Categorical Anomalies

(or RG flows in the SymTFT)

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Generalized symmetries in HEP and CMP

01/08/2025

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Motivation

"Anomalies" are very powerful

['t Hooft '79; Witten '83]

$\mathsf{T}_{\mathsf{UV}} \; : \; G, \omega_{\mathsf{UV}}$

$\downarrow$

$\mathsf{T}_{\mathsf{IR}} \; : \; G, \omega_{\mathsf{IR}}$

$$\boxed{\omega_{\mathsf{UV}} = \omega_{\mathsf{IR}}}$$

$\mathsf{T}_{\mathsf{UV},1} \; : \; G, \omega_{\mathsf{UV},1}$ $\searrow$ T_{IR} $\swarrow$ $\mathsf{T}_{\mathsf{UV},2} \; : \; G, \omega_{\mathsf{UV},2}$

$$\boxed{\omega_{\mathsf{UV},1} = \omega_{\mathsf{UV},2}}$$

Motivation

Caveat: symmetry does NOT need to be the same in IR

- A sub-symmetry can be come trivial
- Emergent (higher-form) symmetries
- $G^{(0)}_{\text{UV}}$ in the IR doesn't act on local operators, but on extended objects

'Phenomenology' of anomaly matching:

- Emergent anomalies. E.g. anomaly free $\mathbb{Z}_4 \longrightarrow$ anomalous $\mathbb{Z}_4/\mathbb{Z}_2 \cong \mathbb{Z}_2$
- UV anomaly of 0-form symmetry $\longrightarrow$ anomaly of "emergent" higher-form symmetry (fractionalization/transmuation [Barkeshli, Bonderson, Cheng, Wang '14; Wang, Wen, Witten '17] [Delmastro, Gomis, Hsin Komargodski '22; Brennan, Cordova, Dumitrescu '22; AA, Benini, Rizi '24; Seiberg, Seifnashri '25]

What is the most general constraint?

Motivation

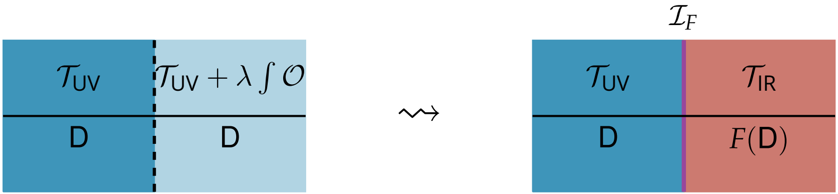
Non-invertible symmetries

- Hard to separate anomaly from symmetry
- Only Yes/No type of definitions: $\mathcal{C}$ anomalous if no trivially gapped realization.
- Not enough for ”anomaly matching”:

When can two anomalous symmetries $\mathcal{C}_{\text{UV}}, \mathcal{C}_{\text{IR}}$ be connected by an RG?

Formulation of anomaly (symmetry) matching?

- Symmetry in QFT has to be thought of as (higher) tensor category.
- The UV symmetry defects remain topological in the IR: fusions and (higher) associativity data are invariant.
- RG interfaces



Motivation

Central concept: **tensor functors**

$$F : \mathcal{C}_{\mathsf{UV}} \to \mathcal{C}_{\mathsf{IR}}$$

- When a tensor functor $F : \mathcal{C}_{\mathsf{UV}} \to \mathcal{C}_{\mathsf{IR}}$ exists?
- Extract some "data" $\mathfrak{A}(\mathcal{C})$

Very complicated (especially w/ non-invertible and/or $d > 2$ ...

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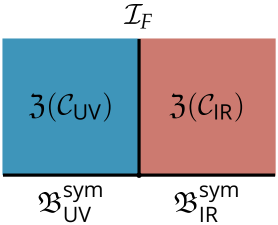
SymTFT !

Out(punch)line

1. Tensor functors are SymTFT interfaces



$$\boxed{\mathfrak{B}^{\text{sym}}_{\text{UV}} \times \mathcal{I}_F = \mathfrak{B}^{\text{sym}}_{\text{IR}}}$$

2. Functor/interface dictionary
3. Normal subcategories and short-exact-sequences

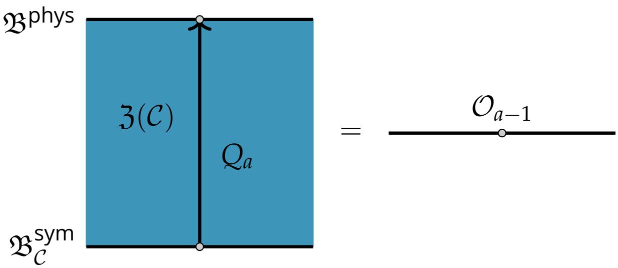
$$\mathcal{N} \to \mathcal{C} \to \mathcal{S}$$

4. Anomalous Simple Categories (ASCies) $\mathcal{S}$: $\mathfrak{A}(\mathcal{C}) = \{\mathcal{S}_1, \mathcal{S}_2, ...\}$
5. Examples:
    - Known
    - Expected
    - New

Symmetry Topological Field Theory (SymTFT)

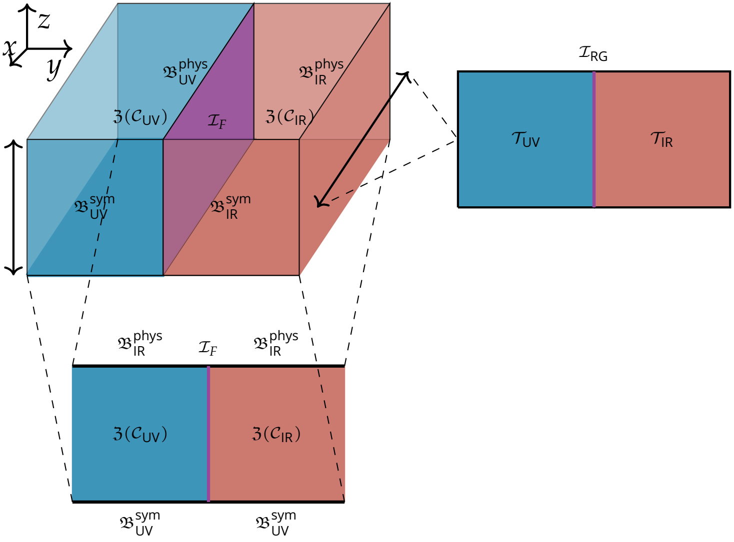
Any symmetry $\mathcal{C}$ in $d-$dim $\cong (\mathfrak{Z}(\mathcal{C}), \mathfrak{B}^{\text{sym}}_{\mathcal{C}})$

- $\mathfrak{Z}(\mathcal{C})$ = (d+1)-dim TQFT = (flat) gauging $\mathcal{C}$ in (d+1)-dim (state-sum).
- $\mathfrak{B}^{\text{sym}}_{\mathcal{C}}$ = topological boundary condition (Lagrangian algebra $\mathcal{L}_{\mathcal{C}}$)



"Isolate symmetry/topology from dynamics"

RG interfaces and RG quiches



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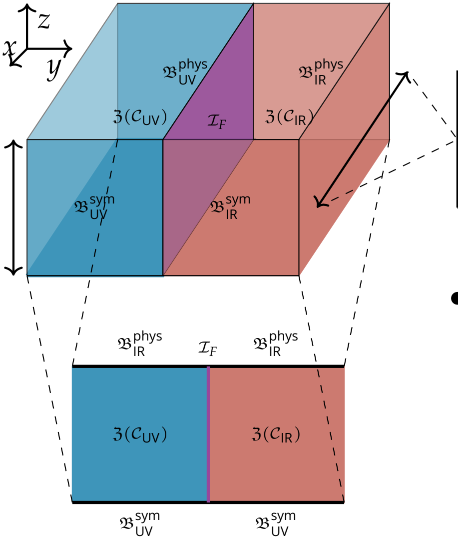
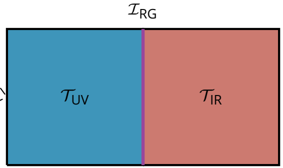
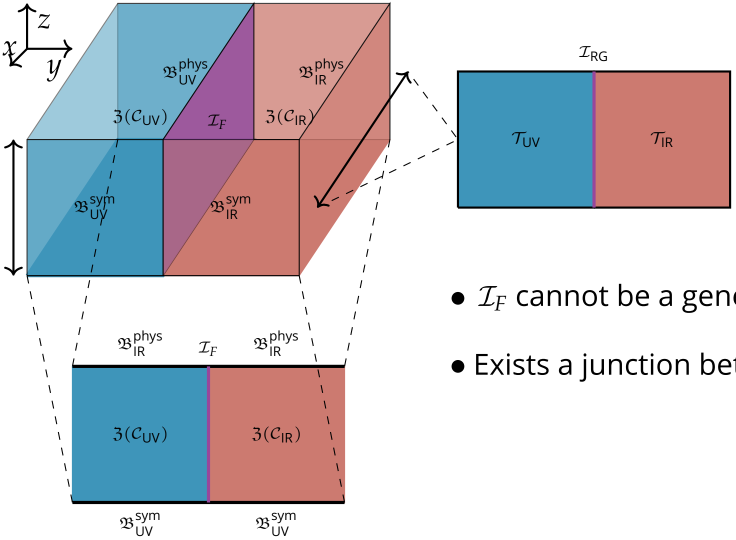




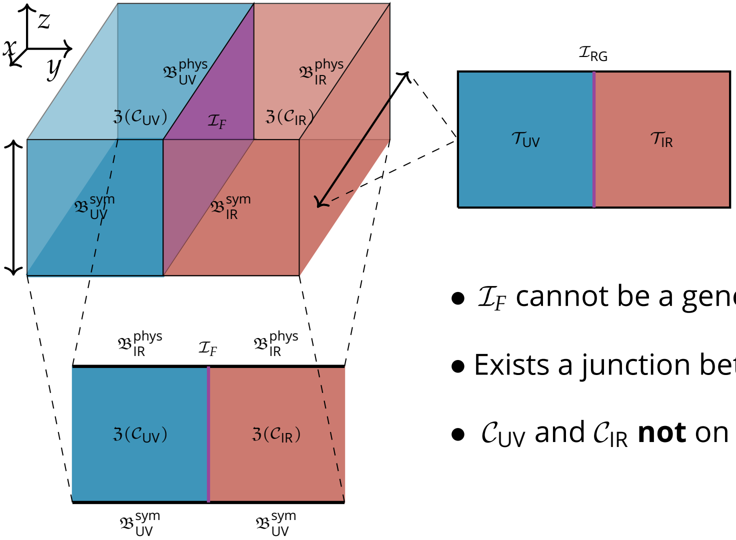
- $\mathcal{I}_F$ cannot be a generic interface

RG interfaces and RG quiches



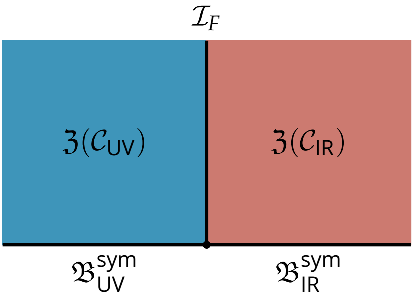
- $\mathcal{I}_F$ cannot be a generic interface
- Exists a junction between $\mathcal{I}_F, \mathfrak{B}^{\text{sym}}_{\text{UV}}, \mathfrak{B}^{\text{sym}}_{\text{IR}}$

RG interfaces and RG quiches



- $\mathcal{I}_F$ cannot be a generic interface
- Exists a junction between $\mathcal{I}_F, \mathfrak{B}^{\text{sym}}_{\text{UV}}, \mathfrak{B}^{\text{sym}}_{\text{IR}}$
- $\mathcal{C}_{\text{UV}}$ and $\mathcal{C}_{\text{IR}}$ **not** on same footing

Symmetry Matching Equation (ME)



$$\boxed{\mathfrak{B}^{\text{sym}}_{\text{UV}} \times \mathcal{I}_F = \mathfrak{B}^{\text{sym}}_{\text{IR}}}$$

- Intuition: $\mathcal{I}_F$ is symmetric under $\mathcal{C}_{\text{UV}}$.
- Action of F on $\text{D} \in \mathcal{C}_{\text{UV}}$ determined by action of $\mathcal{I}_F$ on Q, $\pi_{\text{UV}}(Q) = \text{D}$:

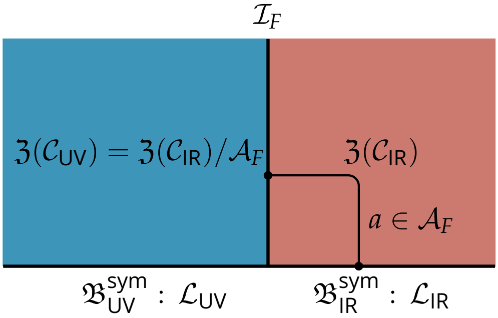
$$F(\text{D}) = \pi_{\text{IR}}\left(\mathcal{I}_F(Q)\right)$$

- The (ME) $\equiv \mathcal{I}_F(\mathcal{L}_{\text{UV}}) = \mathcal{L}_{\text{IR}} \implies$ independence on arbitrariness in Q

Functor/interface dictionary

Injective functors $\longleftrightarrow$ $\mathcal{C}_{\text{UV}}$ acts faithfully in the IR

- $F^* : \mathcal{Q}_{\mathcal{C}_{\text{IR}}} \to \mathcal{Q}_{\mathcal{C}_{\text{UV}}}$ is surjective $\Longrightarrow$ $\mathcal{A} = \ker(F^*)$ trivialized charges.
- $\mathfrak{Z}(\mathcal{C}_{\text{UV}}) = \mathfrak{Z}(\mathcal{C}_{\text{IR}})/\mathcal{A}_F$, $\mathcal{A}_F \subset \mathcal{L}_{\mathcal{C}_{\text{IR}}}$ condensable **electric** algebra.



- $\mathfrak{B}^{\text{sym}}_{\text{UV}} \times \mathcal{I}_F \leftrightarrow \mathcal{L} \in \mathfrak{Z}(\mathcal{C}_{\text{IR}})$ by sequential gauging $\Longrightarrow$ choose $\mathcal{L}_{\text{IR}} = \mathcal{L}$

Functor/interface dictionary

Example: (1+1)d, $\mathcal{C}_{\mathsf{UV}} = \mathsf{Vec}^1_{\mathbb{Z}_2}$.

$$\mathfrak{Z}(\mathcal{C}_{\mathsf{UV}}) = \mathsf{DW}(\mathbb{Z}_2^1) = \frac{2\pi i}{2}\int_{X_3} a \cup db + a \cup \beta(a) \ .$$

Choose

$$\mathfrak{Z}(\mathcal{C}_{\mathsf{IR}}) = \mathsf{DW}(\mathbb{Z}_4) = \frac{2\pi i}{4}\int_{X_3} a \cup db$$

$$e = e^{i\int a}\ , \quad m = e^{i\int b}\ , \quad \theta(e^{n_e} m^{n_m}) = e^{\frac{2\pi i}{4} n_e n_m}$$

We have $\mathfrak{Z}(\mathcal{C}_{\mathsf{UV}}) = \mathfrak{Z}(\mathcal{C}_{\mathsf{IR}})/\mathcal{A}$ with

$$\mathcal{A} = 1 \oplus e^2 m^2$$

The (ME) is satisfied with

$$\mathcal{L}_{\mathsf{IR}} = 1 \oplus e^2 \oplus m^2 \oplus e^2 m^2 \implies \mathcal{C}_{\mathsf{IR}} = \mathsf{Vec}^{\omega}_{\mathbb{Z}_2 \times \mathbb{Z}_2}$$

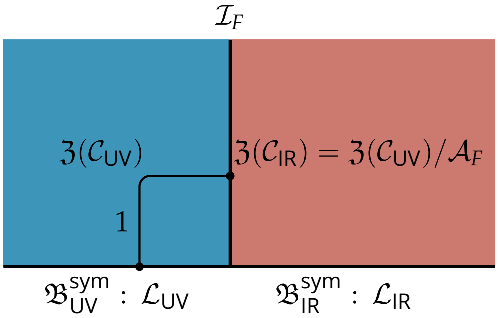
The functor is

$$F : \mathsf{Vec}^1_{\mathbb{Z}_2} \to \mathsf{Vec}^{\omega}_{\mathbb{Z}_2 \times \mathbb{Z}_2}\ , \quad F(\mathsf{D}) = \mathsf{D}_1 \mathsf{D}_2$$

Functor/interface dictionary

Surjective functors $\longleftrightarrow$ **no** emergent symmetries

- $\ker(F)$=trivial subsymmetry in IR
- $\boxed{\mathfrak{Z}(\mathcal{C}_{\text{IR}}) = \mathfrak{Z}(\mathcal{C}_{\text{UV}})/\mathcal{A}_F}$, $\pi_{\text{UV}}(\mathcal{A}_F) = \ker(F)$.
- $\boxed{\mathcal{A}_F \text{ **magnetic** condensable algebra}}$: $\mathcal{A}_F \cap \mathcal{L}_{\text{UV}} = \{1\}$



Functor/interface dictionary

From the (ME) $\mathfrak{B}^{\text{sym}}_{\text{UV}} \times \mathcal{I}_F = \mathfrak{B}^{\text{sym}}_{\text{IR}}$: magnetic $\mathcal{A}_F$ follows from simplicity of $\mathfrak{B}^{\text{sym}}_{\text{UV}} \times \mathcal{I}_F$.

- The trivialized symmetry $\ker(F)$ does **not** determine $\mathcal{A}_F$: $\pi_{\text{UV}}(\mathcal{A}_F^{(i)}) = \ker(F)$.
- Different $\mathcal{A}_F^{(i)} \to$ different $\mathfrak{Z}(\mathcal{C}_{\text{IR}}^{(i)}) = \mathfrak{Z}(\mathcal{C}_{\text{UV}})/\mathcal{A}_F^{(i)} \to$ different $\mathcal{C}_{\text{IR}}^{(i)}$.
- (invertible symm): **Emergent anomalies**
- (non-invertible): More surprising: even fusion rules can be different!!!

Functor/interface dictionary

Example: (1+1)d, $\mathcal{C}_{\mathsf{UV}} = \mathsf{Vec}_{\mathbb{Z}_4}$.

$$\mathcal{L}_{\mathsf{UV}} = 1 \oplus e \oplus e^2 \oplus e^3 \implies \mathsf{D} = \pi_{\mathsf{UV}}(e^x m)$$

To trivialise $\mathbb{Z}_2 \subset \mathbb{Z}_4$, two choices (only two magnetic algebras)

1. $\mathcal{A}_0 = 1 \oplus m^2 \implies \mathsf{Vec}_{\mathbb{Z}_2}$
2. $\mathcal{A}_1 = 1 \oplus e^2 m^2 \implies \mathsf{Vec}^1_{\mathbb{Z}_2}$

Physically, to gap-out $\mathbb{Z}_2 \subset \mathbb{Z}_4$: $\mathcal{O}_e(x)$ heavy while $\mathcal{O}_{e^2}(x)$ light. But

1. $\mathcal{O}_m(x)$ light, while $\mathcal{O}_{em}(x)$ heavy too.
2. $\mathcal{O}_m(x)$ also heavy, while $\mathcal{O}_{em}(x)$ remains light.

Functor/inteface dictionary

Fiber functors $F : \mathcal{C}_{\text{UV}} \to$ (d-1)-Vec

- Physically: RG flow that trivializes the symmetry.
- Surjective $\implies \mathcal{A} \in \mathfrak{Z}(\mathcal{C}_{\text{UV}})$ magnetic.
- $\mathcal{Z}($(d-1)-Vec$)$ =trivial $\implies \mathcal{I}_F$=b.c. $\implies \mathcal{A}_F$=Lagrangian

$\mathcal{C}_{\text{UV}}$ anomalous iff $\nexists$ magnetic Lagrangian algebra

[Kaidi, Nardoni, Zheng; Zhang, Cordova; AA, Benini, Copetti, Galati, Rizi; Cordova, Hsin, Zhang '23]

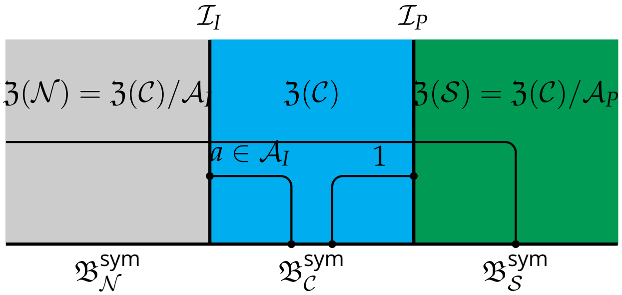
Normal subcategories

A subcategory $\mathcal{N} \subset \mathcal{C}$ is **normal** if

$$\mathcal{N} \xrightarrow{I} \mathcal{C} \xrightarrow{P} \mathcal{S} \ , \qquad \text{im}(I) = \ker(P)$$

- $P \circ I$ is a fiber-functor $\Longrightarrow \mathcal{N}$ is anomaly free
- $\mathcal{N} = \ker(P) \Longrightarrow$ determined by a magnetic algebra $\mathcal{A}_P \in \mathfrak{Z}(\mathcal{C})$

$$\mathcal{A}_I \otimes \mathcal{A}_P \ , \quad \text{Lagrangian algebra}$$



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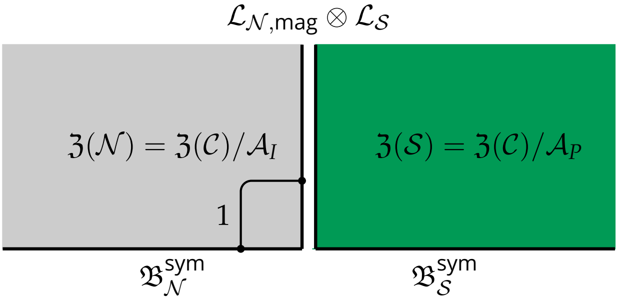

Normal subcategories: physical interpretation

$\mathcal{N}$ is a subgategory that can be gapped out consistently without emergent symmetries

Not all anomaly free subcategories are normal!

Example: $\mathsf{Vec}_{\mathbb{Z}_4}^{\omega=2}$.

$\mathbb{Z}_2 \subset \mathbb{Z}_4$ is anomaly free. But **not** normal!

$\mathfrak{Z}(\mathsf{Vec}_{\mathbb{Z}_4}^2) = \mathsf{DW}(\mathbb{Z}_4^2)$ does not have magnetic algebras $\mathcal{A} \cap \mathcal{L}_{\mathcal{C}} = \{1\} \implies$ no surjective functor $P : \mathsf{Vec}_{\mathbb{Z}_4}^2 \to \mathcal{S}$.

To gap out $\mathbb{Z}_2$:

1. Break explicitly $\mathbb{Z}_4$ down to $\mathbb{Z}_2$ with a deformation.
2. Emergent symmetries. (e.g. in gapped phases)

Anomalous Simple Cateogories (ASCies)

Anomalous Simple Category (ASCy) $\mathcal{S}$ = symmetry with **no** normal subsymmetry

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$$\mathsf{Vec}_{\mathbb{Z}_2}^{1}$$

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$\text{Vec}^1_{\mathbb{Z}_2}$

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$\text{Vec}^1_{\mathbb{Z}_2}$ $\text{Vec}^2_{\mathbb{Z}_4}$

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$\mathsf{Vec}_{\mathbb{Z}_2}^{1}$ $\mathsf{Vec}_{\mathbb{Z}_4}^{2}$ $\mathsf{TY}(\mathbb{Z}_2)$

Anomalous Simple Cateogories (ASCies)

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$\mathrm{Vec}_{\mathbb{Z}_2}^1$ $\mathrm{Vec}_{\mathbb{Z}_4}^2$ $\mathrm{TY}(\mathbb{Z}_2)$ Fib

Anomalous Simple Categories (ASCies)

From any symmetry $\mathcal{C}$: extract ASCies

$$\mathcal{N}_i \to \mathcal{C} \to \mathcal{S}_i$$

$\mathcal{N}_i$ is **maximal** (e.g. avoid $\text{Vec}_{\mathbb{Z}_2} \to \text{Vec}_{\mathbb{Z}_4} \to \text{Vec}^1_{\mathbb{Z}_2}$)

Even for fixed $\mathcal{N}_i$, $\mathcal{S}_i$ is not unique $\Longrightarrow$ list of ASCies

$\mathfrak{A}(\mathcal{C}) = \{\mathcal{S}_1, \mathcal{S}_2, ...\} =$ "quantification" of anomaly of $\mathcal{C}$

SymTFT: look for maximal magnetic algebras $\mathcal{A}_P \in \mathfrak{Z}(\mathcal{C})$:

$$\boxed{\mathfrak{Z}(\mathcal{S}) = \mathfrak{Z}(\mathcal{C})/\mathcal{A}_P}$$

Example: $\mathsf{Vec}_{\mathbb{Z}_8}^{\omega=4}$

$$\mathfrak{Z}(\mathsf{Vec}_{\mathbb{Z}_8}^{\omega=4}) = \mathsf{DW}(\mathbb{Z}_8^4). \qquad \text{Lines } e^{n_e} m^{n_m}:$$

$$\theta(n_e, n_m) = \exp\left(\frac{2\pi i}{8}\left(n_m n_e - \frac{1}{2} n_m^2\right)\right), \qquad \mathcal{L}_{\mathcal{C}} = 1 \oplus e \oplus e^2 \oplus \ldots \oplus e^7$$

Two ('maximal') magnetic algberas $\leftrightarrow$ two surjective functors

1. $\mathcal{A}_{P_1} = 1 \oplus m^4 \Longrightarrow \mathfrak{Z}(\mathsf{Vec}_{\mathbb{Z}_8}^{\omega=4})/\mathcal{A}_{P_1} = \mathfrak{Z}(\mathsf{Vec}_{\mathbb{Z}_4}^{1})$.

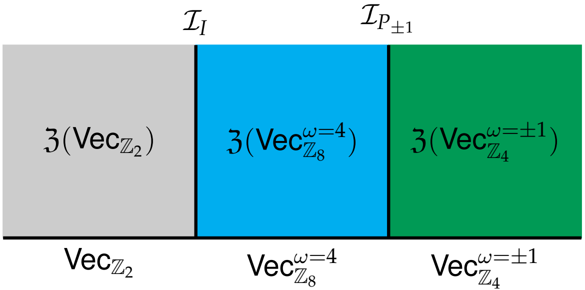
   Kernel determined by $\mathcal{A}_I = 1 \oplus e^2 \oplus e^4 \oplus e^6 \Longrightarrow \mathcal{N} = \mathsf{Vec}_{\mathbb{Z}_2}$.

2. $\mathcal{A}_{P_{-1}} = 1 \oplus e^4 m^4 \Longrightarrow \mathfrak{Z}(\mathsf{Vec}_{\mathbb{Z}_8}^{\omega=4})/\mathcal{A}_{P_{-1}} = \mathfrak{Z}(\mathsf{Vec}_{\mathbb{Z}_4}^{-1})$.

   Kernel determined by $\mathcal{A}_I = 1 \oplus e^2 \oplus e^4 \oplus e^6 \Longrightarrow \mathcal{N} = \mathsf{Vec}_{\mathbb{Z}_2}$.

Example: $\mathsf{Vec}_{\mathbb{Z}_8}^{\omega=4}$

$$\mathfrak{A}(\mathsf{Vec}_{\mathbb{Z}_8}^{\omega=4}) = \left\{\mathsf{Vec}_{\mathbb{Z}_4}^{1}, \mathsf{Vec}_{\mathbb{Z}_4}^{-1}\right\}$$



Upshot: if trivialize $\mathbb{Z}_2 \implies$ anomaly not uniquely determined in the IR!

remark: related with emergent anomaly $\mathsf{Vec}_{\mathbb{Z}_8} \to \mathsf{Vec}_{\mathbb{Z}_4}^{2}$

Example: $\mathsf{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)_{\chi_d,\epsilon=-1}$

$\mathsf{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)_{\chi_d,-1}$ has **Second-obstruction anomaly** [AA, Benini, Copetti, Galati, Rizi '23]

Lines of $\mathfrak{Z}(\mathsf{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)_{\chi_d,\epsilon=-1})$

	$X_{(0,0),\pm 1}$	$X_{(1,0),\pm i}$	$X_{(0,1),\pm i}$	$X_{(1,1),\pm 1}$	$Y_{(0,0),(1,0)}$
θ	1	-1	-1	1	1
d	1	1	1	1	2

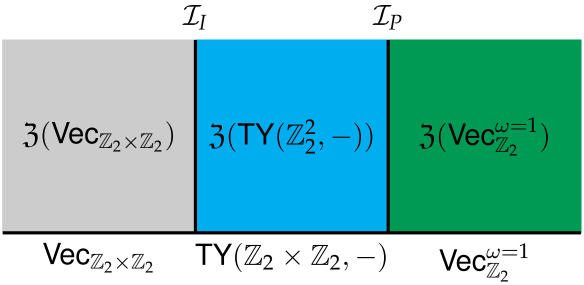
	$Y_{(0,0),(0,1)}$	$Y_{(0,0),(1,1)}$	$Y_{(1,0),(0,1)}$	$Y_{(1,0),(1,1)}$	$Y_{(0,1),(1,1)}$
θ	1	1	1	-1	-1
d	2	2	2	2	2

	$Z_{\rho_1,\pm\zeta_8^3}$	$Z_{\rho_2,\pm\zeta_8}$	$Z_{\rho_3,\pm i}$	$Z_{\rho_4,\pm i}$
θ	$\pm e^{\frac{3\pi i}{4}}$	$\pm e^{\frac{\pi i}{4}}$	$\pm i$	$\pm i$
d	2	2	2	2

$$\mathcal{L}_{\mathsf{TY}(\mathbb{Z}_2\times\mathbb{Z}_2)_{\gamma_d,-}} = 1 \oplus X_{(0,0),-1} \oplus Y_{(0,0),(1,0)} \oplus Y_{(0,0),(0,1)} \oplus Y_{(0,0),(1,1)}$$

Example: $\mathsf{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)_{\gamma_d,-}$

- There is a unique maximal magnetic algebra $\mathcal{A}_P = 1 \oplus X_{(1,1),+1} \oplus Y_{(1,0),(0,1)}$
- $\mathfrak{Z}(\mathsf{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)_{\gamma_d,-})/\mathcal{A}_P = \mathfrak{Z}(\mathsf{Vec}^1_{\mathbb{Z}_2})$
- Kernel of $P : \mathsf{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)_{\gamma_d,-}) \to \mathsf{Vec}^1_{\mathbb{Z}_2}$ given by $\mathcal{A}_I = 1 \oplus X_{(0,0),-1}$



$$\mathfrak{A}(\mathsf{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)_{\gamma_d,-}) = \left\{\mathsf{Vec}^1_{\mathbb{Z}_2}\right\}$$

$$P(\mathcal{D}) = 1 \oplus \eta\,,\ P(a_1) = P(a_2) = 1$$

Example: $\mathsf{TY}(\mathbb{Z}_4)_{\epsilon=+}$

- No fiber functor.
- No duality invariant SPT, but $\exists$ duality invariant TQFT: SSB $\mathbb{Z}_4 \to \mathbb{Z}_2$

	$X_{0,\pm 1}$	$X_{1,\pm\zeta_8}$	$X_{2,\pm 1}$	$X_{3,\pm\zeta_8}$	$Y_{1,0}$	$Y_{2,0}$	$Y_{3,0}$
θ	1	$-i$	1	$-i$	1	1	1
d	1	1	1	1	2	2	2

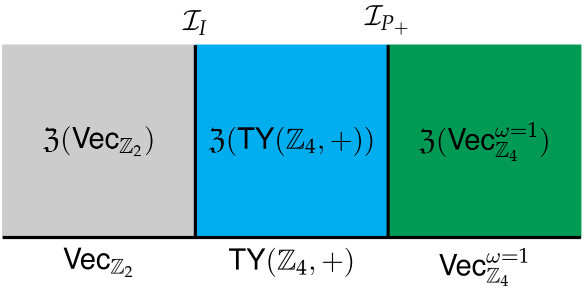
	$Y_{1,2}$	$Y_{1,3}$	$Y_{3,2}$	$Z_{\rho_0,\pm\zeta_{16}}$	$Z_{\rho_1,\pm 1}$	$Z_{\rho_2,\pm\zeta_{16}^{-3}}$	$Z_{\rho_3,\pm 1}$
θ	-1	i	-1	$\pm\zeta_{16}$	± 1	$\pm\zeta_{16}^{-3}$	± 1
d	2	2	2	2	2	2	2

$\mathsf{Vec}_{\mathbb{Z}_4} \subset \mathsf{TY}(\mathbb{Z}_4)$ anomaly free but not normal. $\mathsf{Vec}_{\mathbb{Z}_2}$ also normal!

Two maxiamal magnetic algebras:

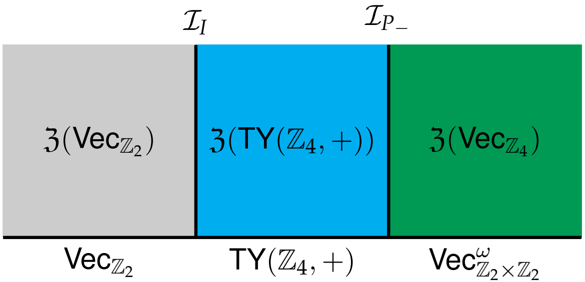
1. $\mathcal{A}_{P_+} = 1 \oplus X_{2,+1} \implies \mathsf{Vec}_{\mathbb{Z}_4}^{\omega=1}$
2. $\mathcal{A}_{P_-} = 1 \oplus X_{2,-1} \implies \mathsf{Vec}_{\mathbb{Z}_2\times\mathbb{Z}_2}^{\omega=\text{mixed}}$

Example: $\mathsf{TY}(\mathbb{Z}_4)_{\epsilon=+}$



$$P_+(\mathcal{D}) = \eta \oplus \eta^3\,, \quad P_+(a) = \eta^2\,, \quad P_+(a^2) = 1$$

Example: $\mathsf{TY}(\mathbb{Z}_4)_{\epsilon=+}$



$$P_-(\mathcal{D}) = \eta_1 \oplus \eta_2\,, \quad P_-(a) = \eta_1\eta_2\,, \quad P_-(a^2) = 1$$

Example: $\mathsf{TY}(\mathbb{Z}_4)_{\epsilon=+}$

$$\mathfrak{A}(\mathsf{TY}(\mathbb{Z}_4)_+) = \left\{\mathsf{Vec}^1_{\mathbb{Z}_4}, \mathsf{Vec}^{\omega}_{\mathbb{Z}_2\times\mathbb{Z}_2}\right\}$$

- Different ASCies have different fusion rules!
- $P_+(a) = \eta^2 \in \mathsf{Vec}^1_{\mathbb{Z}_4}$, $P_-(a) = \eta_1\eta_2 \in \mathsf{Vec}^{\omega}_{\mathbb{Z}_2\times\mathbb{Z}_2}$: both anomalous $\Longrightarrow$ 'duality enforced' emergent anomaly
- Intuitive interpretation: preserving duality $\Longrightarrow$ if $\mathcal{O}_e(x)$ is light, $\mathcal{O}_m(x)$ is also light!

Other topics & future directions

- $d > 2$: matching anomalies with higher-form symmetries. Wang-Wen-Witten, Symmetry fractionalization, transmutation [Seiberg, Seifnashri '25]
- Duality symmetries in (3+1)d
- LSM anomalies [Pace, Aksoy, Lam '25]

Future directions

- Systematics in (2+1)d and (3+1)d
- Math of ASCies?
- Structure of ASCies in higher dimensions
- Continuous symmetries
- Weak symmetries [Sakura's talk]
- Fermionic anomalies