oplus |-\rangle\langle-|$ where $\{|+\rangle, |-\rangle, |f\rangle\}$ are the Cardy states for Ising. Using $\mathcal{N}|\pm\rangle = |f\rangle$, $\mathcal{N}|f\rangle = |+\rangle + |-\rangle$, we conclude that:

$$\mathscr{D}_{IR} = |f\rangle\langle f|\,.$$

# Symmetry and defects III: Folding and Phantom Symmetry

Breaking of vanilla $G$ symmetry bestows several properties on $\mathcal{M}_{\mathscr{D}}$:

(a) $\mathcal{M}_{\mathscr{D}} \simeq G/H$ is an homogeneous space. (Every point on $\mathcal{M}_{\mathscr{D}}$ is equivalent).

(b) The defect free energy $g = \langle \mathscr{D} \rangle$ remains constant on $\mathcal{M}_{\mathscr{D}}$. ($J_0|0\rangle = 0$)

(c) The reflection coefficient [Quella,Runkel,Watts '06] $\mathcal{R}_{\mathscr{D}} \sim \frac{1}{c}\,(1 - \langle T_L\, T_R \rangle_{\mathscr{D}})$ is also constant on $\mathcal{M}_{\mathscr{D}}$. (The symmetry commutes with the stress tensor).

# A mysterious case

Conformal defects classified in a single instance: the (1+1)d Ising CFT [Affleck,Oshikawa '96] .

They come in two **continuous** families:

$$\left\{(D^+, \theta)\,, \quad \theta \in [0, \pi]\right\}, \quad \left\{(N^+, \theta')\,, \quad \theta' \in [0, \pi/2]\right\}.$$

(1) The interval $S^1/\mathbb{Z}_2^C$ is not homogeneous.

(2) Topological lines of Ising are $\mathbb{1} = (D^+, \pi/4)$ , $\eta = (D^+, 3\pi/4)$, $\mathcal{N} = (N^+, \pi/4)$.

(3) $g_{(D^+,\theta)} = 1$, $g_{(N^+,\theta')} = \sqrt{2}$, but $\mathcal{R}_{(D^+,\theta)} = \cos^2(2\theta)$, $\mathcal{R}_{(N^+,\theta')} = \cos^2(2\theta')$.

These families do form defect conformal manifolds.

**However**:

(a) The Ising CFT has **no continuous symmetry** which can be broken.

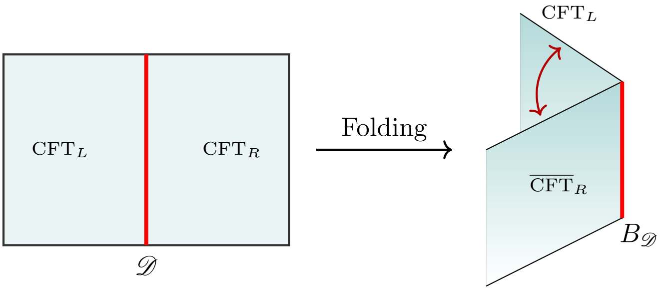
(b) (1) and (3) are *not* allowed by the symmetry breaking physics.

**Question**:

(1) Is there any mechanism guaranteeing these defect conformal manifolds?

(2) Are these cases common or fine tuning?

We can describe Ising defects as conformal boundaries in Ising$^2$, via the folding trick:



It is well known that:

$$\text{Ising}^2 \qquad c = 1 \,\text{on orbifold branch}, \; R_{\text{orb}} = \sqrt{2}\,.$$

This theory has a **continuous**, **non-invertible** cosine symmetry [Thorngren,Wang '21]

$$L^{(m)}_{\theta} = 2\cos\left(-2\theta \int \frac{\star dX}{2\pi}\right), \quad L^{(w)}_{\theta'} = 2\cos\left(2i\theta' \int \frac{dX}{2\pi}\right) \,.$$

$$L^{(m/w)}_{\theta_1} \times L^{(m/w)}_{\theta_2} = L^{(m/w)}_{\theta_1+\theta_2} + L^{(m/w)}_{\theta_1-\theta_2}\,.$$

With this normalization $\theta \in [0,\pi]$, $\theta' \in [0,\pi/2]$: Ising defects' families express the breaking of the cosine symmetry!

An alternative perspective is useful. Primaries of the Ising CFT are:

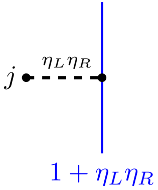
$$\mathcal{H}_{\mathbb{1}} : \mathbb{1}_{0,0}\,, \quad \epsilon_{\frac{1}{2},\frac{1}{2}}\,, \quad \sigma_{\frac{1}{16},\frac{1}{16}}\,; \qquad\qquad \mathcal{H}_{\eta} : \varphi_{\frac{1}{2},0}\,, \quad \bar{\varphi}_{0,\frac{1}{2}}\,, \quad \mu_{\frac{1}{16},\frac{1}{16}}$$

The folded theory has non-local currents:

$$j = \varphi_L\, \varphi_R\,, \qquad \bar{j} = \bar{\varphi}_L\, \bar{\varphi}_R\,, \qquad j, \bar{j} \in \mathcal{H}_{\eta_L \eta_R}\,.$$

These define the cosine symmetry operators by dressing an $\eta_L\eta_R$-invariant topological line:

$$L_\theta = (1 + \eta_L \eta_R) \exp\left( i\theta \int j \right)\,,$$




We dub this a **phantom symmetry**, as it is not a symmetry of the single CFT.

Phantom symmetries are present in a variety of RCFTs:

**c=1**

| | $R = \sqrt{2}n$ | $R = \sqrt{2}/n$ |
|---|---|---|
| $\left(\frac{1}{2}, \frac{1}{2}\right)$ | $V_{\pm 2n, 0}$ | $V_{0, \pm n}$ |
| $\left(\frac{1}{2}, 0\right)$ | $V_{n, \frac{1}{2n}}, \; V_{-n, -\frac{1}{2n}}$ | $V_{\frac{1}{n}, \frac{n}{2}}, \; V_{-\frac{1}{n}, -\frac{n}{2}}$ |
| $\left(0, \frac{1}{2}\right)$ | $V_{-n, \frac{1}{2n}}, \; V_{n, -\frac{1}{2n}}$ | $V_{\frac{1}{n}, -\frac{n}{2}}, \; V_{-\frac{1}{n}, \frac{n}{2}}$ |
| $\eta$ | $\left(0, \frac{1}{2n}\right), \; \left(0, -\frac{1}{2n}\right)$ | $\left(\frac{1}{n}, 0\right), \; \left(-\frac{1}{n}, 0\right)$ if $n$ even<br>$\left(\frac{1}{n}, \frac{1}{2}\right), \; \left(-\frac{1}{n}, \frac{1}{2}\right)$ if $n$ odd |

**WZW**

| | |
|---|---|
| $h = 1/2$ | $SU(2)_2$, $SU(4)_1$, $\mathrm{Spin}(n)_1$ , |
| $h = 1/4$ | $SU(2)_1$, |
| $h = 3/4$ | $SU(2)_3$, $SU(6)_1$, $\mathrm{USp}(6)_1$, $\mathrm{Spin}(12)_1$, $(\mathrm{E}_7)_1$ |

Can extend to cosets. E.g. 3Potts/$\mathrm{SU}(3)_1$ have interfaces with $\mathfrak{su}(2)$ phantom symmetry.

# The reflection coefficient

A phantom symmetry $j = \psi_L \psi_R$ commutes only with the combination $T = T_L + T_R$.

On the other hand the (2,0) operators:

$$W^+ \simeq c_R T_L - c_L T_R \in \mathcal{H}_{\mathbb{1}}\,, \quad W^- \simeq h_R \partial\psi_L\,\psi_R - h_L \psi_L\,\partial\psi_R \in \mathcal{H}_\eta\,,$$

form a doublet under the phantom symmetry:

$$[j_0, W^+] = nW^-\,, \quad [j_0, W^-] = -nW^+\,, \quad n \in \mathbb{Z}\,.$$

The reflection coefficient is computed by a 2pf of $W^+$. Defining

$$\langle 0|W^{\pm}(z)\overline{W}^{\pm}(\bar{w})|B\rangle = \frac{\omega_B^{\pm}}{(z-w)^4}\,.$$

we have $\mathcal{R} = (c_L^2 + c_R^2 + 2c_L c_R \omega_B^+)/(c_L + c_R)^2$ and:

$$\omega_B^+(\varepsilon) = \cos^2(n\varepsilon)\omega_B^+(0) + s\sin^2(n\varepsilon)\omega_B^-(0)\,.$$

Choosing a “nice” reference $B$ gives simple expressions.

(a) Transmissive $|B\rangle = |\mathcal{L}\rangle$:

$$\mathcal{R}(\varepsilon) = \frac{1 + s q^{\mathcal{L}}_{\psi_L} q^{\mathcal{L}}_{\psi_R}}{2} \sin^2(n\varepsilon)\,.$$

(b) Reflective $|B\rangle = |B_L\rangle|B_R\rangle$:

$$\mathcal{R}(\varepsilon) = 1 - \frac{2c_L c_R}{(c_L + c_R)^2}\left(1 \mp \frac{\beta_L \beta_R}{k}\right)\sin^2(n\varepsilon)\,,$$

$$\langle 0|\psi_{L/R}(z)\overline{\psi}_{L/R}(\overline{w})|B_0\rangle = \beta_{L/R}/(z-w)^{2h_{L/R}}\,.$$

## Symmetry and Defects IV: Symmetry Breaking and Modulation

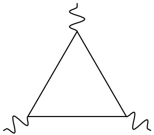
We now consider symmetry-breaking defects, in which case $t(x)$ is nontrivial. See also Shuhei's talk

Breaking a continuous symmetry $G$ defines a family of defect $\mathscr{D}_\sigma$ by the deformation:

$$i \int_{\mathscr{D}} \mathrm{Tr}\, \sigma\, t(x) \,, \quad g \in G = e^{i\sigma} \,.$$

For boundary conditions $\mathscr{B}$, if $G$ suffers from an 't Hooft anomaly (e.g. the $SU(N_f)$ symmetry of $N_f$ Weyl fermions),

$$Z[A + d_A \lambda] = e^{i \int \omega(A,\lambda)} Z[A] \,,$$



then $\mathscr{B} = \mathscr{B}_\sigma$ must break the symmetry.

We call this breaking **anomaly-enforced** [CC '25] , see also Shuhei's talk

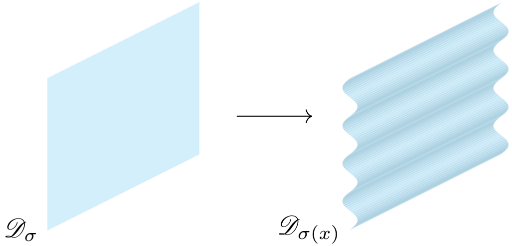
A natural question is whether an anomaly-enforced breaking fundamentally differs from a vanilla one.

To answer this we would like to couple the bulk + defect system to a gauge field $A$.

Naively this is not possible, as $A \to A + d_A\lambda$ gives rise to a boundary term

$$i \int_{\mathscr{B}} \mathrm{Tr}\, \lambda(x) t(x)\,.$$

This can be circumvented provided we consider coupling the defect to a **modulated** coupling $\sigma(x)$.




The bulk + boundary system can be made gauge invariant by a non-linear transformation for $\sigma$:

$$A \to A + d\lambda\,, \quad \sigma \to \sigma - \lambda\,.$$

The defect free-energy now depends on $A$, $\sigma$ and the invariant combination $\omega_A = g^{-1}(d + A)g \quad .$

For anomalous symmetries in the presence of a boundary, the Wess-Zumino consistency condition is violated by a boundary term:

$$\delta_{\lambda_1}\omega(A,\lambda_2) - \delta_{\lambda_2}\omega(A,\lambda_1) - \omega(A,[\lambda_1,\lambda_2]) = d\beta(\lambda_1,\lambda_2,A)\,.$$

The presence of a boundary term forces the symmetry breaking [Jensen, Yarom '19] .

However, for modulated defects, $\beta$ can be cancelled by the modulated free energy $F_{\mathscr{B}}$:

$$\beta(\lambda_1,\lambda_2,A) = \delta_{\lambda_1}F_{\mathscr{B}}(\lambda_2,A) - \delta_{\lambda_2}F_{\mathscr{B}}(\lambda_1,A)\,.$$

This fixes universal, anomaly-induced terms in $F_{\mathscr{B}}$.

## Boundary Transport and SPT pumping

Consider $G = U(1)$ the anomalies:

$$\omega_{(1+1)} = \frac{\chi_{(1+1)}}{2\pi} \int d\lambda\, A\,, \qquad \omega_{(3+1)} = \frac{\chi_{(3+1)}}{24\pi^2} \int d\lambda A\, dA\,,$$

Fix:

$$F_{\mathscr{B},(1+1)} = \frac{\chi_{(1+1)}}{2\pi} \int \sigma(A + d\sigma) + \ldots\,, \quad F_{\mathscr{B},(3+1)} = \frac{\chi_{(3+1)}}{24\pi^2} \int \sigma(A + d\sigma)dA + \ldots$$

Give rise to the following (Hall) boundary currents:

$$Q_{\mathscr{B})} = \chi_{(1+1)} \frac{\sigma}{2\pi}\,, \qquad \mathcal{J}^i_{\mathscr{B}} = \chi_{(3+1)} \frac{\sigma}{8\pi^2} \epsilon^{ijk} F_{jk}\,.$$

As we wind around the circle $\sigma \to \sigma + 2\pi$ charge is deposited on the boundary.

This is a Thouless-pump phenomenon and correspond to the stacking of $U(1)$ SPTs

$$i\chi_{(1+1)} \int_{\mathscr{B}} A\,, \qquad\qquad i\frac{\chi}{4\pi} \int_{\mathscr{B}} AdA\,,$$

Which describe Integer Quantum Hall states in (0+1) and (2+1) dimensions.

This shows a deep interplay between bulk 't Hooft anomalies and the topology of families of defects related by anomaly-enforced symmetry breaking.


Thank you!

- How do symmetry-refined versions of defect entropy interplay with the possible representations of symmetry on $\mathscr{D}$? ([Karch,Kusuki,Ooguri,Sun,Wang '23] for recent studies of defect entropy and [Choi, Rayhaun, Zheng '24] [Heymann,Quella '24] [Kusuki,Murciano,Ooguri,Pal '24] [Bastida,Das,Sierra,Molina-Vilaplana '24] symmetry resolved entropy)

- Does (generalized) symmetry allow to constrain/bootstrap defect fusion rules? [Bachas,Brunner '07] [Konechny '15] [Soderberg '21] [Diatlyk,Khanchandani,Popov,Wang '24] [Kravchuck,Radcliffe,Sinha '24] .

- "Anomalies in the space of couplings" [Cordova,Freed,Lam,Seiberg '19] for defect RG flows? Relation with [Debray,Devalapurkar,Krulewski,Liu,Pacheco-Tallaj,Thorngren '23] ?

- Application to lattice impurities (generalized Kondo)? How is the representation of symmetry on the defect encoded in the lattice formulation?