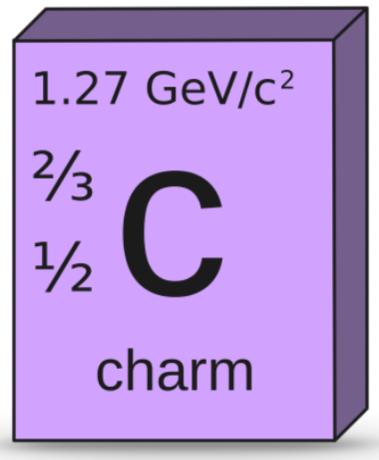


Recent progress on inclusive decays of heavy quarks

秦 溱

华中科技大学

第22届全国重味物理和CP破坏研讨会

北京大学，2025年10月24-28日

Inclusive decays of heavy quarks

❖ **Fully inclusive heavy hadron decays**

➡ $H_b \to X, H_c \to X$ ➡ Lifetime

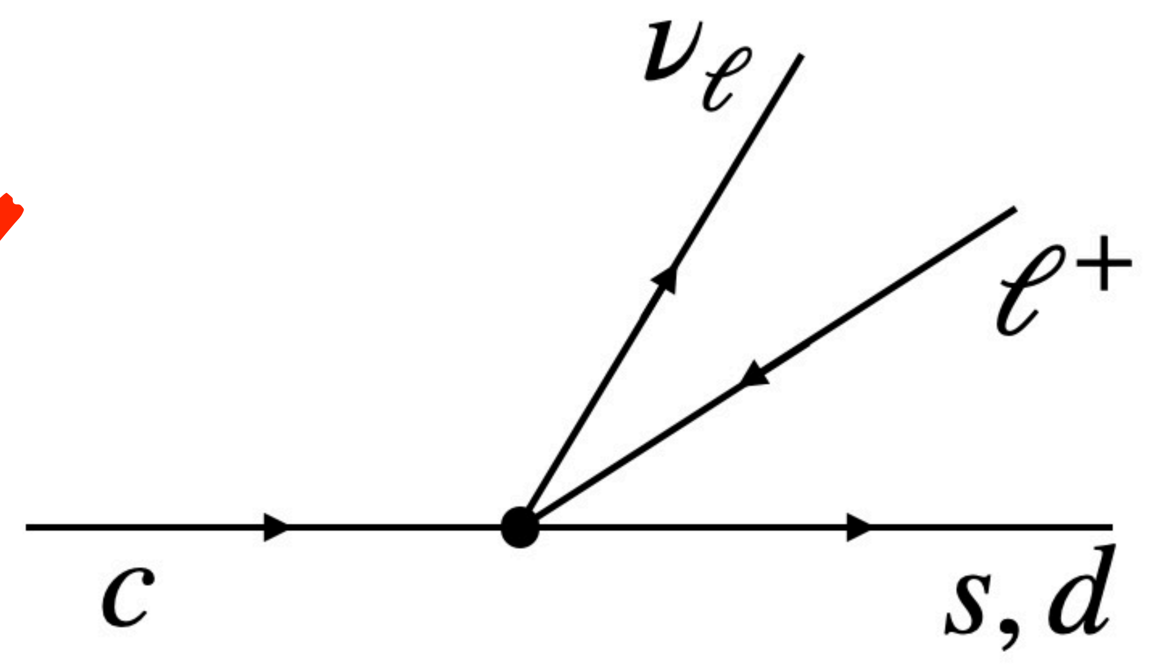
❖ **Semi-inclusive heavy hadron decays**

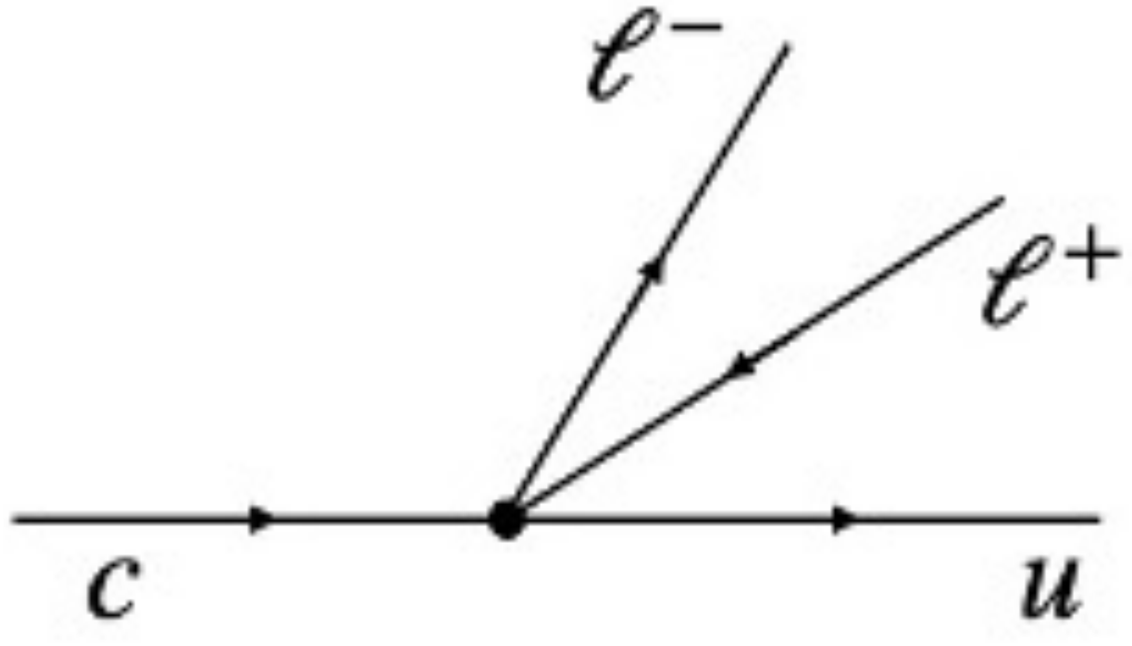
➡ $H_c \to \ell^+ X_{s,d}$ ($H_c \to \ell^+ \nu_\ell X_{s,d}$, only ℓ^+ is detected) ✔

➡ $H_b \to \ell^- X_{c,u}$ ($H_b \to \ell^- \bar{\nu}_\ell X_{c,u}$)

➡ $H_c \to \ell^+ \ell^- X_u, \gamma X_u$

➡ $H_b \to \ell^+ \ell^- X_{s,d}, \gamma X_{s,d}$





Why inclusive decays?

❖ Compared to exclusive decays

➡ Better theoretical control

More important with more powerful experiments (BESIII, STCF)

❖ More specific reasons

➡ Determine fundamental SM parameters —— CKM matrix elements

➡ Precise test of the SM ——search for new physics

➡ Test of heavy quark expansion

Why inclusive decays?

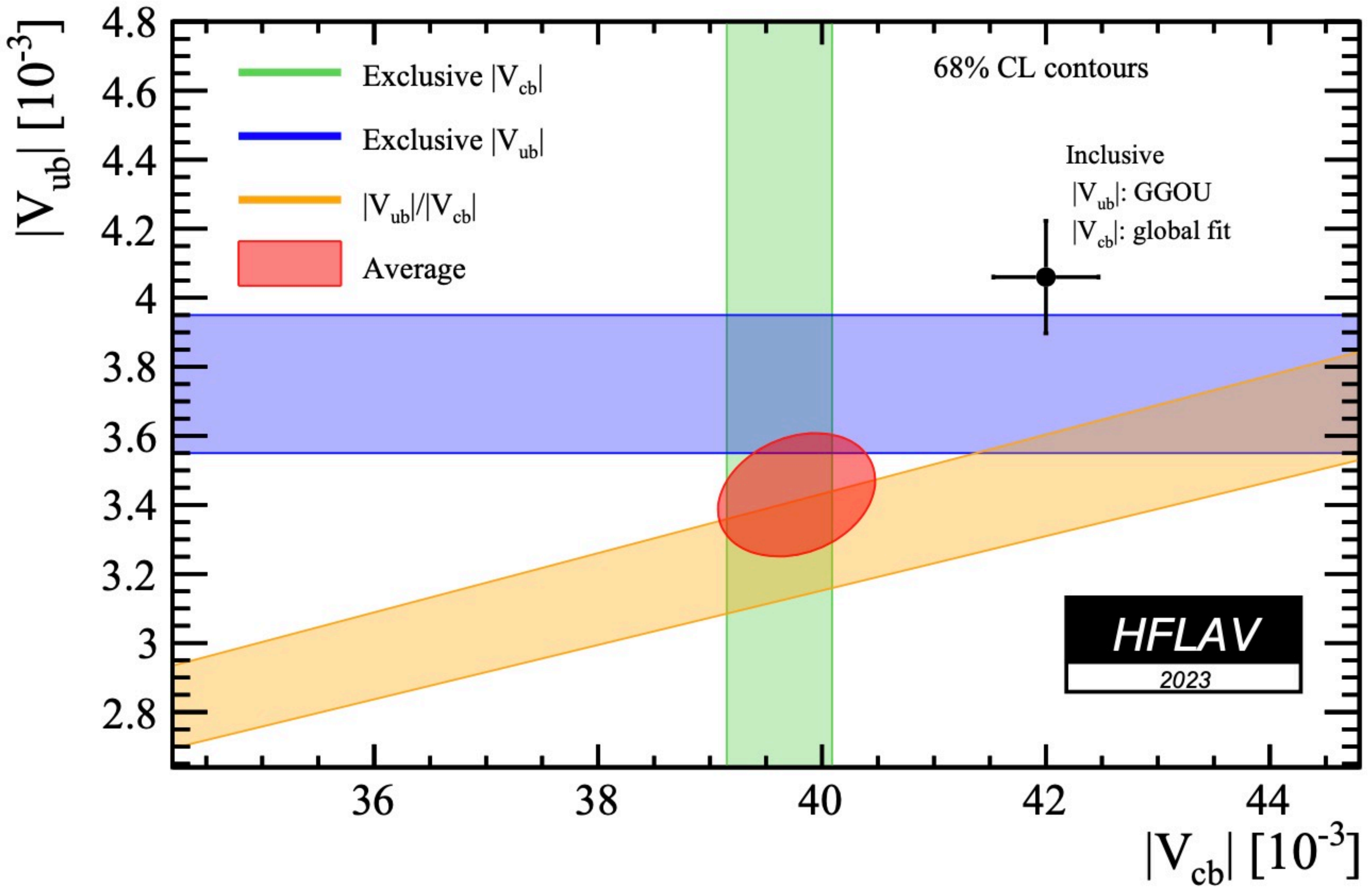
❖ **Determine fundamental SM parameters —— CKM matrix elements**

➡ **Test the CKM mechanism by comparison values determined by other values**

❖ **Flavor puzzle.** V_{cb}, V_{ub}: inclusive vs exclusive

➡ **Check the experiment and theory frameworks**

✦ **Cross check:** Test $V_{cs,cd}$: inclusive vs exclusive



Why inclusive decays?

❖ **Precise test of the SM ——search for new physics**

❖ **Flavor puzzle.** $b \to s$ anomalies: P_5' in $B \to K^* \ell\ell$

✦ **Key issue:** First-principle calculation of long-distance penguin in this channel is still missing

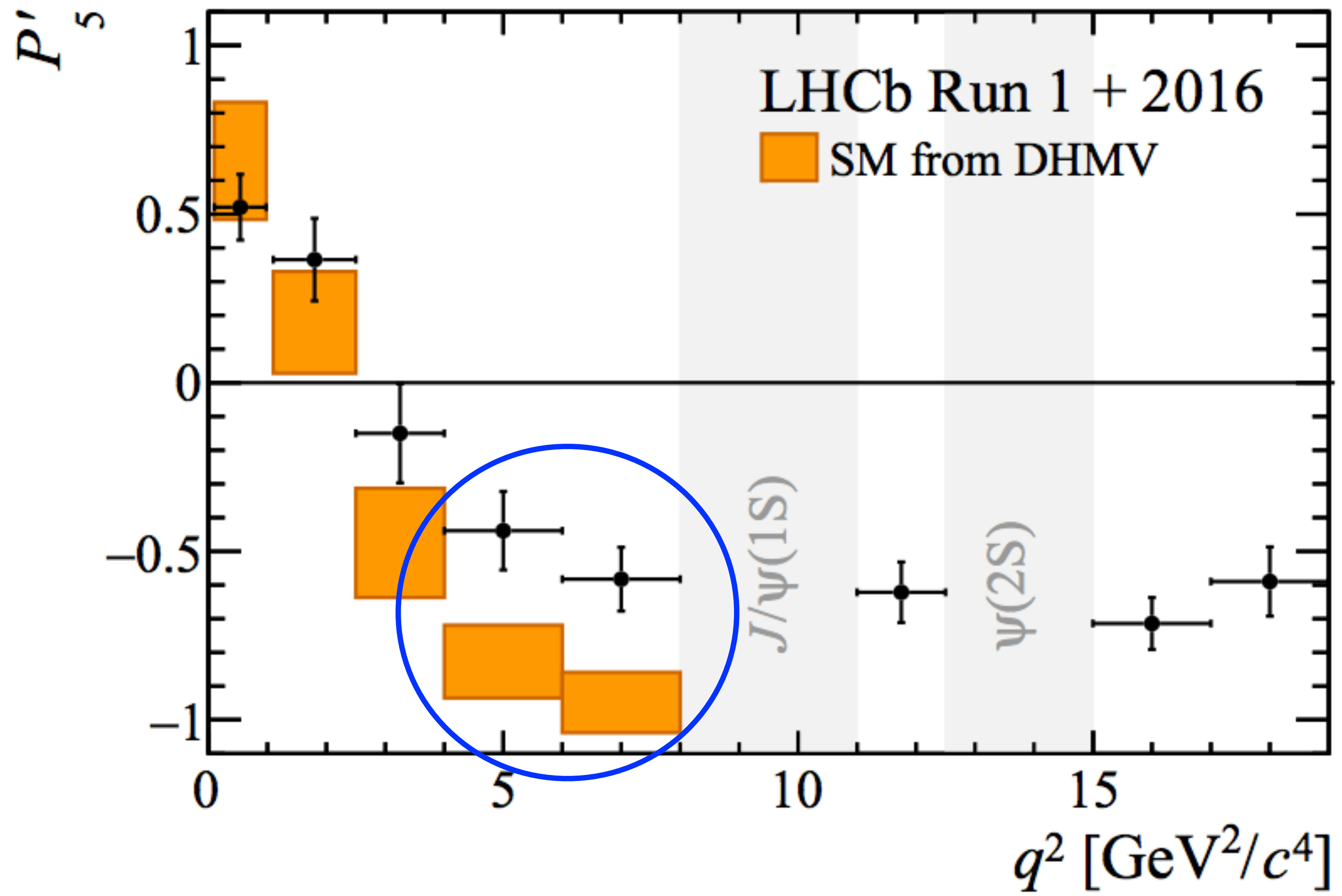
➡ Only achieved in $B \to \gamma\gamma$

[**QQ**,Shen,Wang,Wang, PRL, '23]

✦ **Solution**: Test FCNC inclusive decays

➡ Inclusive $B \to X_s \ell\ell$

[Huber,Hurth,Jenkins,Lunghi,**QQ**,Vos, JHEP, '19,'20,'24]



[LHCb, '20]

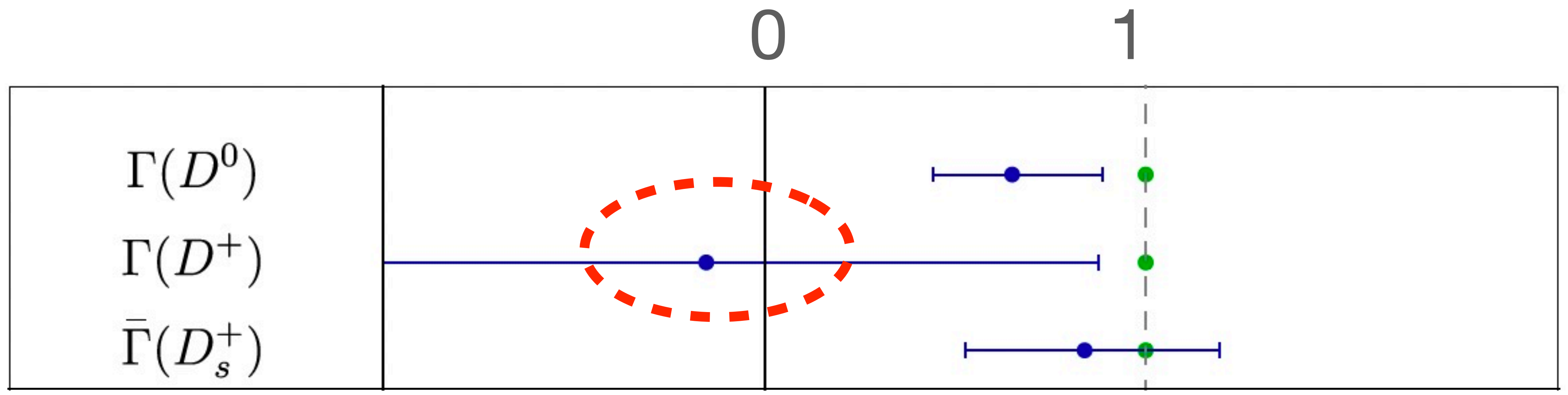
Why inclusive decays?

❖ **Test of heavy quark expansion**

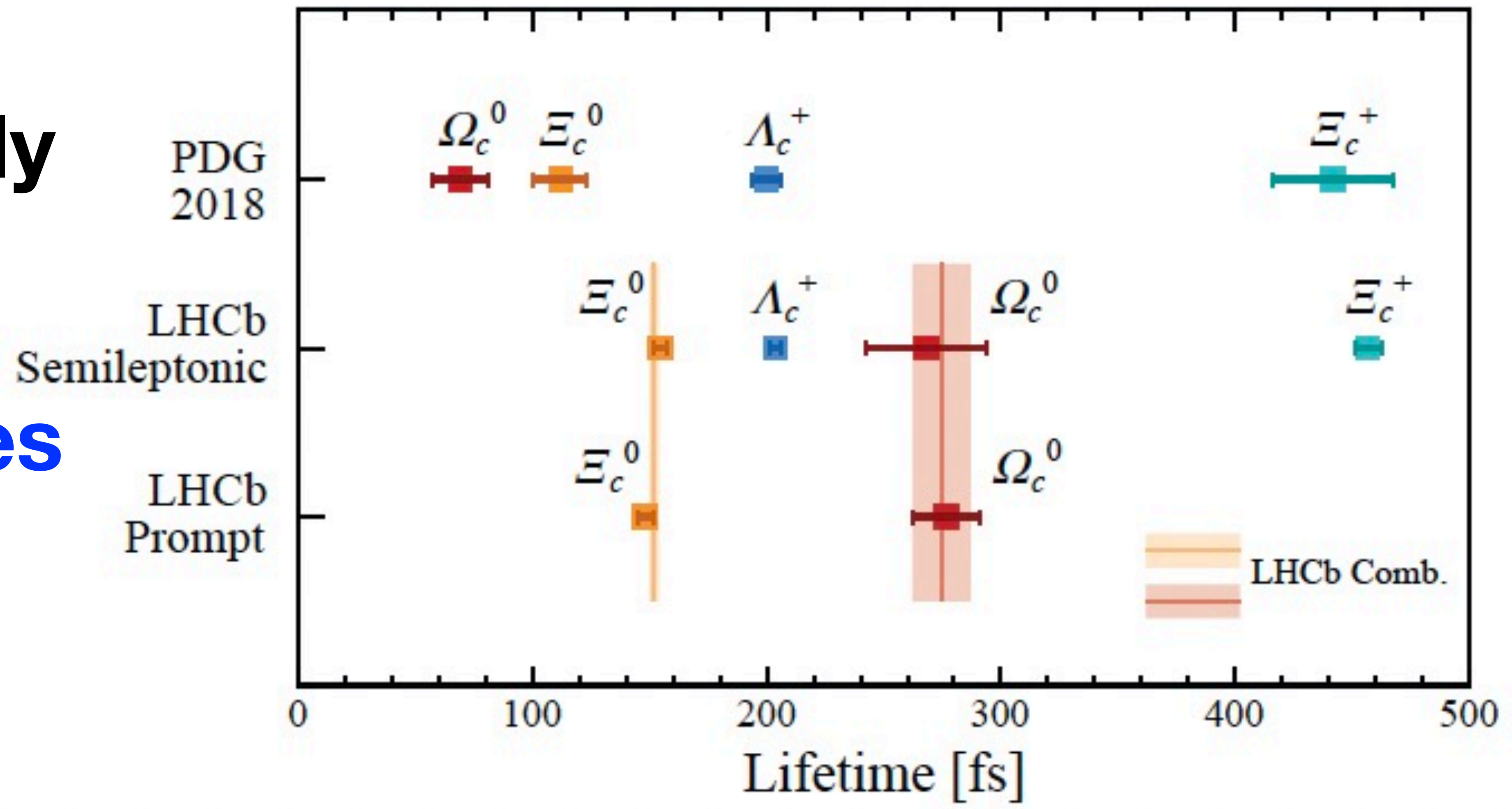
➡ **Semi-inclusive decay spectra and hadron lifetimes rely on identical HQE parameters**

➡ **Extraction from inclusive spectra and apply in lifetimes**

❖ **This works well for bottom, but for charm …**



[Lenz et al, ’22]



$$\mathcal{O}(1/m_c^3) \Rightarrow \tau(\Xi_c^+) > \tau(\Lambda_c^+) > \tau(\Xi_c^0) > \tau(\boldsymbol{\Omega_c^0}),$$
$$\mathcal{O}(1/m_c^4) \Rightarrow \tau(\boldsymbol{\Omega_c^0}) > \tau(\Xi_c^+) > \tau(\Lambda_c^+) > \tau(\Xi_c^0),$$
$$\mathcal{O}(1/m_c^4) \text{ with } \alpha \Rightarrow \tau(\Xi_c^+) > \tau(\boldsymbol{\Omega_c^0}) > \tau(\Lambda_c^+) > \tau(\Xi_c^0).$$

[Cheng, ’21]

Again a more precise experimental determination of μ_π^2 from fits to semileptonic D^+, D^0 and D_s^+ meson decays – as it was done for the B^+ and B^0 decays – would be very desirable.

[Lenz et al, ’22]

Theoretical Framework

Theoretical framework

❖ **Optical theorem**

$$\sum \langle D | H | X \rangle \langle X | H | D \rangle \propto \mathrm{Im} \int d^4 x e^{-ip_D \cdot x} \langle D | T\{H(x)H(0)\} | D \rangle$$

❖ **Operator product expansion (OPE)**

➡ Short distance $x \sim 1/m_c$

➡ Dynamical fluctuation in D meson $\sim \Lambda_{\mathrm{QCD}}$

$$T\{H(x)H(0)\} = \sum_n C_n(x) O_n(0) \to 1 + \frac{\Lambda_{\mathrm{QCD}}}{m_c} + \frac{\Lambda_{\mathrm{QCD}}^2}{m_c^2} + \ldots$$

Systematic OPE in HQET.

Theoretical framework

- **Heavy quark effective theory**

$$h_v(x) \equiv e^{-im_c v\cdot x}\frac{1+\gamma\cdot v}{2}c(x) \qquad v = (1,0,0,0)$$

Subtract the big intrinsic momentum,

Leave only ~Λ_{QCD} degrees of freedom.

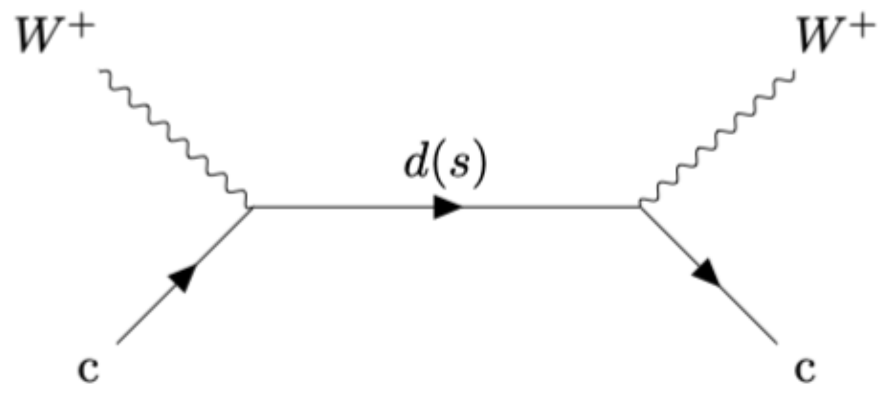
$$L \ni \bar{h}_v iv\cdot Dh_v$$
$$-\bar{h}_v\frac{D_\perp^2}{2m_c}h_v - a(\mu)g\bar{h}_v\frac{\sigma\cdot G}{4m_c}h_v + \dots$$

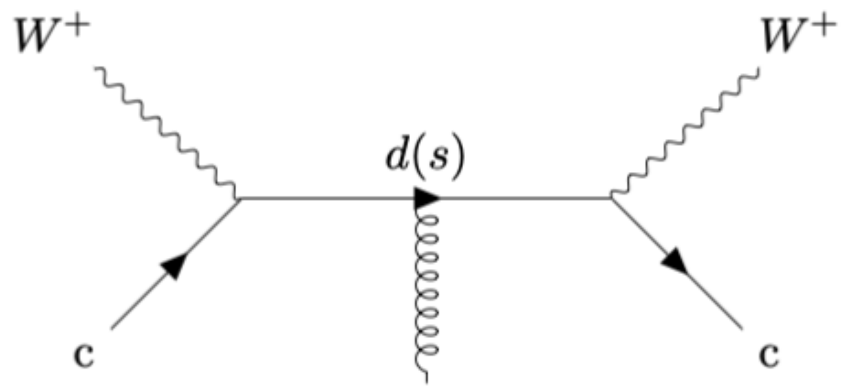
Similar to $\dfrac{m}{\sqrt{1-v^2}} = m + \dfrac{1}{2}mv^2 + \dots$

Theoretical framework

❖ **OPE and Perturbative QCD:** Systematical expansion of $\alpha_s(m_c)$ and $\Lambda_{\rm QCD}/m_c$

$$\langle T\{H(x)H(0)\}\rangle = \sum_n C_n(x)\langle O_n(0)\rangle$$





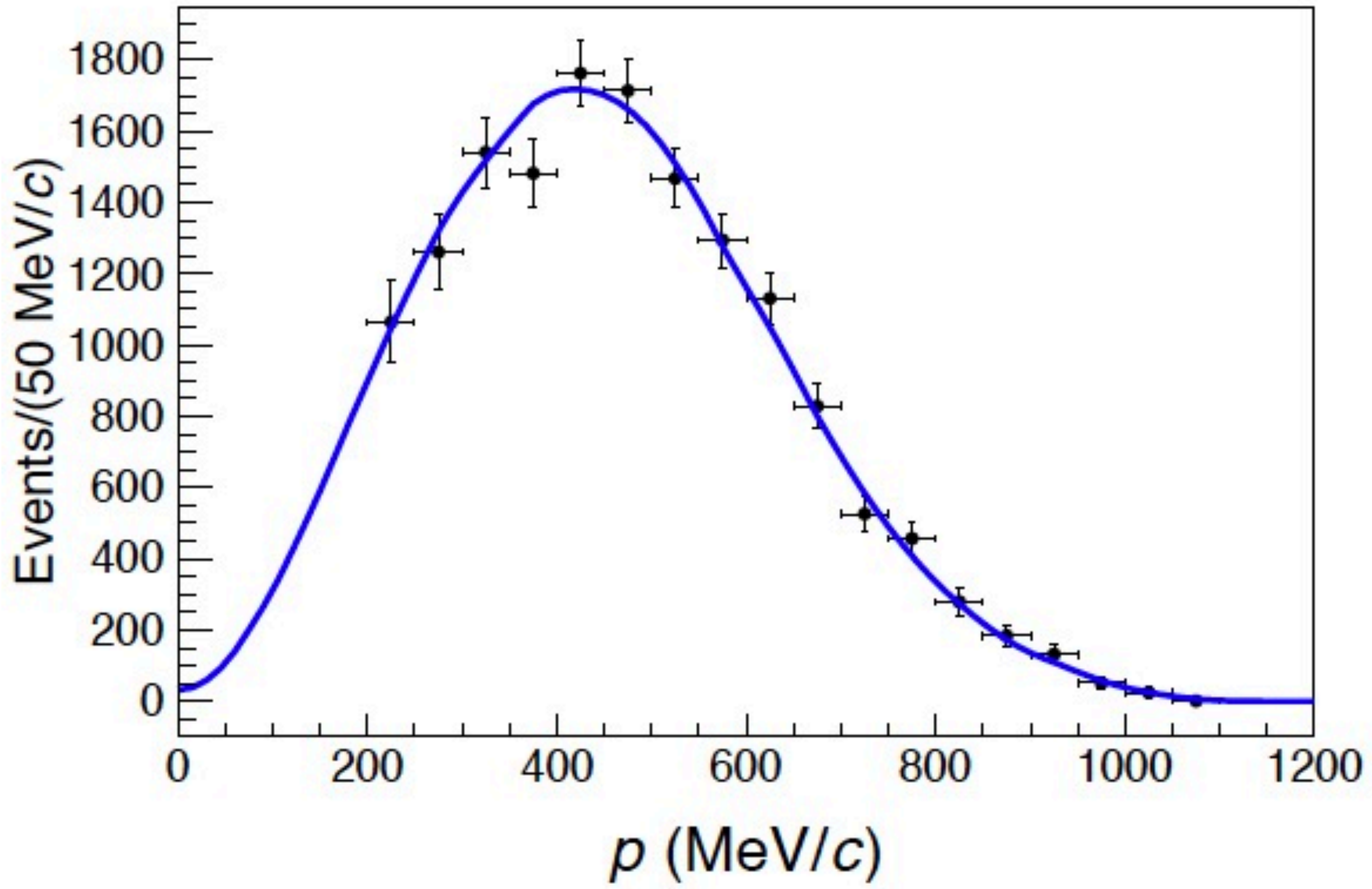
- LO: $\alpha_s^0(m_c)$
- NLO: $\alpha_s(m_c)$
- NNLO: $\alpha_s^2(m_c)$
- ...

- Dim-3: $\bar{h}_v h_v$ $(\bar{c}\gamma^\mu c)$ **partonic decay rate**
- Dim-5: $\bar{h}_v D_\perp^2 h_v,\ g\bar{h}_v\sigma\cdot G h_v$.
- Dim-6: $\bar{h}_v D_\mu(v\cdot D)D^\mu h_v,\ (\bar{h}_v\Gamma_1 q)(\bar{q}\Gamma_2 h_v),\ldots$
- ...

Theory versus Experiment

❖ **Electron energy spectrum** ($y \equiv 2E_e/m_c$)

$$
\begin{aligned}
\frac{1}{\Gamma_0}\frac{d\Gamma}{dy} &= 12(1-y)y^2\theta(1-y) \\
&+\frac{2\mu_\pi^2}{m_c^2}\Big[-10y^3\theta(1-y)+2\delta(1-y)\Big] \\
&-\frac{2\mu_G^2}{3m_c^2}\Big[6y^2(6-5y)\theta(1-y)\Big]+\mathcal{O}(\alpha_s,\frac{\Lambda^3}{m_c^3})
\end{aligned}
$$



❖ Up to finite power, the obtained spectrum is **NOT** the experimental spectrum

➡ Observables require integration over final states

➡ $\Gamma = \int \frac{d\Gamma}{dy}dy\,,\ \langle E_\ell^n\rangle = \frac{1}{\Gamma}\int \frac{d\Gamma}{dy}E_\ell^n\,dy$ (n=1,2,3,4) are the **observables**

➡ Shape function —— infinite power summation? [Neubert, ’93]

Precent Progresses on Bottom Decays

Charged-current bottom decays

❖ **Kolya: An automatic calculation package for $\bar{B} \to \ell^- \bar{\nu}_\ell X_c$** [Fael, Milutinb, Vos, '24]

➡ Input favored parameters

➡ Output numerical predictions

$$\Delta\Gamma_{\rm sl}(E_{\rm cut}) = \int_{E_l \geq E_{\rm cut}} \frac{\mathrm{d}\Gamma}{\mathrm{d}E_l}\,\mathrm{d}E_l\,, \qquad \Delta\Gamma_{\rm sl}(q^2_{\rm cut}) = \int_{q^2 \geq q^2_{\rm cut}} \frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}\,\mathrm{d}q^2.$$

$$\ell_1(E_{\rm cut}) = \langle E_l \rangle_{E_l \geq E_{\rm cut}}\,, \qquad \ell_n(E_{\rm cut}) = \Big\langle (E_l - \langle E_l \rangle)^n \Big\rangle_{E_l \geq E_{\rm cut}}$$

$$h_1(E_{\rm cut}) = \langle M_X^2 \rangle_{E_l \geq E_{\rm cut}}\,, \qquad h_n(E_{\rm cut}) = \Big\langle (M_X^2 - \langle M_X^2 \rangle)^n \Big\rangle_{E_l \geq E_{\rm cut}}$$

$$q_1(q^2_{\rm cut}) = \langle q^2 \rangle_{q^2 \geq q^2_{\rm cut}}\,, \qquad q_n(q^2_{\rm cut}) = \Big\langle (q^2 - \langle q^2 \rangle)^n \Big\rangle_{q^2 \geq q^2_{\rm cut}}$$

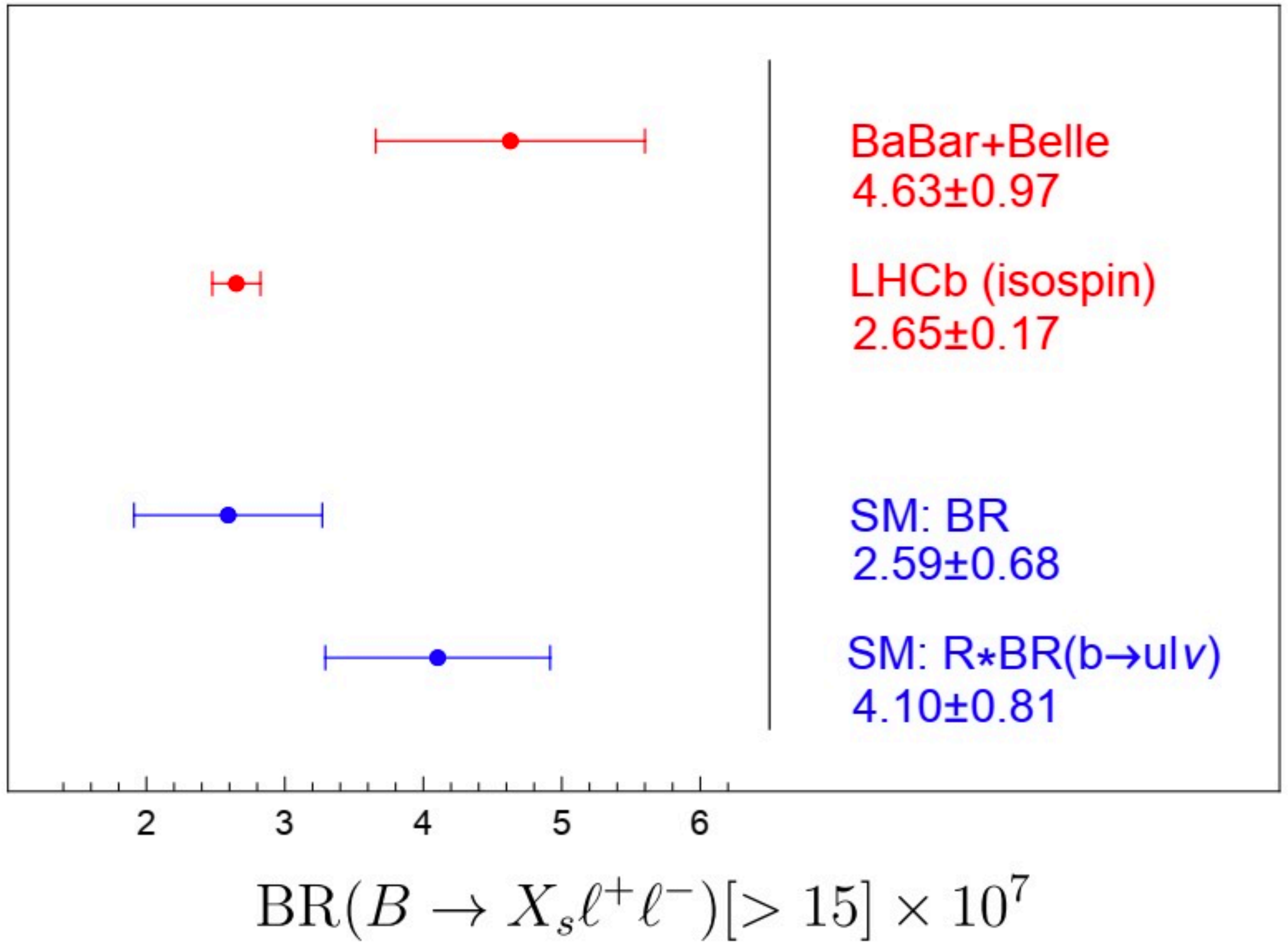
$\Gamma_{\rm sl}$	tree	α_s	α_s^2	α_s^3
Partonic		[34]	[35–38]	[39]
μ_π^2, μ_G^2	[1, 2]	[40–43]		
ρ_D^3, ρ_{LS}^3	[44]	[45]		
$1/m_b^4, 1/m_b^5$	[23, 26–28]			
$q_n(q^2_{\rm cut})$	tree	α_s	α_s^2	
Partonic		[45, 46]	[47]	
μ_G^2, μ_π^2	[1, 2]	[41, 42]		
ρ_D^3, ρ_{LS}	[44]	[45]		
$1/m_b^4, 1/m_b^5$	[23, 28]			
$\ell_n(E_{\rm cut}), h_n(E_{\rm cut})$	tree	α_s	$\alpha_s^2\beta_0$	α_s^2
Partonic		[46, 48, 49]	[46]	[33]*
μ_G^2, μ_π^2	[1, 2]	[42, 50]		
ρ_D^3	[44]			
$1/m_b^4, 1/m_b^5$	[26–28]			

Neutral-current bottom decays

❖ **Updated $\bar{B} \to X_s\mu^+\mu^-$ results for LHCb** [Huber,Hurth,Jenkins,Lunghi,**QQ**,Vos, JHEP, ’19,’20,’24]

➡ Sum-of-exclusive for the high-$m^2_{\mu\mu}$ region

➡ Dominated by K, K^*, also contributed by S-wave $K\pi$, tail effects from $K^*(1410,1430)$

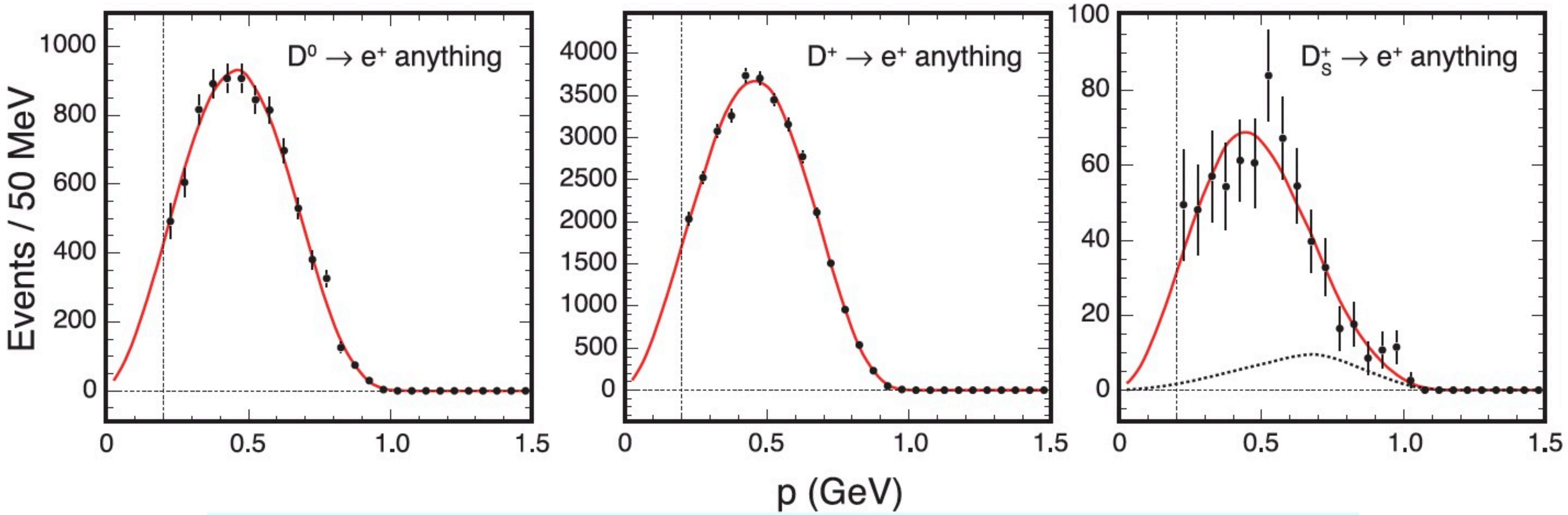


Precent Progresses on Charm Decays

Experimental status

CLEO measurements

$D^0 \to e^+X$ $\qquad$ $D^+ \to e^+X$ $\qquad$ $D_s^+ \to e^+X$

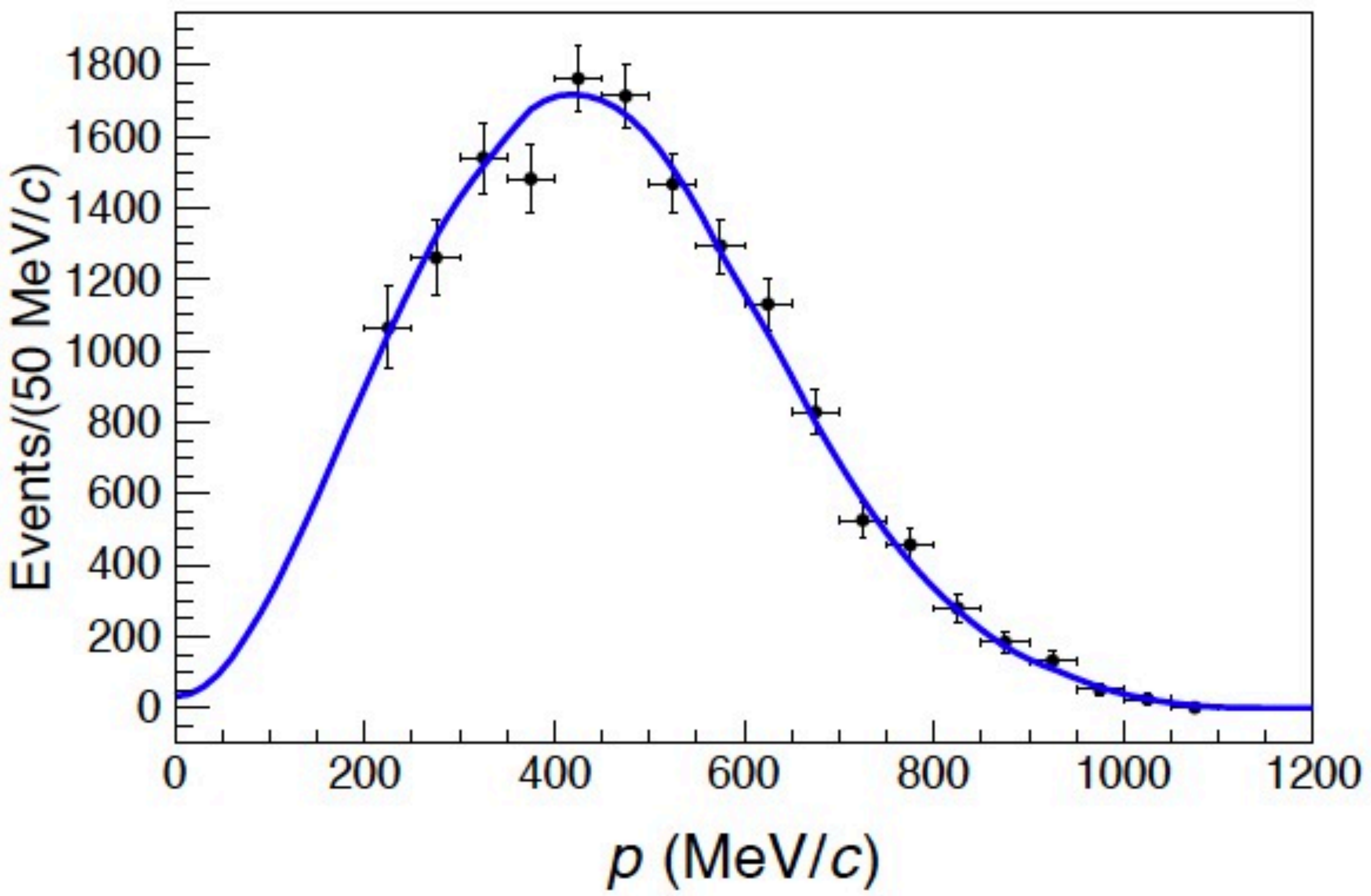


$$\mathcal{B}(D^0 \to Xe^+\nu_e) = (6.46 \pm 0.09 \pm 0.11)\%,$$
$$\mathcal{B}(D^+ \to Xe^+\nu_e) = (16.13 \pm 0.10 \pm 0.29)\%,$$
$$\mathcal{B}(D_s^+ \to Xe^+\nu_e) = (6.52 \pm 0.39 \pm 0.15)\%,$$

[CLEO, '09]

BESIII measurements

$D_s^+ \to e^+X$



$$B(D_s^+ \to Xe^+\nu_e) = (6.30 \pm 0.13 \pm 0.10)\,\%$$

[BESIII, '21]

2% precision!

Phenomenological status

PHYSICAL REVIEW D **104**, 012003 (2021)

Measurement of the absolute branching fraction of inclusive semielectronic D_s^+ decays

(BESIII Collaboration)

$$\frac{\Gamma(D_s^+ \to X e^+ \nu_e)}{\Gamma(D^0 \to X e^+ \nu_e)}$$
$$= 0.790 \pm 0.016(\text{stat.}) \pm 0.011(\text{syst.}) \pm 0.016(\text{ext.}).$$

where the external systematic uncertainty includes the total uncertainty from $\mathcal{B}(D^0 0 \to X e^+ \nu_e)$, the D^0 and D_s^+ lifetimes, and $\mathcal{B}(D_s^+ \to \tau^+ \nu_\tau)$. This result is in agreement with the prediction of $\frac{\Gamma(D_s^+ \to X e^+ \nu)}{\Gamma(D^0 0 \to X e^+ \nu)} = 0.813$ from Ref. [18],

[18] M. Gronau and J. L. Rosner, Phys. Rev. D **83**, 034025 (2011).

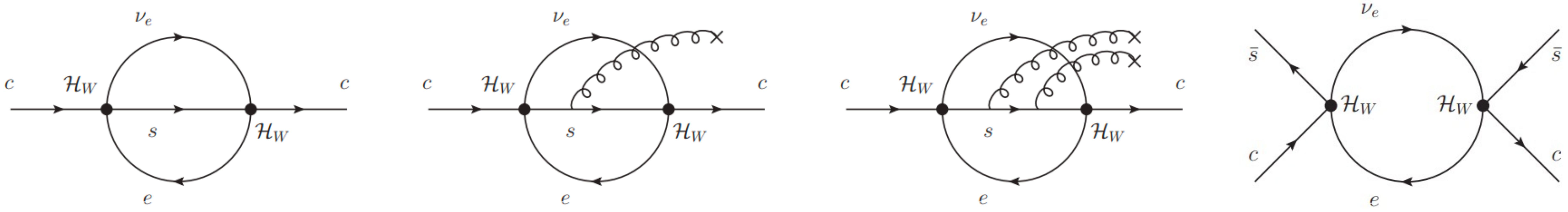
A model calculation basically with only phase-space effects.

Question: convergent expansion of $\alpha_s(m_c)$ and Λ_{QCD}/m_c?

Heavy quark expansion for charm

❖ **Heavy quark expansion up to dimension-7 operators (LO)** [Fael,Mannel,Vos, ’19]

➡ QCD and HQET operator matching by calculating quark-gluon diagrams

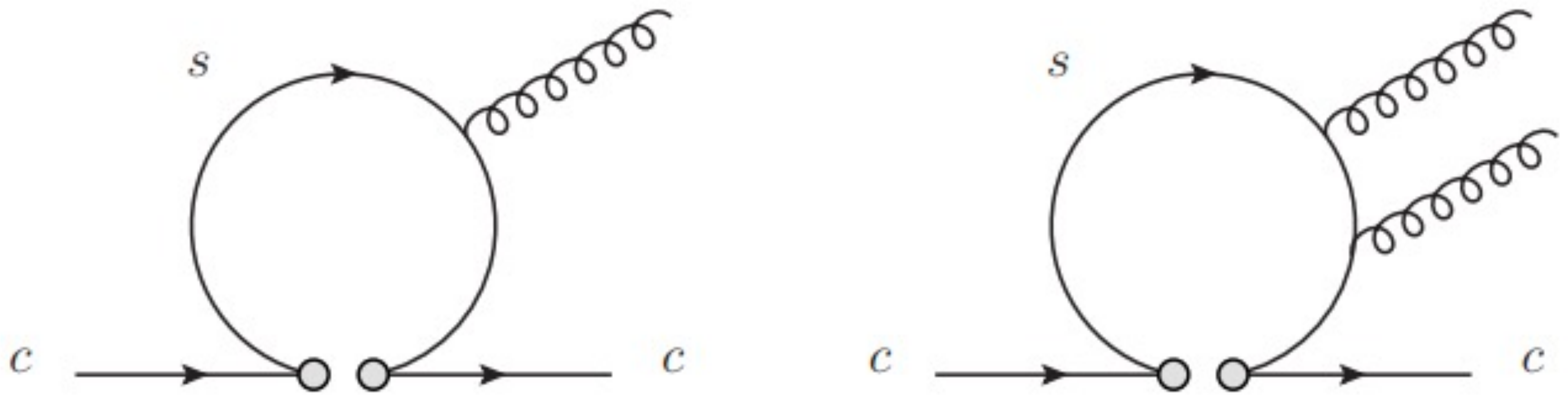


➡ Free to choose HQET operator basis, but RPI ones are favored

➡ $c \to d$ is like $b \to u$, $c \to s$ **is not** like $b \to c$: m_s appears in operators, not WCs

➡ **Operator mixing at LO**: 4q operators to 2q operators

$$C_i^{2q}(\mu) = C_i^{2q}(m_c) + \log\left(\frac{\mu}{m_c}\right) \sum_j \hat{\gamma}_{ij}^T C_j^{4q}(m_c)$$



Pheno-1. Determine the HQE parameters

[Shao,Huang,**QQ**,2502.05901]

Theoretical results

❖ Theoretical results for <u>total decay rate</u> and energy moments (NNLO & $\Lambda^3_{\text{QCD}}/m_c^3$)

NLO analytical integration

NNLO numerical results provided by authors of [Chen,Chen,Guan,Ma,'23]

$$\Gamma_{D_i\to X_q} = \hat{\Gamma}_0\,|V_{cq}|^2\, m_c^5\Big\{1+\frac{\alpha_s(\mu)}{\pi}\frac{2}{3}\left(\frac{25}{4}-\pi^2\right)+\frac{\alpha_s^2(\mu)}{\pi^2}\left[\frac{\beta_0}{4}\frac{2}{3}\left(\frac{25}{4}-\pi^2\right)\log\left(\frac{\mu^2}{m_c^2}\right)+2.14690n_l-29.88311\right]$$
$$-8\rho\delta_{sq}-\frac{1}{2}\frac{\mu_\pi^2(D_i)}{m_c^2}-\frac{3}{2}\frac{\mu_G^2(D_i)}{m_c^2}+\left(6+8\log\left(\frac{\mu^2}{m_c^2}\right)\right)\frac{\rho_D^3(D_i)}{m_c^3}+\frac{\tau_0(D_i\to X_q)}{m_c^3}+\ldots\Big\}$$

Dim-5, $\Lambda^2_{\text{QCD}}/m_c^2$

Dim-6, $\Lambda^3_{\text{QCD}}/m_c^3$

❖ For Dim-6 4-quark operator contributions, practically difficult to extract

➡ Vacuum-Insertion-Approximation (VIA), $\tau_0 = 0$

➡ HQET Sum Rules [King, Lenz, Piscopo, Rauh,'19]

Theoretical results

❖ Theoretical results for total decay rate and energy moments (NNLO & $\Lambda^3_{\rm QCD}/m_c^3$)

$$\langle E_e\rangle_{D_i} = \frac{\hat\Gamma_0}{\Gamma_{D_i}}\sum_{q=d,s}|V_{cq}|^2\, m_c^6\left[\frac{3}{10}+\frac{\alpha_s}{\pi}a_1^{(1)}+\frac{\alpha_s^2}{\pi^2}a_1^{(2)}-3\rho\delta_{sq}-\frac{1}{2}\frac{\mu_G^2(D_i)}{m_c^2}+\frac{139}{30}\frac{\rho_D^3\,(D_i)}{m_c^3}+\frac{3}{10}\frac{\rho_{LS}^3\,(D_i)}{m_c^3}+...\right],$$

$$\langle E_e^2\rangle_{D_i} = \frac{\hat\Gamma_0}{\Gamma_{D_i}}\sum_{q=d,s}|V_{cq}|^2\, m_c^7\left[\frac{1}{10}+\frac{\alpha_s}{\pi}a_2^{(1)}+\frac{\alpha_s^2}{\pi^2}a_2^{(2)}-\frac{6}{5}\rho\delta_{sq}+\frac{1}{12}\frac{\mu_\pi^2(D_i)}{m_c^2}-\frac{11}{60}\frac{\mu_G^2(D_i)}{m_c^2}+\frac{17}{6}\frac{\rho_D^3\,(D_i)}{m_c^3}\right.$$
$$\left.+\frac{7}{30}\frac{\rho_{LS}^3\,(D_i)}{m_c^3}+...\right],$$

$$\left\langle E_e^3\right\rangle_{D_i} = \frac{\hat\Gamma_0}{\Gamma_{D_i}}\sum_{q=d,s}|V_{cq}|^2\, m_c^8\left[\frac{1}{28}+\frac{\alpha_s}{\pi}a_3^{(1)}+\frac{\alpha_s^2}{\pi^2}a_3^{(2)}-\frac{1}{2}\rho\delta_{sq}+\frac{1}{14}\frac{\mu_\pi^2\,(D_i)}{m_c^2}-\frac{1}{14}\frac{\mu_G^2\,(D_i)}{m_c^2}+\frac{223}{140}\frac{\rho_D^3\,(D_i)}{m_c^3}\right.$$
$$\left.+\frac{1}{7}\frac{\rho_{LS}^3\,(D_i)}{m_c^3}+...\right],$$

$$\left\langle E_e^4\right\rangle_{D_i} = \frac{\hat\Gamma_0}{\Gamma_{D_i}}\sum_{q=d,s}|V_{cq}|^2\, m_c^9\left[\frac{3}{224}+\frac{\alpha_s}{\pi}a_4^{(1)}+\frac{\alpha_s^2}{\pi^2}a_4^{(2)}-\frac{3}{14}\rho\delta_{sq}+\frac{3}{64}\frac{\mu_\pi^2\,(D_i)}{m_c^2}-\frac{13}{448}\frac{\mu_G^2\,(D_i)}{m_c^2}+\frac{481}{560}\frac{\rho_D^3\,(D_i)}{m_c^3}\right.$$
$$\left.+\frac{9}{112}\frac{\rho_{LS}^3\,(D_i)}{m_c^3}+...\right],$$

Mass scheme

$$\boxed{\Gamma = m_c^5(\Gamma^{(0)} + \alpha_s\Gamma^{(1)} + \alpha_s^2\Gamma^{(2)}) = \left(\overline{m}_c(1 + \alpha_s\overline{m}^{(1)} + \alpha_s^2\overline{m}^{(2)})\right)^5 (\Gamma^{(0)} + \alpha_s\Gamma^{(1)} + \alpha_s^2\Gamma^{(2)})}$$

❖ **Pole mass scheme** (suffering from renormalon)

$\Gamma/\Gamma_{\mathrm{LO}} = 1 - 0.77\alpha_s - 2.38\alpha_s^2 - 10.73\alpha_s^3 \approx$ 1 - 30% - 36% - 62%

Not convergent, negative at NNNLO!

❖ $\overline{\mathrm{MS}}$ **mass scheme**

[Melnikov,van Ritbergen, ’99]

$\Gamma/\Gamma_{\mathrm{LO}} = 1 + 1.35\alpha_s + 3.02\alpha_s^2 + 7.69\alpha_s^3 \approx$ 1 + 52% + 46% + 44%

Convergent, but very slowly!

❖ **1S mass scheme** (half of J/ψ mass)

[Hoang,Ligeti,Manohar, ’98; Hoang,Teubner, ’99]

$\Gamma/\Gamma_{\mathrm{LO}} \approx$ 1 - 13.1% - 4.8% + 1.8%

Answer: convergent expansion of $\alpha_s(m_c)$!

❖ **Kinetic mass scheme**

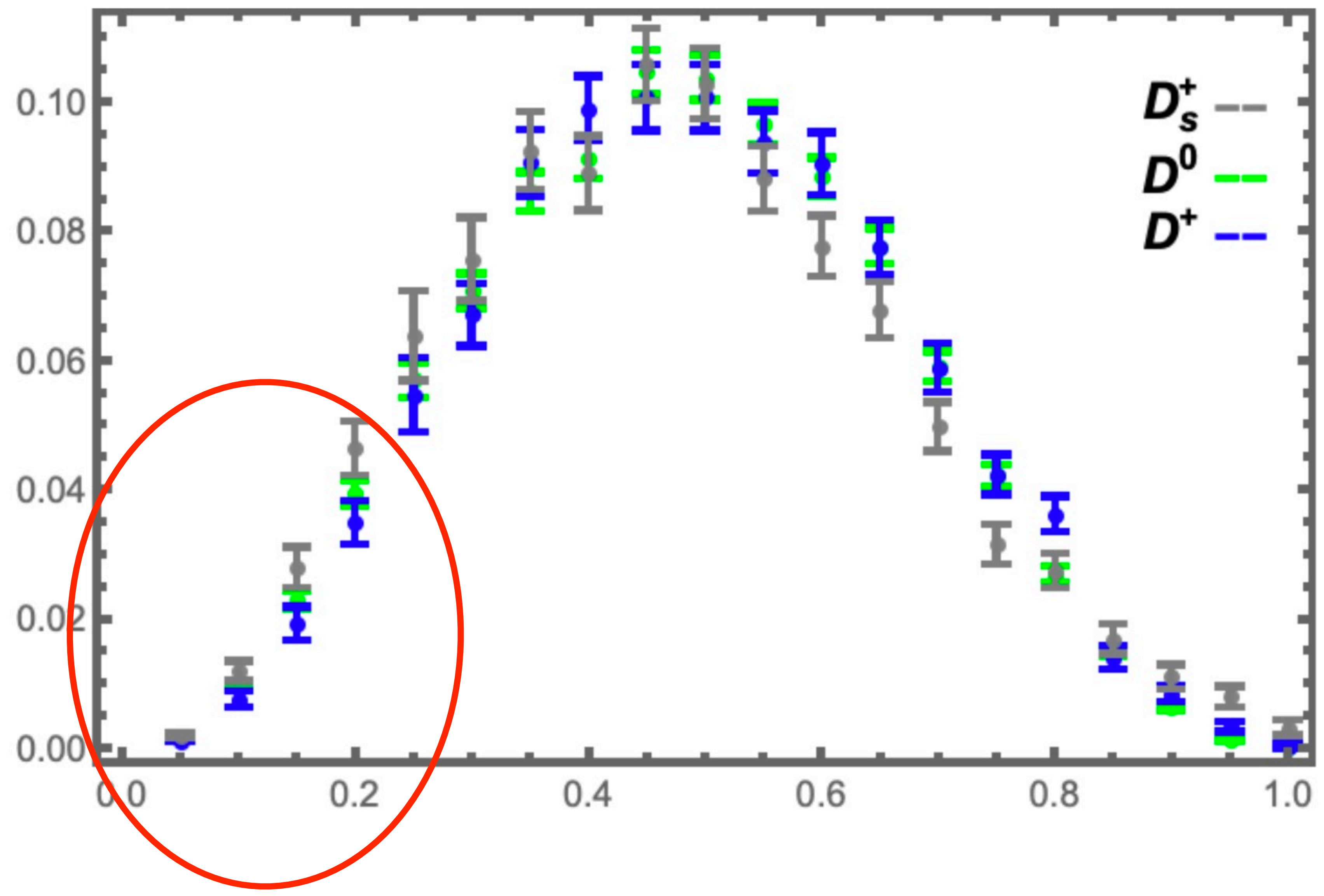
[Fael,Schönwald, Steinhauser, ’20]

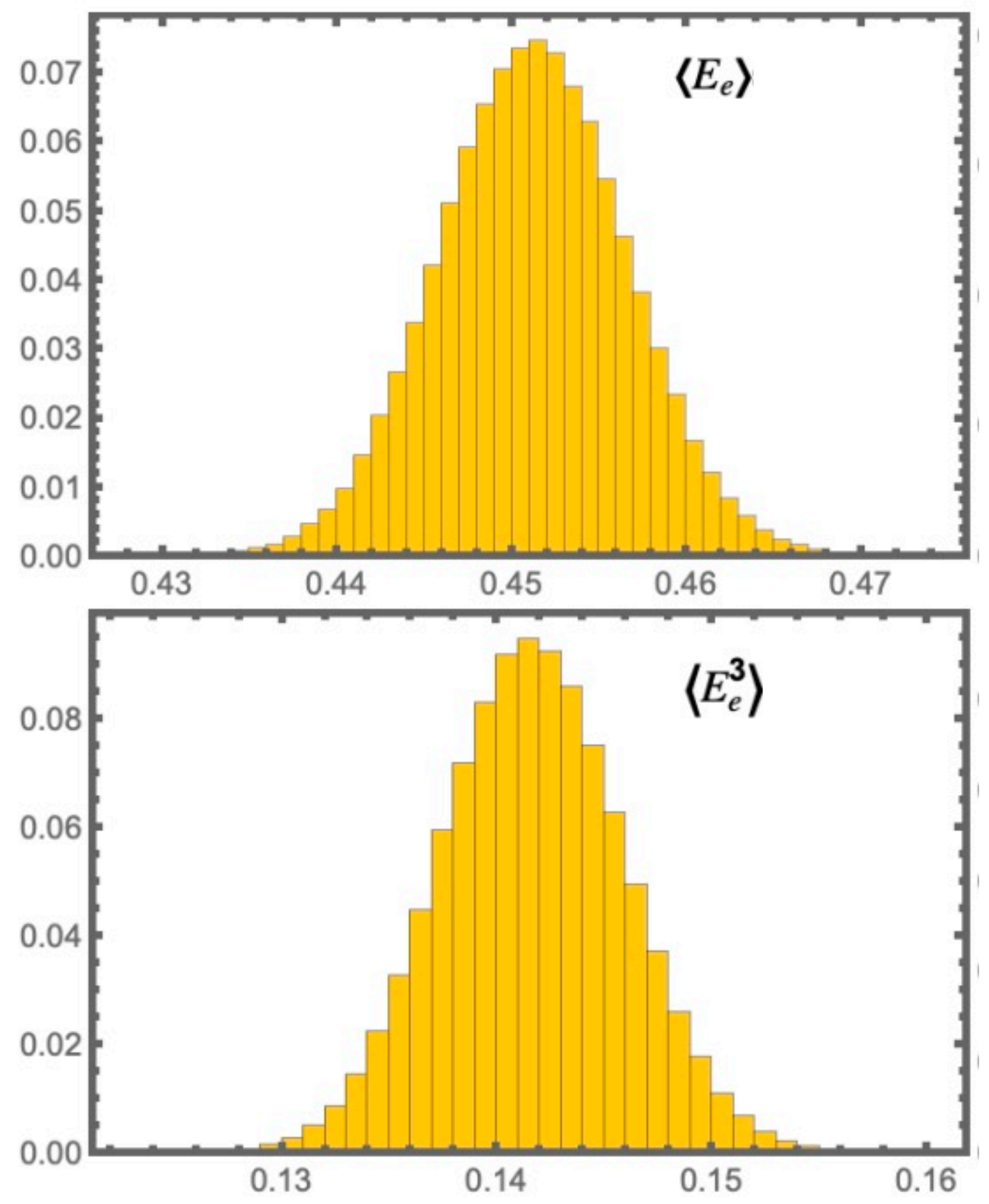
Not work for charm, because it requires expansion of μ^2/m_c^2 and $\alpha_s(\mu)$

Experimental data

$$\frac{d\Gamma}{dy} = ay^2(1+by)(1-y)$$

Original data → **Extrapolation** [Gambino,Kamenik, '10] → **All-bin distribution** → **Lorentz boost** → **Rest frame** → **MC Simulation** → **Electron energy moments**





$$\langle E_e\rangle^{D_s}_{exp} = 0.437(6)\mathrm{GeV}, \quad \langle E_e^2\rangle^{D_s}_{exp} = 0.220(5)\mathrm{GeV}^2$$
$$\langle E_e\rangle^{D^0}_{exp} = 0.462(5)\mathrm{GeV}, \quad \langle E_e^2\rangle^{D^0}_{exp} = 0.242(5)\mathrm{GeV}^2$$
$$\langle E_e\rangle^{D^+}_{exp} = 0.455(4)\mathrm{GeV}, \quad \langle E_e^2\rangle^{D^+}_{exp} = 0.236(4)\mathrm{GeV}^2$$
$$\langle E_e^3\rangle^{D_s}_{exp} = 0.121(4)\mathrm{GeV}^3, \quad \langle E_e^4\rangle^{D_s}_{exp} = 0.072(3)\mathrm{GeV}^4$$
$$\langle E_e^3\rangle^{D^0}_{exp} = 0.138(4)\mathrm{GeV}^3, \quad \langle E_e^4\rangle^{D^0}_{exp} = 0.084(3)\mathrm{GeV}^4$$
$$\langle E_e^3\rangle^{D^+}_{exp} = 0.134(3)\mathrm{GeV}^3, \quad \langle E_e^4\rangle^{D^+}_{exp} = 0.081(3)\mathrm{GeV}^4$$

Statistical uncertainties uncorrelated,
Systematic uncertainties fully correlated.

Global fit

❖ Global fit in two mass schemes, each with **Scenario 1** ($\Lambda^2_{\rm QCD}/m_c^2$) and **Scenario 2** ($\Lambda^3_{\rm QCD}/m_c^3$)

4-q operator contributions vanish under VIA, $\tau_0 = 0$.

$\overline{\rm MS}$ scheme	χ^2/d.o.f.	D_i	$\mu_\pi^2/{\rm GeV}^2$	$\mu_G^2/{\rm GeV}^2$	$\rho_{\rm D}^3/{\rm GeV}^3$	$\rho_{LS}^3/{\rm GeV}^3$
Scenario 1	4.5	$D^{0,+}$	0.09 ± 0.01	0.27 ± 0.14	-	-
		D_s	0.09 ± 0.02	0.39 ± 0.12	-	-
Scenario 2	2.1	$D^{0,+}$	0.11 ± 0.02	0.26 ± 0.14	-0.002 ± 0.002	0.003 ± 0.002
		D_s	0.12 ± 0.02	0.38 ± 0.13	-0.003 ± 0.002	0.005 ± 0.002

1S scheme	χ^2/d.o.f.	D_i	$\mu_\pi^2/{\rm GeV}^2$	$\mu_G^2/{\rm GeV}^2$	$\rho_{\rm D}^3/{\rm GeV}^3$	$\rho_{LS}^3/{\rm GeV}^3$
Scenario 1	4.9	$D^{0,+}$	0.04 ± 0.01	0.33 ± 0.02	-	-
		D_s	0.06 ± 0.02	0.44 ± 0.02	-	-
Scenario 2	0.33	$D^{0,+}$	0.09 ± 0.02	0.32 ± 0.02	-0.003 ± 0.002	0.004 ± 0.002
		D_s	0.11 ± 0.02	0.43 ± 0.02	-0.004 ± 0.002	0.005 ± 0.002

Reliable perturbative calculation ensures a good fit!

Differences between Scenarios 1 and 2 as systematic uncertainties.

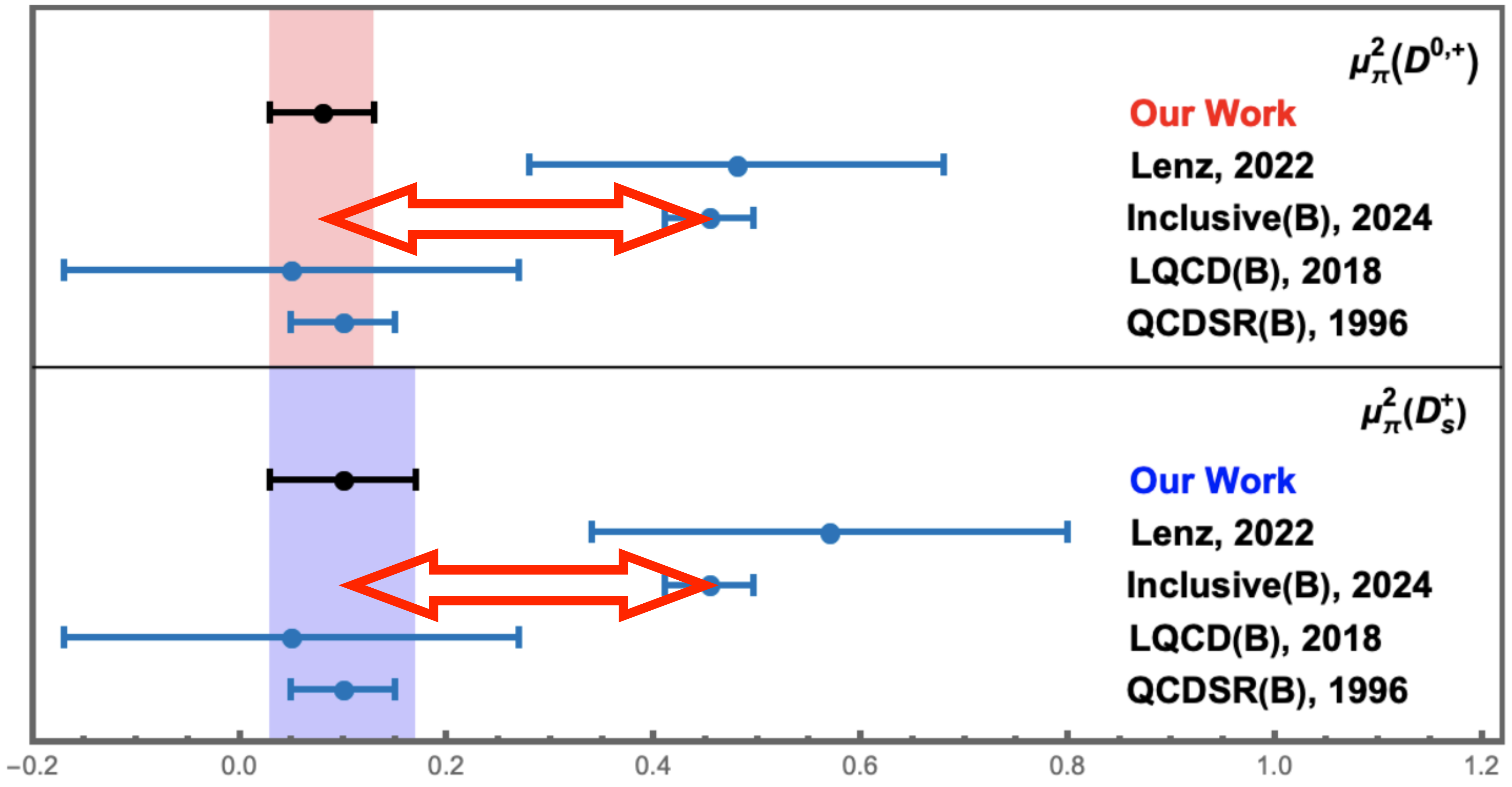
Global fit

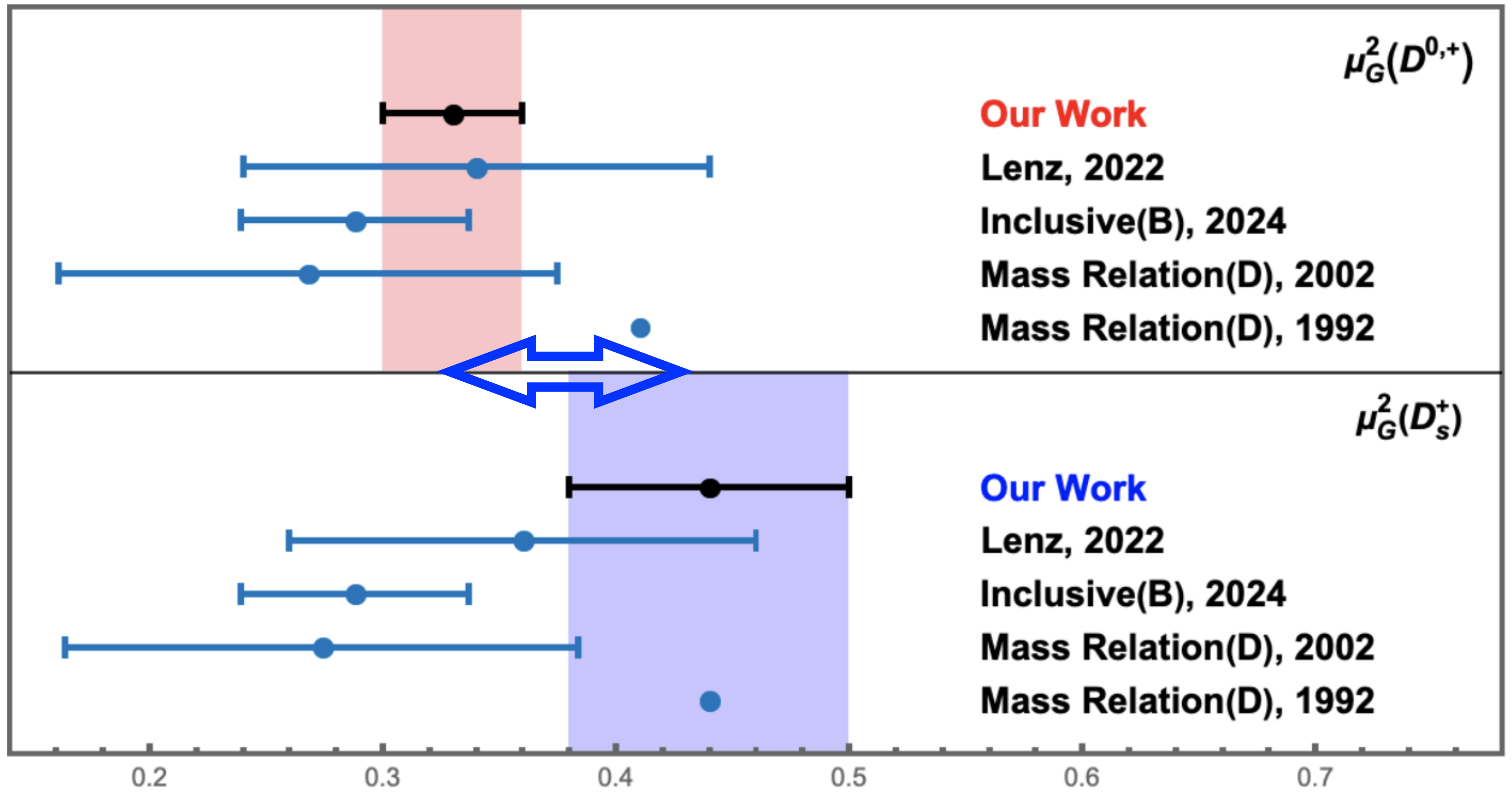
❖ Extracted nonperturbative HQE parameters

$$\mu_\pi^2(D^{0,+}) = (0.08 \pm 0.05)\mathrm{GeV}^2, \qquad \mu_\pi^2(D_s^+) = (0.10 \pm 0.07)\mathrm{GeV}^2,$$
$$\mu_G^2(D^{0,+}) = (0.33 \pm 0.03)\mathrm{GeV}^2, \qquad \mu_G^2(D_s^+) = (0.44 \pm 0.06)\mathrm{GeV}^2,$$
$$\rho_D^3(D^{0,+}) = (-0.003 \pm 0.002)\mathrm{GeV}^3, \qquad \rho_D^3(D_s^+) = (-0.004 \pm 0.002)\mathrm{GeV}^3,$$
$$\rho_{LS}^3(D^{0,+}) = (0.004 \pm 0.002)\mathrm{GeV}^3, \qquad \rho_{LS}^3(D_s^+) = (0.005 \pm 0.002)\mathrm{GeV}^3,$$

Sizable breaking effects of flavor SU(3) symmetry and heavy quark symmetry.

$\boxed{\mu_\pi^2}$

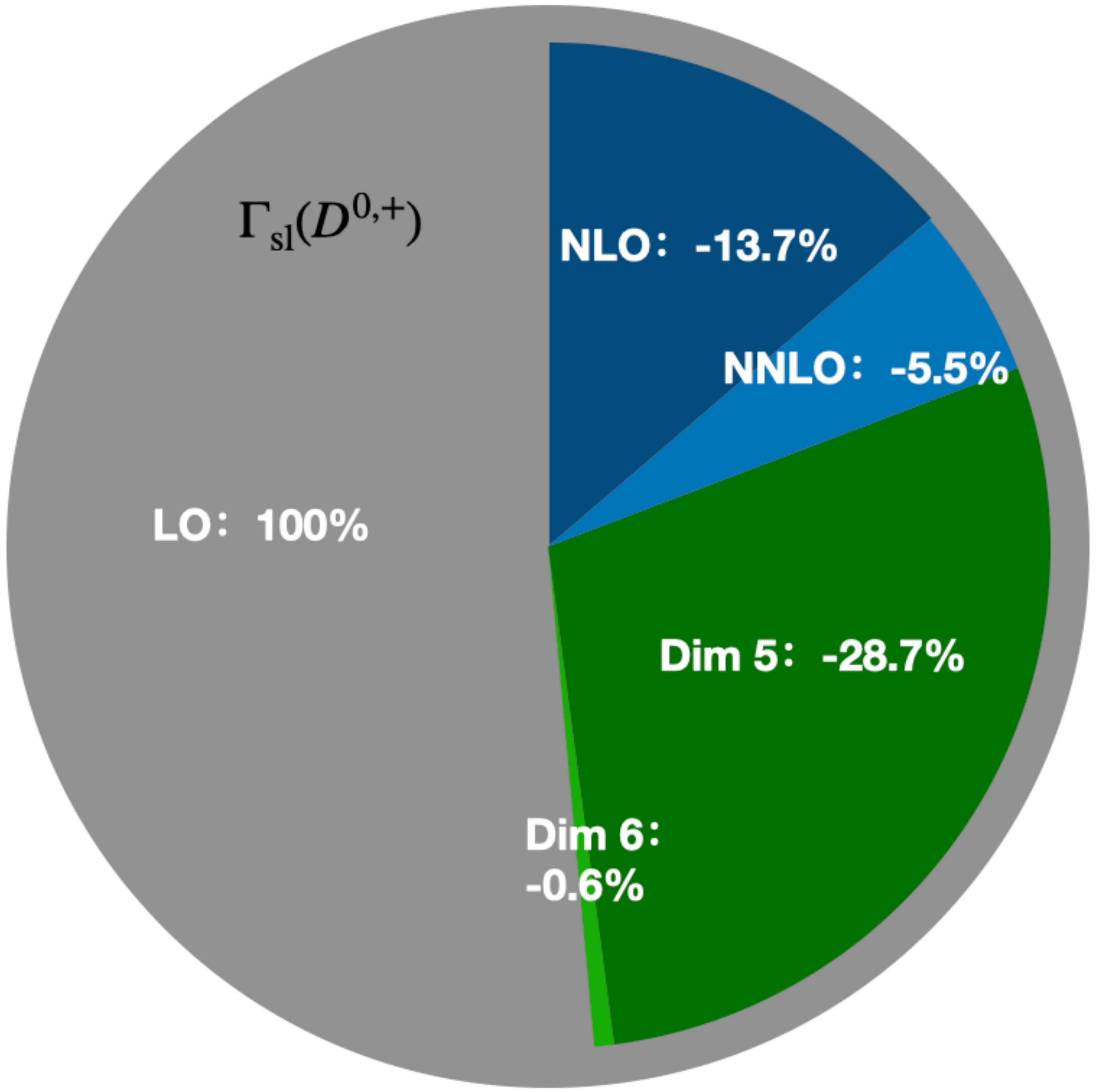


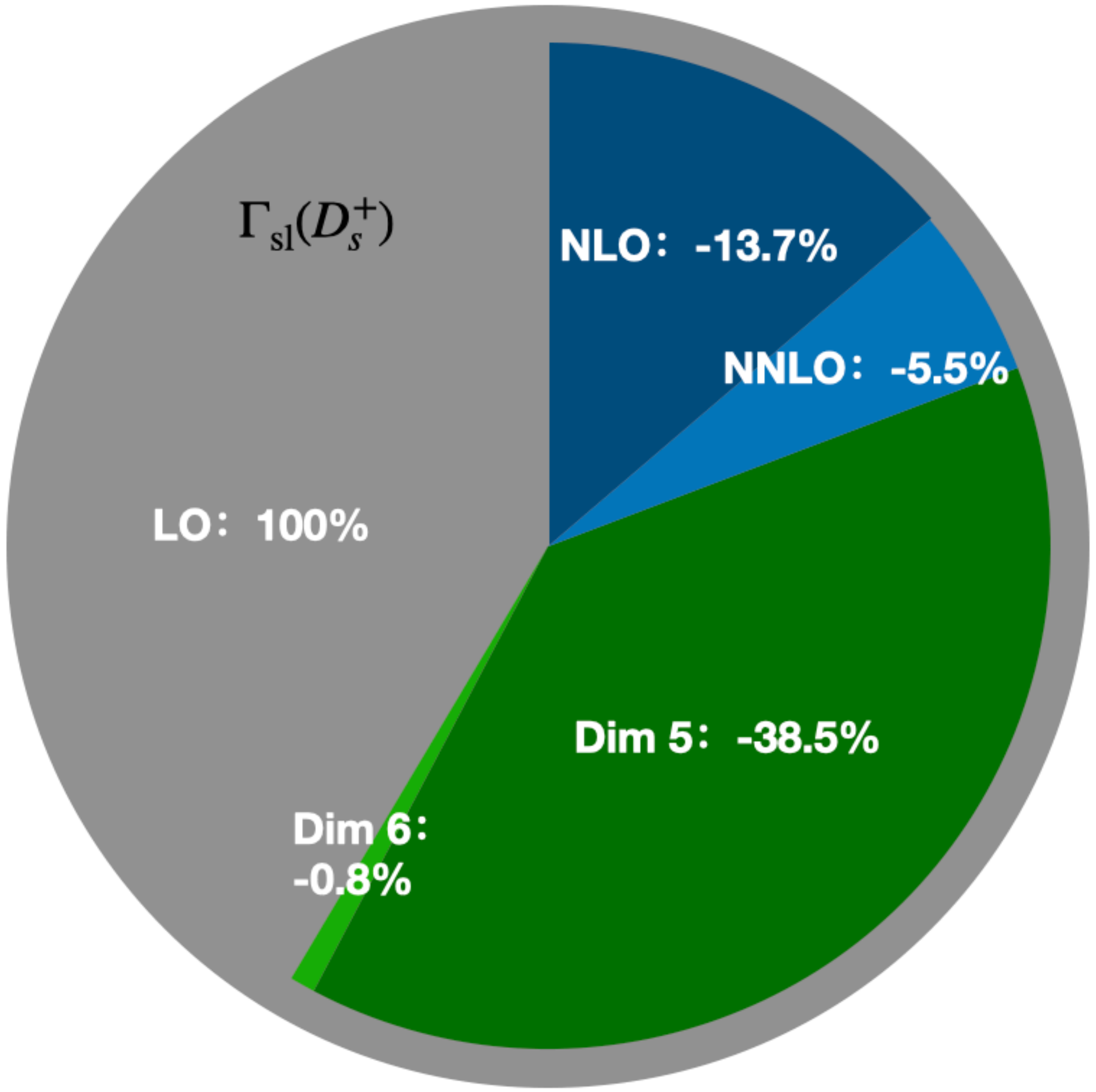


$\boxed{\mu_G^2}$

Convergence

❖ Various contributions to inclusive $D^{0,+}$ and D_s^+ decay widths





Convergent expansions of $\alpha_s(m_c)$ and $\Lambda_{\rm QCD}/m_c$!

Pheno-2. Determine the CKM matrix elements

[Shao,Feng,Liu,**QQ,**Sun,Yu,2509.11404]

Determine the CKM matrix elements

- Treat the $V_{cs,cd}$ as unknown parameters, instead of input
- Difficult for inclusive measurements to separate X_s and X_d $\Rightarrow$ **Sum of exclusive channels**

D^+ decays		D^0 decays		D_s decays	
Mode	BR(%)	Mode	BR(%)	Mode	BR(%)
$D^+ \to \bar{K}^0 e^+\nu_e$	8.72 ± 0.09	$D^0 \to K^- e^+\nu_e$	3.549 ± 0.026	$D_s \to \phi e^+\nu_e$	2.34 ± 0.12
$(K^-\pi^+)_{\bar{K}^*(892)^0} e^+\nu_e$	3.54 ± 0.09 [23]	$(\bar{K}^0\pi^-)_{\text{S-wave}} e^+\nu_e$	0.079 ± 0.017	$D_s \to \eta e^+\nu_e$	2.27 ± 0.06
$(K^-\pi^+)_{\text{S-wave}} e^+\nu_e$	0.228 ± 0.011	$D^0 \to K^*(892)^- e^+\nu_e$	2.04 ± 0.047 [24]	$D_s \to \eta' e^+\nu_e$	0.81 ± 0.04
$D^+ \to \bar{K}_1(1270)^0 e^+\nu_e$	0.230 ± 0.026	$D^0 \to \bar{K}_1(1270)^- e^+\nu_e$	0.101 ± 0.018	$D_s \to f_0(980) e' \nu_e$	0.164 ± 0.013
$D^+ \to \eta e^+\nu_e$	0.111 ± 0.007	$D^0 \to \pi^- e^+\nu_e$	0.291 ± 0.004	$D_s \to K^0 e^+\nu_e$	0.288 ± 0.026
$D^+ \to \pi^0 e^+\nu_e$	0.372 ± 0.017	$D^0 \to \rho(770)^- e^+\nu_e$	0.145 ± 0.007	$D_s \to K^*(892)^0 e^+\nu_e$	0.205 ± 0.020
$D^+ \to \pi^+\pi^- e^+\nu_e$	0.245 ± 0.008	$D^0 \to a(980)^- e^+\nu_e$	$0.0133^{+0.0034}_{-0.0030}$		
$D^+ \to \pi^0\pi^0 e^+\nu_e$	0.0315 ± 0.0027 [25]				
$D^+ \to \omega e^+\nu_e$	0.169 ± 0.011				
$D^+ \to \eta' e^+\nu_e$	0.020 ± 0.004				
$D^+ \to a(980)^0 e^+\nu_e$	0.017 ± 0.008				

	D^+	D^0	D_s
$X_s e^+\nu_e$	$14.60 \pm 0.16\%$	$5.81 \pm 0.06\%$	$5.58 \pm 0.14\%$
$X_d e^+\nu_e$	$0.96 \pm 0.03\%$	$0.45 \pm 0.01\%$	$0.49 \pm 0.03\%$

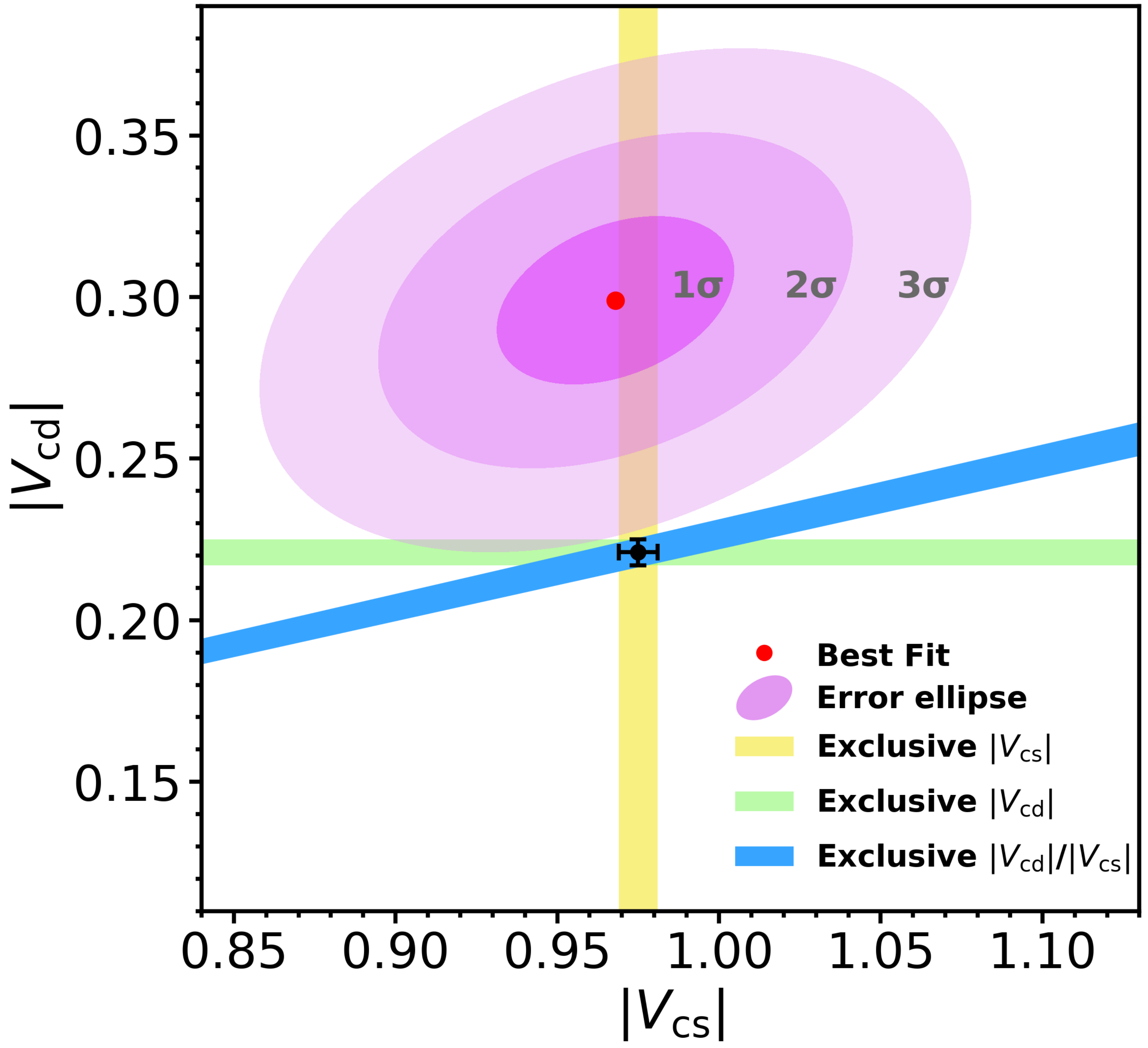
$$B(D^0 \to X_{d+s} e^+\nu_e) = 0.0636(15),$$
$$B(D^+ \to X_{d+s} e^+\nu_e) = 0.1602(32),$$
$$B(D_s \to X_{d+s} e^+\nu_e) = 0.0631(14),$$

Compatible within 1-2σ

Determine the CKM matrix elements

S1 $(X_{s,d})$	χ^2/d.o.f.	$\lvert V_{cs}\rvert$	$\lvert V_{cd}\rvert$
Scenario 1	3.24	$0.977 \pm 0.022 \pm 0.026$	$0.252 \pm 0.006 \pm 0.006$
Scenario 2	1.12	$0.969 \pm 0.022 \pm 0.026$	$0.253 \pm 0.006 \pm 0.006$
S2 (X_s)	χ^2/d.o.f.	$\lvert V_{cs}\rvert$	$\lvert V_{cd}\rvert$
Scenario 1	0.43	$0.974 \pm 0.022 \pm 0.026$	$0.279 \pm 0.031 \pm 0.007$
Scenario 2	0.29	$0.960 \pm 0.021 \pm 0.026$	$0.278 \pm 0.031 \pm 0.006$
S3 (X_d)	χ^2/d.o.f.	$\lvert V_{cs}\rvert$	$\lvert V_{cd}\rvert$
Scenario 1	4.08	$0.982 \pm 0.023 \pm 0.027$	$0.253 \pm 0.006 \pm 0.005$
Scenario 2	1.24	$0.974 \pm 0.022 \pm 0.026$	$0.254 \pm 0.006 \pm 0.006$

$$|V_{cs}| = 0.968 \pm 0.022 \pm 0.026 \pm 0.014$$
$$|V_{cd}| = 0.299 \pm 0.025 \pm 0.007 \pm 0.002$$



Inclusive vs exclusive: 3σ tension!

Summary and Prospect

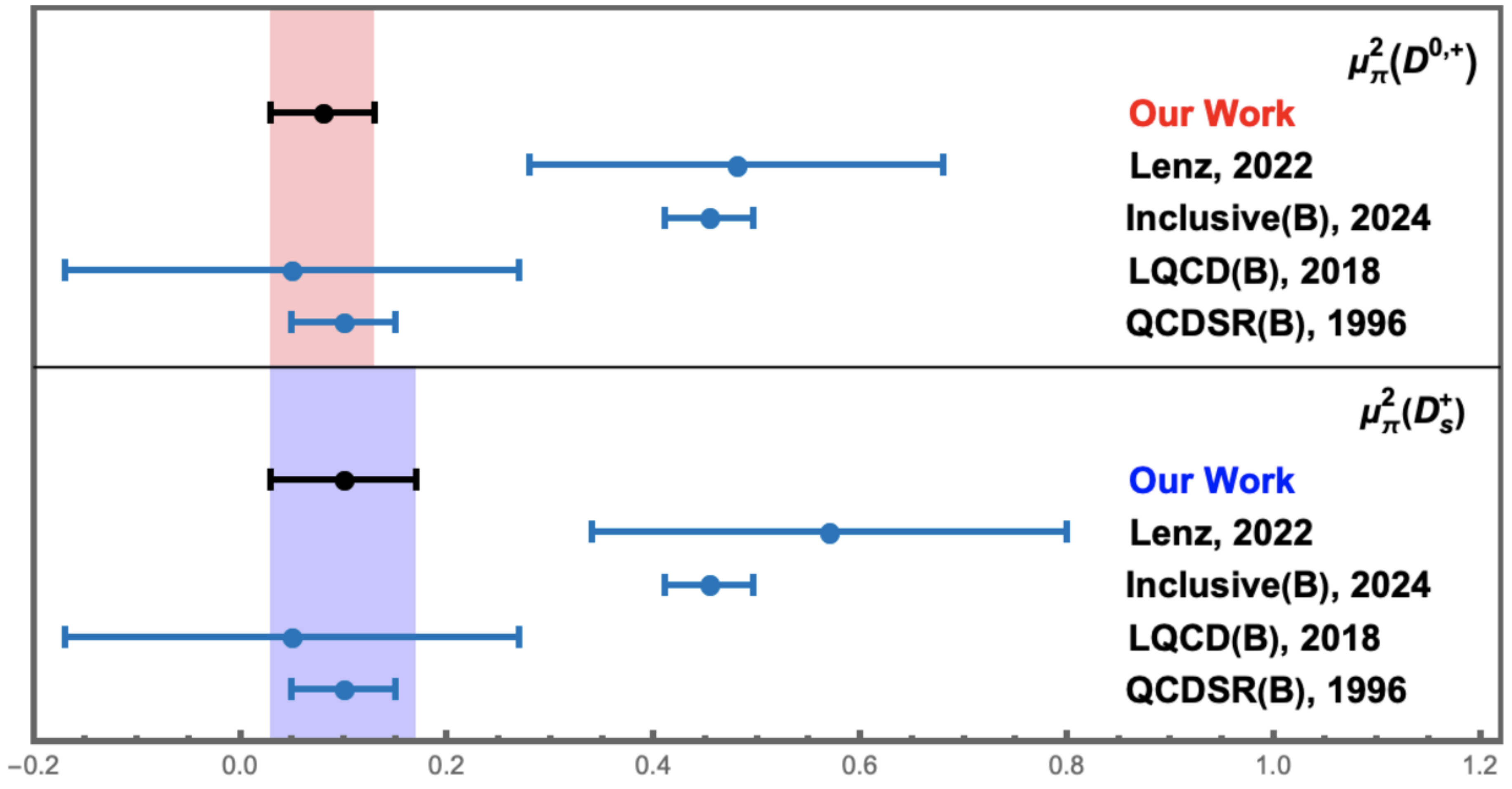
❖ Theoretical studies of inclusive bottom decays are in very good shape.

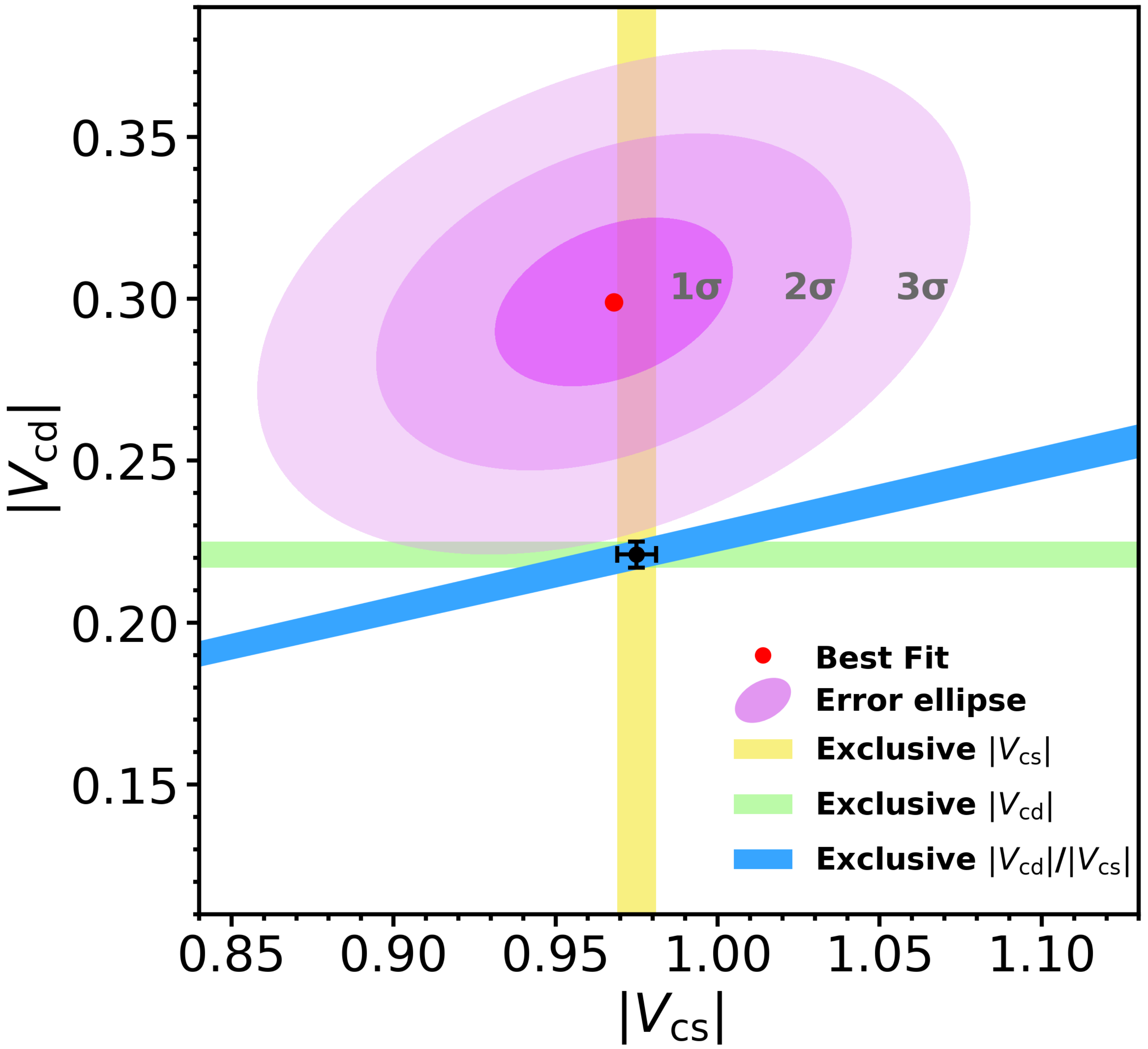
❖ α_s**-expansion and heavy quark expansion** are **valid** in inclusive charm decays (with an proper mass scheme chosen)

❖ For the **first time**, we determine

➡ D meson **HQE parameters** from first principle

➡ **CKM matrix elements** $V_{cs,cd}$ (inclusive)





Summary and Prospect

❖ **Possible theoretical improvements**

➡ Include higher order radiative corrections, $\mathcal{O}(\alpha_s^3)$

➡ Include higher power corrections, complete dimension-6 and -7 operator

➡ Include charmed baryons in the study

➡ ……

❖ **Possible experimental improvements**

➡ Laboratory frame -> rest frame

➡ Direct measurements of the moments

➡ Separate X_d, X_s

Thank you!

Backup

Heavy quark expansion

❖ **Heavy quark expansion up to dimension-7 operators (LO)** [Fael,Mannel,Vos, '19]

➡ LO results for decay widths, $\langle E_e^n \rangle$, $\langle (q^2)^n \rangle$ are given ($q^2 \equiv (p_e + p_\nu)^2$)

$$\frac{\Gamma(D \to X_s \ell \nu)}{\Gamma_0} = \left(1 - 8\rho - 10\rho^2\right)\mu_3 + (-2 - 8\rho)\frac{\mu_G^2}{m_c^2} + 6\frac{\tilde{\rho}_D^3}{m_c^3} + \frac{16}{9}\frac{r_G^4}{m_c^4} + \frac{32}{9}\frac{r_E^4}{m_c^4} - \frac{34}{3}\frac{s_B^4}{m_c^4} + \frac{74}{9}\frac{s_E^4}{m_c^4} + \frac{47}{36}\frac{s_{qB}^4}{m_c^4} + \frac{\tau_0}{m_c^3}$$

$\mathcal{Y}_1 \equiv E_e/m_c$

$$\mathcal{Y}_1 = \frac{3}{5} - 6\rho - 23\rho^2 - (1 + 16\rho)\frac{\mu_G^2}{m_c^2} + \frac{139}{15}\frac{\rho_D^3}{m_c^3} + \frac{3}{5}\frac{\rho_{LS}^3}{m_c^3} + \frac{503}{90}\frac{\delta\rho_D^3}{m_c^3} + \frac{3}{10}\frac{\delta\rho_{LS}^3}{m_c^3} + \frac{1271}{180}\frac{r_G^4}{m_c^4} - \frac{208}{45}\frac{r_E^4}{m_c^4} - \frac{682}{45}\frac{s_B^4}{m_c^4} + \frac{203}{15}\frac{s_E^4}{m_c^4} + \frac{283}{180}\frac{s_{qB}^4}{m_c^4} + \frac{1}{4}\frac{\delta_{G2}^4}{m_c^4} + \frac{\tau_0}{m_c^3} + \frac{\tau_m}{m_c^4} + \frac{\tau_\epsilon}{m_c^4}$$

❖ **Leading-order Dim-7 operator WCs** ($\Lambda_{\rm QCD}^3/m_c^3$ correction) [Finauri, '25]

➡ only for $b \to c$ spectrum

Perturbative expansion

❖ α_s **corrections to Dim-3 operator WCs** (c-quark decay)

➡ NLO for $b \to c$ spectrum; zero-mass limit to $b \to u$ [Fazio,Neubert, '99; Capdevila,Gambino,Nandi, '21]

➡ NNLO for $b \to u$ (total width and spectrum) [Ritbergen '99; Brucherseifer,Caola,Melnikov, '13]

➡ **NNNLO** for $c \to s/d$ total width

◉ Analytical by expansion of $\delta = (1 - m_q/m_c)$ (10% uncertainty) [Fael,Schonwald,Steinhauser, '20]

◉ Analytical at leading color (95% contribution) [Chen,Li,Li,Wang,Wu, '23]

◉ Numerical full contribution (AMFlow) [Chen,Chen,Guan,Ma, '23]

Perturbative expansion

❖ α_s **corrections to Dim-5 and Dim-6 operator WCs** $(\Lambda^2_{\rm QCD}/m_c^2$ correction)

➡ NLO for $b \to c$ spectrum; zero-mass limit to $b \to u$ [Alberti,(Ewerth),Gambino,Nandi, '12,'13; Capdevila,Gambino,Nandi, '21]

➡ NLO for $c \to s/d$ Darwin operator (total width) [Moreno '21]

Mass scheme transformation

$$m_c = \overline{m}_c\,(\mu)\Big[1 + \frac{\alpha_s\,(\mu)}{\pi}\left(\frac{4}{3} + \log\left(\frac{\mu^2}{\overline{m}_c^2}\right)\right) + \frac{\alpha_s^2\,(\mu)}{\pi^2}\frac{1}{288}\Big(112\pi^2 + 2905 + 16\pi^2\log(4) - 48\zeta(3)$$
$$- 12(2n_f - 45)\log^2\left(\frac{\mu^2}{\overline{m}_c^2}\right) - 4(26n_f - 519)\log\left(\frac{\mu^2}{\overline{m}_c^2}\right) - 2\left(71 + 8\pi^2\right)n_f\Big) + \mathcal{O}(\alpha_s^3)\Big]$$

$$m_c = m_{c,1\mathrm{S}} + m_{c,1\mathrm{S}}\frac{\alpha_s(\mu)^2 C_F^2}{8}\left\{1 + \frac{\alpha_s}{\pi}\left[\left(-\log\left(\alpha_s(\mu)m_{c,1\mathrm{S}}C_F/\mu\right) + \frac{11}{6}\right)\beta_0 - 4 + \frac{\pi}{8}C_F\alpha_s\right] + ...\right\}$$

Weak-annihilation Uncertainty

❖ Redo the fits, adopting the HQET SR calculation for the weak annihilation contributions

$$\tau_0(D_d \to X_d) = \tau_0(D_s \to X_s) = \tau_{\text{val}} = (-0.18 \pm 0.65)\ \text{GeV}^3$$

[King, Lenz, Piscopo, Rauh,'19]

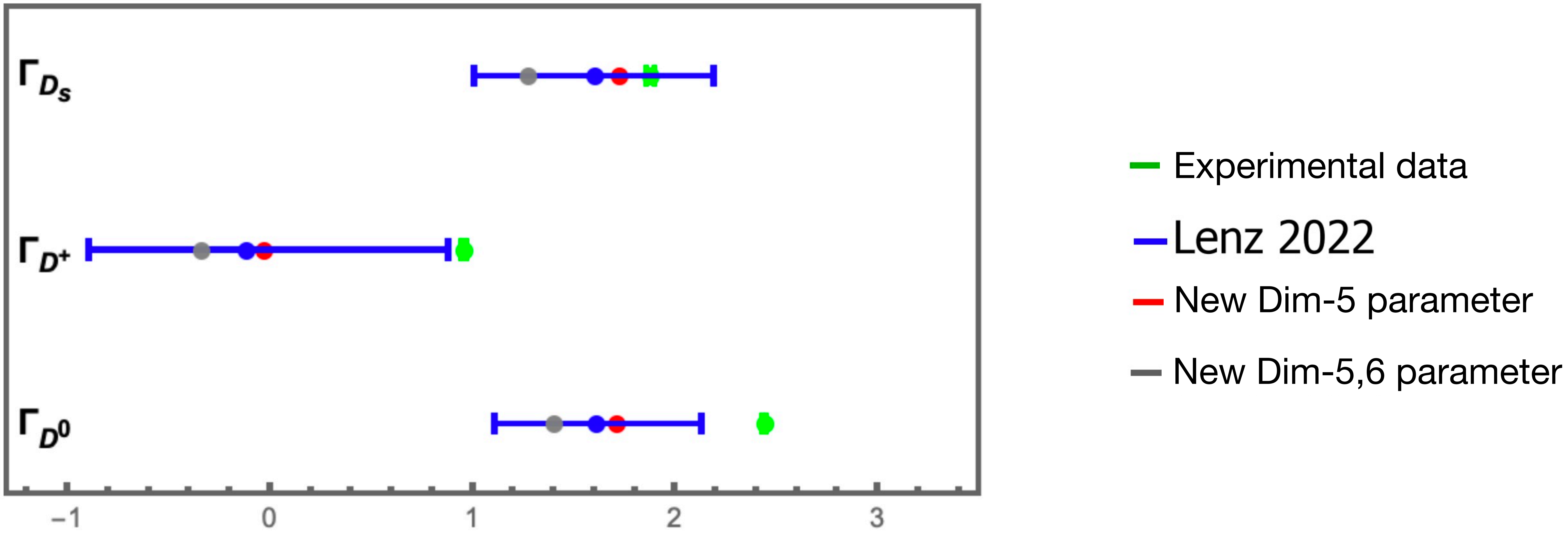
$$\tau_0(D_u \to X_{d,s}) = \tau_0(D_d \to X_s) = \tau_0(D_s \to X_d) = \tau_{\text{nonval}} = (0.45 \pm 2.10)\ \text{GeV}^3$$

$$\chi^2(\boldsymbol{\theta}) = \sum_{l=u,d,s} \sum_{i=1}^{5} \sum_{j=1}^{5} (y_{i,l} - \eta_{i,l})\, V^{-1}_{ij,l}\, (y_{j,l} - \eta_{j,l}) + \left(\frac{\tau_{\text{nonval}} - (-0.18)}{0.65}\right)^2 + \left(\frac{\tau_{\text{val}} - 0.45}{2.10}\right)^2$$

$$\mu_\pi^2(D^{0,+}) = 0.08\text{GeV}^2\ ,\ \mu_G^2(D^{0,+}) = 0.33\text{GeV}^2\ ,\ \rho_D^3(D^{0,+}) = -0.003\text{GeV}^3\ ,\ \rho_{LS}^3(D^{0,+}) = 0.004\text{GeV}^3$$
$$\mu_\pi^2(D_s) = 0.15\text{GeV}^2\ ,\ \mu_G^2(D_s) = 0.38\text{GeV}^2\ ,\ \rho_D^3(D_s) = -0.005\text{GeV}^3\ ,\ \rho_{LS}^3(D_s) = 0.006\text{GeV}^3\ ,$$
$$\tau_{\text{val}} = -0.11\text{GeV}^3,\ \tau_{\text{nonval}} = 0.002\text{GeV}^3\ .$$

The best-fit values only slightly change.

Modification of D Lifetime Prediction



Determine the CKM matrix elements

- Data choice strategy: all inclusive X_{s+d} data are used
  - **S1**: Sum-of-exclusive X_s data, plus sum-of-exclusive X_d data,
  - **S2**: Sum-of-exclusive X_s data
  - **S3**: Sum-of-exclusive X_d data
- For each strategy, two scenarios for weak-annihilation contributions
  - **Scenario 1**: VIA, $\tau_0 = 0$
  - **Scenario 2**: HQET SR, input the previous best-fit values $\tau_{\rm val} = -0.11 \mathrm{GeV}^3, \ \tau_{\rm nonval} = 0.002 \mathrm{GeV}^3$.

Robust test of the fit