1st August 2025

Spontaneously Non-uniform Entangled Phases

Apoorv Tiwari

Assistant Professor

Centre for Quantum Mathematics &

Danish Institute for Advanced Study,

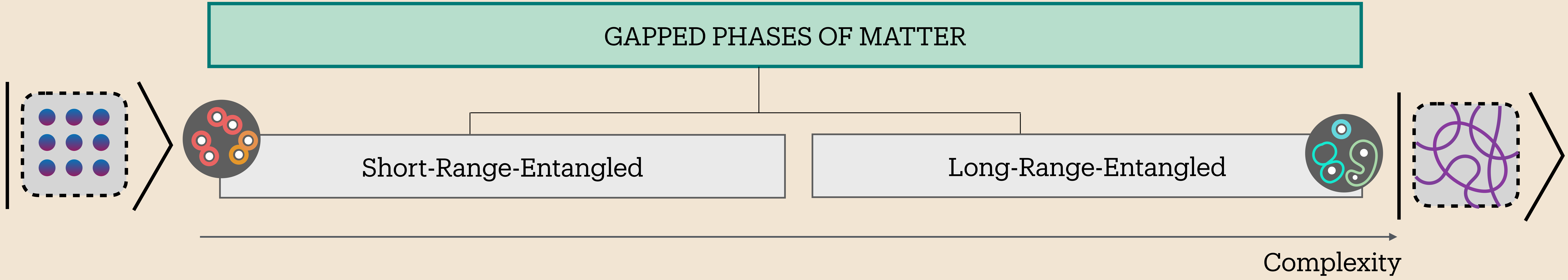
University of Southern Denmark

Symmetry and Organisation of Phases

Symmetry and Organisation of Phases

GAPPED PHASES OF MATTER

Symmetry and Organisation of Phases



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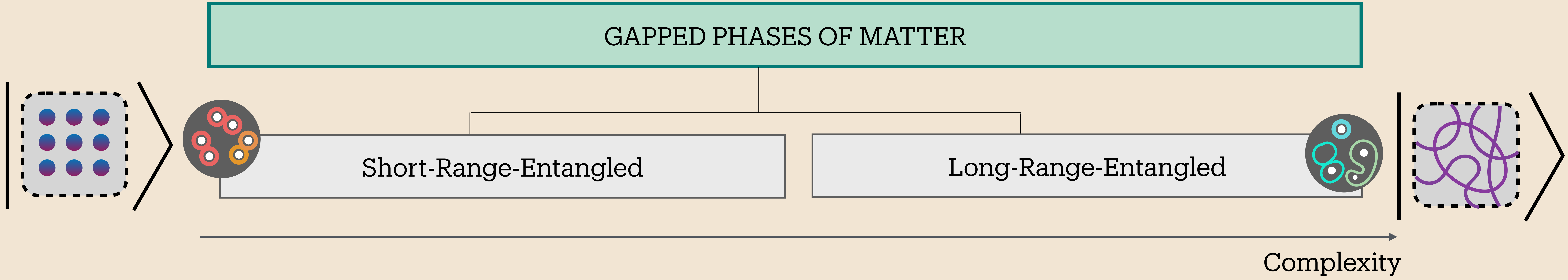

Conventional Symmetry Broken
→ **E.g. Ising Ferromagnet, Dimer States**

Conventional Symmetry Preserving
→ **E.g. Paramagnets, Topological Insulators**

(Often) Explained by Conventional Symmetries.

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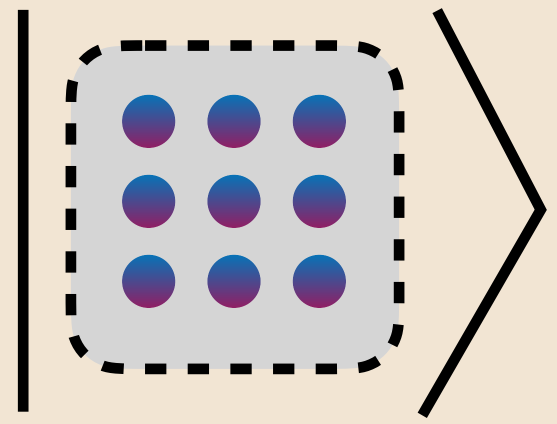
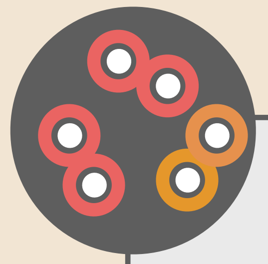
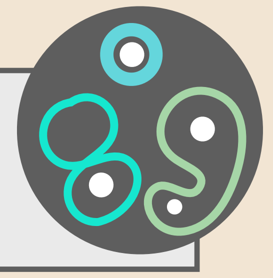
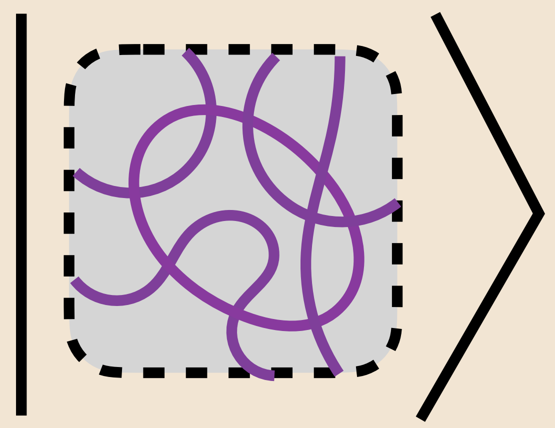




Short-Range-Entangled

Long-Range-Entangled





Complexity

Conventional Symmetry Broken

→ **E.g. Ising Ferromagnet, Dimer States**

Conventional Symmetry Preserving

→ **E.g. Paramagnets, Topological Insulators**

(Often) Explained by Conventional Symmetries.

Topological Order

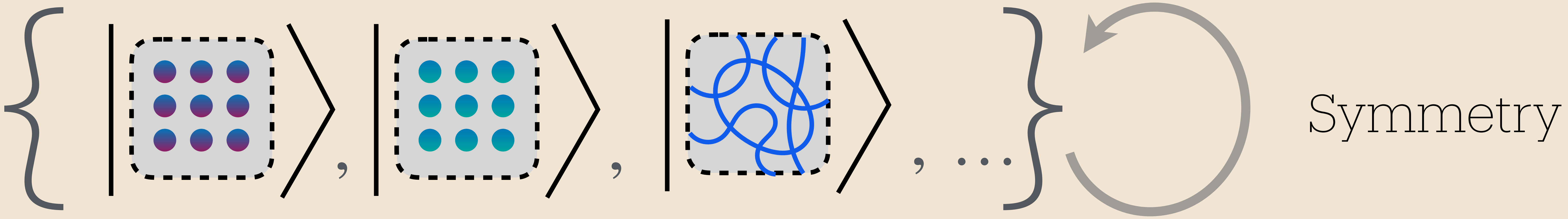
→ **E.g., Toric Code**

→ **E.g., Spin Liquids, Hall Fluids**

Explained by Generalized Symmetries.

Main Message

- Non-invertible symmetries can support phases with ground states with distinct (non-uniform) entanglement patterns.



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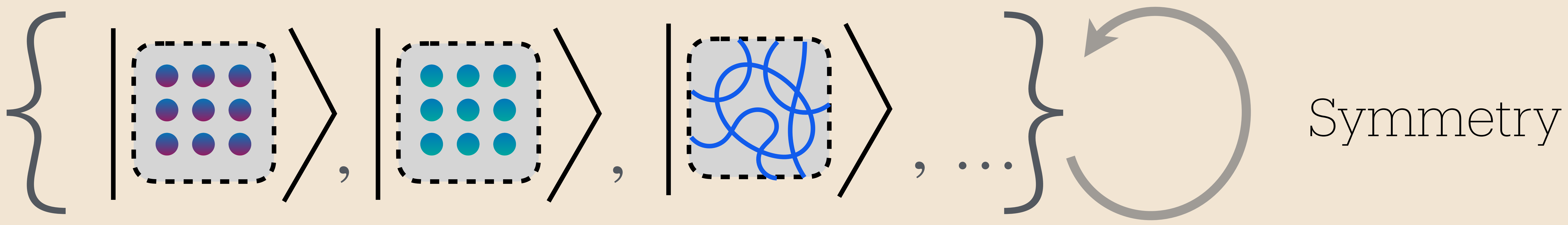

I will call these "Spontaneously non-uniform entangled phases" (SNE Phases).

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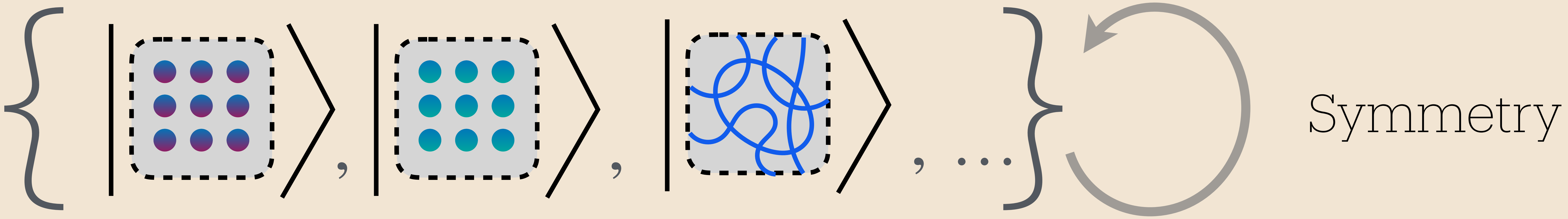


I will call these "Spontaneously non-uniform entangled phases" (SNE Phases).

- This talk: I will describe such phases, associated transitions and lattice models using the SymTFT.

Based on:

- arXiv: 2502.20440 [Gapped Phases w/ Fusion 2-categorical symmetries]

  With Lakshya Bhardwaj, Sakura Schafer-Nameki & Alison Warman

- arXiv: 2503.12699 [Gapless Phases w/ Fusion 2-categorical symmetries]

  With Lakshya Bhardwaj, Yuhan Gai, Shengjie Huang, Kansei Inamura,
  Sakura Schafer-Nameki & Alison Warman

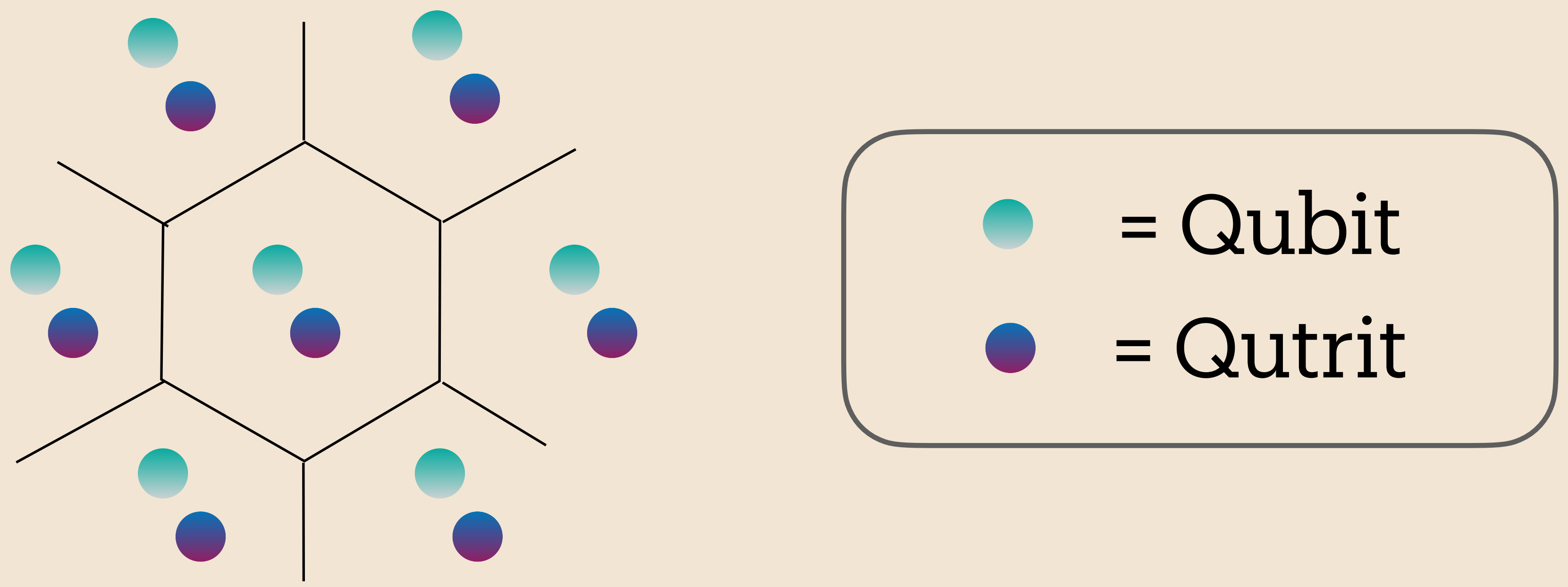
- arXiv: 2506.09177 [Lattice Models w/ Fusion 2-categorical symmetries]

  With Shengjie Huang, Kansei Inamura & Sakura Schafer-Nameki

Intuitive Picture

Intuitive Picture

S_3 symmetric Lattice Model



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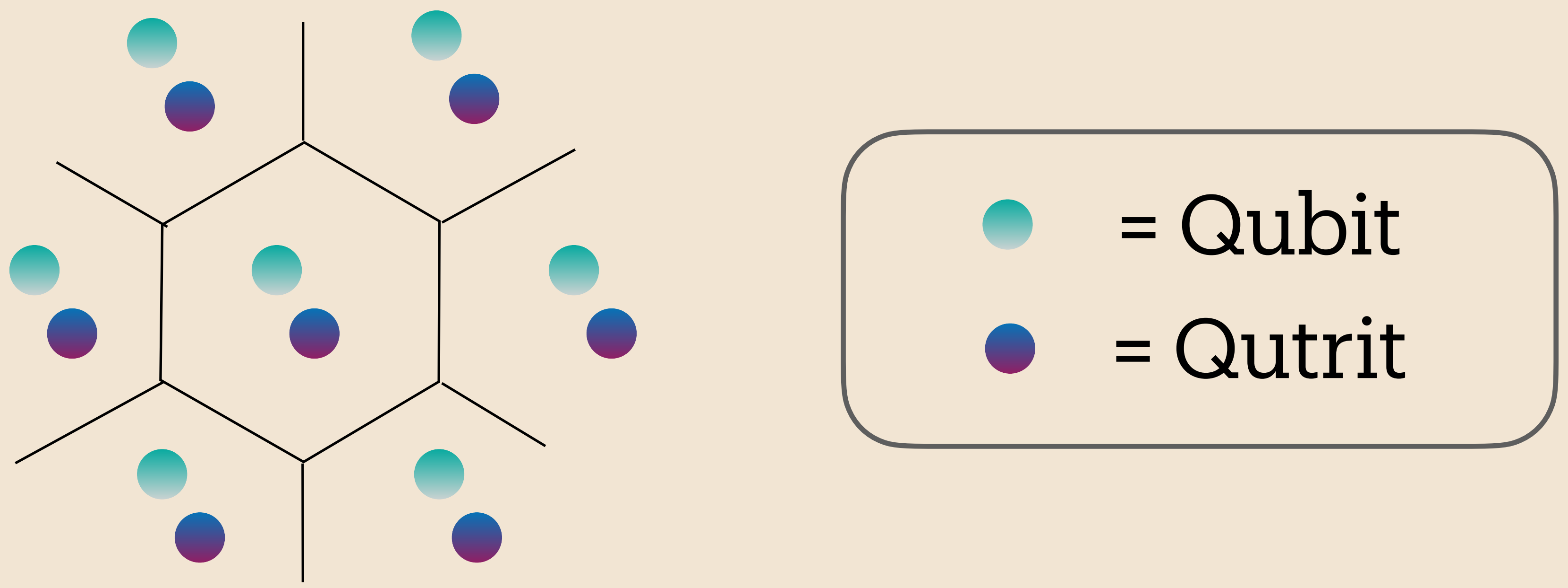
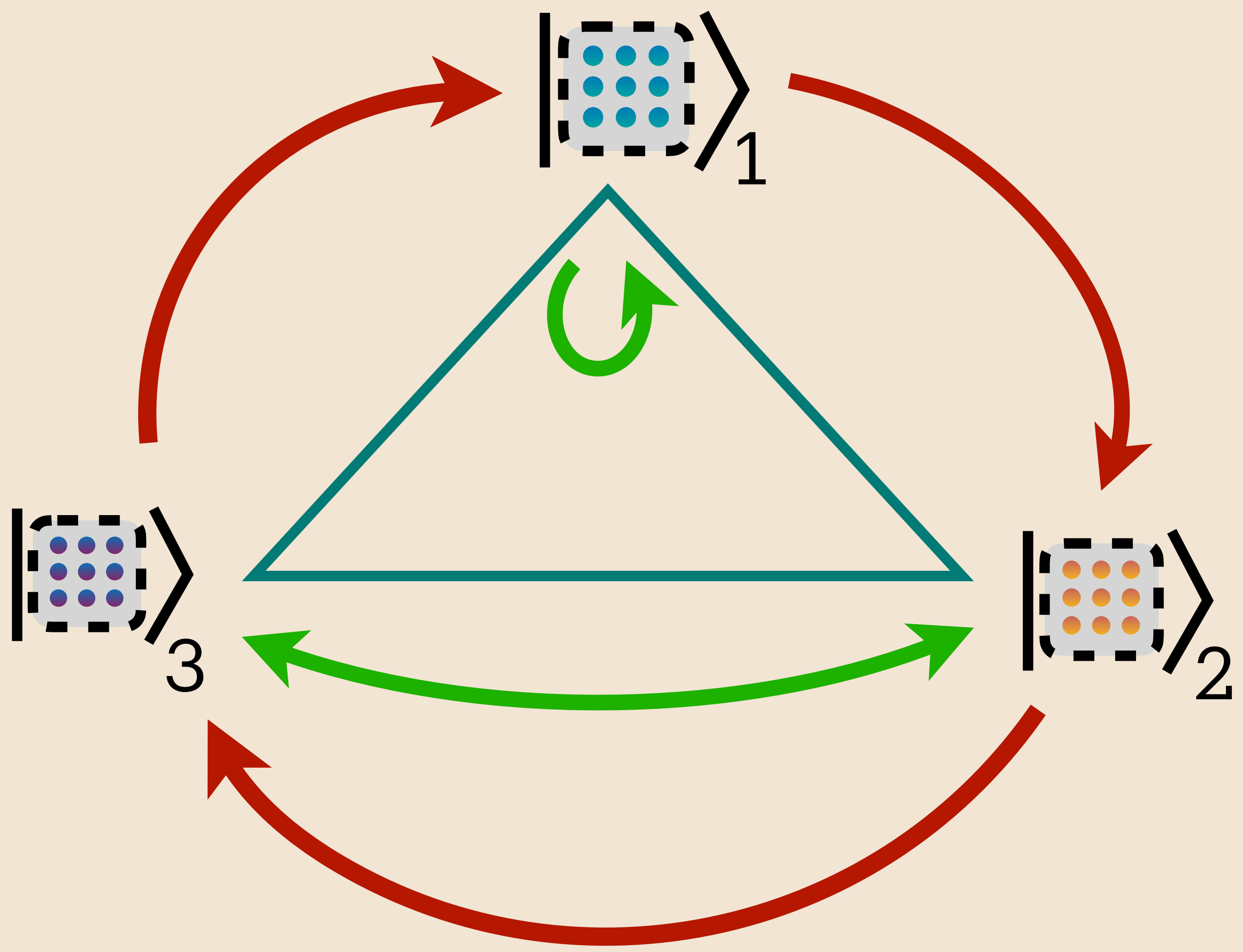


Ground states and symmetry action



$\mathbb{Z}_3$ SSB Phase

Intuitive Picture

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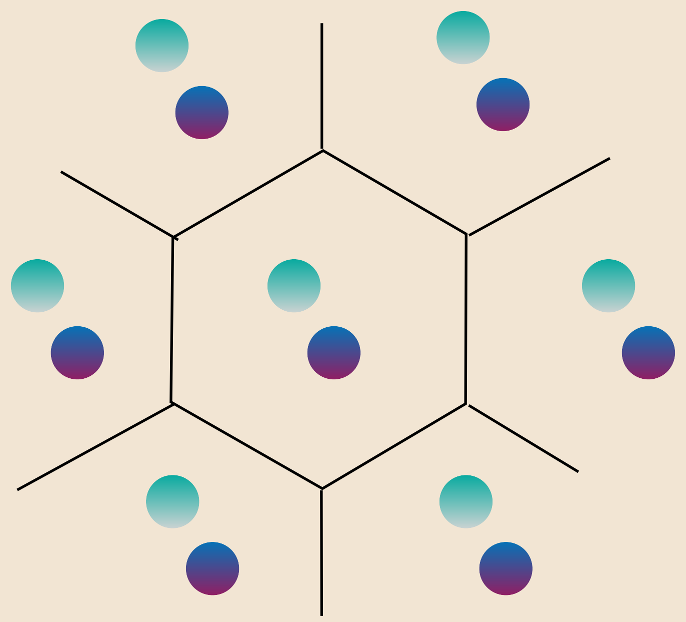
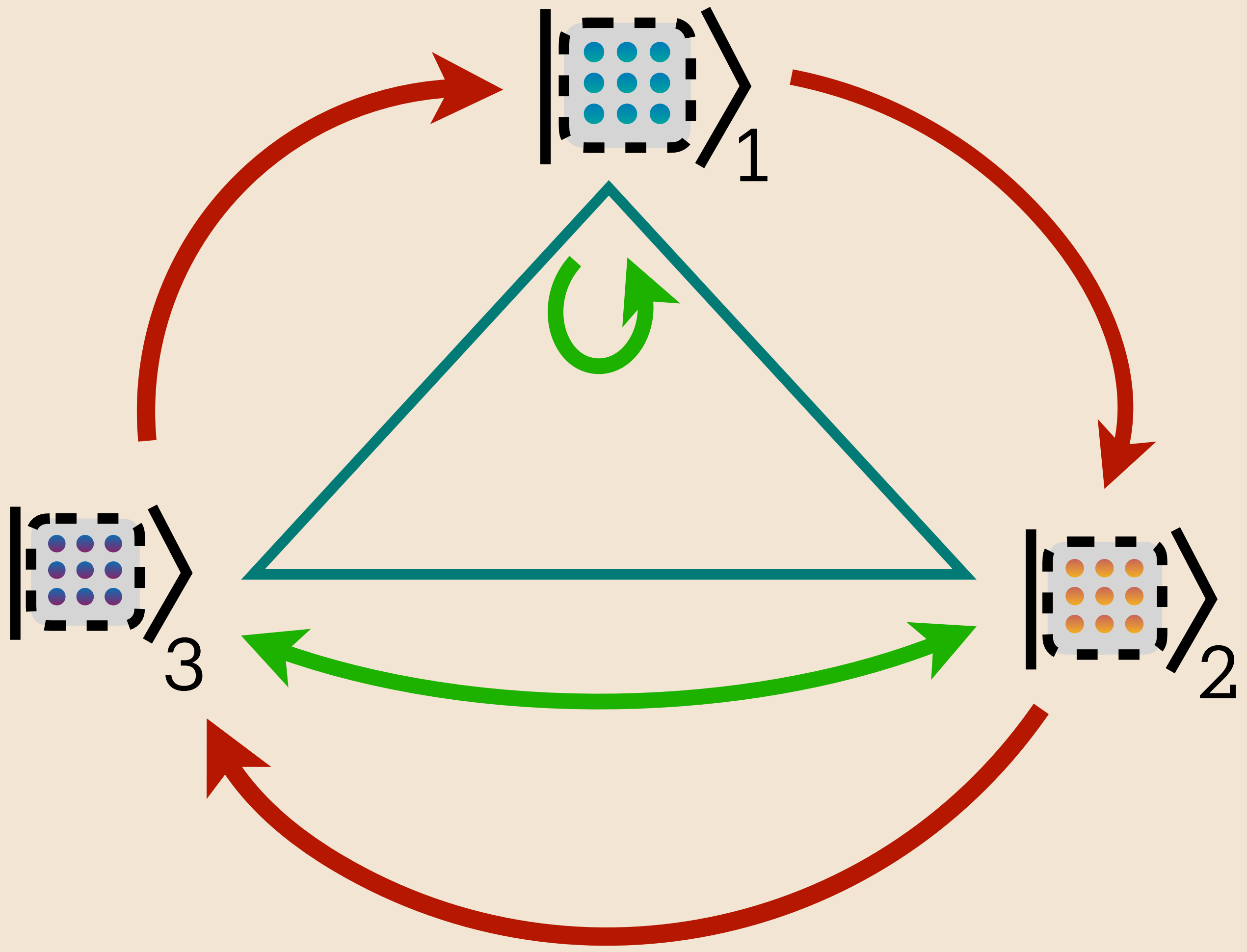


= Qubit
= Qutrit

Ground states and symmetry action



$\mathbb{Z}_3$ SSB Phase

Gauging non-normal $\mathbb{Z}_2$ subgroup symmetry

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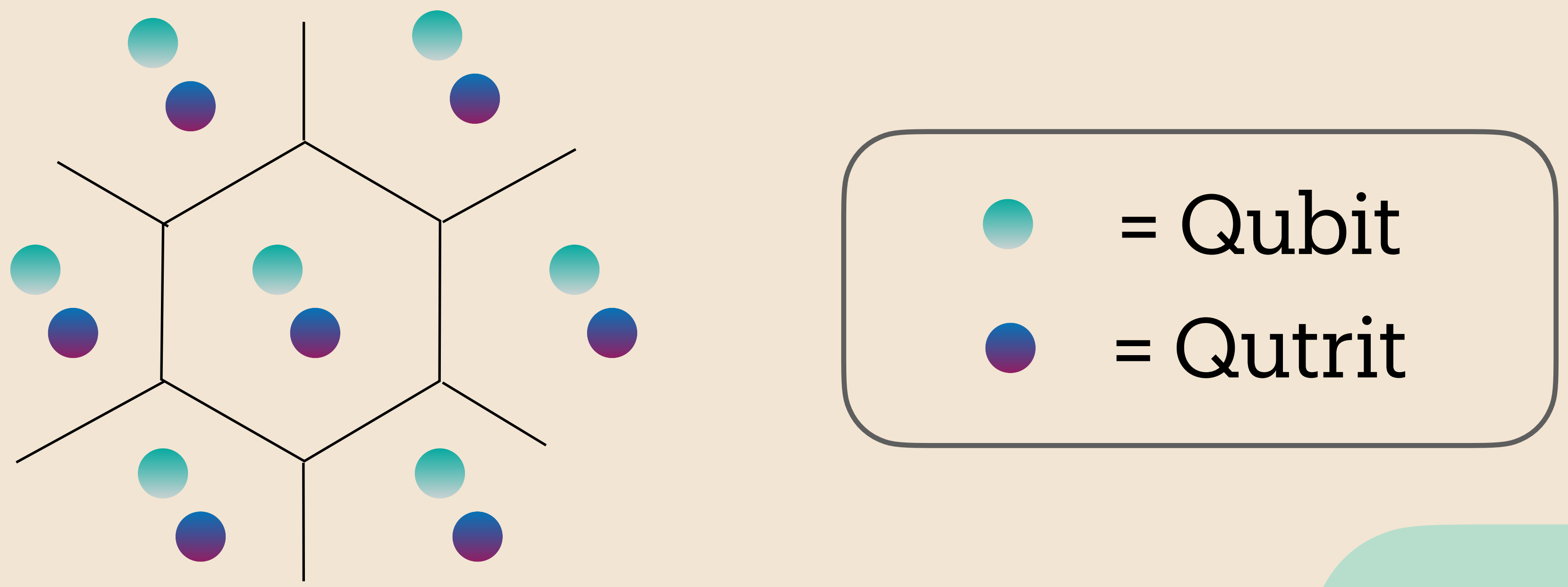

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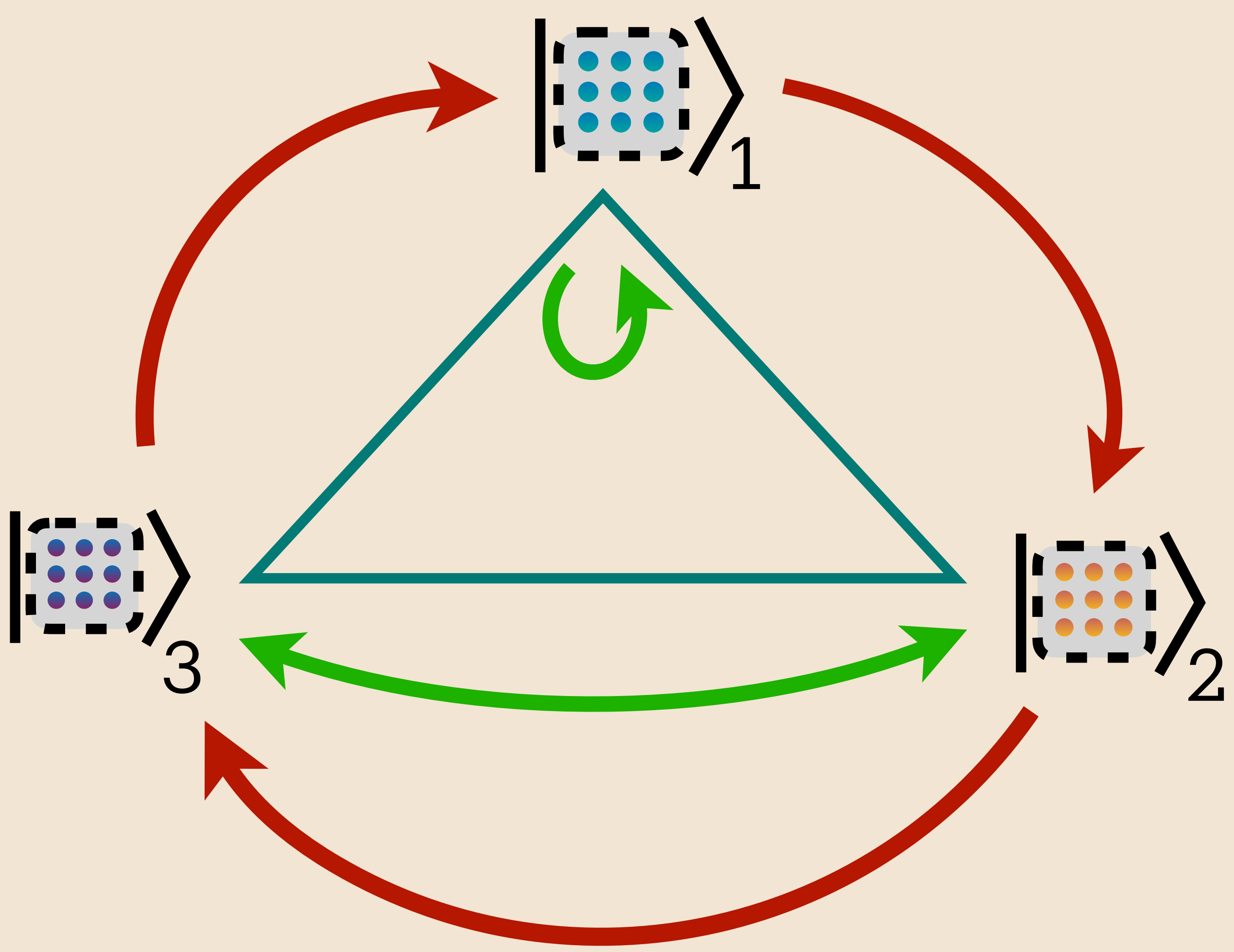

$\mathbb{Z}_3$ SSB Phase

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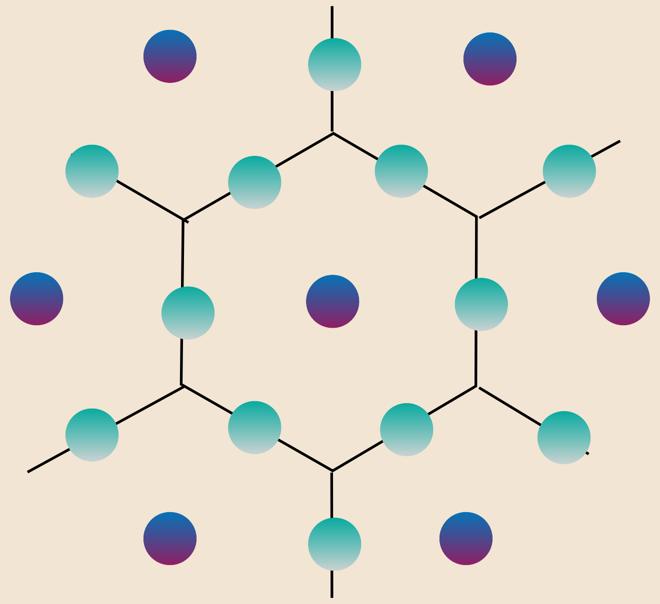
2Rep($\mathbb{G}$) symmetric Lattice Model

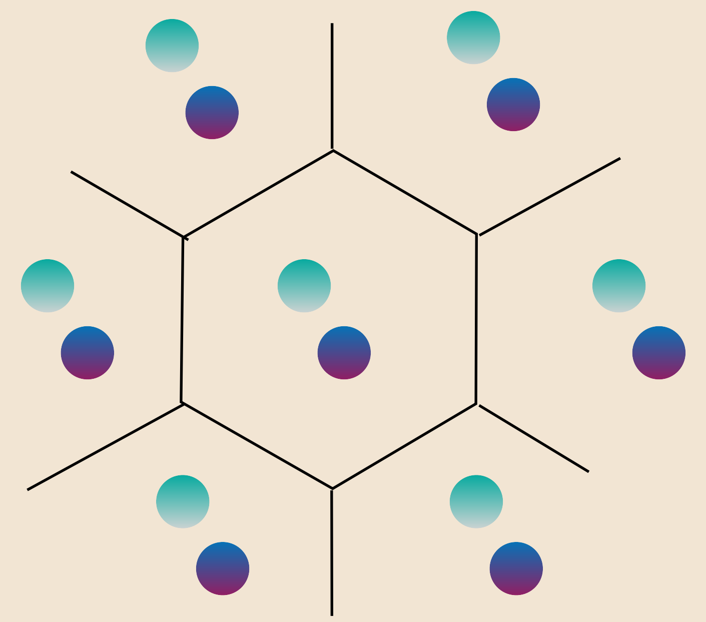


[Delcamp, Tiwari '23], [Inamura, Huang, Tiwari, Nameki '25]

$* \ \mathbb{G} = \mathbb{Z}_3^{(1)} \rtimes \mathbb{Z}_2^{(0)}$

Intuitive Picture

S_3 symmetric Lattice Model



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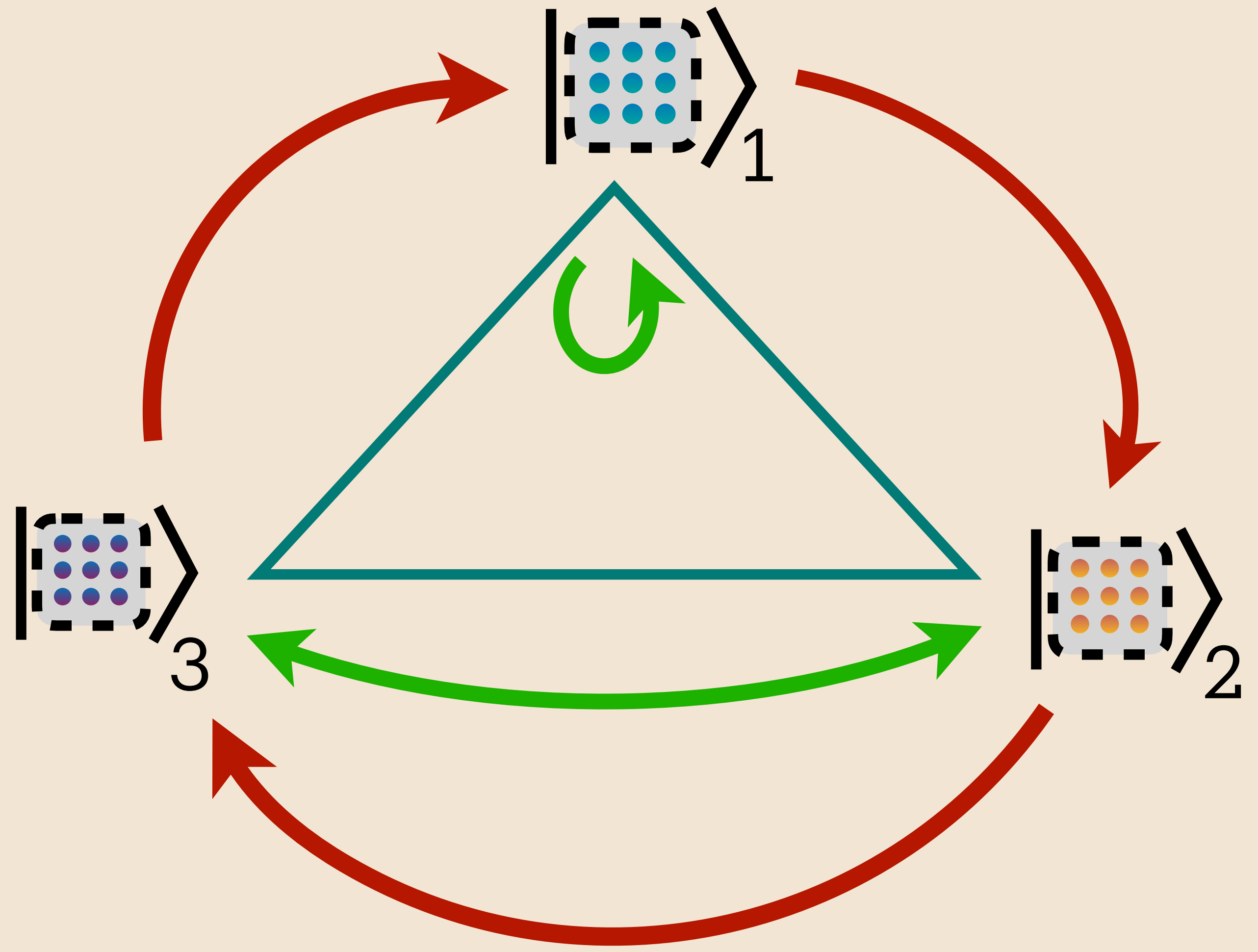

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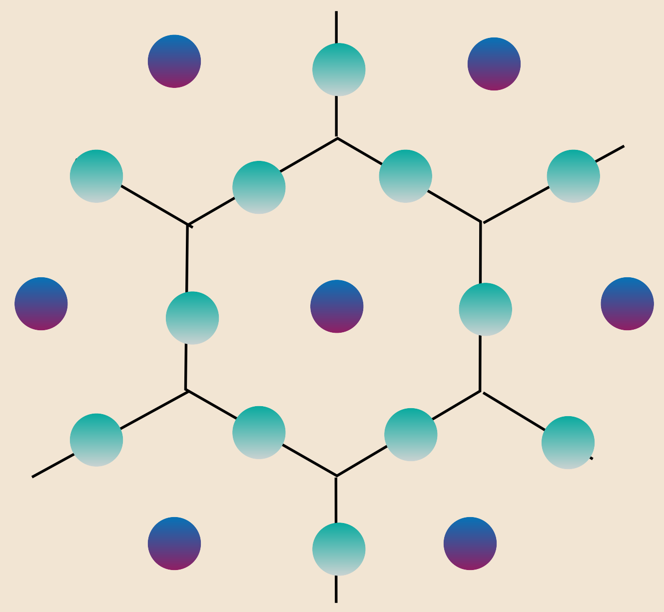

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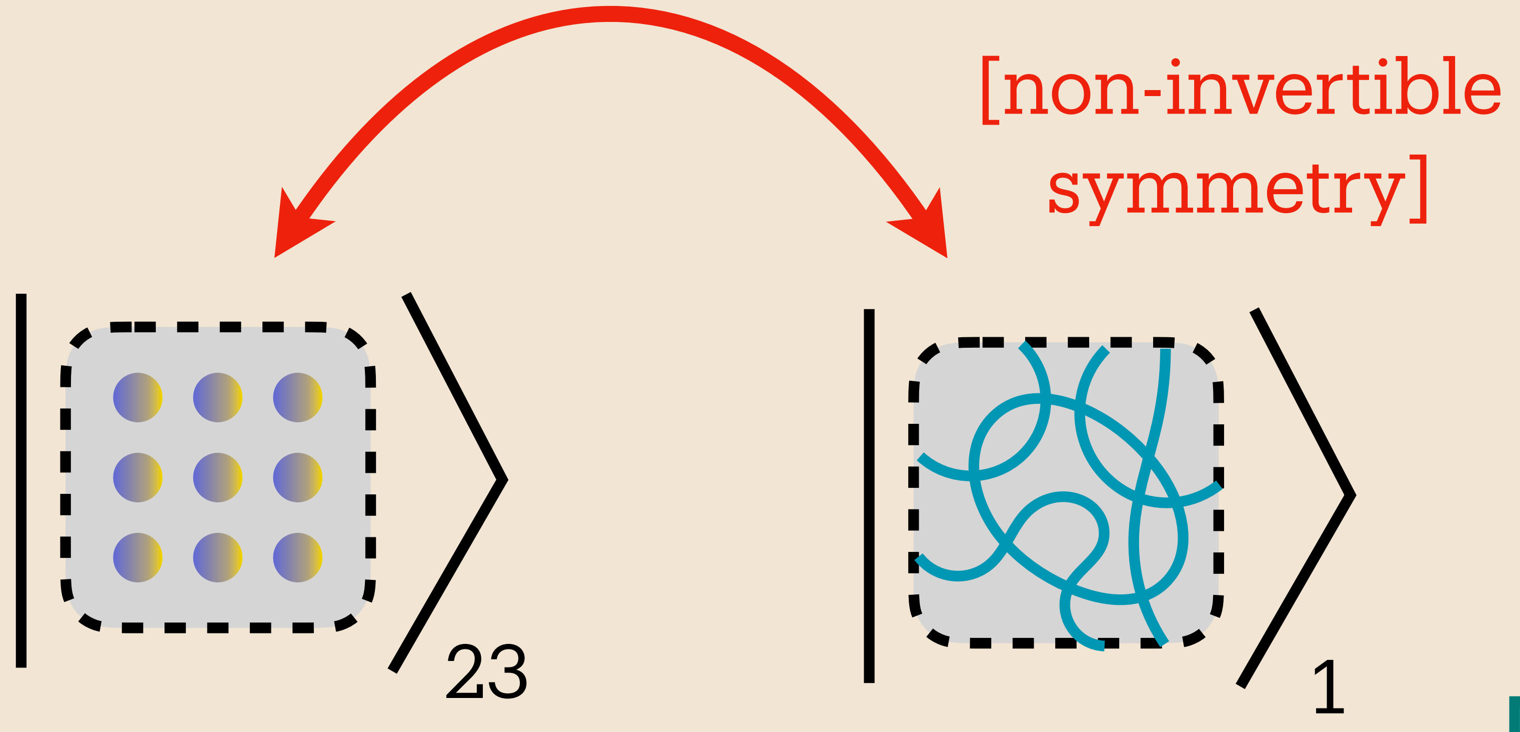


[1-form Symmetry Preserving]

[1-form Symmetry Breaking]

2Rep($\mathbb{G}$)
SSB Phase

* $\mathbb{G} = \mathbb{Z}_3^{(1)} \rtimes \mathbb{Z}_2^{(0)}$

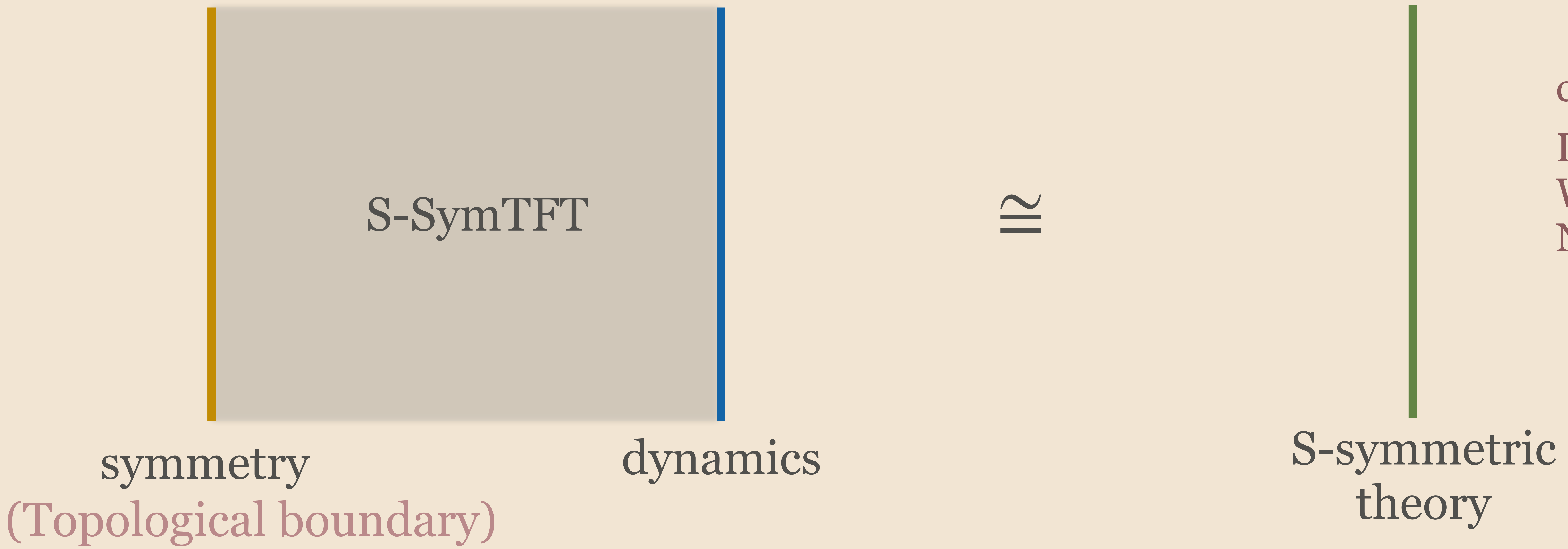
Plan:

- Overview of Symmetry Topological Field Theory (SymTFT)
- SNE Phase from SymTFT Based on arXiv:2502.20440
- A second order transition involving the SNE Phase Based on arXiv:2503.12699
- Lattice realisation of SNE Phase Based on arXiv:2506.09177

Symmetry Topological Field Theory (SymTFT)

*Symmetry TFT = Topological Holography = Symmetry TO

A theoretical gadget that separates the dynamical properties of a quantum theory from its symmetry structure.

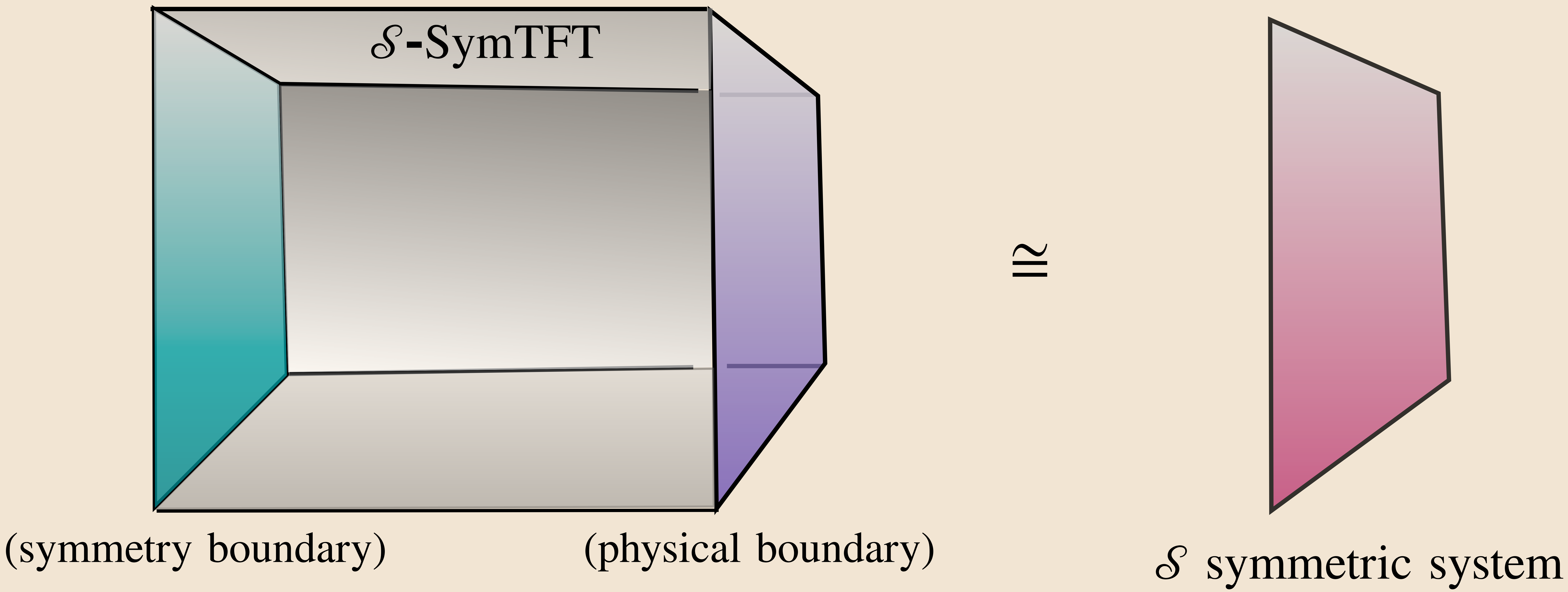


cf Talks by:

Ling-Yan Hung, Jiahua Tian, Wenjie Ji and Sakura Schafer-Nameki.

... [Witten];[Freed, Moore, Teleman]; [Apruzzi et al]; [Gaiotto, Kulp]; [Bhardwaj et al]; [Thorngren, Wang]; [Ji, Wen]; [Chatterjee, Wen]; [Moradi, Moosavian, AT]; [Kaidi, Ohmori, Zheng], [Lootens, Delcamp, Ortiz, Verstraete]; [Assen, Mong, Fendley]; [Lichtman, et al] ...

SymTFT overview



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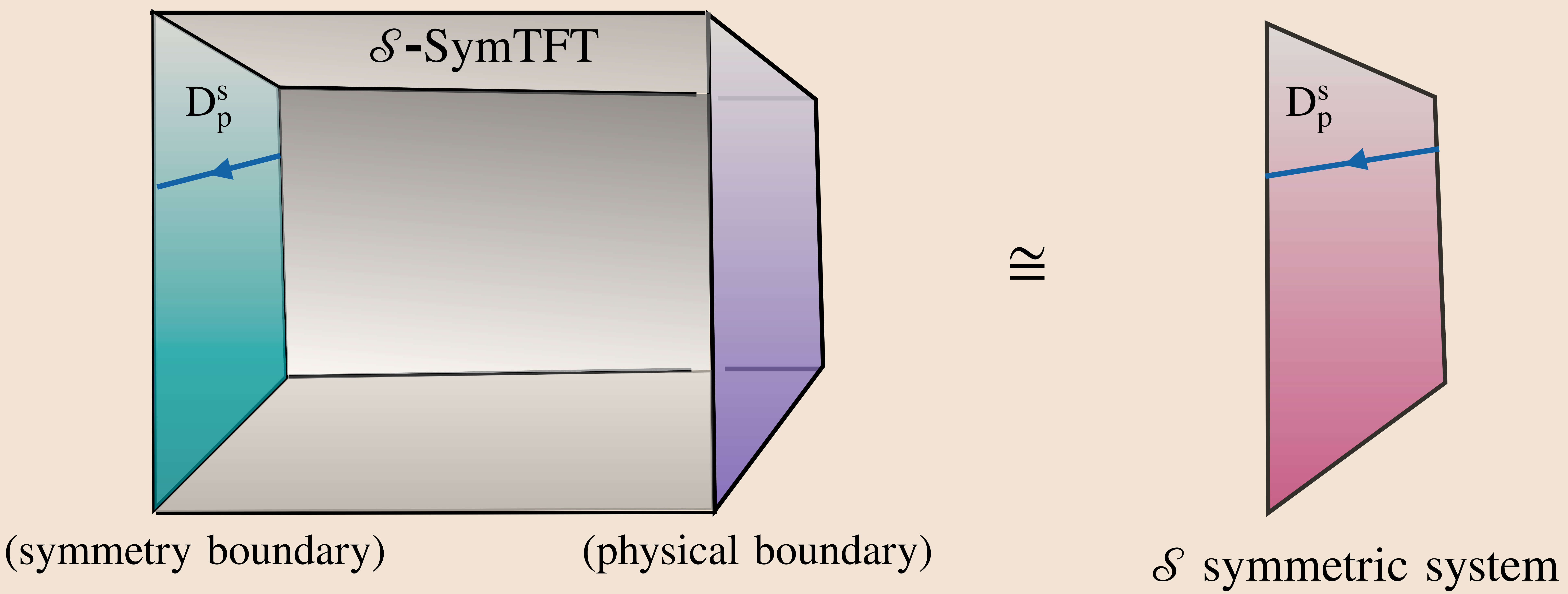

- **Symmetries:** **Defects on the symmetry boundary.**

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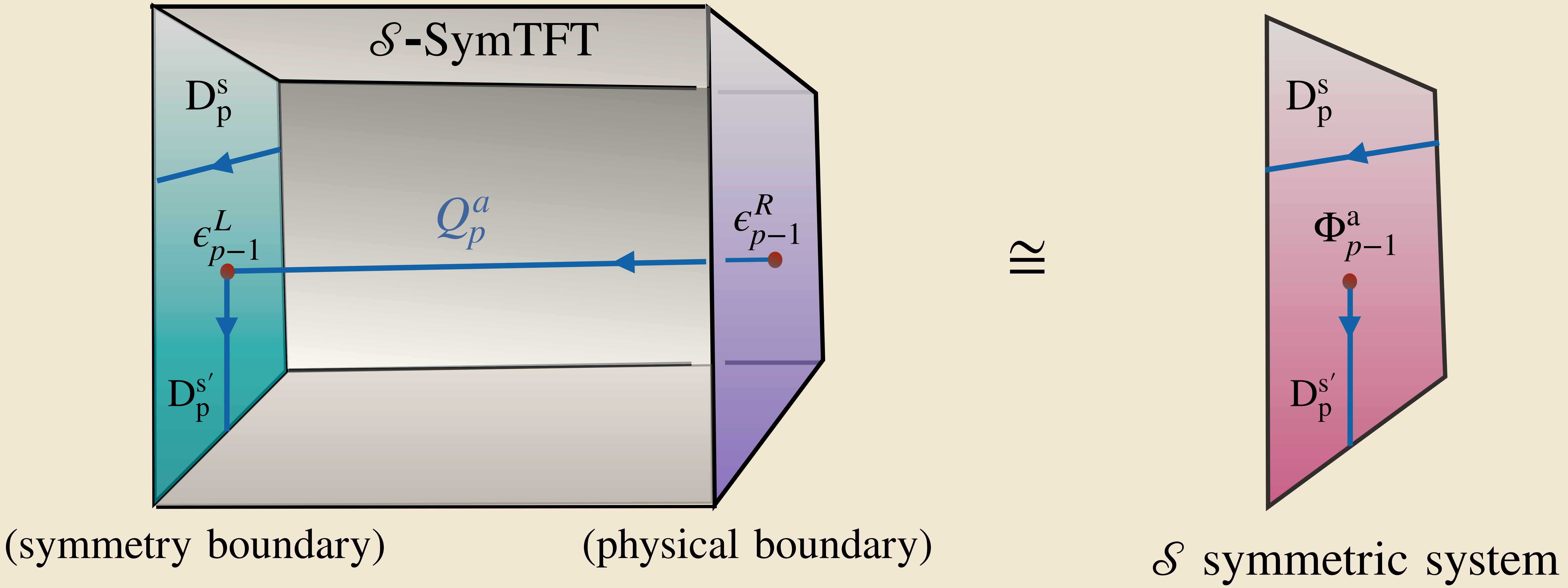

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- **Charged operators/multiplets: Classified by SymTFT defects.**

[Lin, Okada, Seifnashri, Tachikawa], [Bhardwaj, Schafer-Nameki], [Bartsch, Bullimore, Grigoletto, Pearson]

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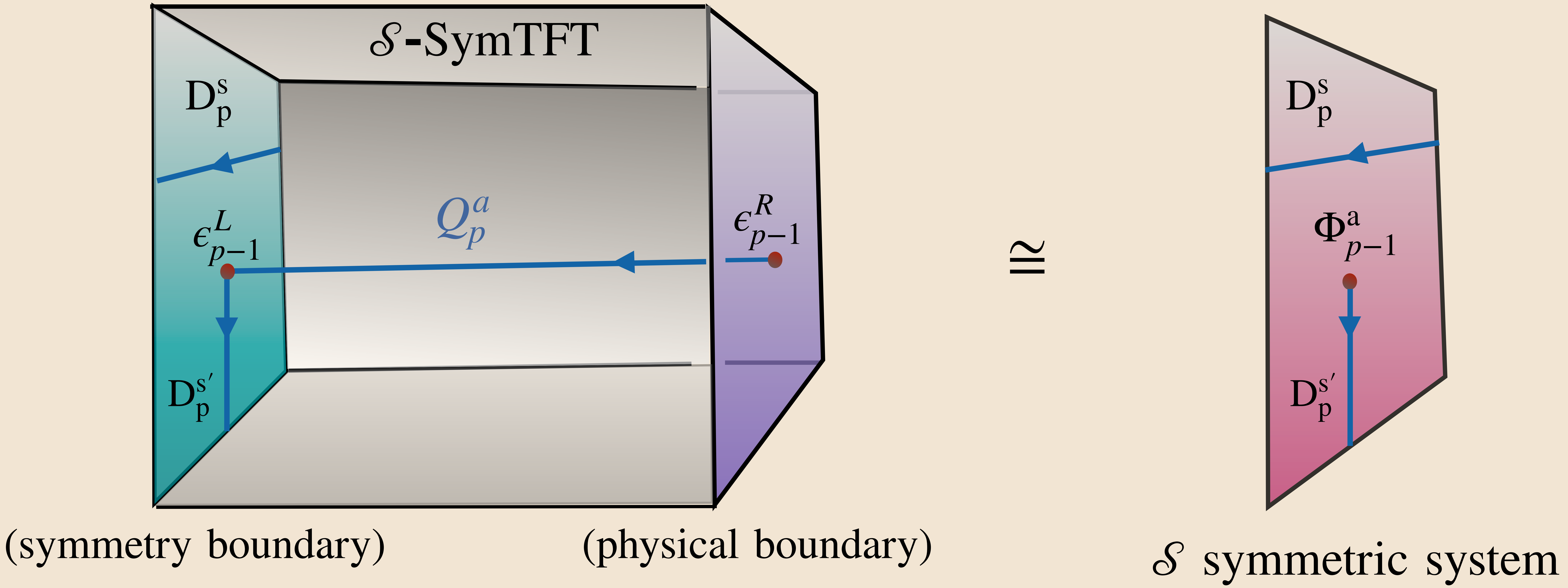

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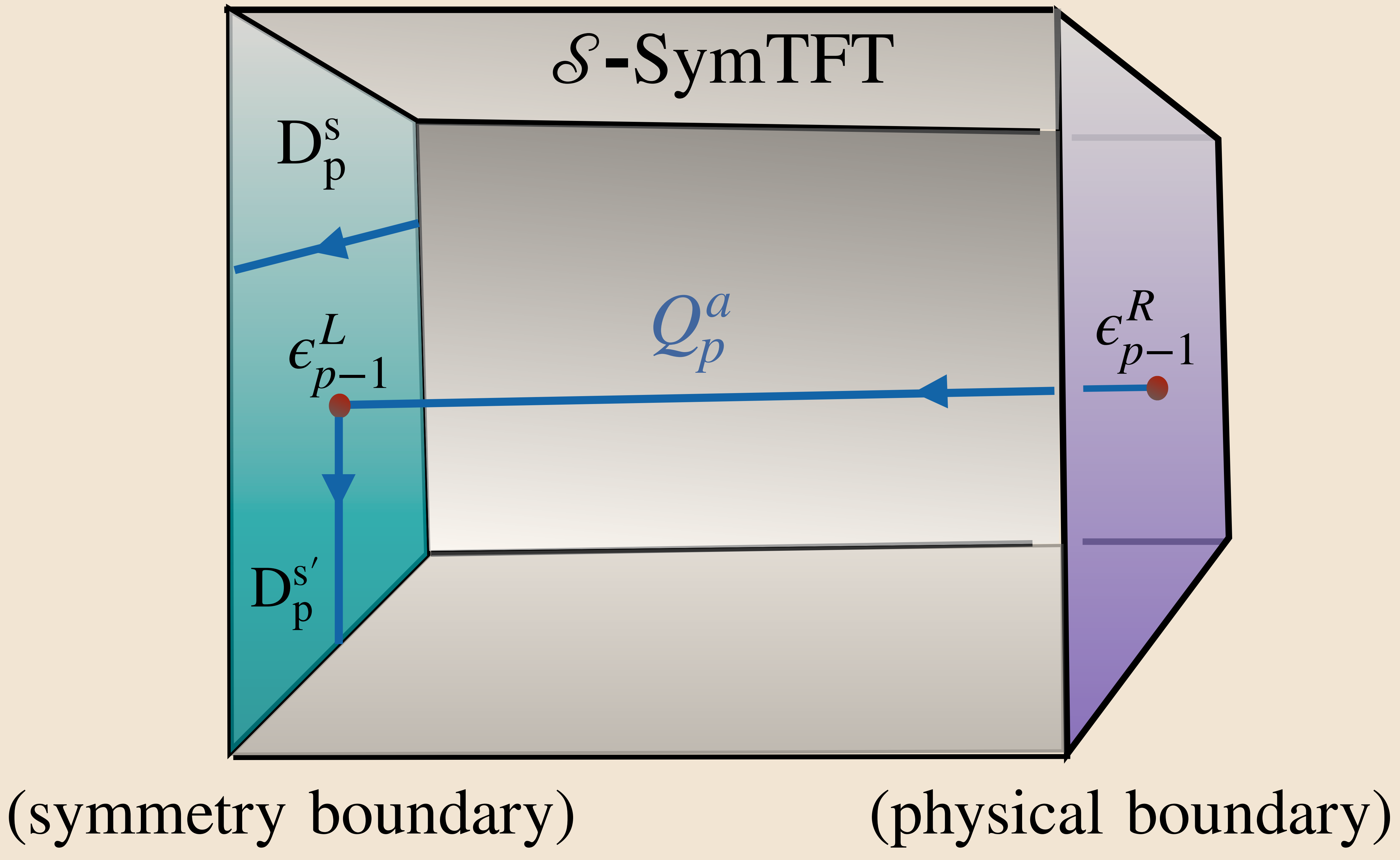

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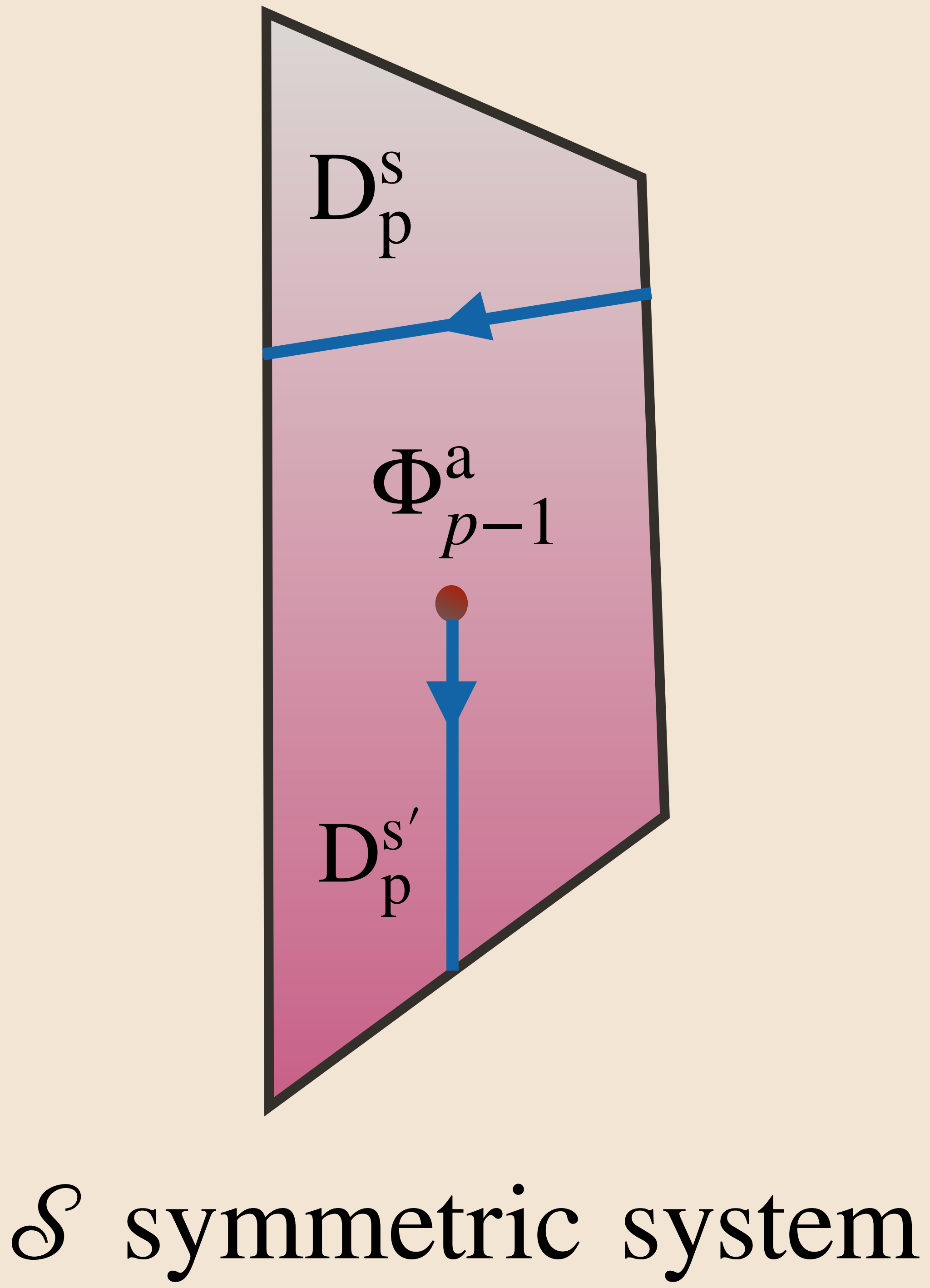

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- **Generalized gauging: Changing symmetry boundary.**

[Lin, Okada, Seifnashri, Tachikawa], [Bhardwaj, Schafer-Nameki], [Bartsch, Bullimore, Grigoletto, Pearson]

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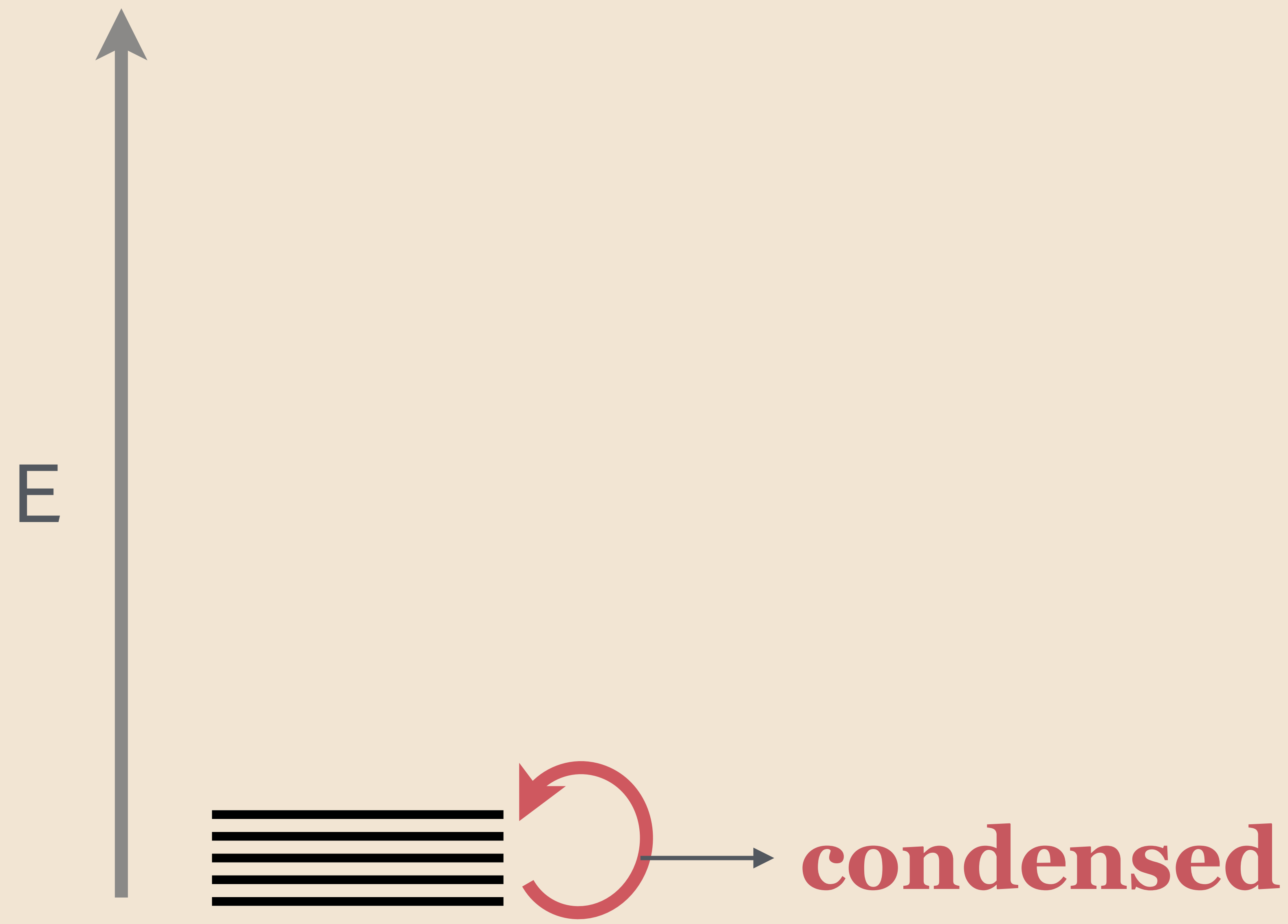

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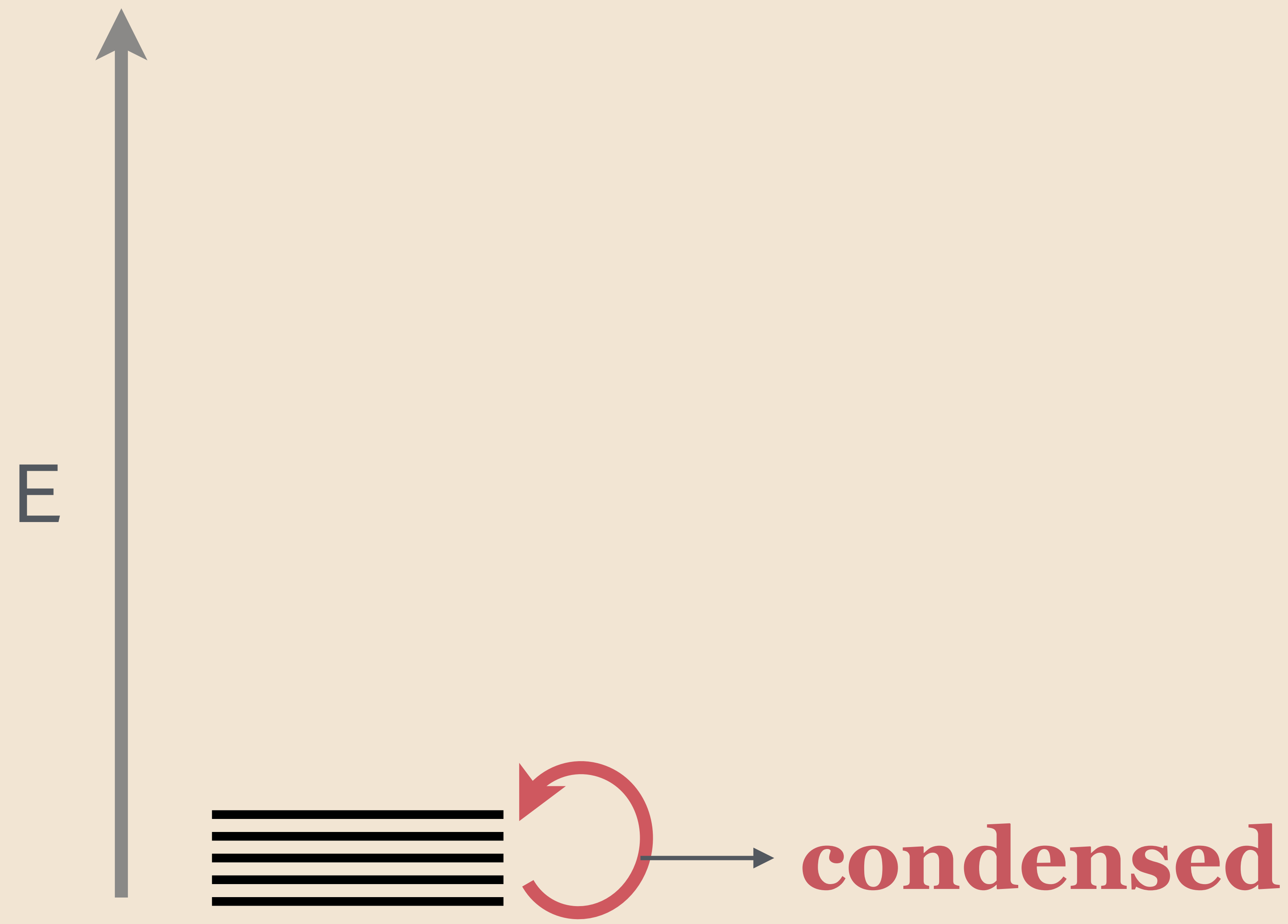

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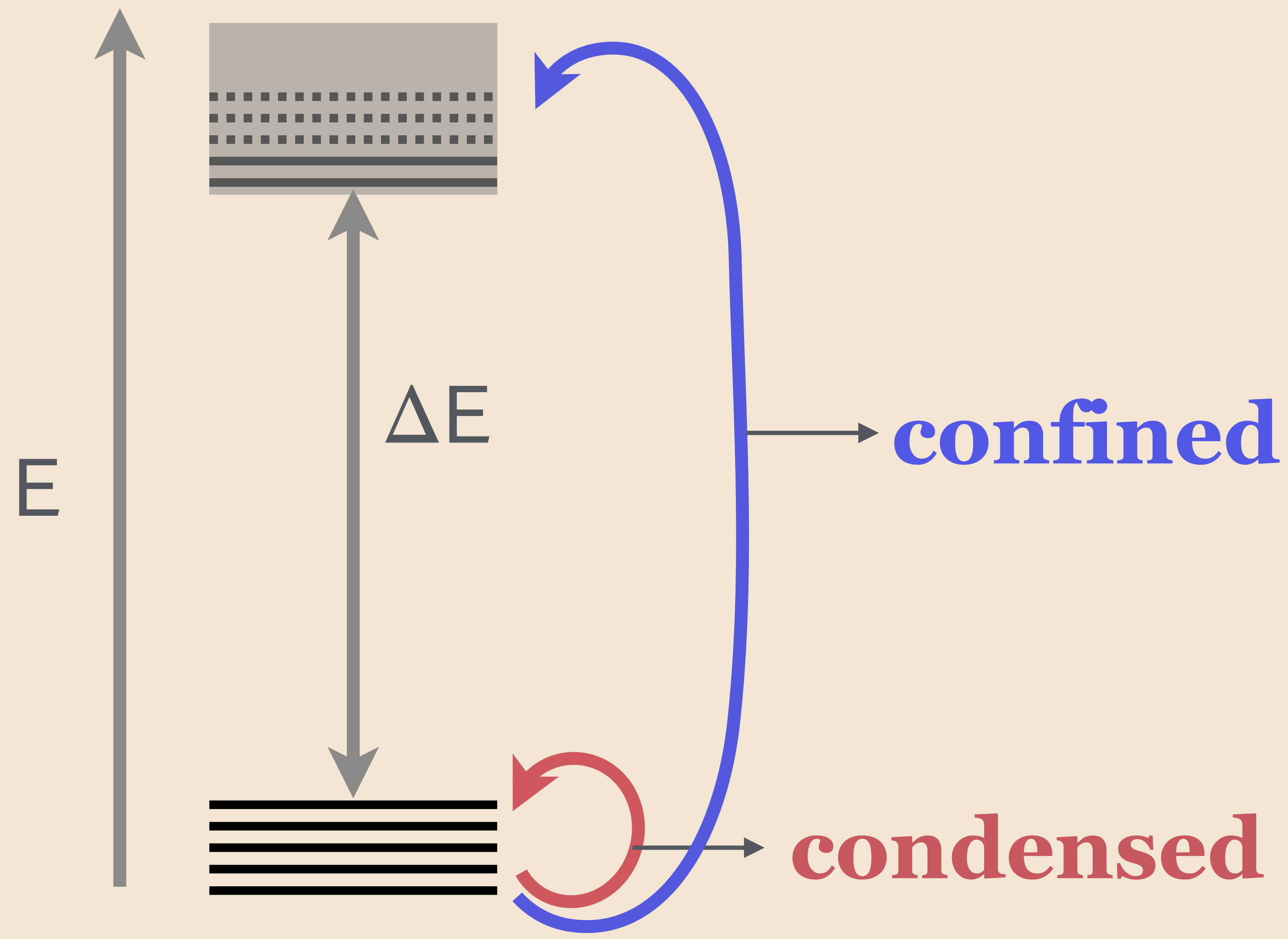

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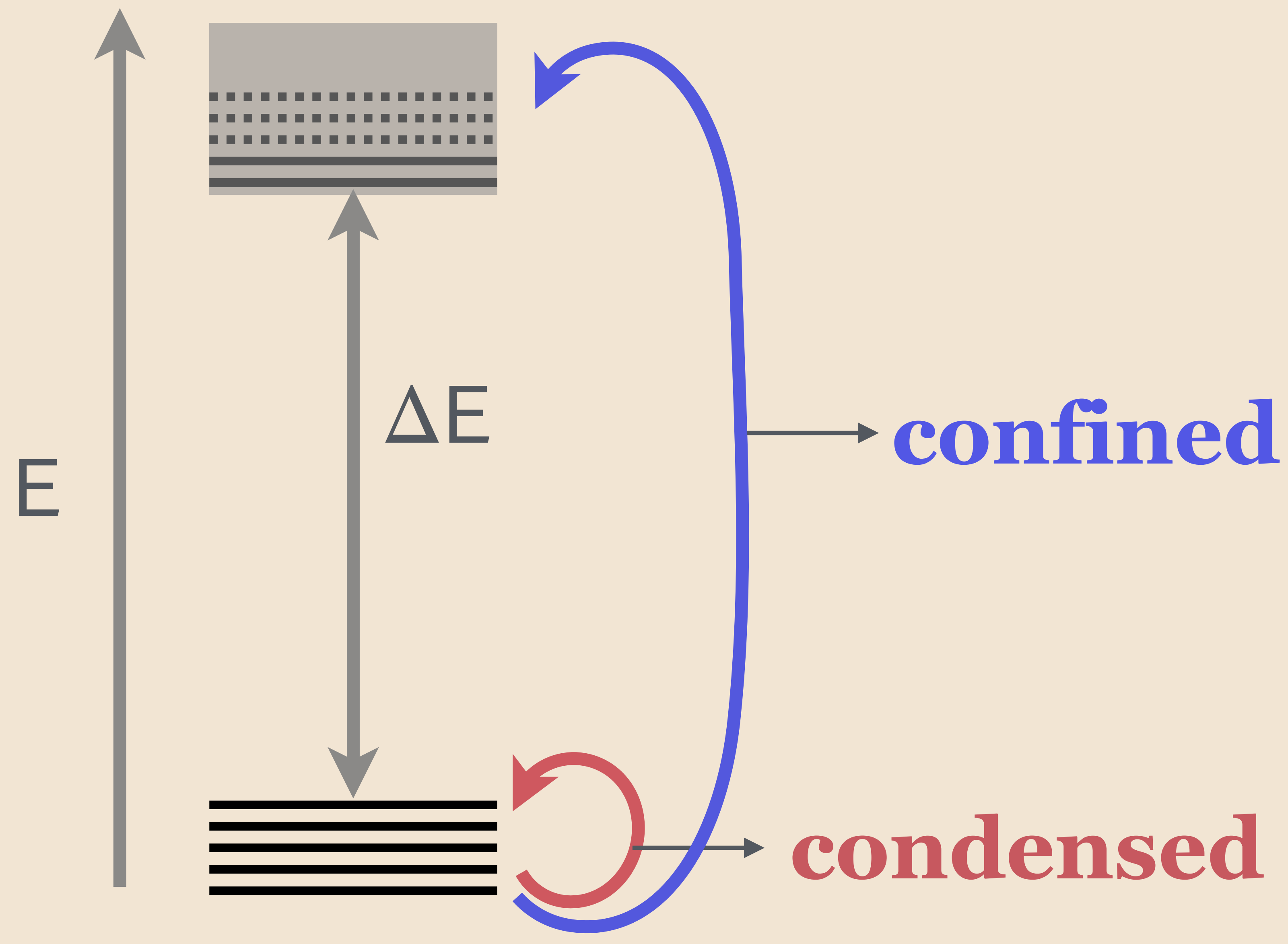

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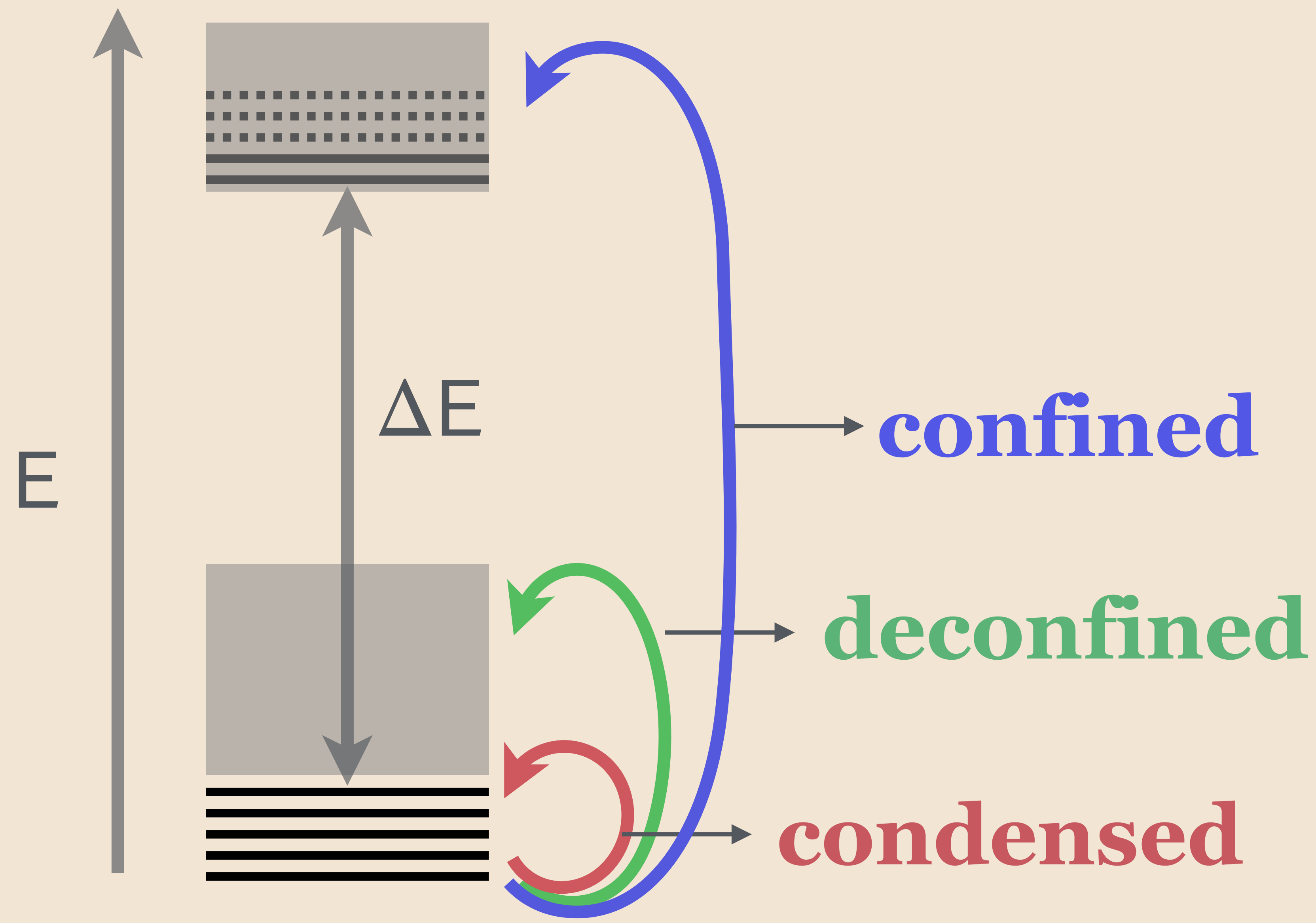

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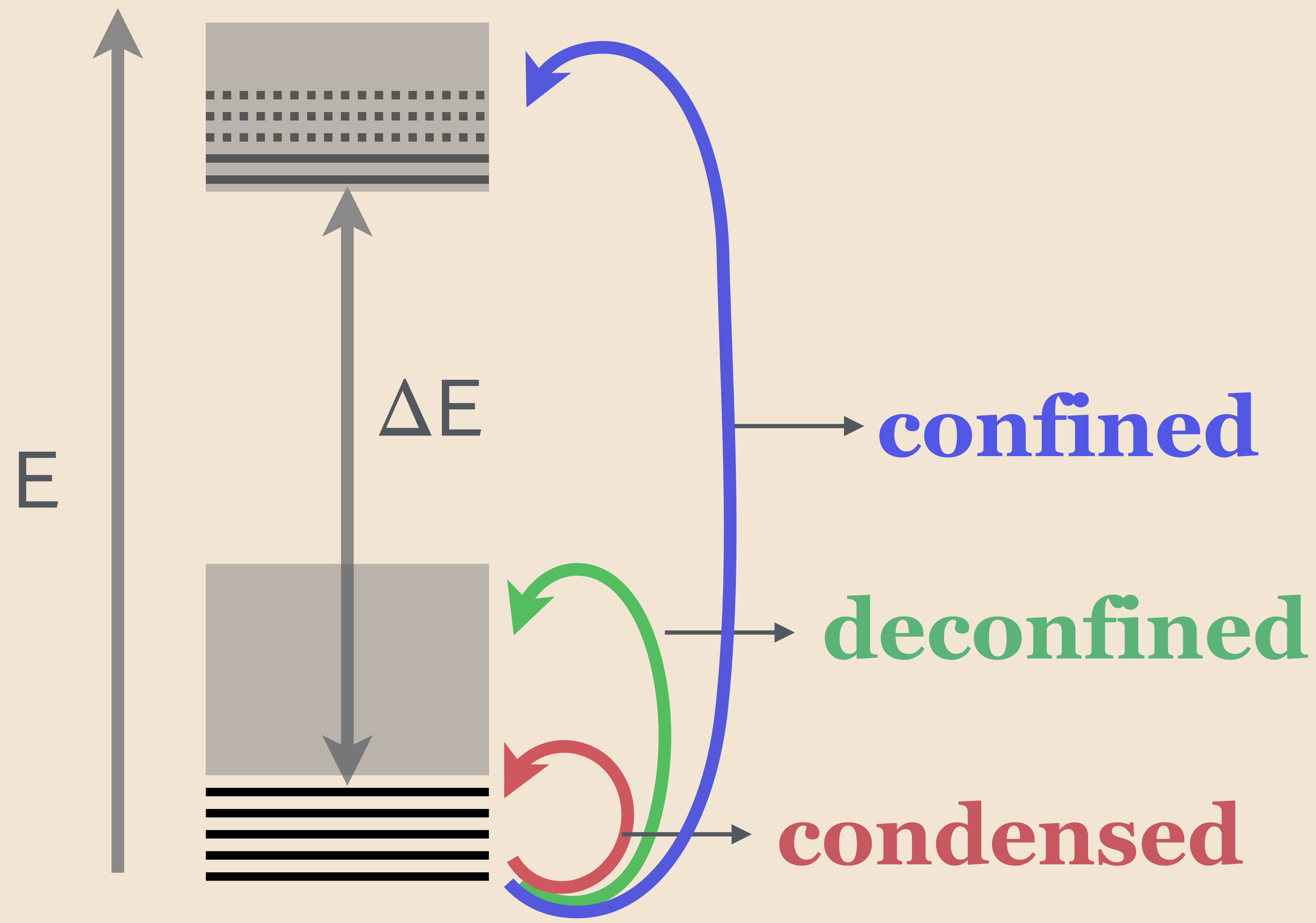

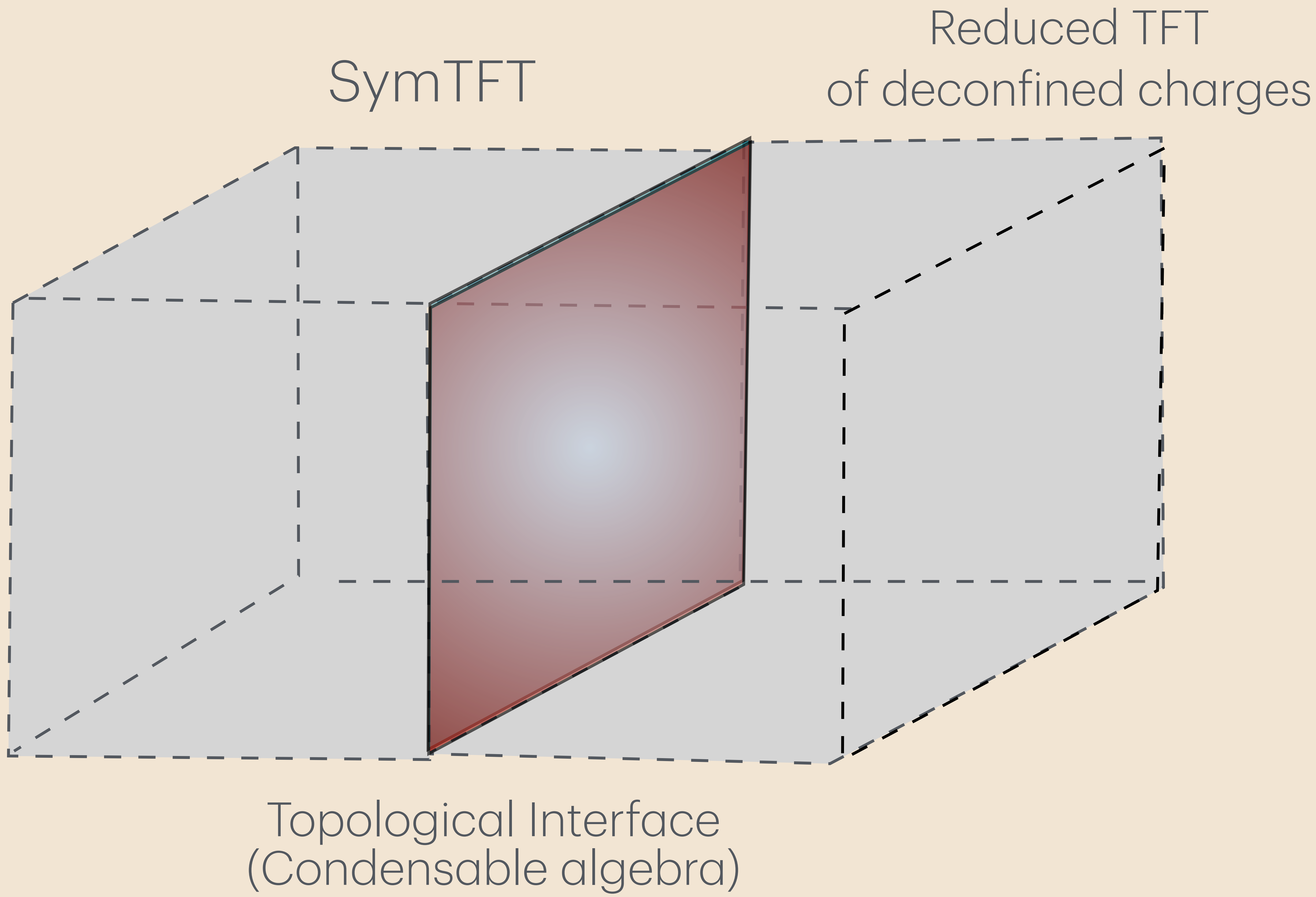

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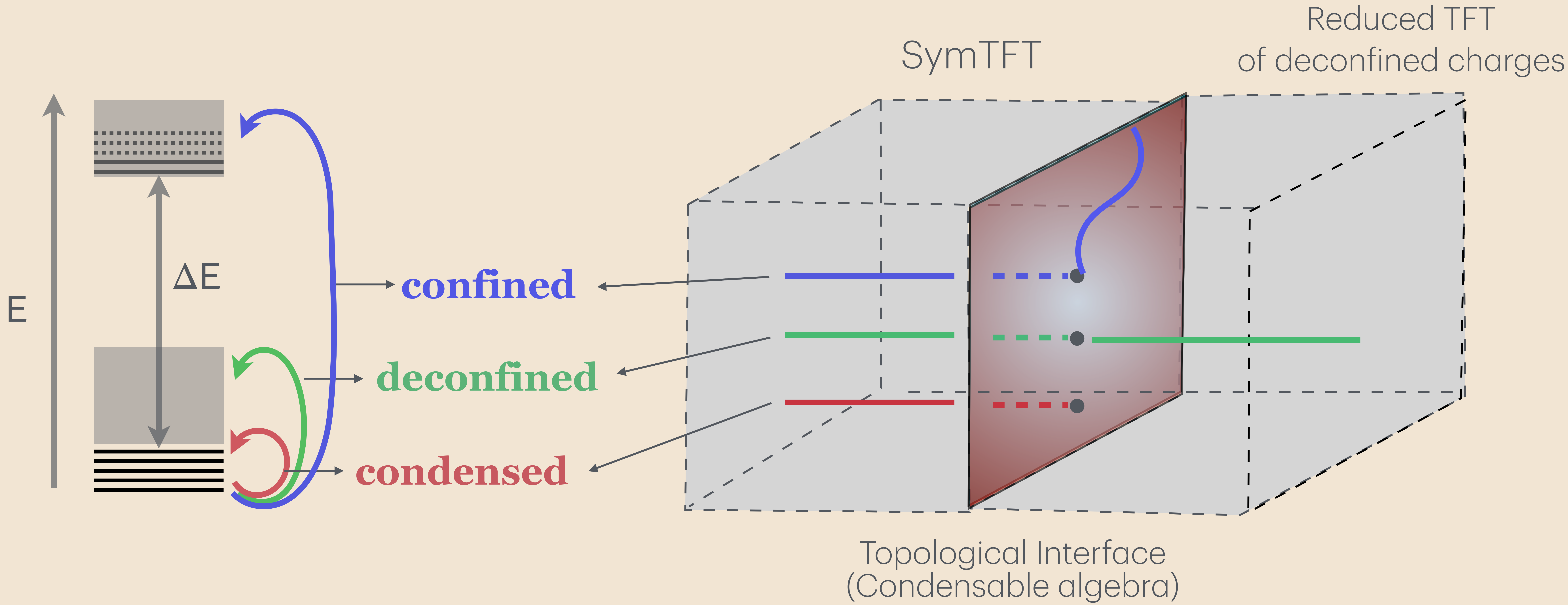

SymTFT for Fusion 2-categorical symmetries

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(SymTFT is $Z(2\mathrm{Vec}_{G}^{\pi})$)

Emergent Fermion type

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SymTFT for Fusion 2-categorical symmetries

This talk

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- **Topological Interfaces & Boundaries of** $Z(2\mathrm{Vec}_G)$ **:**

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  1. **Codimension-1 defects: All condensation defects**
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  3. **Codimension-3 defects:** $\mathcal{Q}_1^{R}$
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* $[g]$ = Conjugacy class, Z_g = Centraliser group of $g \in [g]$, $R \in \mathrm{Rep}(G)$

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  3. **Codimension-3 defects:** $\mathcal{Q}_1^{\mathrm{R}}$

- **Topological Interfaces & Boundaries of $Z(2\mathrm{Vec}_{\mathrm{G}})$:**

  1. **Math approach: Based on etale algebras.** **[Xu, Decoppet ' 24, talk by Zhihao Zhang]**
  2. **Physics based approach: Based on gauging boundary conditions.**
     [Bhardwaj, Nameki, Pajer, Tiwari, Warman, Wu ' 24]

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Gapped Boundaries via gauging

- Gapped boundaries are constructible by generalized gauging on the Dirichlet boundary ($\mathfrak{B}_{\text{Dir}}$) i.e., $\mathfrak{B}^{\text{T}}_{\text{Neu(H)}} = \left[\mathfrak{B}_{\text{Dir}} \boxtimes \text{T}\right]/H$.

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- Non-minimal boundaries : $\mathbf{T}$ is a non-trivial topological order.

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- **Non-minimal boundaries** : T is a non-trivial topological order.

<u>Example</u>:

$$\mathfrak{B}^{\text{T}}_{\text{Neu}(\mathbb{Z}_2)} = \frac{\mathfrak{B}_{\text{Dir}(\mathbb{Z}_2)} \boxtimes \text{Toric Code}}{\mathbb{Z}_2}$$

1. As symmetry boundaries: non-minimal $\mathbb{Z}_2$ 1-form symmetry ($\{1, \psi, \bar{\psi}, \psi\bar{\psi}, \sigma\bar{\sigma}\}$)
2. As physical boundaries:
   i. Non-minimal $\mathbb{Z}_2$ 0-form preserving phase (Toric code)
   ii. Non-minimal $\mathbb{Z}_2$ 1-form breaking phase (Doubled Ising topological order)

S_3 SymTFT

- SymTFT(2Vec(S_3)) = SymTFT(2Rep($\mathbb{G}$)) = 4d S_3 Dijkgraaf-Witten Theory

$\rightarrow \mathbb{G} = \mathbb{Z}_3^{(1)} \rtimes \mathbb{Z}_2^{(0)}$

S_3 conventions
Group structure: $S_3 = \langle a\,,b\ |\, a^3 = b^2 = 1\,, bab = a^2 \rangle$
Conjugacy classes: $[\mathrm{id}]\,, [a] = \{a\,, a^2\}\,, [b] = \{b\,, ab\,, a^2 b\}$
Irreducible Representations: $\{1\,, P\,, E\}$
Fusion of irreps: $P \otimes P = 1\,, \quad P \otimes E = E \otimes P = 1 \oplus P \oplus E$

- Bulk defects:

Surfaces	Lines
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$Q_2^{[b]}$	Q_1^{E}

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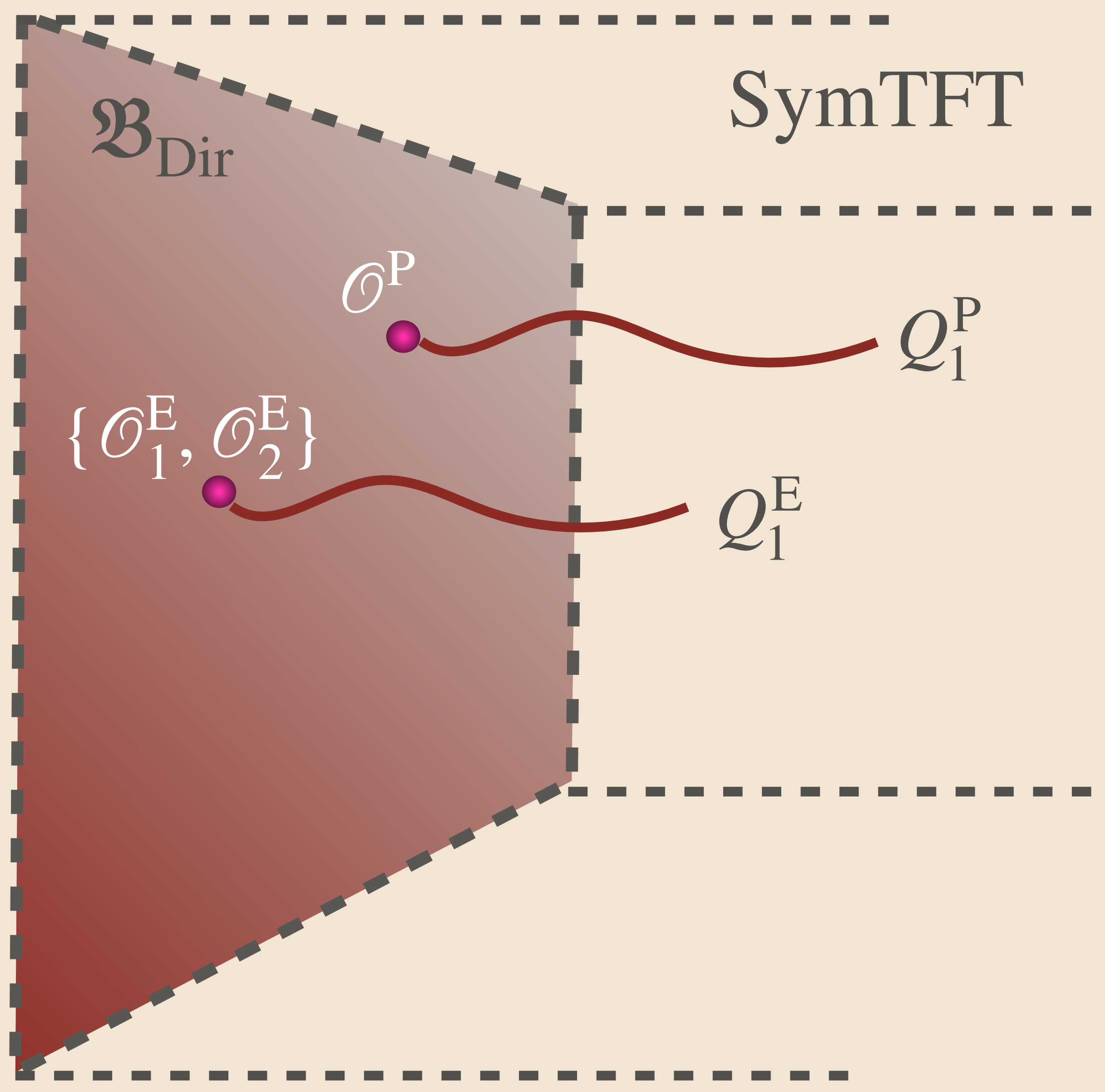
Organize the generalized charges. Other defects obtained by condensation.

Dirichlet and Neumann(Z_2) Boundaries of S3 SymTFT

Dirichlet and Neumann(Z_2) Boundaries of S3 SymTFT

Dirichlet boundary



Dirichlet and Neumann(Z_2) Boundaries of S3 SymTFT

Dirichlet
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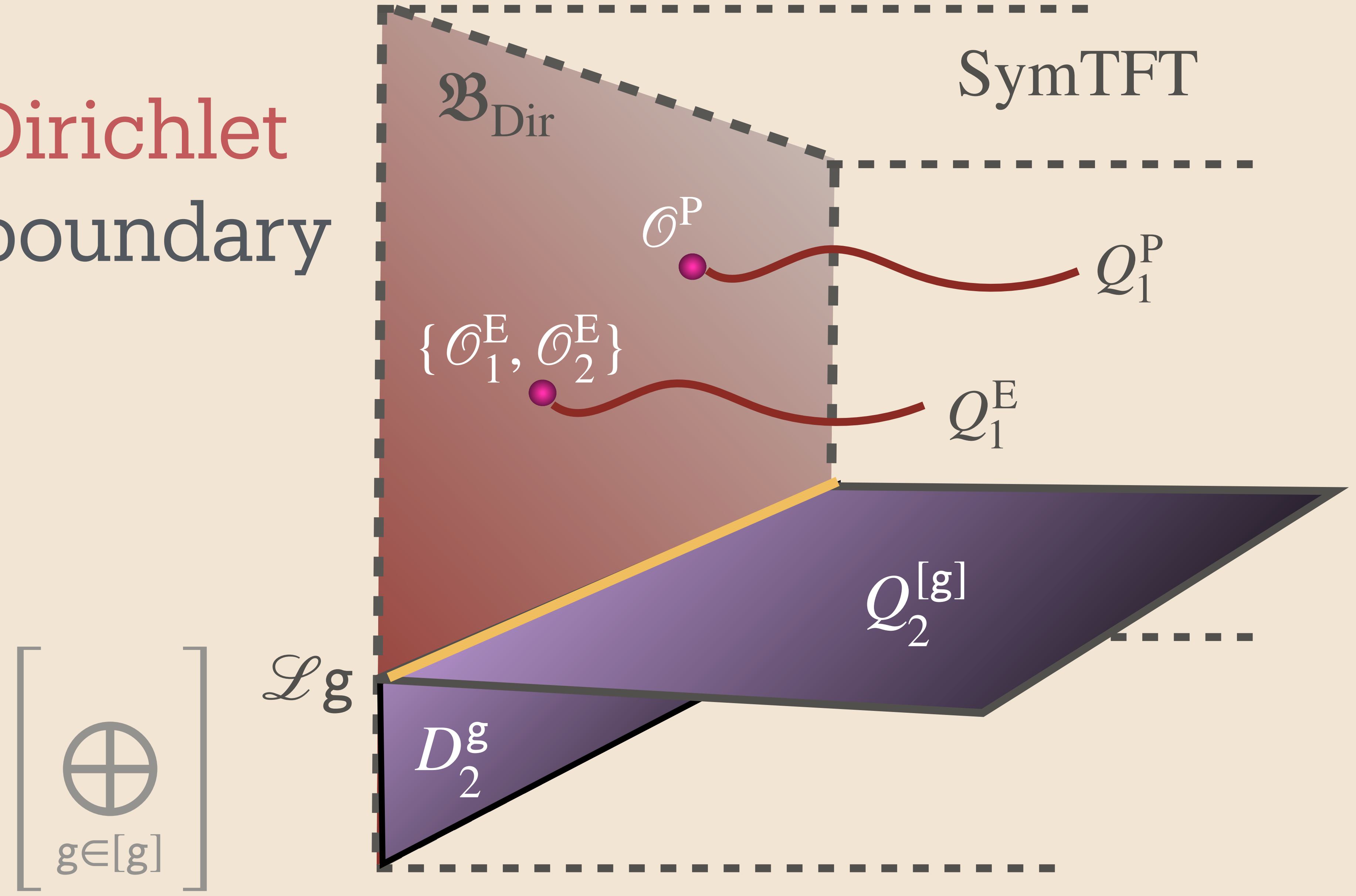
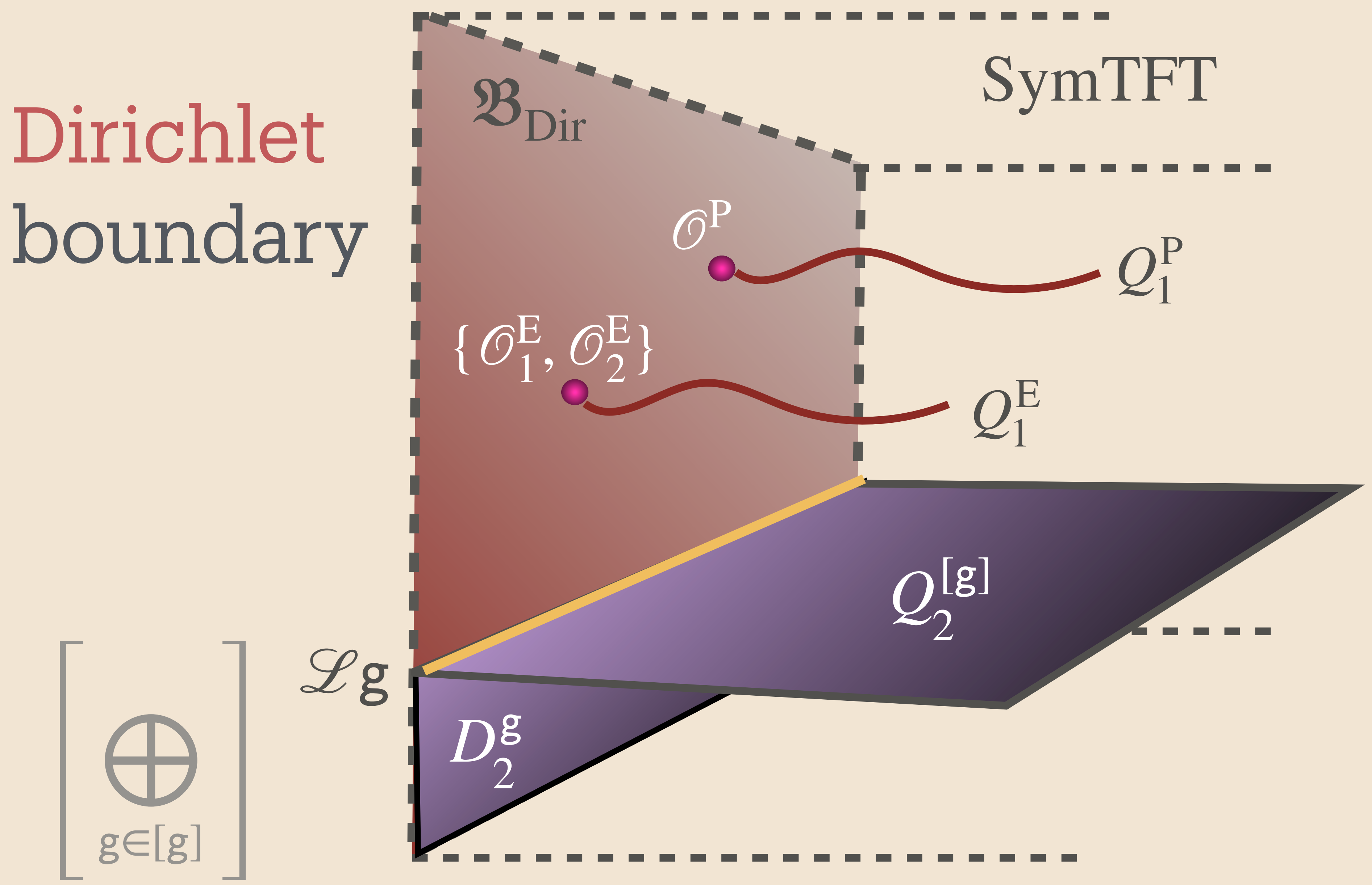


Dirichlet and Neumann(Z_2) Boundaries of S3 SymTFT



Boundary defect category $\mathbf{2Vec}_{S_3}$ generated by $\mathbf{D}_2^{\mathbf{g}}$.

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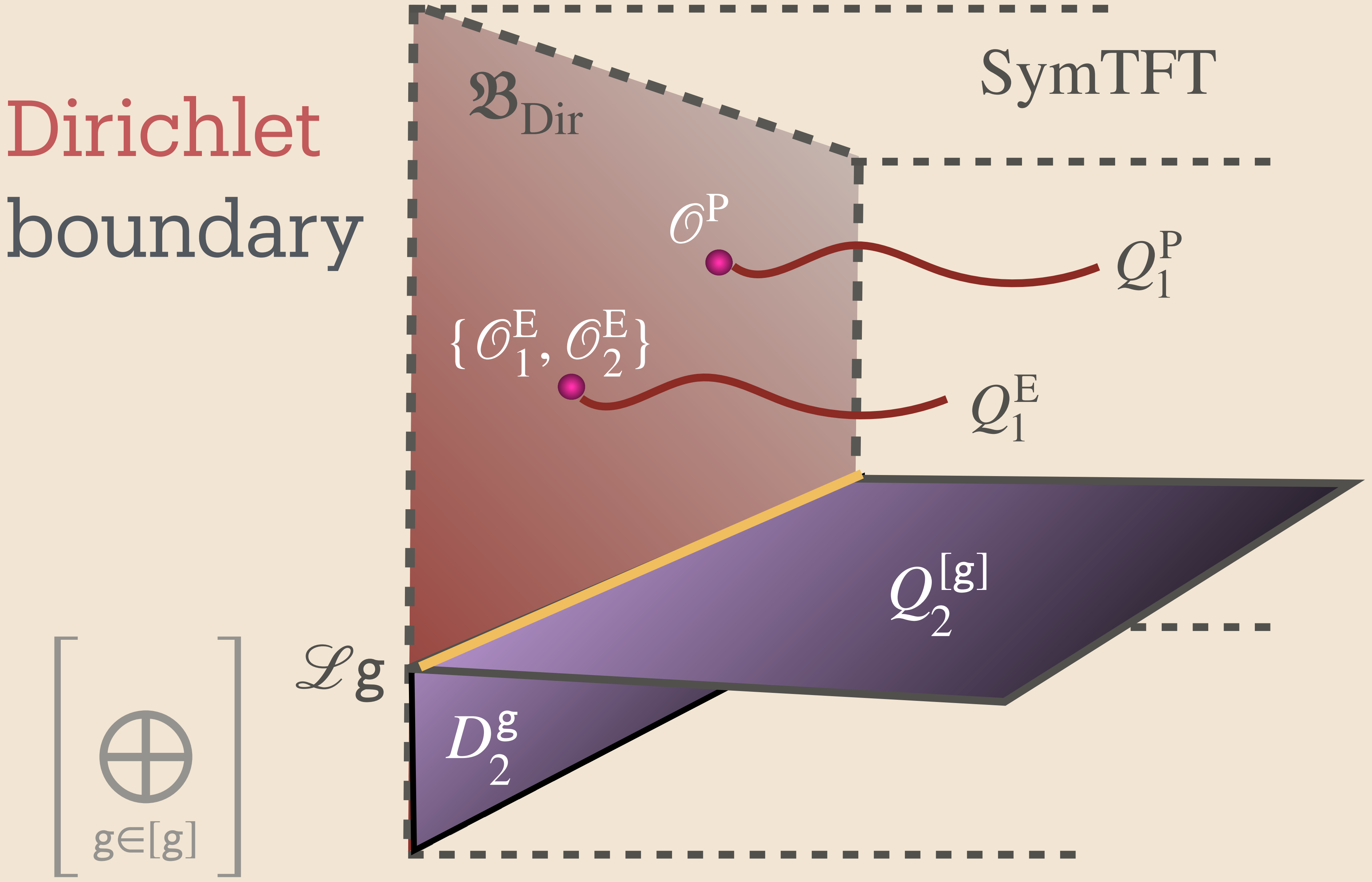

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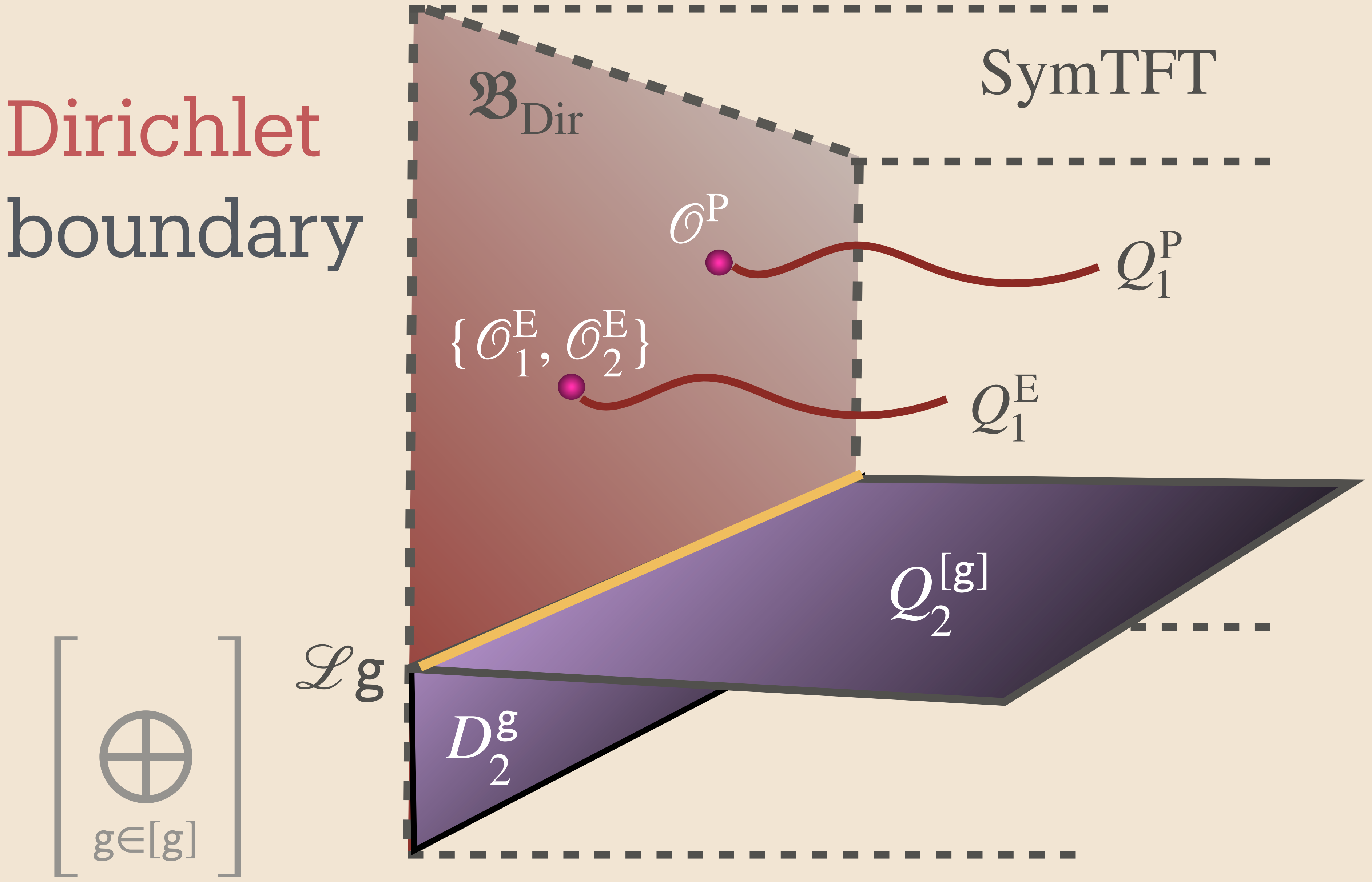
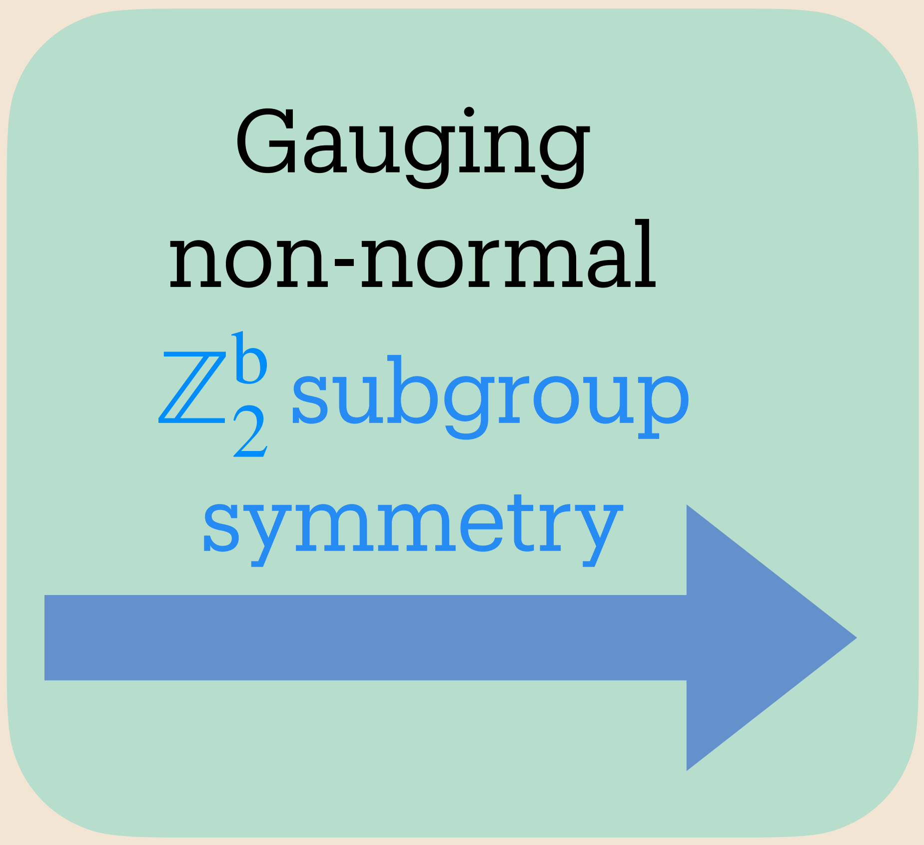
Dirichlet
boundary





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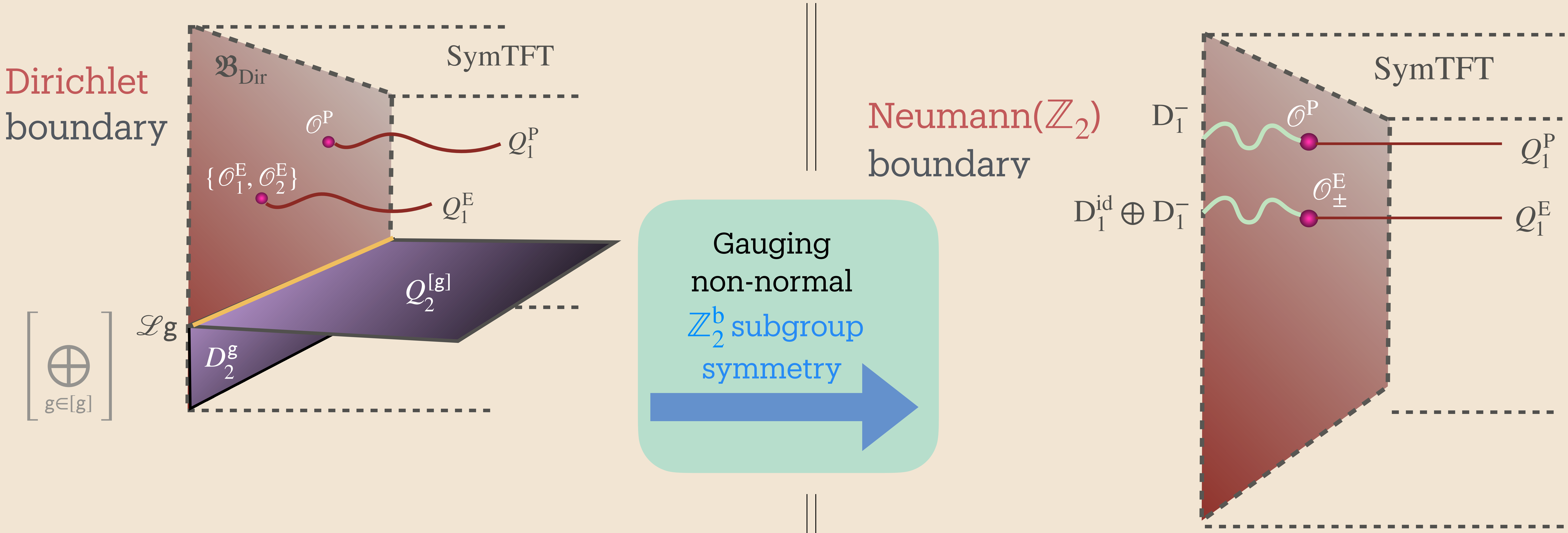

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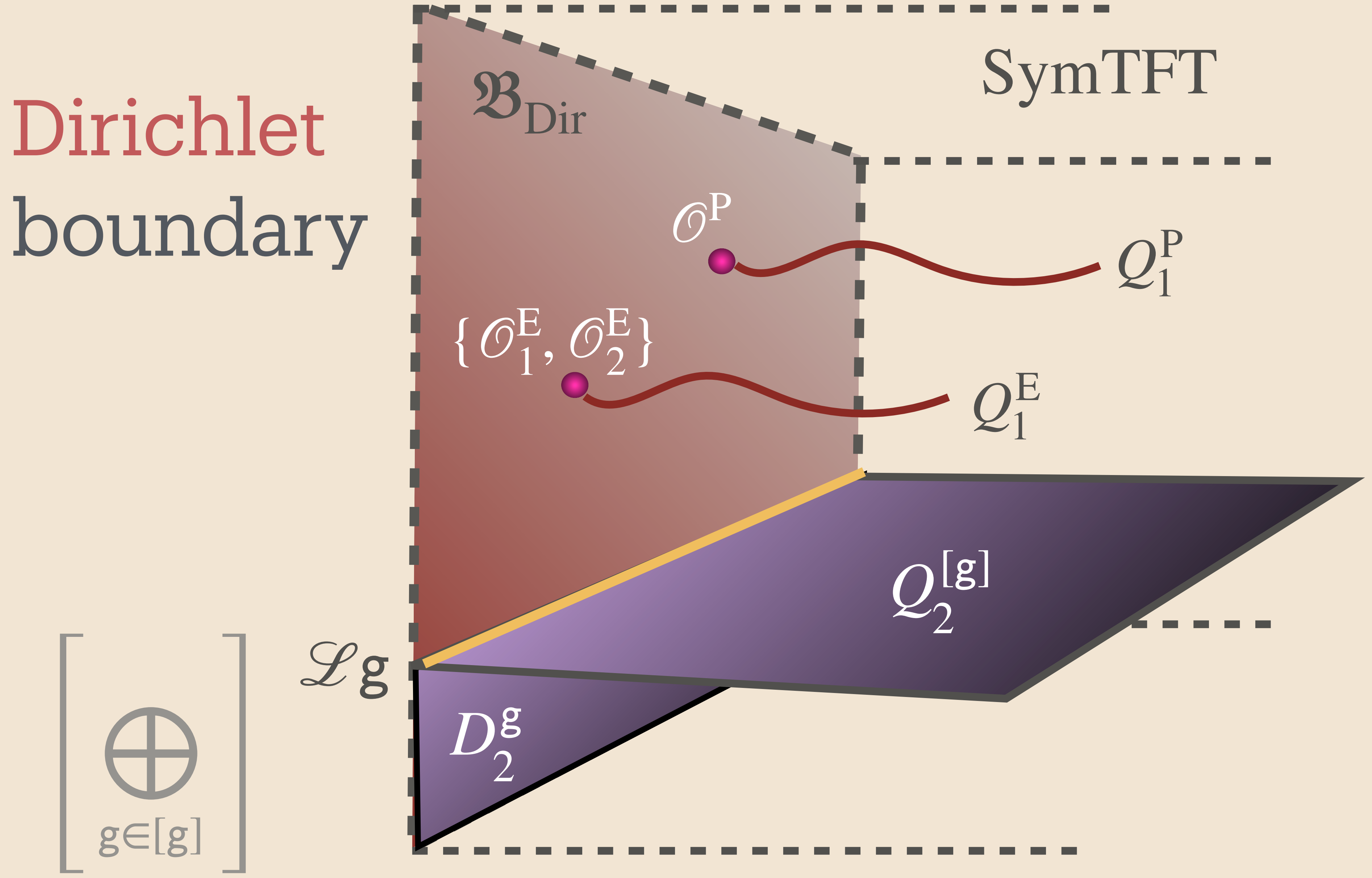

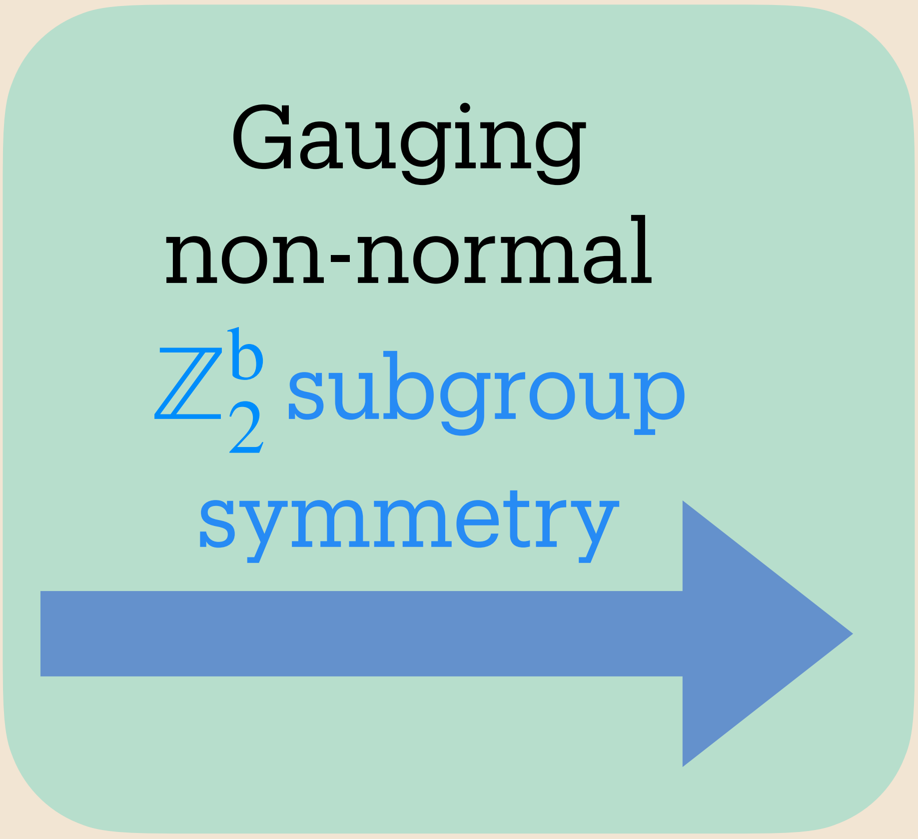

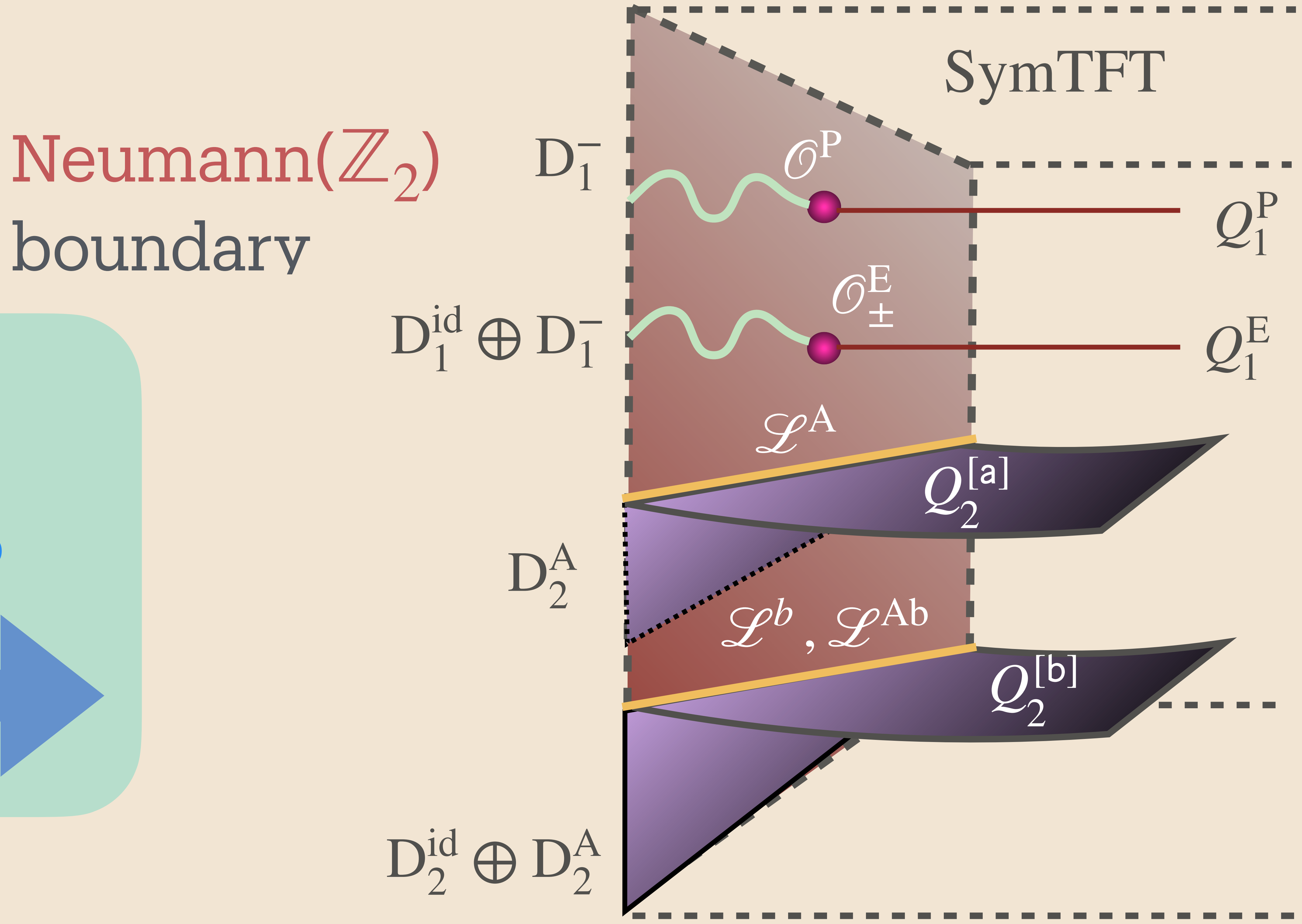

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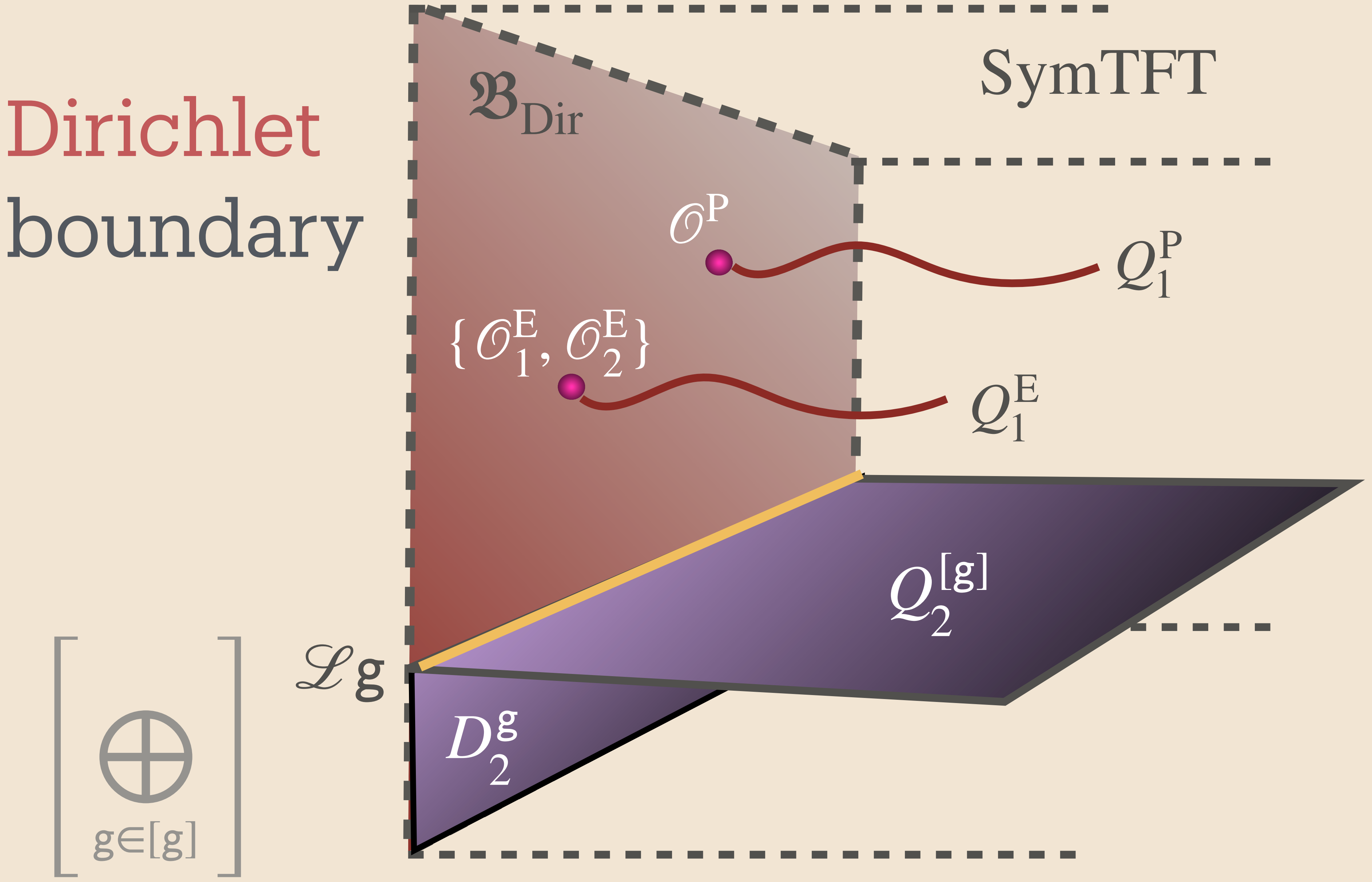
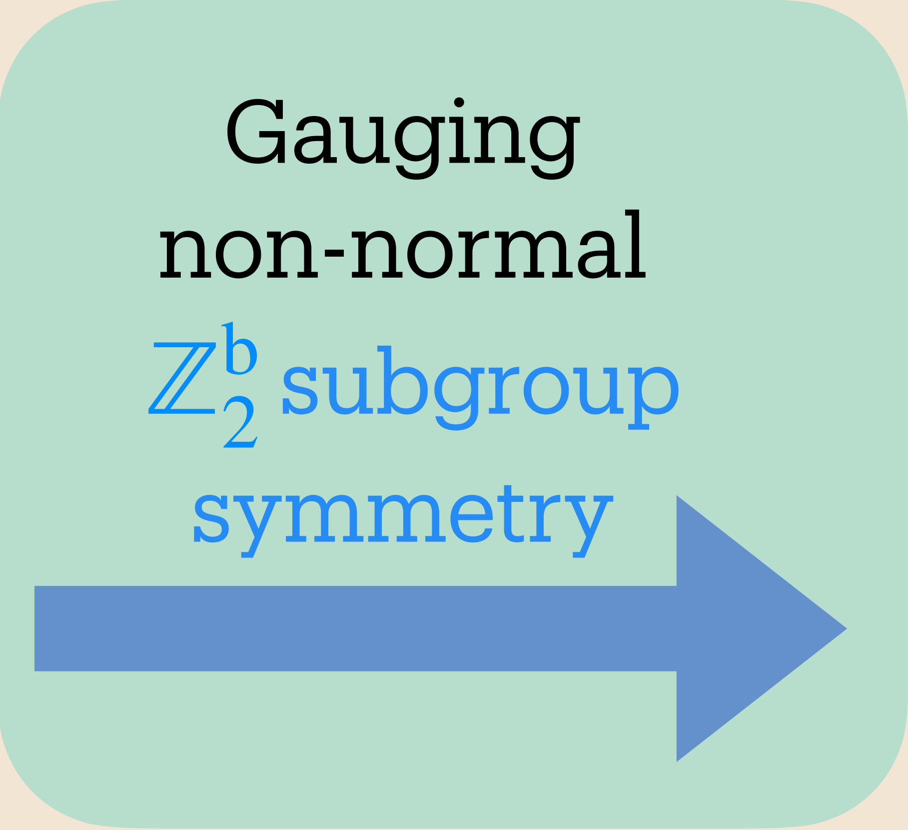
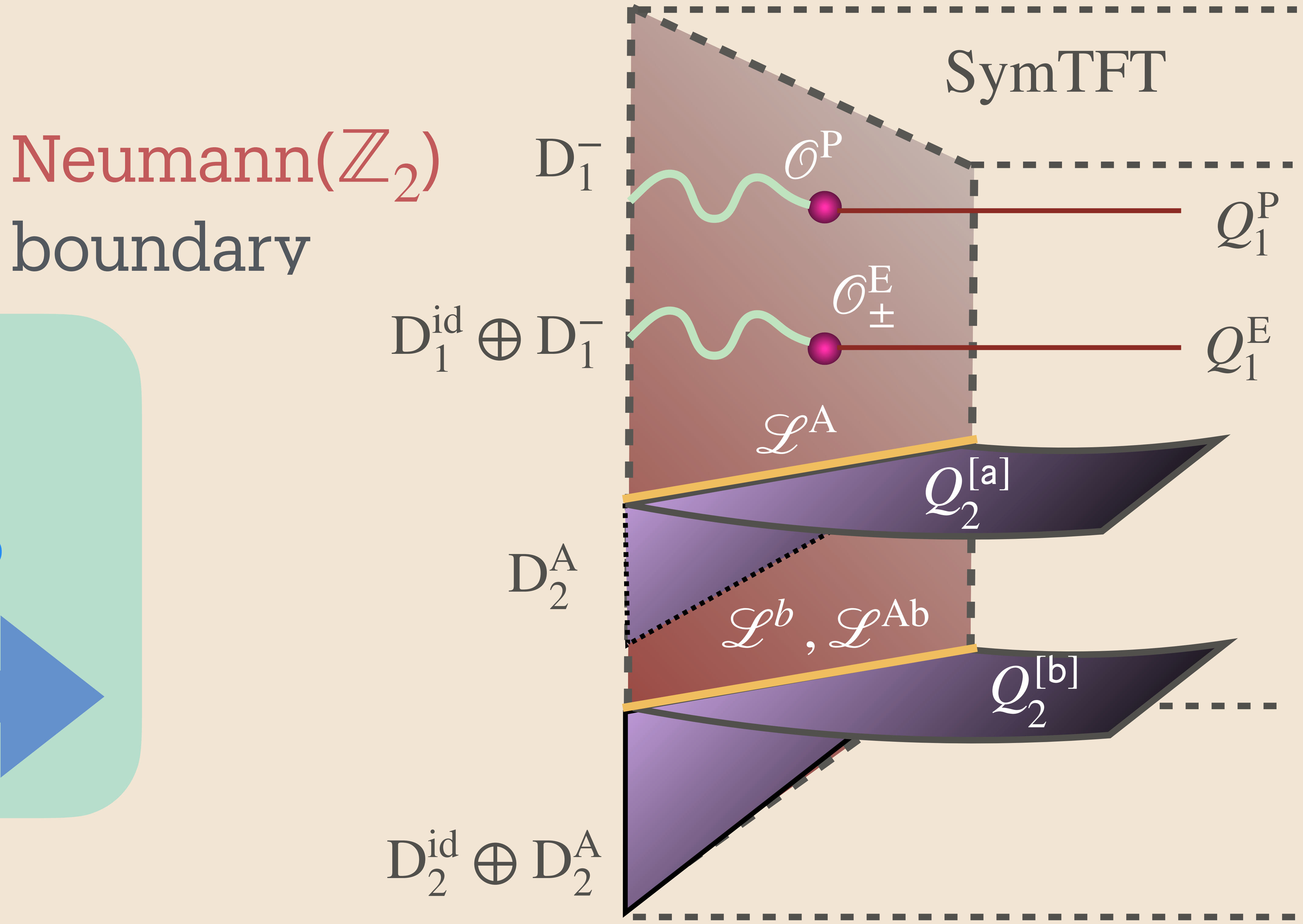






Boundary defect category $\mathbf{2Vec}_{S_3}$ generated by D_2^{g}.

Boundary defect category $\mathbf{2Rep}(\mathbb{Z}_3^{(1)} \rtimes \mathbb{Z}_2^{(0)})$:

$$\text{D}_2^{\text{A}} \otimes \text{D}_2^{\text{A}} = \text{D}_2^{\text{A}} \oplus \text{D}_2^{\text{Cond}}$$

$$\text{D}_2^{\text{A}} \otimes \text{D}_2^{\text{Cond}} = 2\text{D}_2^{\text{A}}$$

$$\text{D}_1^- \otimes \text{D}_1^- = \text{D}_1^{\text{id}}$$

* Additional choice of discrete torsion

Summary of generalized charges

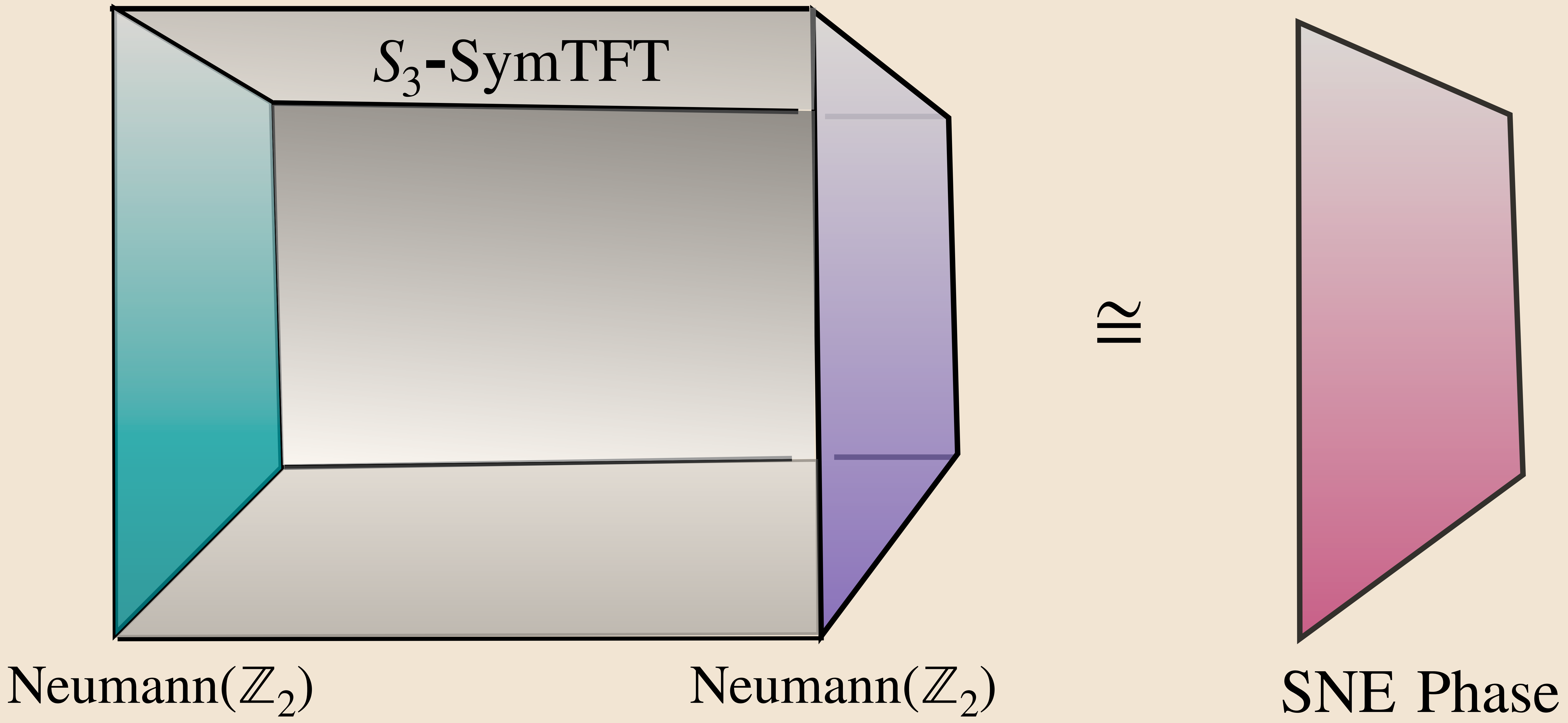
	$\mathfrak{B}_{\text{sym}} = \mathfrak{B}_{\text{Dir}}$	$\mathfrak{B}_{\text{sym}} = \mathfrak{B}_{\text{Neu}(\mathbb{Z}_2)}$
Q_1^{P}	$\mathcal{O}^{\text{P}}$	$(\mathcal{O}^{\text{P}}, \text{D}_1^-)$
Q_1^{E}	$\mathcal{O}^{\text{E}} = \{\mathcal{O}_1^{\text{E}}, \mathcal{O}_2^{\text{E}}\}$	$\{\mathcal{O}_+^{\text{E}}, (\mathcal{O}_-^{\text{E}}, \text{D}_1^-)\}$
$Q_2^{[\text{a}]}$	$\{(\mathcal{L}^{\text{a}}, \text{D}_2^{\text{a}}), (\mathcal{L}^{\text{a}^2}, \text{D}_2^{\text{a}^2})\}$	$(\mathcal{L}^{\text{A}}, \text{D}_2^{\text{A}})$
$Q_2^{[\text{b}]}$	$\{(\mathcal{L}^{\text{b}}, \text{D}_2^{\text{b}}), (\mathcal{L}^{\text{ab}}, \text{D}_2^{\text{ab}}), (\mathcal{L}^{\text{a}^2\text{b}}, \text{D}_2^{\text{a}^2\text{b}})\}$	$\{\mathcal{L}^{\text{b}}, (\mathcal{L}^{\text{Ab}}, \text{D}_2^{\text{Ab}})\}$

* Relative (twisted sector) point and line operators denoted as $(\mathcal{O}, D_1^x)$ and $(\mathcal{L}, D_2^y)$ respectively.

SNE Phase

Spontaneously non-uniform entangled phase

Spontaneously non-uniform entangled phase



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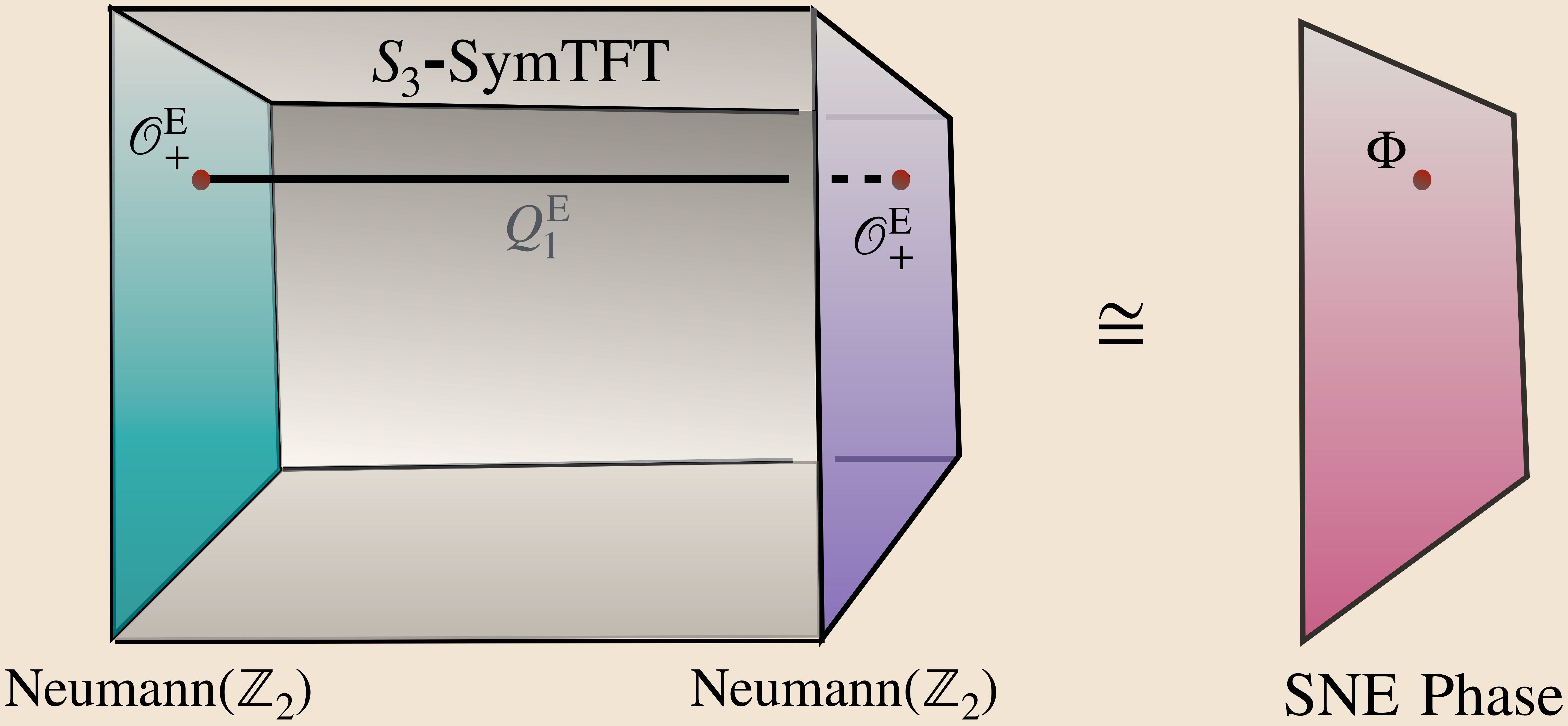

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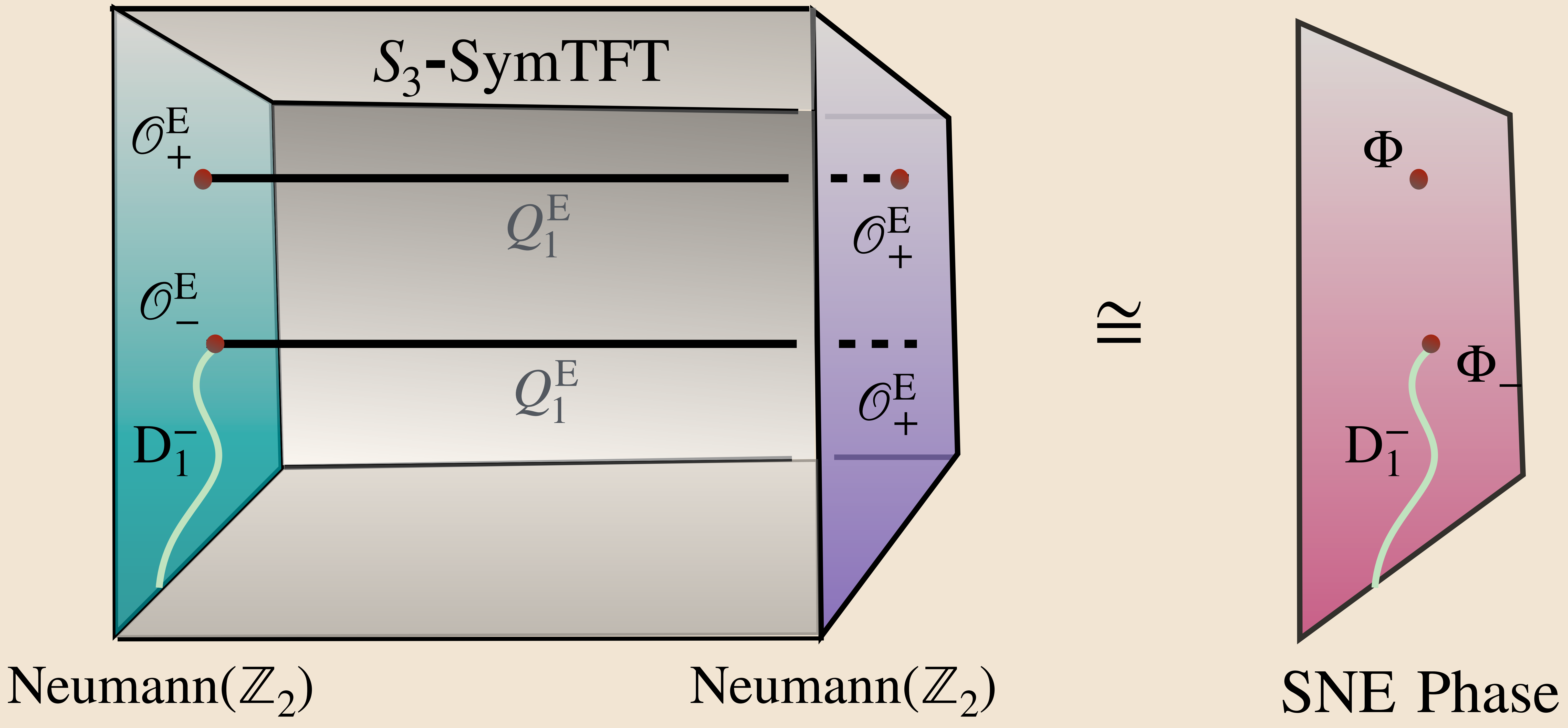
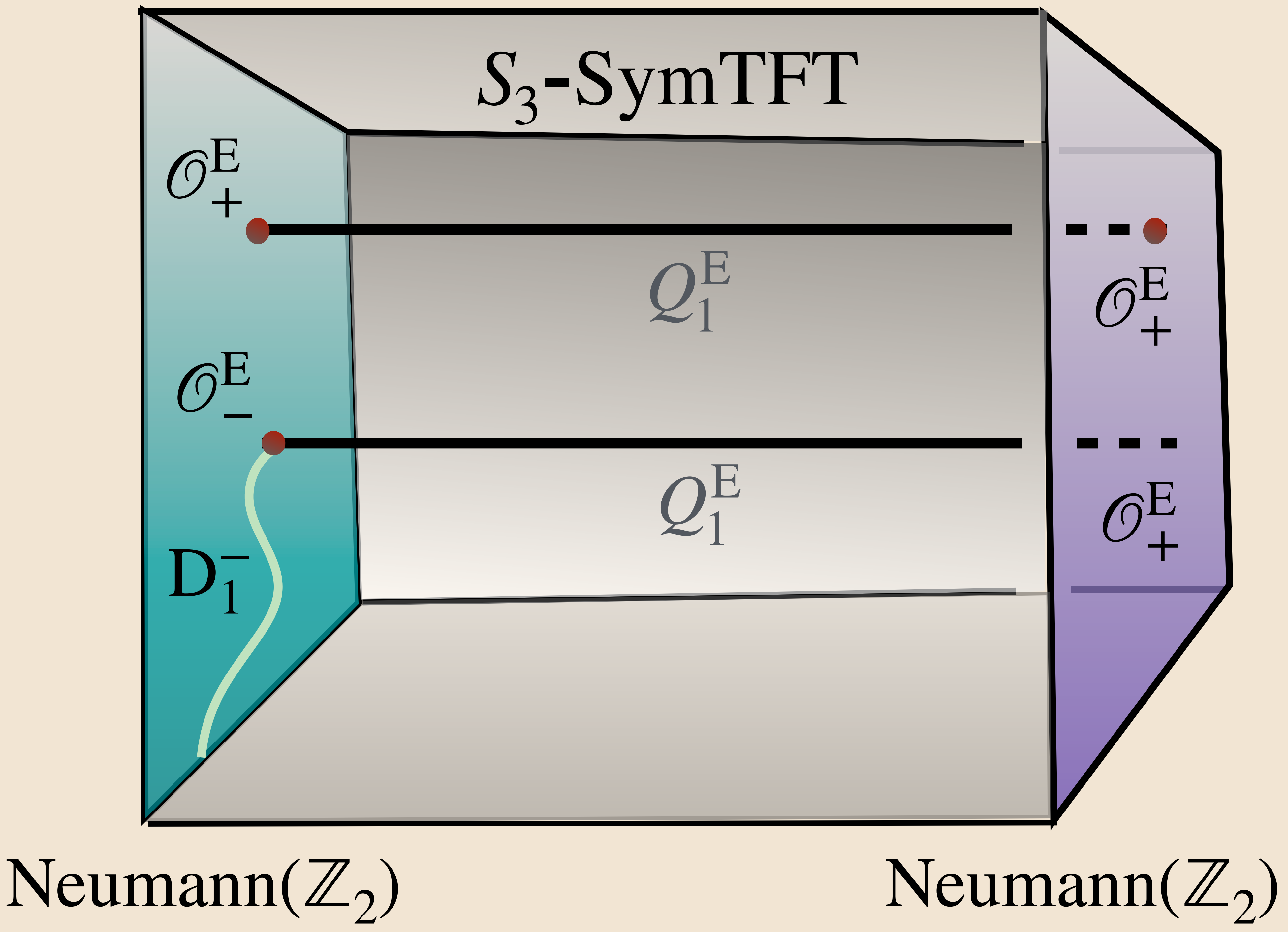
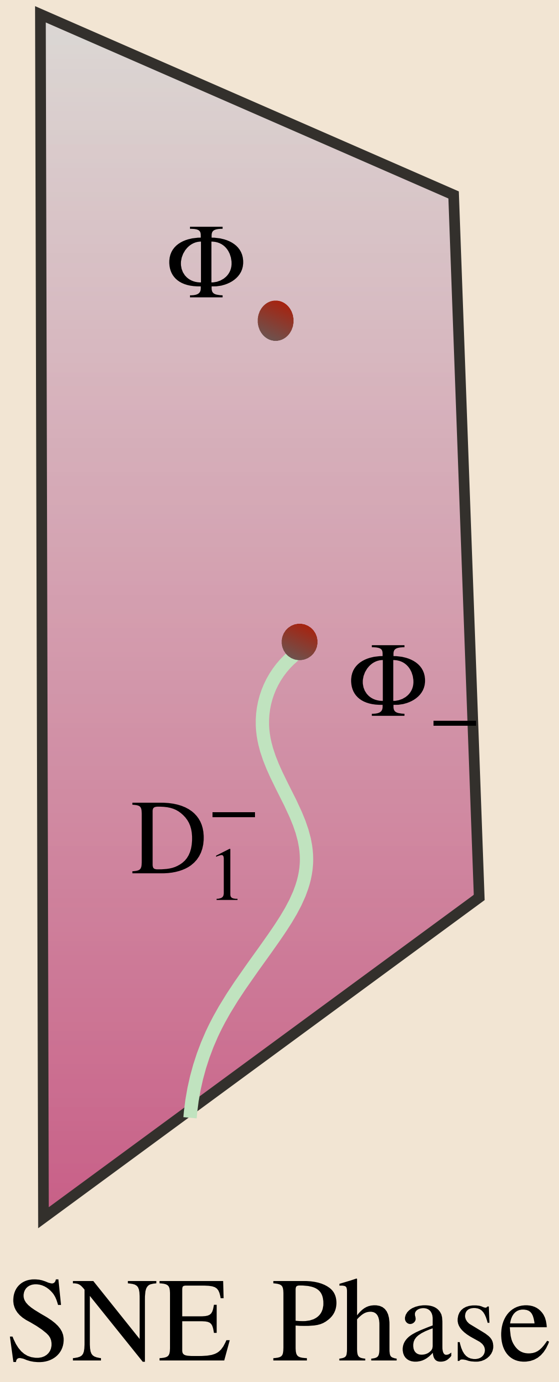


Spontaneously non-uniform entangled phase



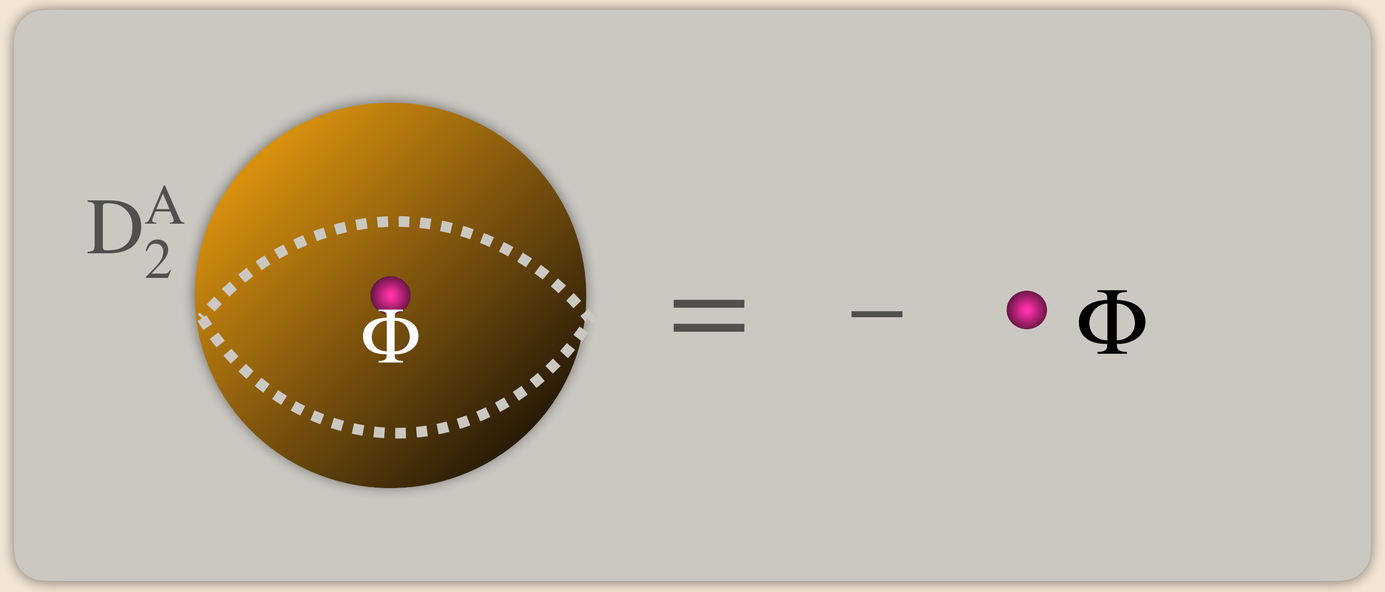
$\cong$



Algebra of local operators

$$\Phi \times \Phi = \Phi + 2$$
$$\Phi \times \Phi_- = -\Phi_-$$

Symmetry action



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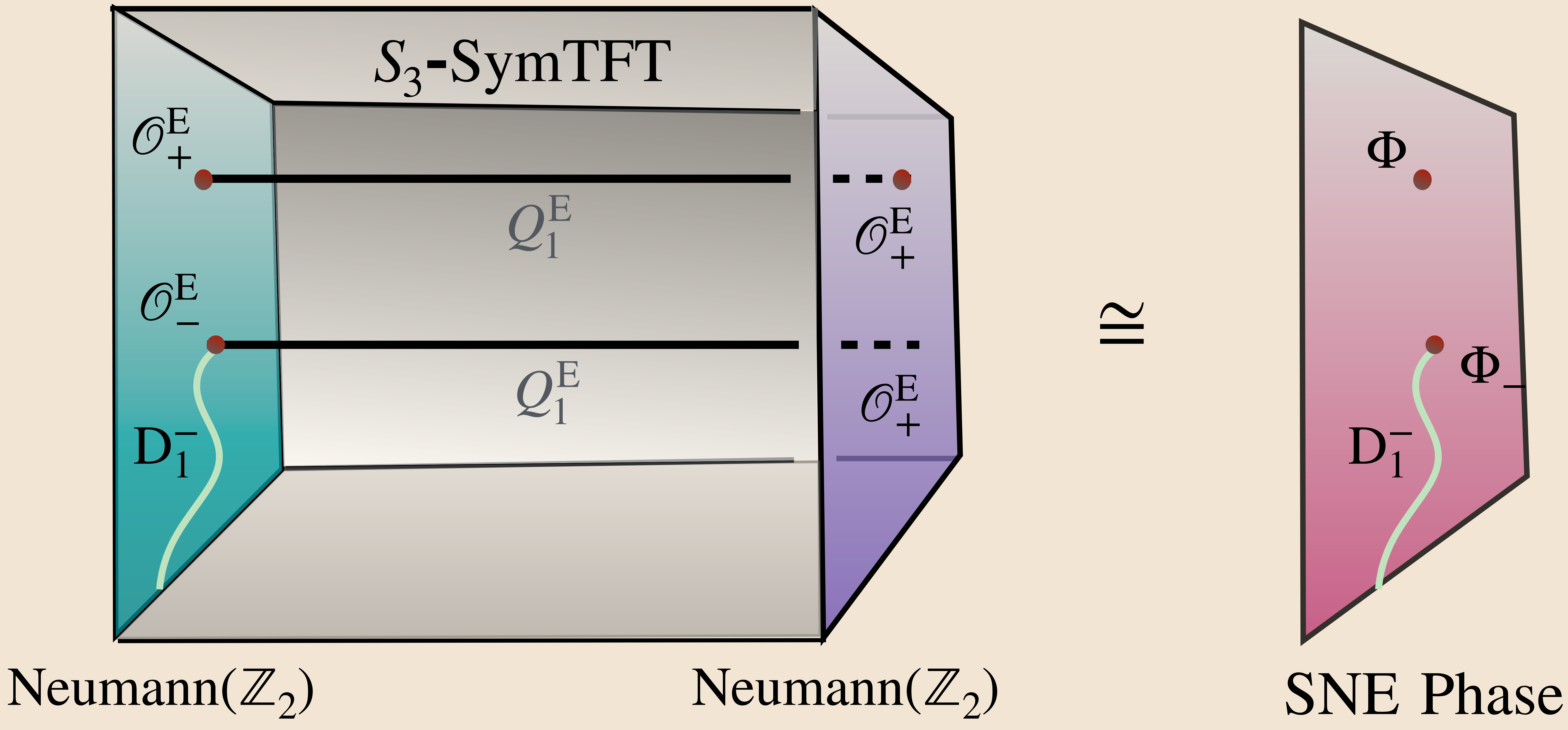

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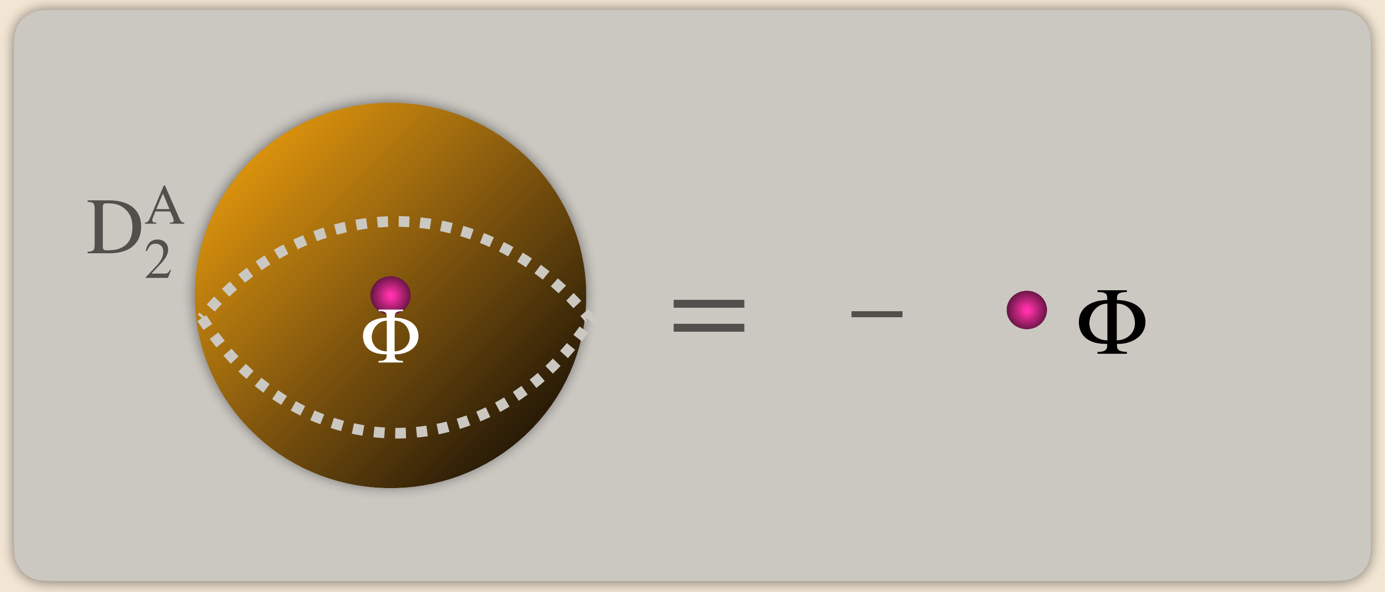
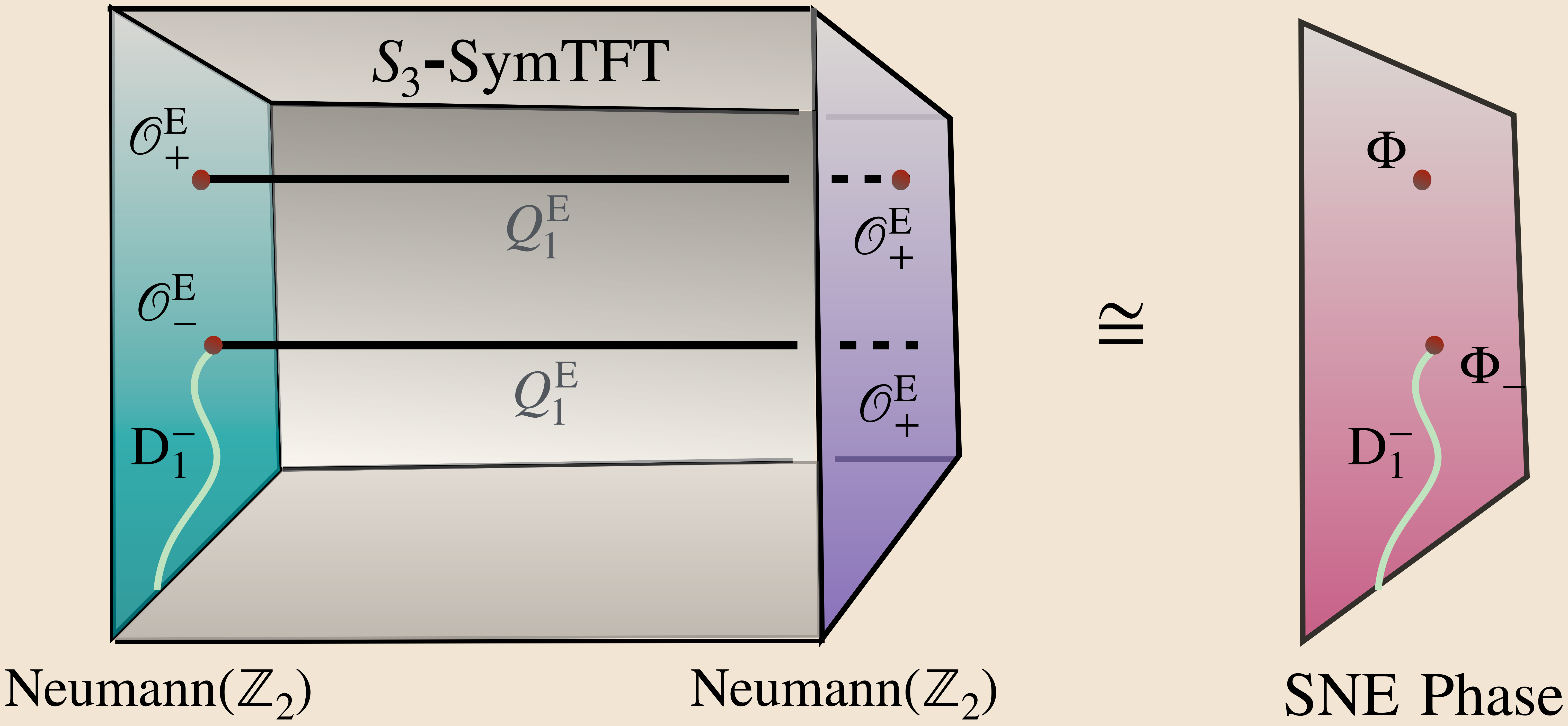


Vacua (idempotents) are:

$$v_0 = (2 - \Phi)/3\,,$$
$$v_1 = (1 + \Phi)/3$$

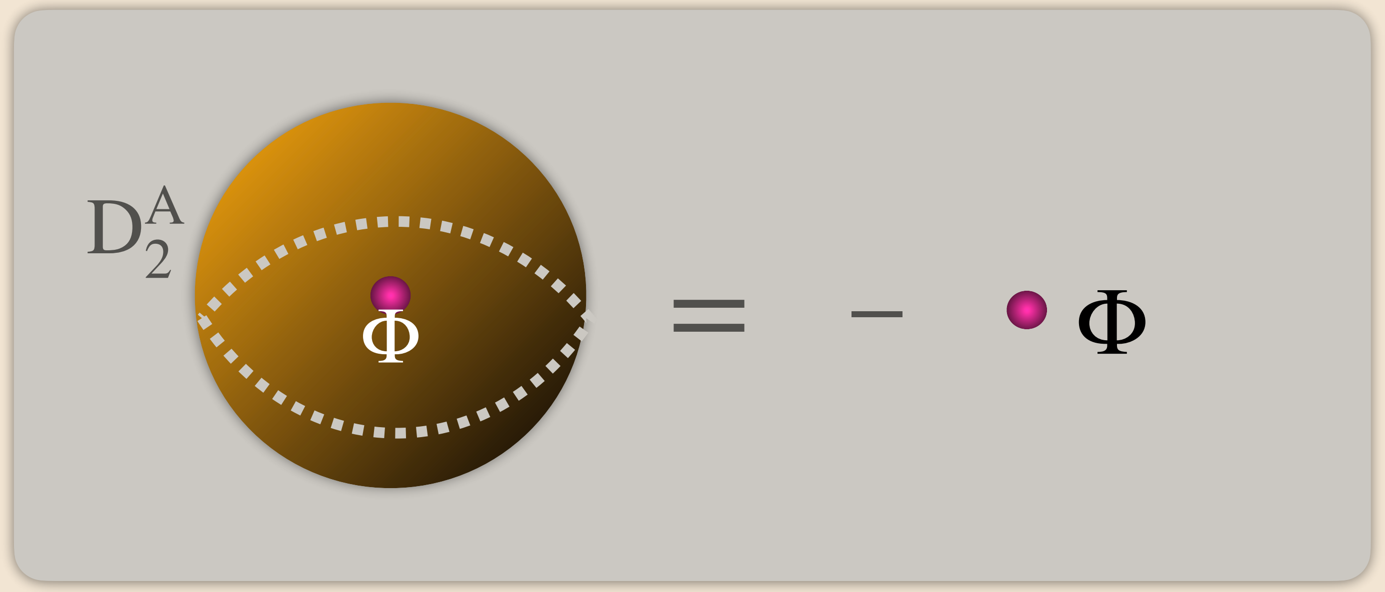
Spontaneously non-uniform entangled phase



Algebra of local operators

$$\Phi \times \Phi = \Phi + 2$$
$$\Phi \times \Phi_{-} = -\Phi_{-}$$

Symmetry action



Vacua (idempotents) are:

$$\mathrm{v}_0 = (2 - \Phi)/3\,,$$
$$\mathrm{v}_1 = (1 + \Phi)/3$$

Z2 1-form properties:

$\Phi_{-} \cdot \mathrm{v}_0 = \mathrm{v}_0$ (Z2 1-form preserving)

$\Phi_{-} \cdot \mathrm{v}_1 = 0$ (Z2 1-form breaking)

Spontaneously non-uniform entangled phase

Spontaneously non-uniform entangled phase

Non-invertible symmetry action:

$$\mathrm{D}_2^{\mathrm{A}} : (\mathrm{v}_0\,, \mathrm{v}_1) \longrightarrow (\mathrm{v}_0 + 2\mathrm{v}_1\,, \mathrm{v}_0)$$

$$\mathrm{D}_2^{\mathrm{A}} = 1_{00} \oplus \mathrm{B}_{01} \oplus \overline{\mathrm{B}}_{10}\,.$$

Spontaneously non-uniform entangled phase

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Sanity check:

$$\mathrm{D}_2^{\mathrm{A}} \otimes \mathrm{D}_2^{\mathrm{A}} = \overbrace{(1_{00} \oplus \mathrm{B}_{01} \oplus \overline{\mathrm{B}}_{10})}^{\mathrm{D}_2^{\mathrm{A}}} \oplus \overbrace{(\mathrm{B}\overline{\mathrm{B}})_{00} \oplus (\overline{\mathrm{B}}\mathrm{B})_{11}}^{\mathrm{D}_2^{\mathrm{Cond}}}\,.$$
$$= \mathrm{D}_2^{\mathrm{A}} \oplus \mathrm{D}_2^{\mathrm{Cond}}\,.$$

Spontaneously non-uniform entangled phase

Non-invertible symmetry action:

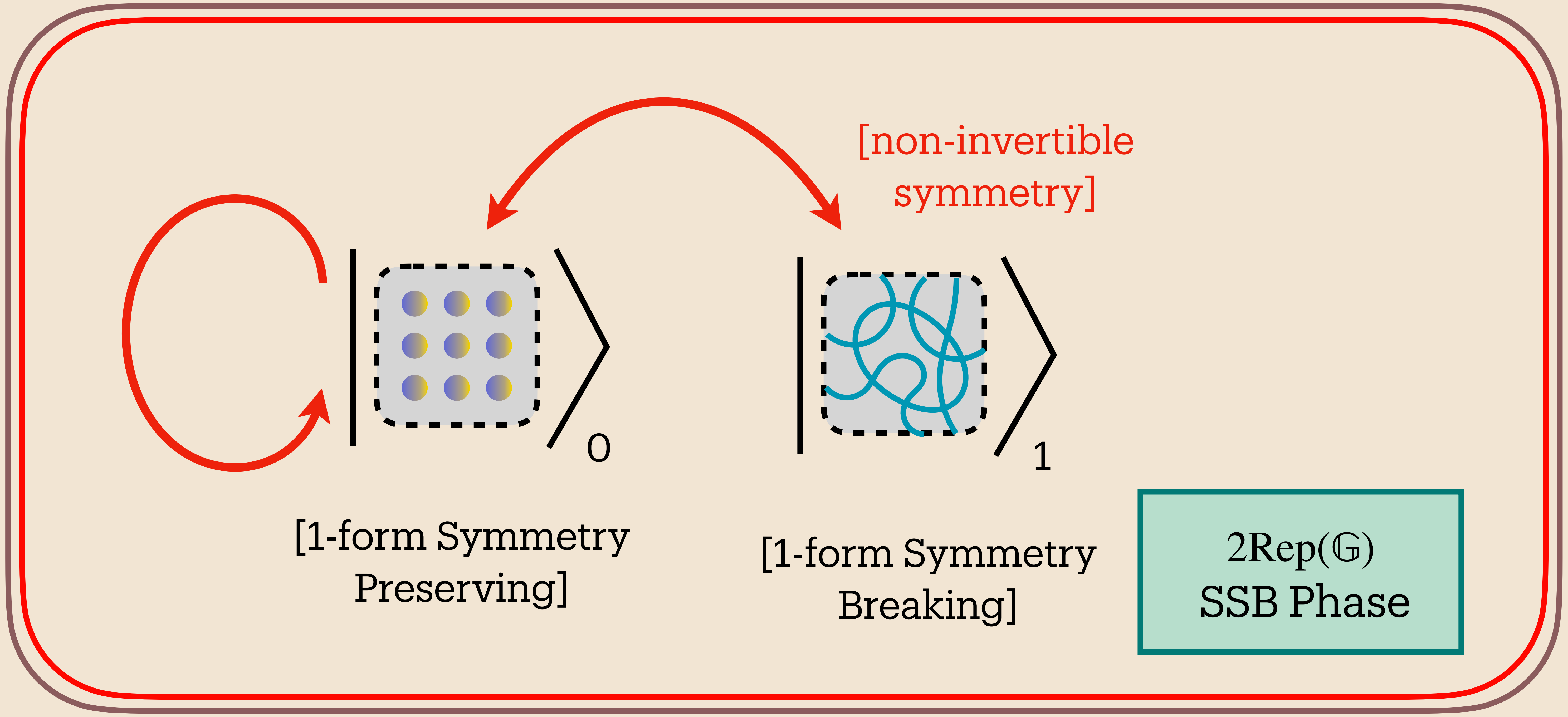
$$D_2^A : (v_0\,, v_1) \longrightarrow (v_0 + 2v_1\,, v_0)$$

$$D_2^A = 1_{00} \oplus B_{01} \oplus \overline{B}_{10}\,.$$

Sanity check:

$$D_2^A \otimes D_2^A = \overbrace{(1_{00} \oplus B_{01} \oplus \overline{B}_{10})}^{D_2^A} \oplus \overbrace{(B\overline{B})_{00} \oplus (\overline{B}B)_{11}}^{D_2^{\mathrm{Cond}}}\,.$$

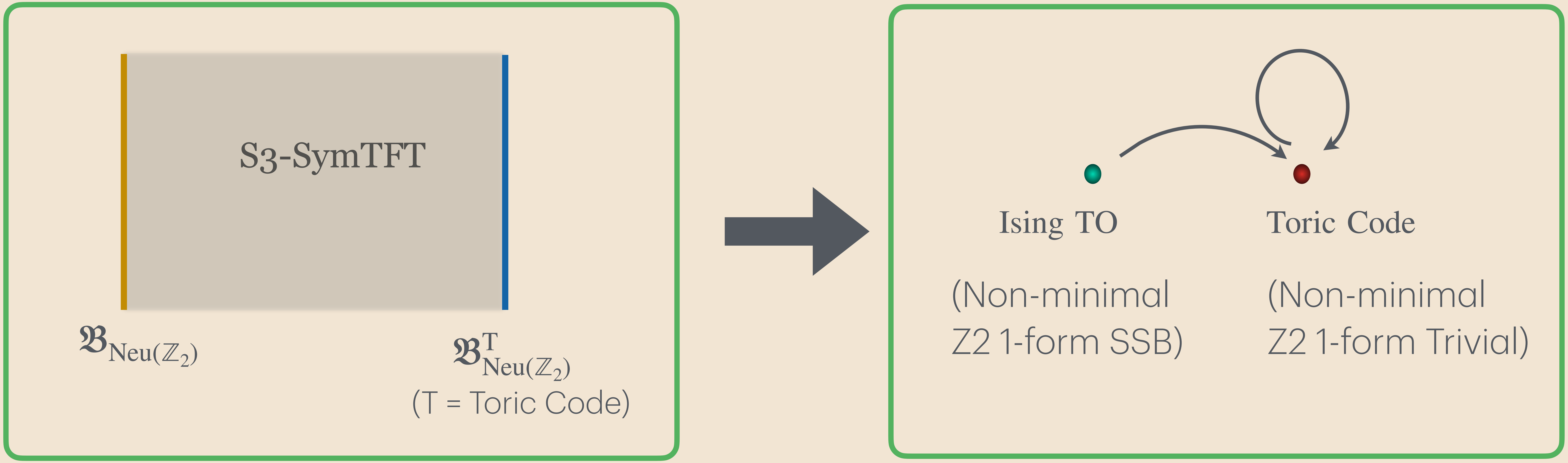
$$= D_2^A \oplus D_2^{\mathrm{Cond}}\,.$$



Non-minimal SNE Phases

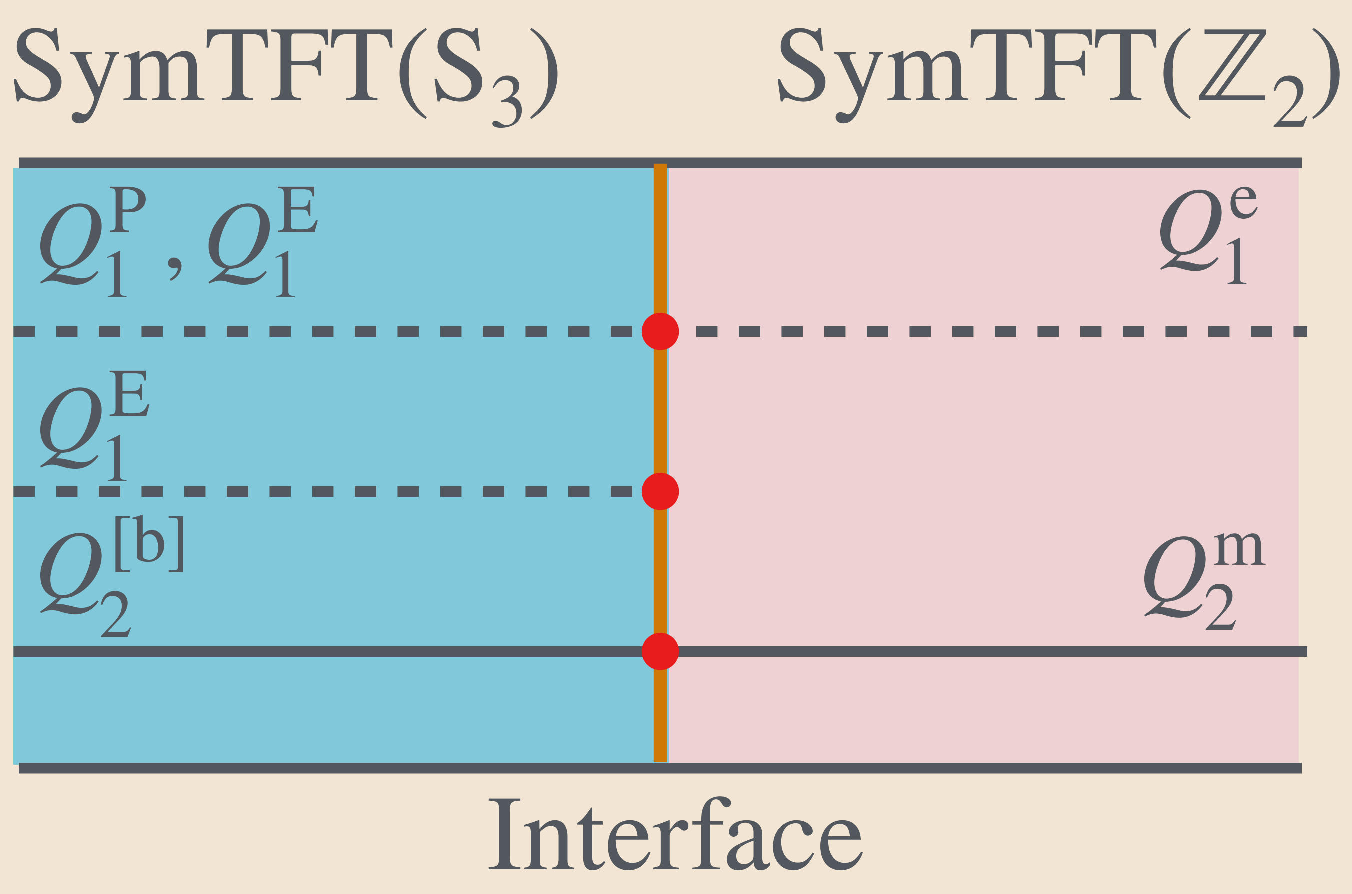
SymTFT setup

Non-minimal SNE Phase



A 2nd order transition involving SNE phase

- **Consider the Q_1^{E} condensed interface in the S3 SymTFT**



implements a condensation to the Z2 SymTFT (3+1d Toric Code)

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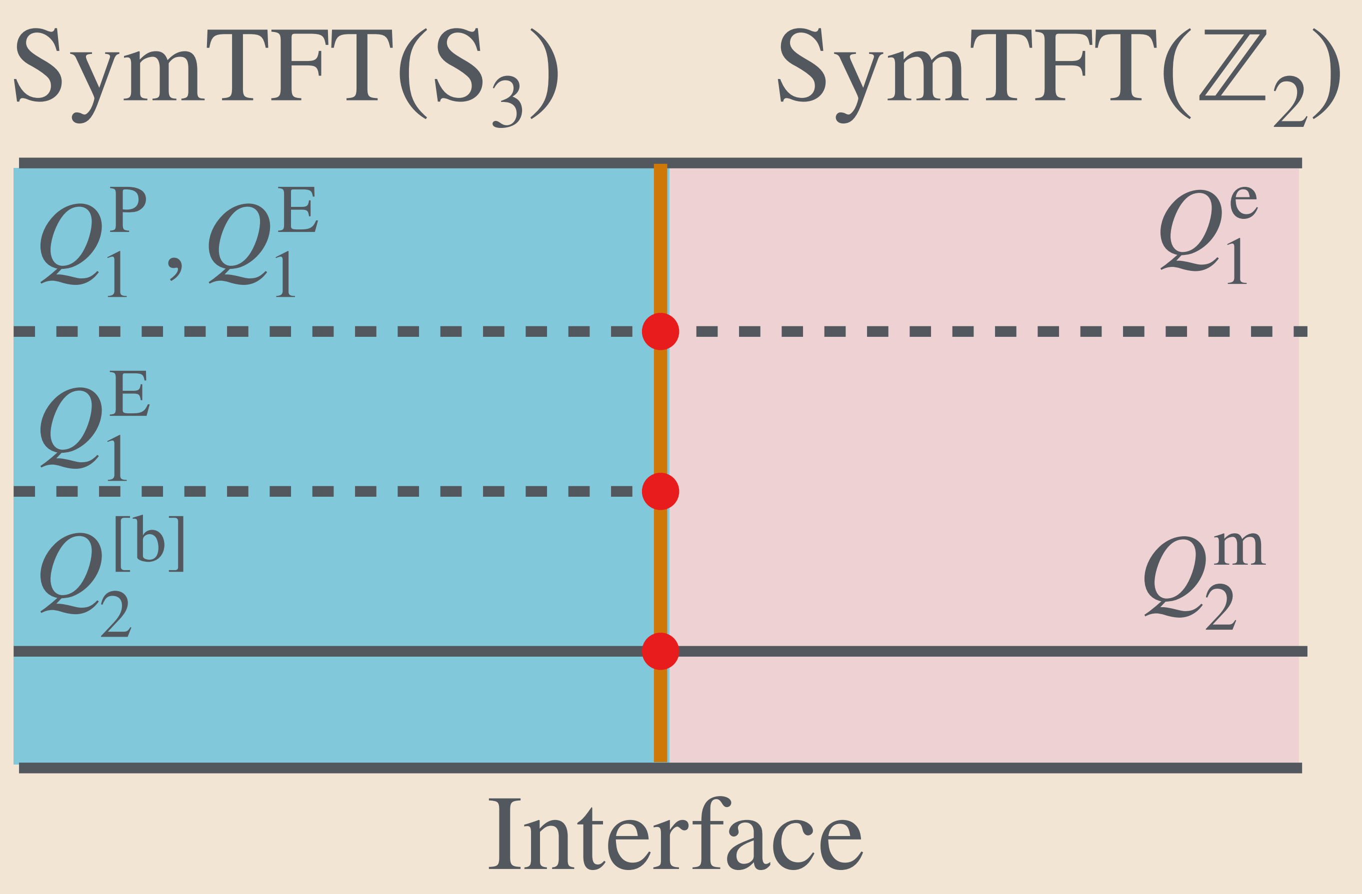
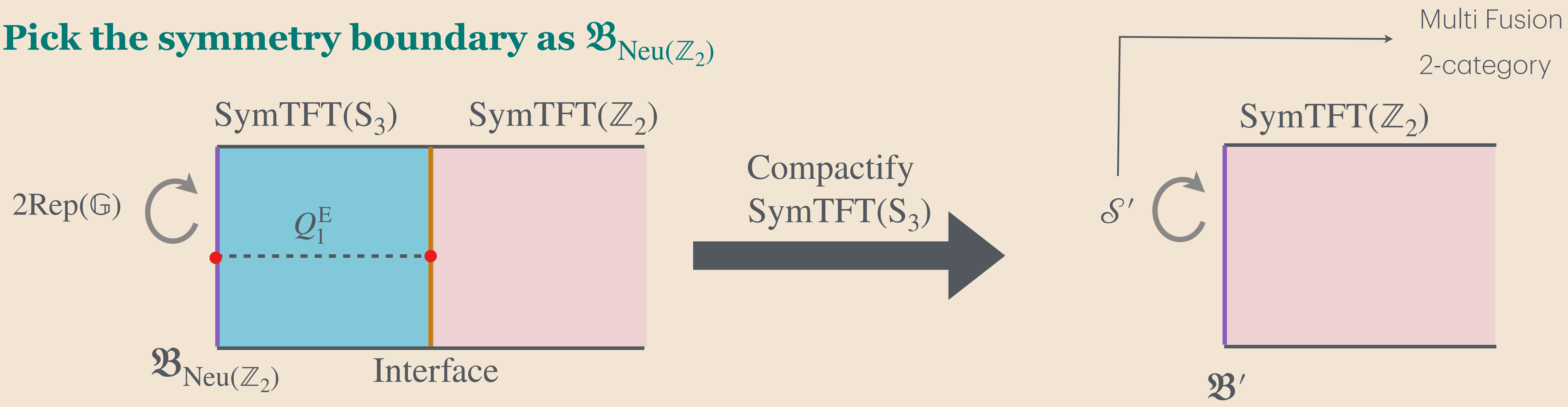


implements a condensation to the Z2 SymTFT (3+1d Toric Code)

- **Pick the symmetry boundary as $\mathfrak{B}_{\mathrm{Neu}(\mathbb{Z}_2)}$**



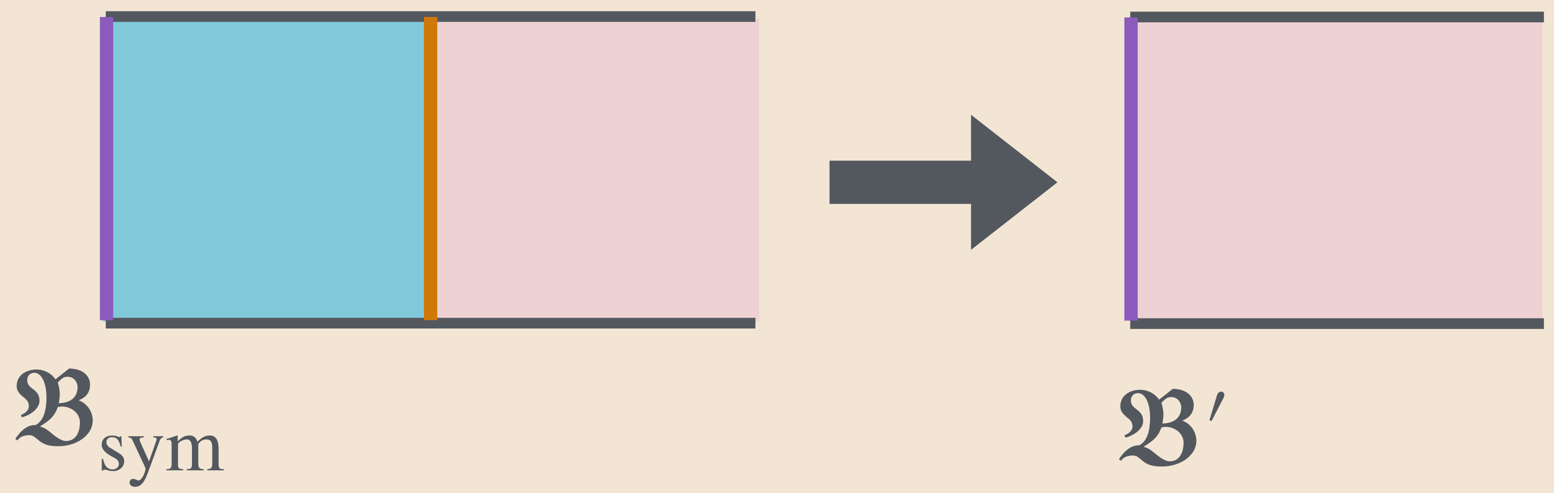
Provides a monoidal 2-functor $\Phi : 2\mathrm{Rep}(\mathbb{G}) \longrightarrow \mathcal{S}'$

- In the present case, $\mathfrak{B}'$ is the decomposable boundary of the Z2 SymTFT

$$\mathfrak{B}' = (\mathfrak{B}_{\mathrm{m}})_0 \boxplus (\mathfrak{B}_{\mathrm{e}})_1$$

on which the multi 2-fusion category of defects is

$$\mathcal{S}' = \begin{pmatrix} 2\mathrm{Rep}(\mathbb{Z}_2) & 2\mathrm{Vec} \\ 2\mathrm{Vec} & 2\mathrm{Vec}(\mathbb{Z}_2) \end{pmatrix}.$$



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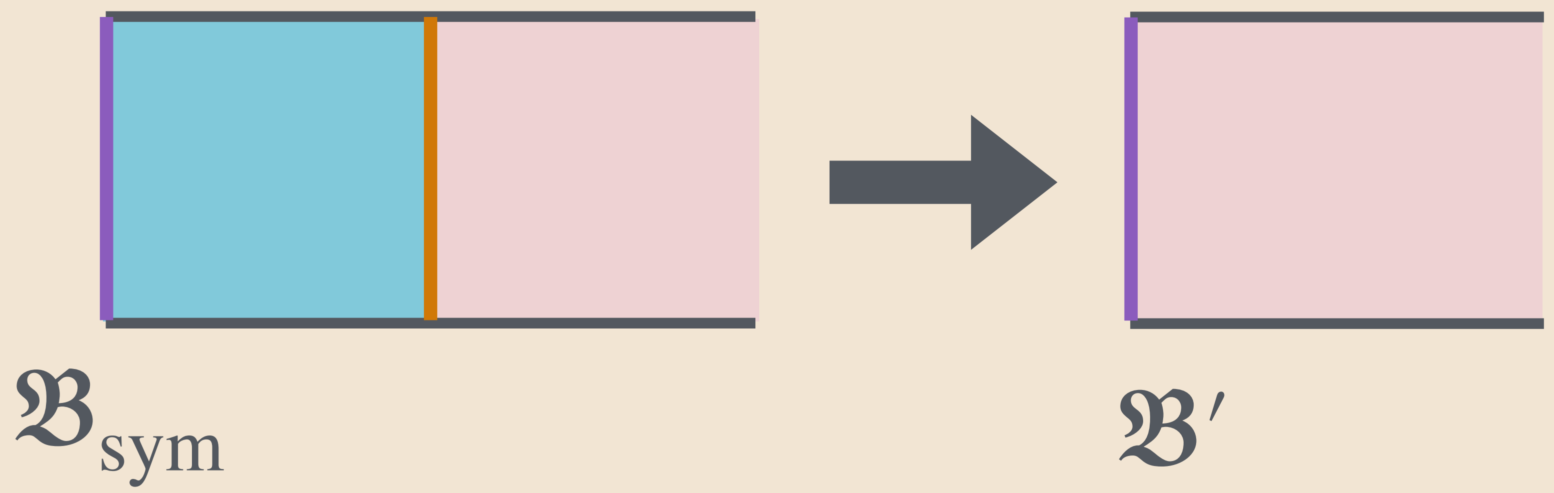

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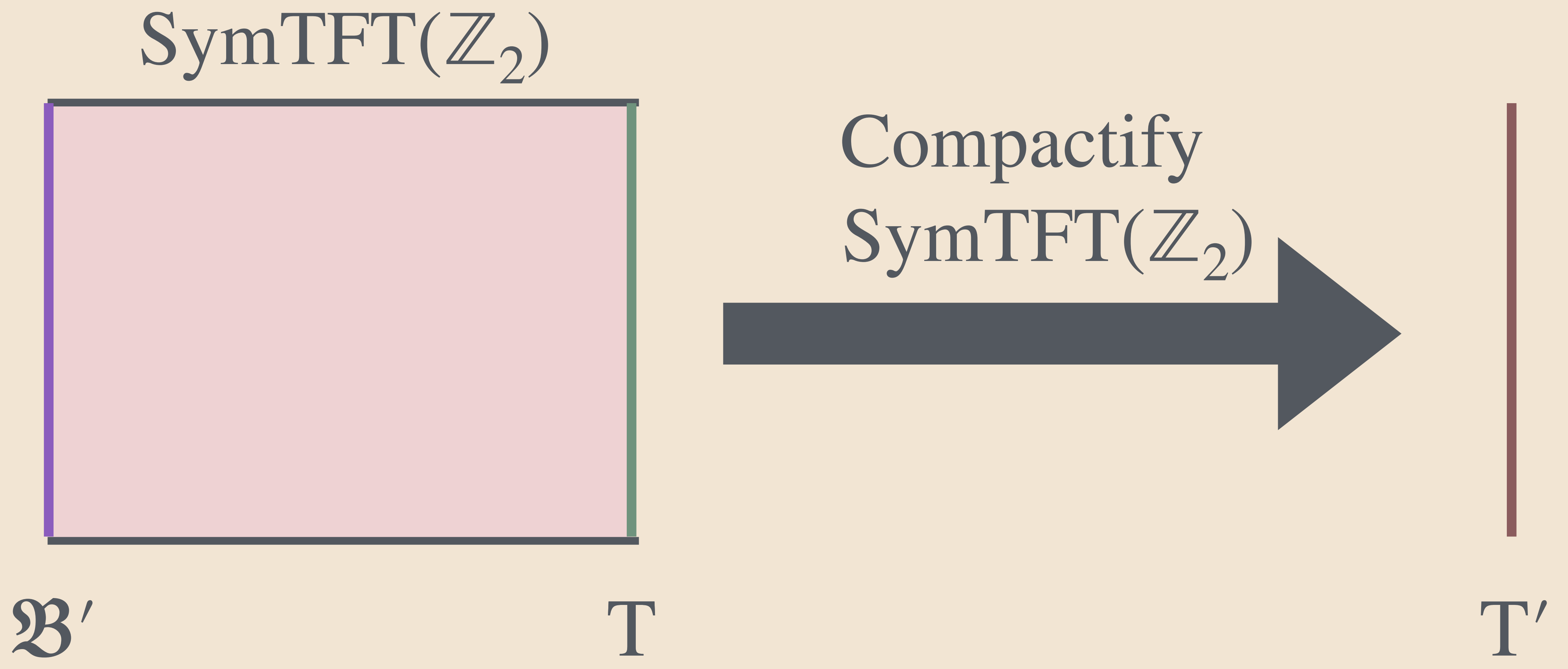
- $2\mathrm{Rep}(\mathbb{Z}_3^{(1)} \rtimes \mathbb{Z}_2^{(0)})$ is realized on $\mathfrak{B}'$ via the monoidal 2-functor Φ

$$\Phi(D_1^-) = (\mathrm{D}_1^-)_0 \oplus (\mathrm{D}_1^{\mathrm{id}})_1$$

$$\Phi(\mathrm{D}_2^{\mathrm{A}}) = \mathrm{B}_{01} \oplus \mathrm{B}_{10} \oplus 1_{11}\,.$$

- Inserting a $\mathbb{Z}_2$ symm. gapless theory $\mathbf{T}$ on the physical boundary produces the phase

$$\mathrm{T}' = \frac{\mathrm{T}}{\mathbb{Z}_2^{(0)}} \boxplus \mathrm{T}$$



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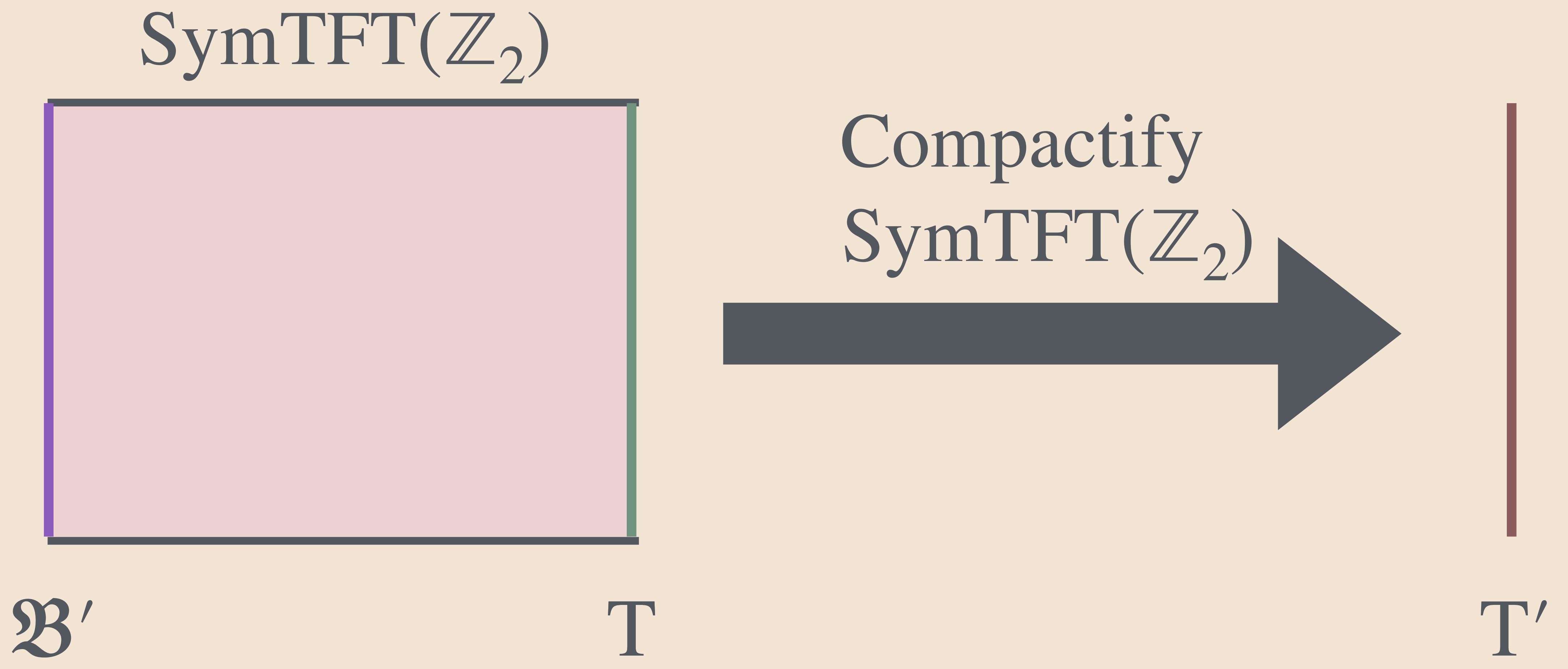
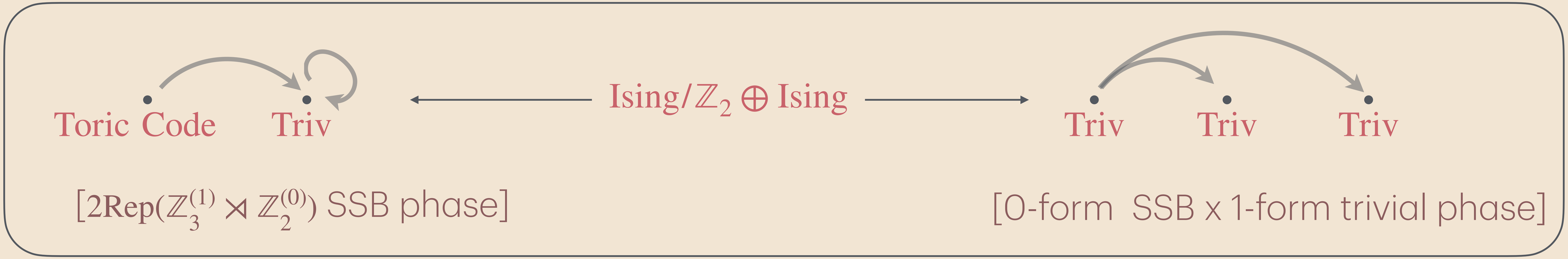


- Picking $\mathbf{T} = \mathbf{Ising}$, produces the transition $\mathbf{Ising}/\mathbb{Z}_2 \oplus \mathbf{Ising}$ transition between



Lattice models

[Inamura, Huang, Tiwari, Nameki ’25]

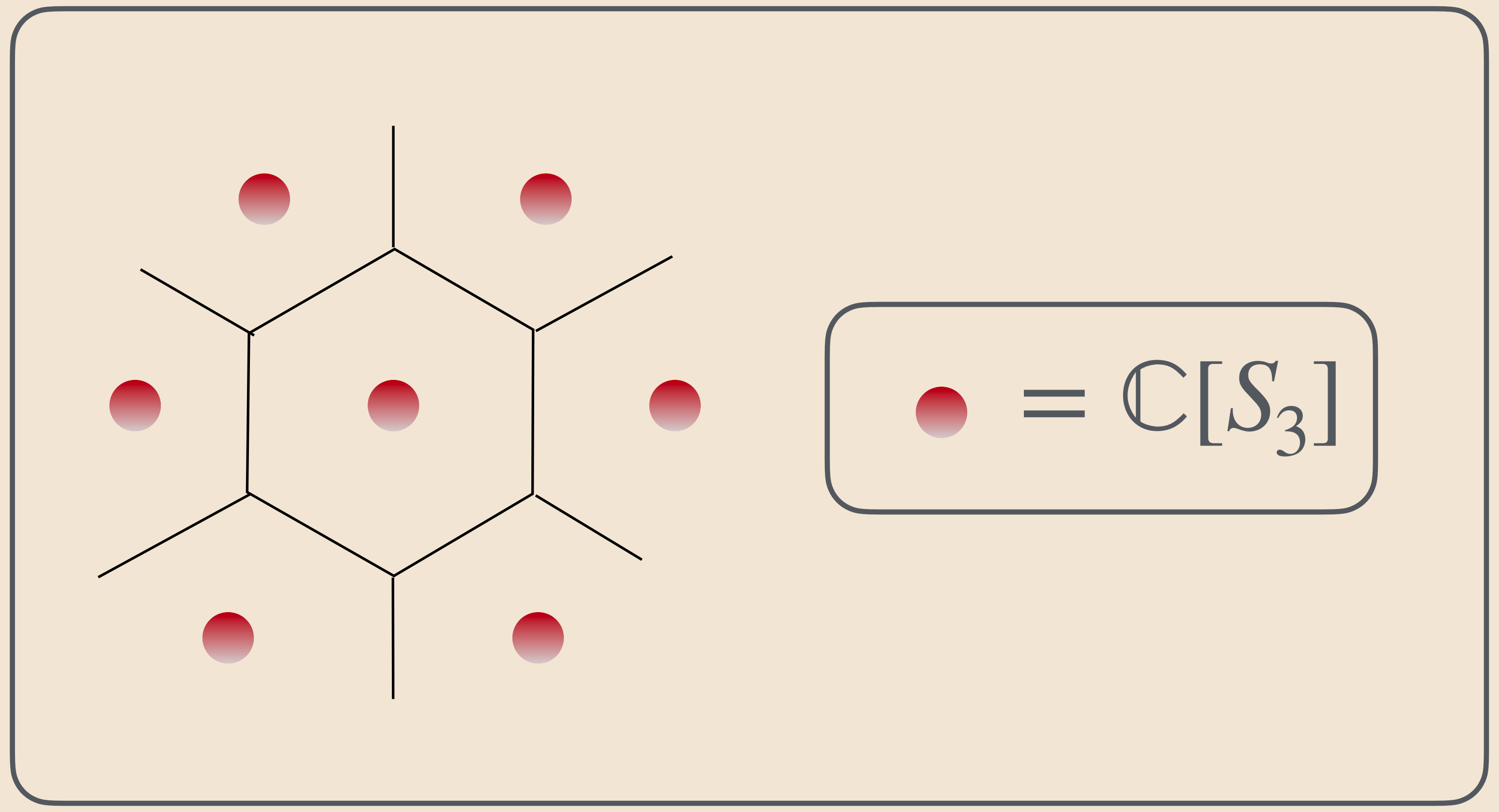
S_3 symmetric Lattice Model → Non-minimal 2Rep($\mathbb{G}$) symmetric Lattice Model

[Inamura, Huang, Tiwari, Nameki ’25]

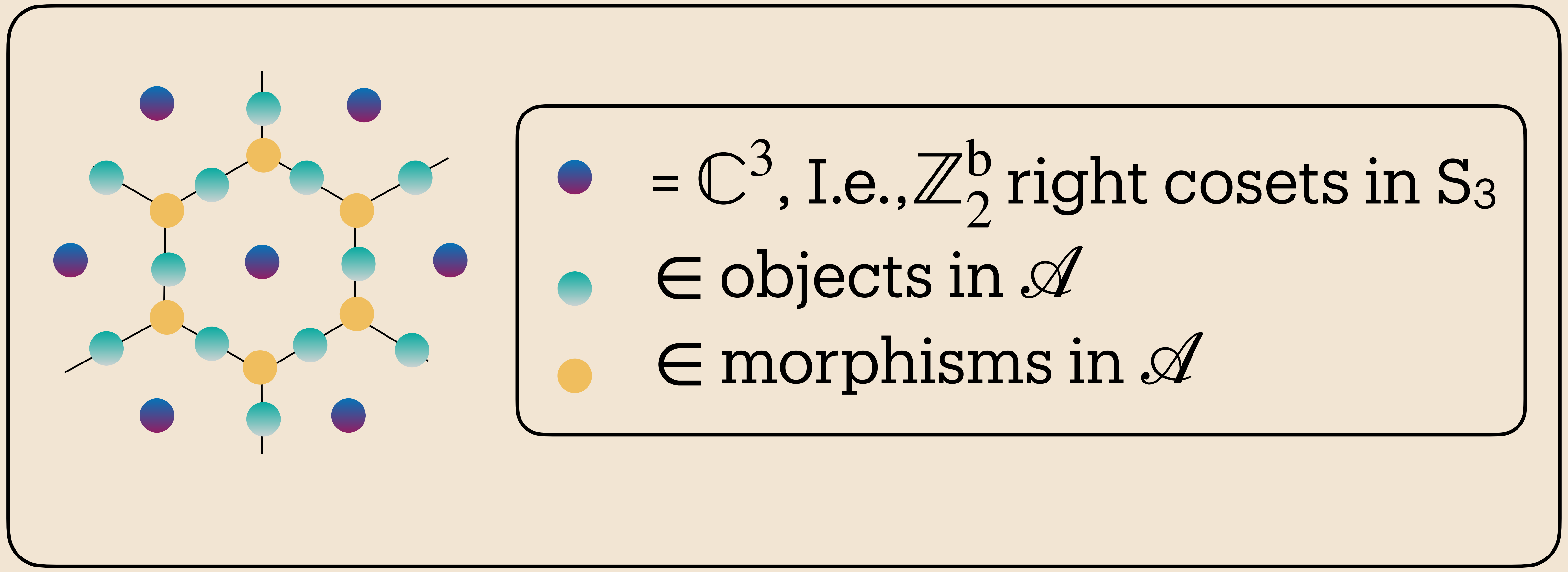
S_3 symmetric Lattice Model

Non-minimal 2Rep($\mathbb{G}$) symmetric Lattice Model

Generalized gauging via $\mathscr{A} = \bigoplus_{h \in H} \mathscr{A}_h$



[Minimal gauging when $\mathscr{A} = \mathrm{Vec}_H^{\omega}$]



[Inamura, Huang, Tiwari, Nameki ’25]

S_3 symmetric Lattice Model

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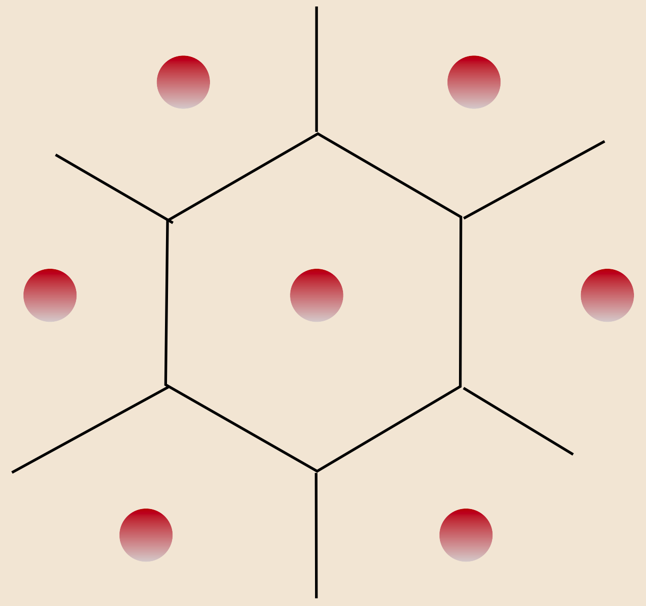
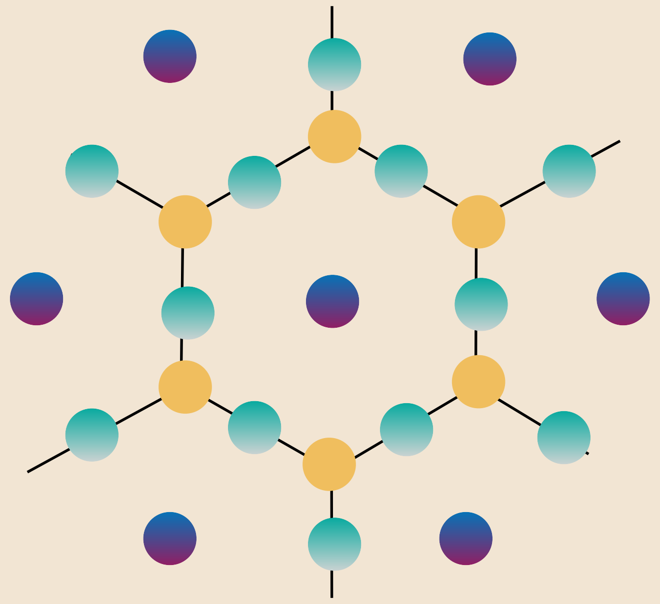
[Minimal gauging when $\mathscr{A} = \mathrm{Vec}^{\omega}_{H}$]



● $= \mathbb{C}[S_3]$

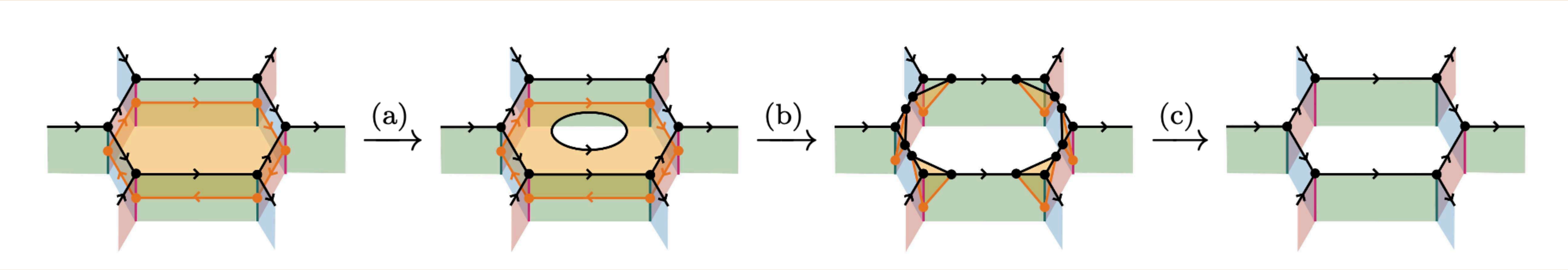


● $= \mathbb{C}^3$, I.e., $\mathbb{Z}_2^b$ right cosets in S_3

● $\in$ objects in $\mathscr{A}$

● $\in$ morphisms in $\mathscr{A}$

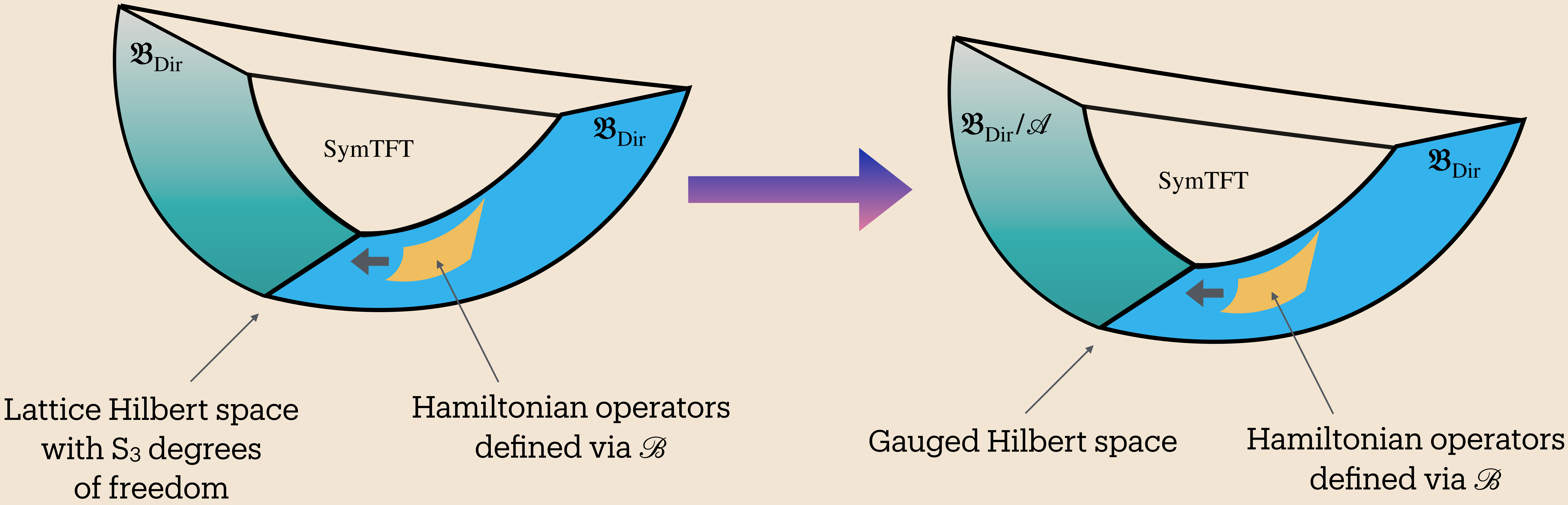
- Hamiltonian: The state spaces admit a right action of 2Vec(S3), using which Hamiltonian operators can be defined as:



defining the Hamiltonian via an algebra $\mathscr{B} = \bigoplus_{k \in K} \mathscr{B}_k$ produces a fixed-point limit of a gapped phase.

[Inamura, Huang, Tiwari, Nameki ’25]

Schematic SymTFT Picture



Lattice model for non-minimal SNE Phase

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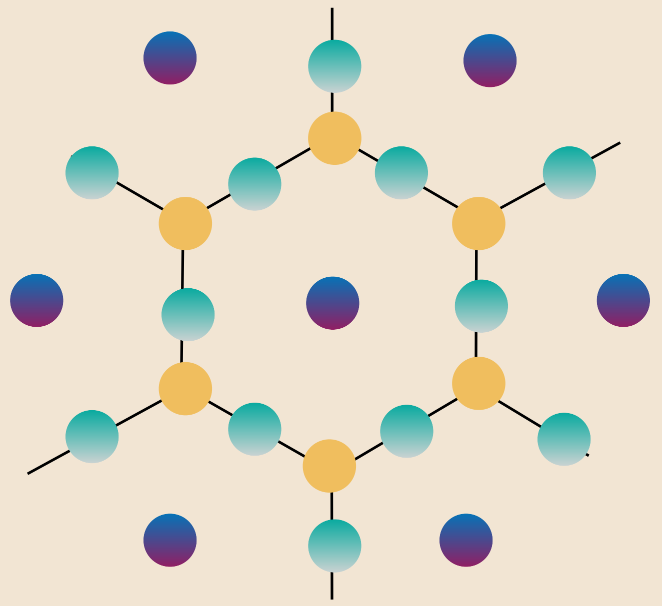

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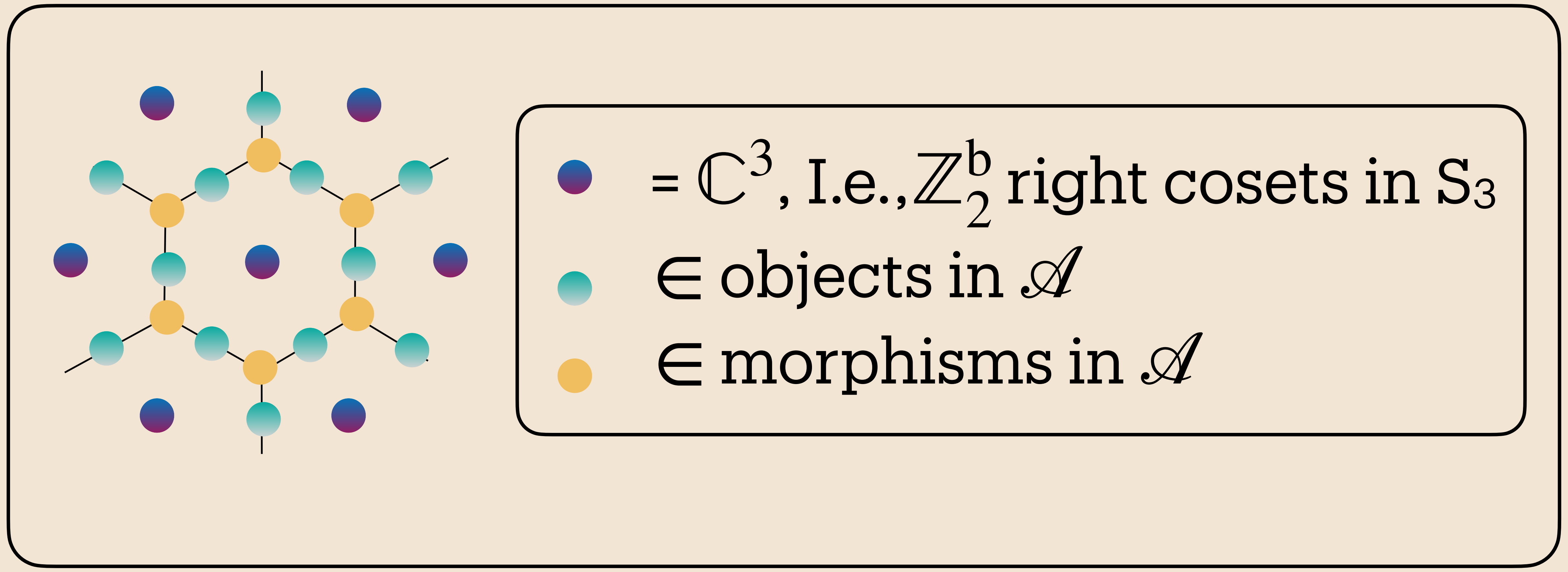
Lattice model for non-minimal SNE Phase

- Let us choose $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_b$.

- Consider the basis $|a_i^q\rangle$, with $q = 0,1,2$ on the plaquettes. We define operators

$$Z_i |a^q\rangle = e^{2\pi i q/3} |a^q\rangle$$
$$X_i |a^q\rangle = |a^{q+1 \bmod 3}\rangle$$
$$\Gamma_i |a^q\rangle = |a^{-q \bmod 3}\rangle$$



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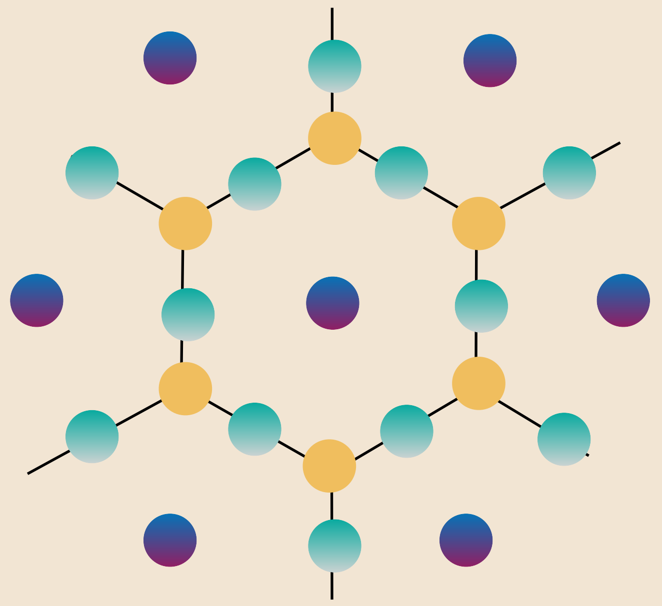
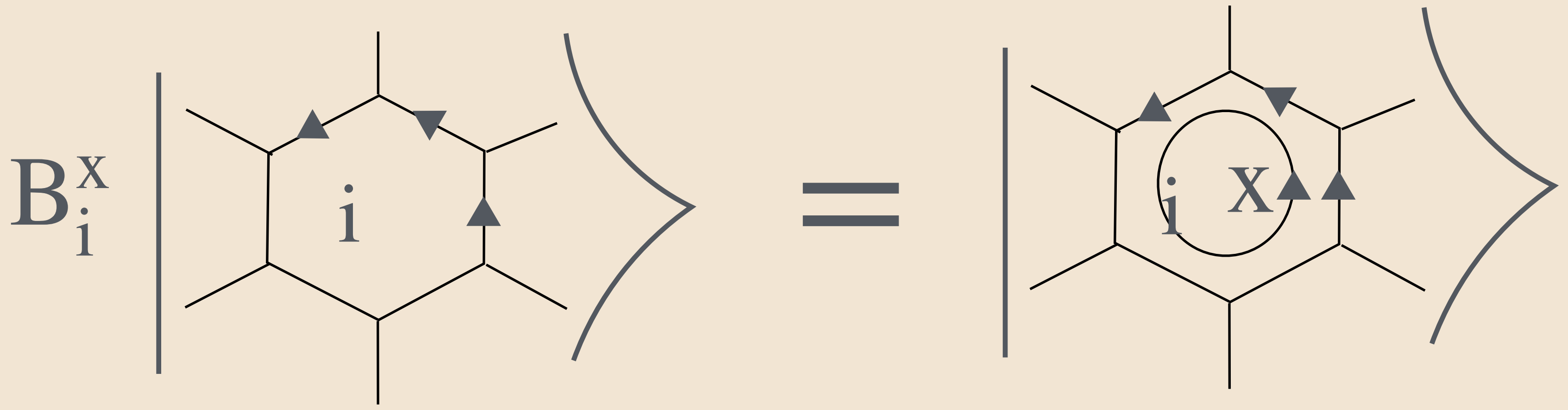
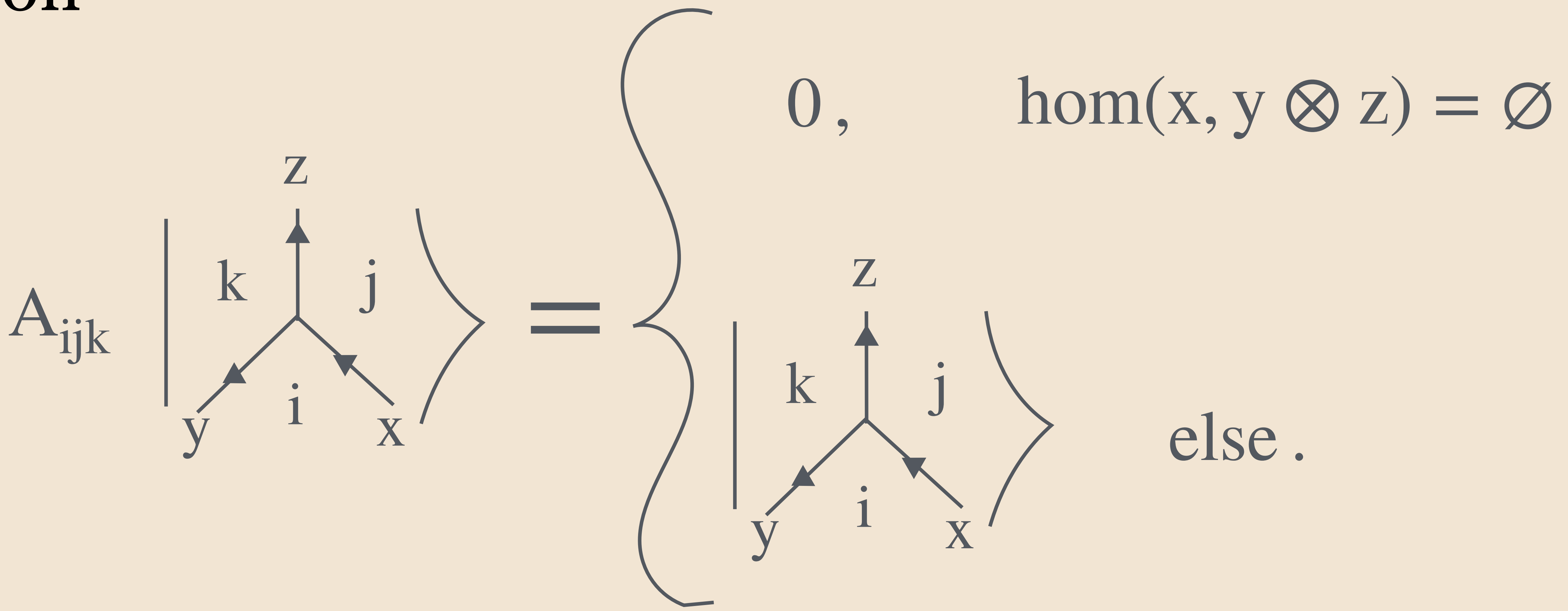


- (dark purple dot) $= \mathbb{C}^3$, I.e., $\mathbb{Z}_2^b$ right cosets in S_3
- (teal dot) $\in$ objects in $\mathscr{A}$
- (yellow dot) $\in$ morphisms in $\mathscr{A}$

- Consider the standard $\mathscr{A}$- Levin Wen operators on the edges and vertices of the triangulation



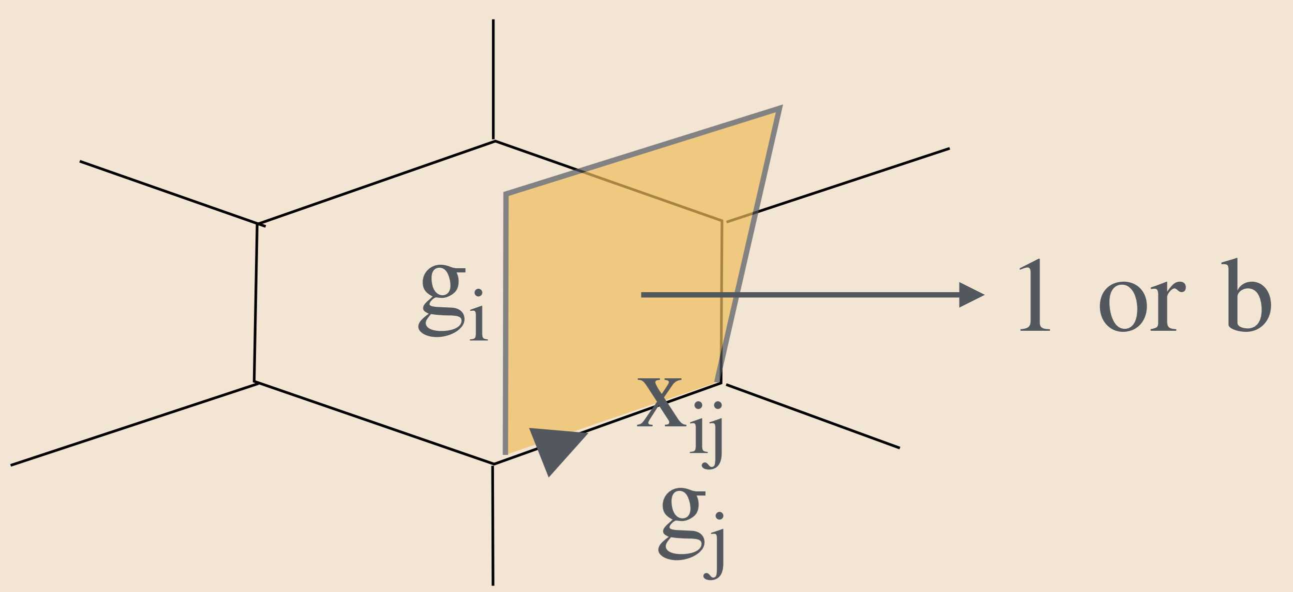


- Let us choose $\mathscr{B} = \mathrm{Vec}_{\mathbb{Z}_2^{\mathrm{b}}}$. The corresponding fixed-point Hamiltonian is

$$\mathscr{H}_{\mathrm{SNEP}} = -\sum_{\mathrm{ij}} \mathrm{P}_{\mathrm{ij}} - \frac{1}{2}\sum_{\mathrm{i}} \left[\mathscr{O}_{\mathrm{i}}^{1} + \mathscr{O}_{\mathrm{i}}^{\mathrm{b}}\right]$$

- Let us choose $\mathscr{B} = \mathrm{Vec}_{\mathbb{Z}_2^b}$. The corresponding fixed-point Hamiltonian is

$$\mathscr{H}_{\mathrm{SNEP}} = -\sum_{ij} P_{ij} - \frac{1}{2}\sum_{i} \left[\mathcal{O}_i^1 + \mathcal{O}_i^b\right]$$



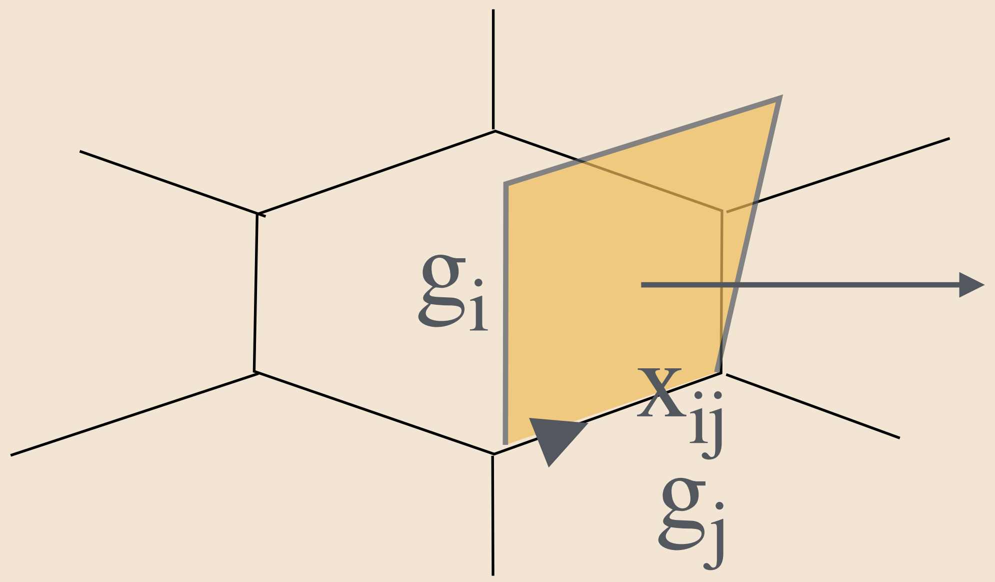
Projects onto:

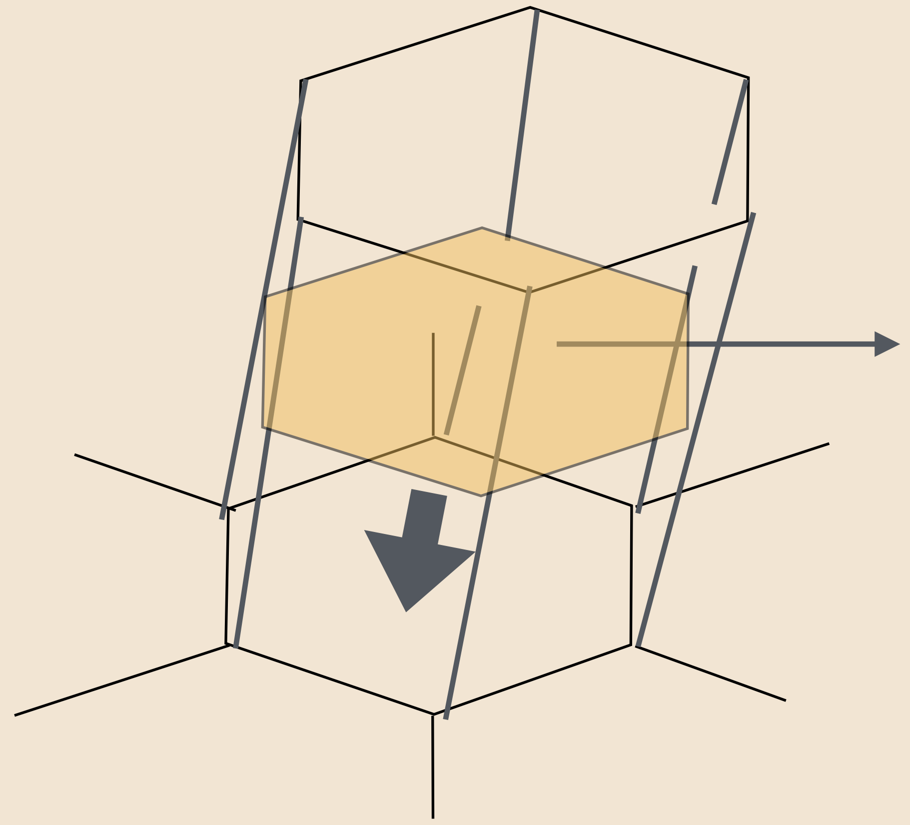
$(g_i, x_{ij}, g_j) \in \{(a^q, x, a^q) \,|\, x \in \mathscr{A}_1\} \sqcup \{(a^q, x, a^{-q}) \,|\, x \in \mathscr{A}_b\}$

$$P_{ij} = (P_{\mathscr{A}_1})_{ij} \otimes \frac{1 + Z_iZ_j^{-1} + Z_i^{-1}Z_j}{3} + (P_{\mathscr{A}_b})_{ij} \otimes \frac{1 + Z_iZ_j + Z_i^{-1}Z_j^{-1}}{3}$$

- Let us choose $\mathscr{B} = \mathrm{Vec}_{\mathbb{Z}_2^b}$. The corresponding fixed-point Hamiltonian is

$$\mathscr{H}_{\mathrm{SNEP}} = -\sum_{ij} P_{ij} - \frac{1}{2}\sum_i \left[\mathscr{O}_i^1 + \mathscr{O}_i^b\right]$$





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$$\mathscr{O}_i^1 = \sum_{x \in \mathscr{A}_1} \frac{d_x}{\mathscr{D}} B_i^x, \qquad \mathscr{O}_i^b = \left[\sum_{x \in \mathscr{A}_b} \frac{d_x}{\mathscr{D}} B_i^x\right] \Gamma_i,$$

$$\Gamma_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

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Pauli-x in q=1,2 subspace

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- Symmetry operators can be constructed concretely that reproduce the 2Rep(2-group) action

$$|GS_0\rangle = |Z(\mathscr{A}_1)\rangle\,, \quad |GS_1\rangle = |Z(\mathscr{A})\rangle \qquad \mathscr{U}_A|GS_0\rangle = 2|GS_1\rangle\,, \quad \mathscr{U}_A|GS_1\rangle = |GS_1\rangle + |GS_0\rangle\,,$$

Summary

- Spontaneous non-uniform entangled (SNE) phases are qualitatively new kinds of gapped phases protected by non-invertible (non-condensation) 2 fusion categorical symmetries.

- The SymTFT is a useful framework to organise various aspects of these phases as well as construct lattice models.

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Thank you!!