BEYOND BOLTZMANN-GIBBS: A NONADDITIVE STATISTICAL STUDY OF ENERGY LOSS IN HIGH-ENERGY COLLISIONS

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UJK Uniwersytet Jana Kochanowskiego w Kielcach

Are extreme deviations rare?

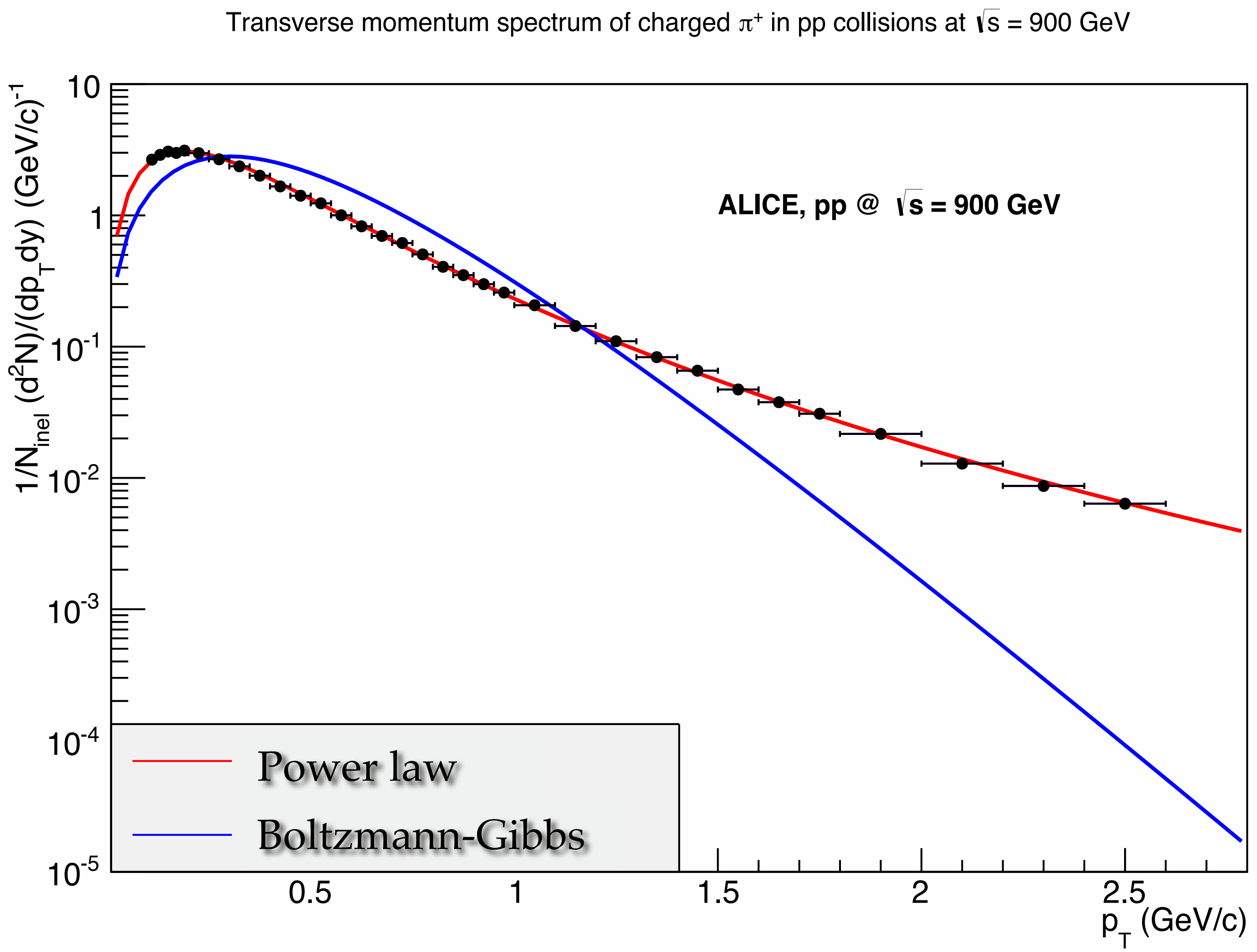
Our everyday experience says - yes

Whether it a massive flood, a stock market crash, or a huge measurement error

But they happen far more often than our intuition suggests

Earthquakes, word frequency, ………….., high-energy collisions

Power law vs exponential



Adapted from a presentation by Jean Cleymans

Phenomenological (power-law)

$$\left.\frac{d^2N}{dp_{\mathrm{T}}\,dy}\right|_{y=0,\ \mu=0} = gV\frac{p_{\mathrm{T}}m_{\mathrm{T}}}{(2\pi)^2}\left[1+(q-1)\frac{m_{\mathrm{T}}}{T}\right]^{-\frac{q}{q-1}}$$

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$q \to 1$

Boltzmann-Gibbs Maxwell-Boltzmann

$$\left.\frac{d^2N}{dp_T\,dy}\right|_{y=0,\mu=0} = gV\frac{p_{\mathrm{T}}m_{\mathrm{T}}}{(2\pi)^2}e^{-\frac{m_{\mathrm{T}}}{T}}$$

Fluctuating ambience, long-range correlations, finite size, fractal dynamics

G. Wilk, Z. Włodarczyk, Phys. Rev. Lett. 84, 2770(2000)

A. Deppman, E. Megías, D. P. Menezes, Phys. Rev. D 101 (2020) 034019

J. S. Lima, A. Deppman, Phys. Rev. E 101 (2020) 040102 040102 (R)

W. M. Alberico, A. Lavagno, Eur. Phys. J. A 40 (2009) 313

- Such scenarios need a beyond-Boltzmann-Gibbs perspective, and nonadditive statistics is one such framework

C. Tsallis, J. Stat. Phys. 52, 479 (1988)

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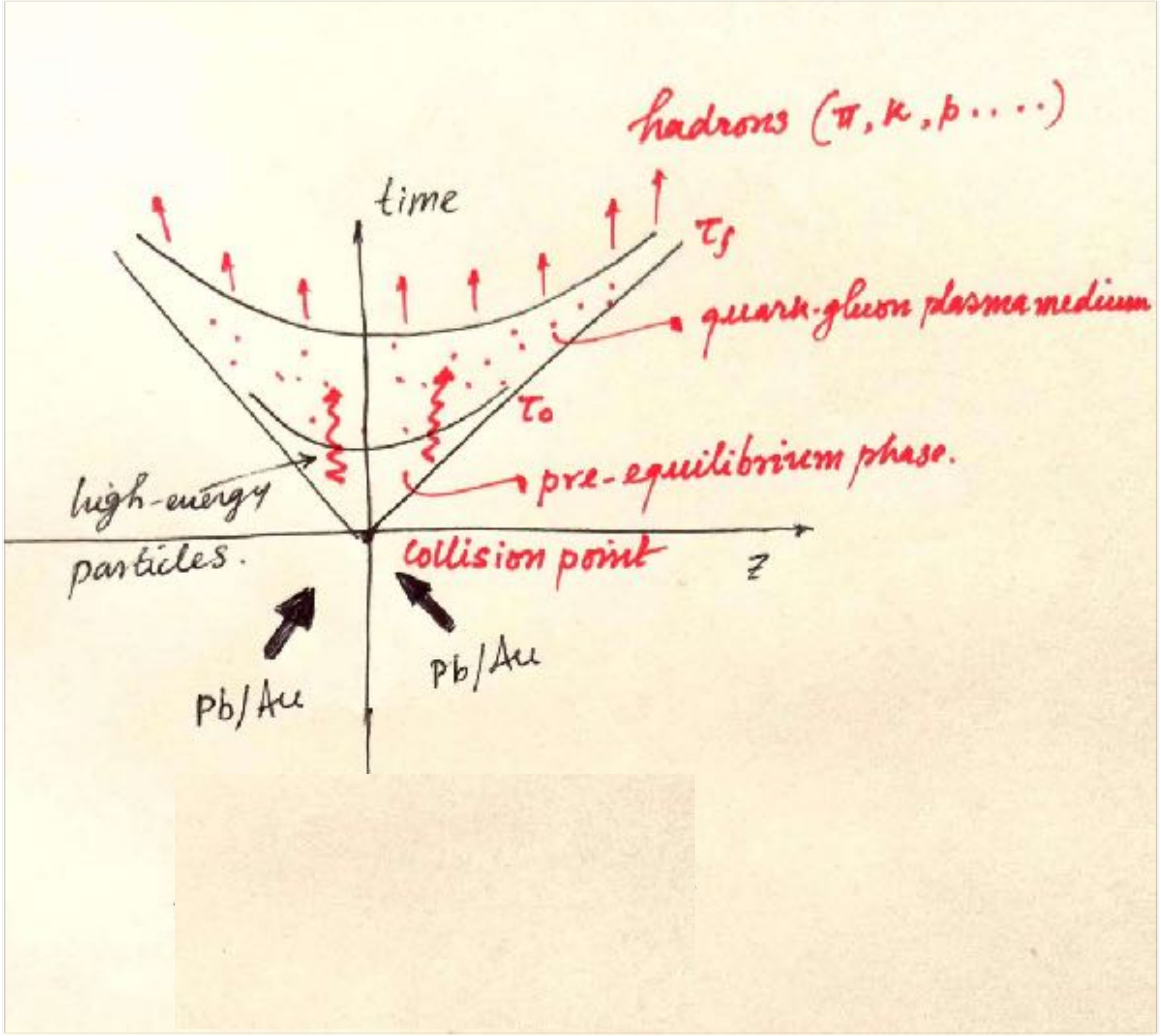
- In this lecture we discuss the transport of energetic particles through the QGP in the framework of nonadditive statistics


hadrons (π, K, p....)
time
τf
quark-gluon plasma medium
τ0
pre-equilibrium phase.
high-energy
particles.
collision point
z
Pb/Au
Pb/Au

- QGP: a study of evolutions

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- Evolution 1: that of the high-energy particles (e.g. the heavy quarks) inside the QGP; dictated by the Boltzmann Transport Equation (BTE), the Fokker-Planck Equation (FPE)……

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- Evolution 2: that of the background medium, dictated by the hydrodynamic equation

- QGP: a study of evolutions

- Evolution 1: that of the high-energy particles (e.g. the heavy quarks) inside the QGP; dictated by the Boltzmann Transport Equation (BTE), the Fokker-Planck Equation (FPE)……

- Evolution 2: that of the background medium, dictated by the hydrodynamic equation

We study evolution 1

The Boltzmann Transport Equation

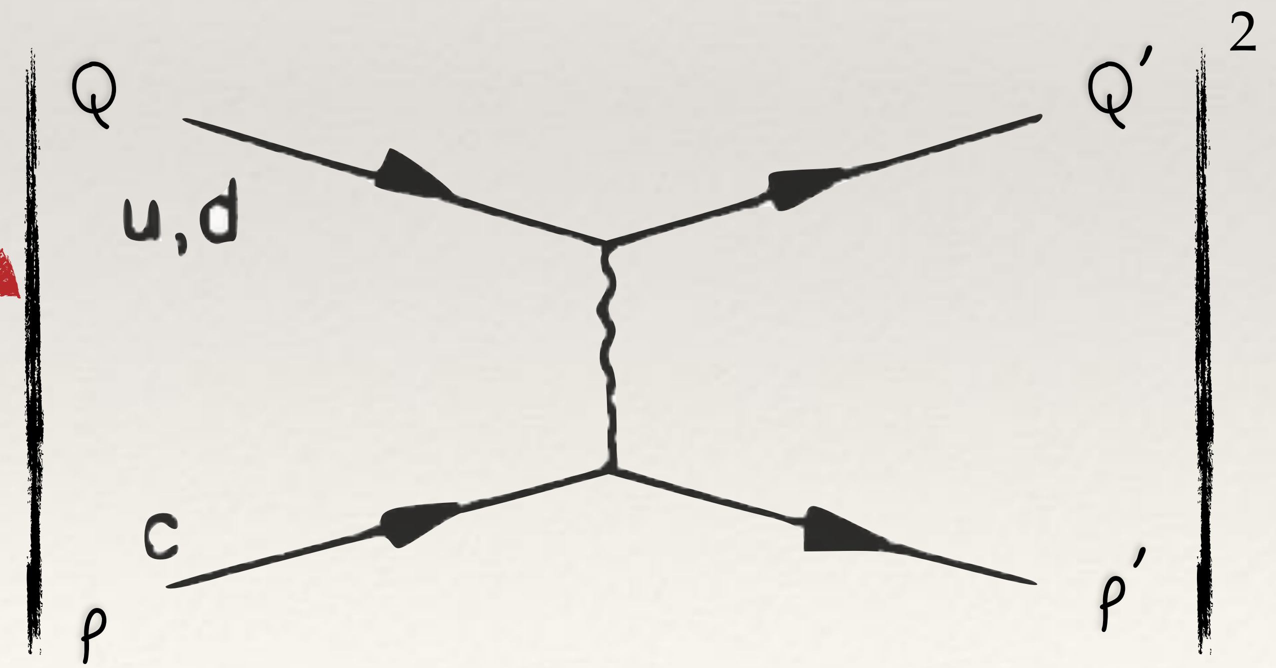
$$\frac{\partial f}{\partial t} + \overrightarrow{v} \,.\, \overrightarrow{\nabla}_r f + \overrightarrow{F} \,.\, \overrightarrow{\nabla}_p f = C[f]$$

$$\text{BTE} \xrightarrow[\textit{Hom. plasma, space av.}]{\textit{small momentum transfer}} \text{FPE}$$

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial p_i}\left(A_i f + \frac{\partial}{\partial p_j} B_{ij} f\right) = -\overrightarrow{\nabla_p} \,.\, \overrightarrow{\mathbb{P}}$$

FPE: drag and diffusion

$$A_i = \int \text{dynamics} \otimes \text{phase space} \times (p_i - p_i')$$
$$= \langle\langle (p_i - p_i') \rangle\rangle$$
$$B_{ij} = \langle\langle (p_i - p_i')(p_j - p_j') \rangle\rangle$$



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- Molecular chaos

$$F^{(2)} = f^{(1)}_{\text{HQ}} \times f^{(1)}_{\text{bath}}$$

FPE: exponential stationary solution?

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D.B. Walton and J. Rafelski Phys. Rev. Lett. 84, 31 (2000)

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Ansatz

$$f = f_{\rm st} = \exp[-\Phi(p; T, q)]$$

FPE: exponential stationary solution?

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$$A_i f + \frac{\partial}{\partial p_j} B_{ij} f = 0$$

FPE: exponential stationary solution?

$$f = f_{\mathrm{st}} = \exp[-\Phi(p; T, q)]$$

$$A_i f + \frac{\partial}{\partial p_j} B_{ij} f = 0$$

$$A_i(\vec{p}, T) = B_{ij}(\vec{p}, T) \frac{\partial \Phi(\vec{p})}{\partial p_j} - \frac{\partial B_{ij}(\vec{p}, T)}{\partial p_j}$$

"Fluctuation-dissipation relation"

FPE: exponential stationary solution?

We know the stationary distribution if we know the transport coefficients

Do we know the transport coefficients?

Yes……

B. Svetitsky Phys Rev D 37, 2484 (1988)

S.K. Das, J. Alam, P. Mohanty, Phys Rev C 80, 054916 (2009)

Recapitulation

$$A_i = \int \text{dynamics} \otimes \text{phase space} \times (p_i - p_i')$$

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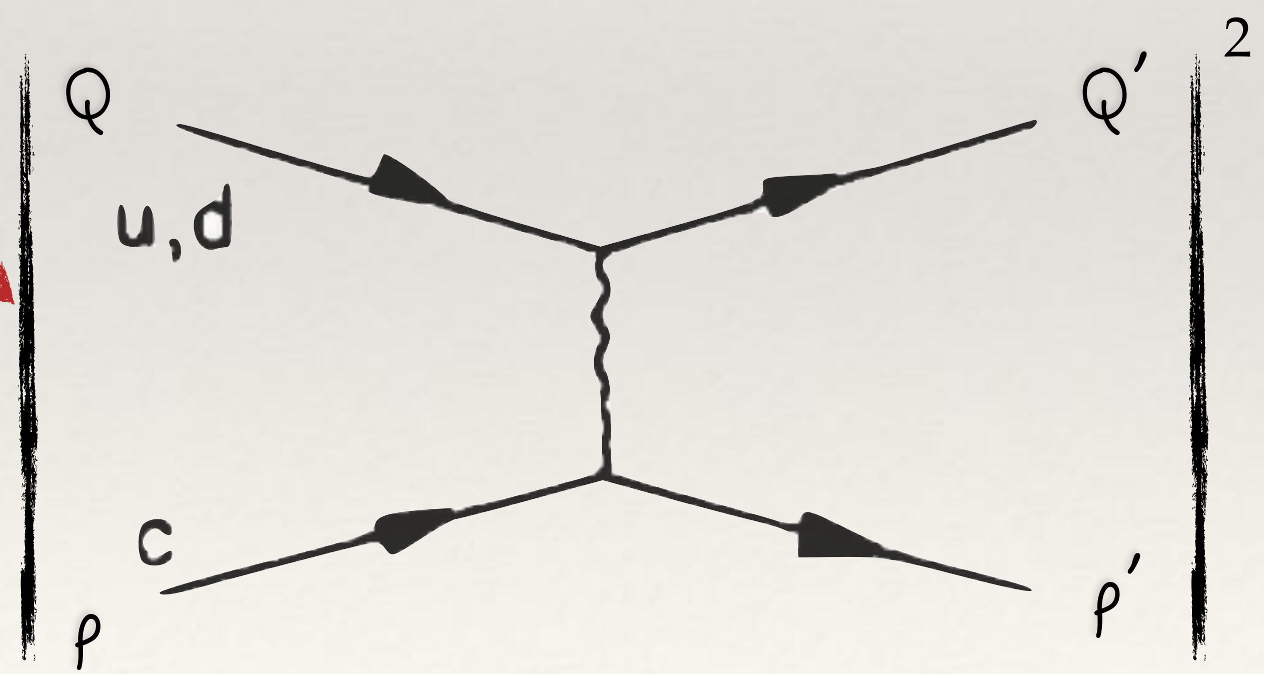

FPE: exponential stationary solution?

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No. This function is similar to the one obtained in nonadditive statistics.

$$f_{\rm st} = \exp[-\Phi(p;T,q)] = \exp\left[-\frac{1}{q-1}\ln\left\{1+(q-1)\frac{E(p)}{T}\right\}\right]; \quad q \neq 1$$

D.B. Walton and J. Rafelski Phys. Rev. Lett. 84, 31 (2000)

S. Mazumder, T. Bhattacharyya, J. Alam, Phys Rev D 89, 014002 (2014)

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$$\lim_{q\to 1} f_{\text{st}} = f_{\text{BG}} = \exp\left\{-\frac{E(p)}{T}\right\}$$

D.B. Walton and J. Rafelski Phys. Rev. Lett. 84, 31 (2000)

S. Mazumder, T. Bhattacharyya, J. Alam, Phys Rev D 89, 014002 (2014)

Walton and Rafelski investigate if the stationary distribution may be exponential and conclude that it is rather given by a nonadditive distribution.

Another equation: same stationary solution

Generalized BTE/FPE (gBTE/gFPE) with a power-law stationary solution

How to generalize?

- Molecular chaos

$$F^{(2)} = f^{(1)}_{\rm HQ} \times f^{(1)}_{\rm bath} = e^{\ln[f^{(1)}_{\rm HQ}]} \times e^{\ln[f^{(1)}_{\rm bath}]}$$

How to generalize?

- Generalize molecular chaos

$$F^{(2)} = f^{(1)}_{\mathrm{HQ}} \otimes_q f^{(1)}_{\mathrm{bath}} = e_q^{\ln_q[f^{(1)}_{\mathrm{HQ}}]} \times e_q^{\ln_q[f^{(1)}_{\mathrm{bath}}]}$$

$$e_q(x) = [1-(q-1)x]_+^{1/(1-q)} \; ; \; \ln_q(x) = \frac{1-x^{1-q}}{q-1}$$

T. Osada and G. Wilk, Phys. Rev. C 77, 044903 (2008)

- Also consider energy and momentum conservation

How to generalize?

$$\frac{\partial f^q}{\partial t} + \overrightarrow{v} \, . \, \overrightarrow{\nabla}_r f^q + \overrightarrow{F} \, . \, \overrightarrow{\nabla}_p f^q = C_q[f]$$

gBTE in the RTA

$$\frac{\partial f^q}{\partial t} = -\frac{f - f_{\mathrm{eq}}}{\tau}$$

gBTE in the RTA

$$\frac{\partial f^q}{\partial t} = -\frac{f - f_{\text{eq}}}{\tau}$$

- An approximate iterative analytical closed solution of *g*BTE in the RTA is proposed

$$f_i = f_{i-1} + \epsilon_i \ , \ i \in \mathbb{Z}^{>}$$

$$\epsilon_i = \frac{f_{i-1}}{\sum_{r=0}^{i} f_{\text{eq}}^{r} f_{i-1}^{i-r}} \left(f_{i-1}^{i+1-q}(K \quad \theta) + \sum_{r=0}^{i} \frac{f_{\text{eq}}^{r} f_{i-1}^{i-r}}{r+1-q} \right).$$

T. Bhattacharyya, Physica A 624, 128910 1-12 (2023)

T. Bhattacharyya, M. Rybczyński, and Z. Włodarczyk, Phys. Lett. B 872, 140076 (2026) gBTE $\xrightarrow{\textit{small momentum transfer}}$ gFPE / PPE

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial p_i}\left(A_{i,q} f\right) + \frac{\partial}{\partial p_i}\left(\frac{\partial}{\partial p_j} B_{ij,q} f^{2-q}\right)$$

A.R. Plastino and A. Plastino Physica A 222, 347 (1995)

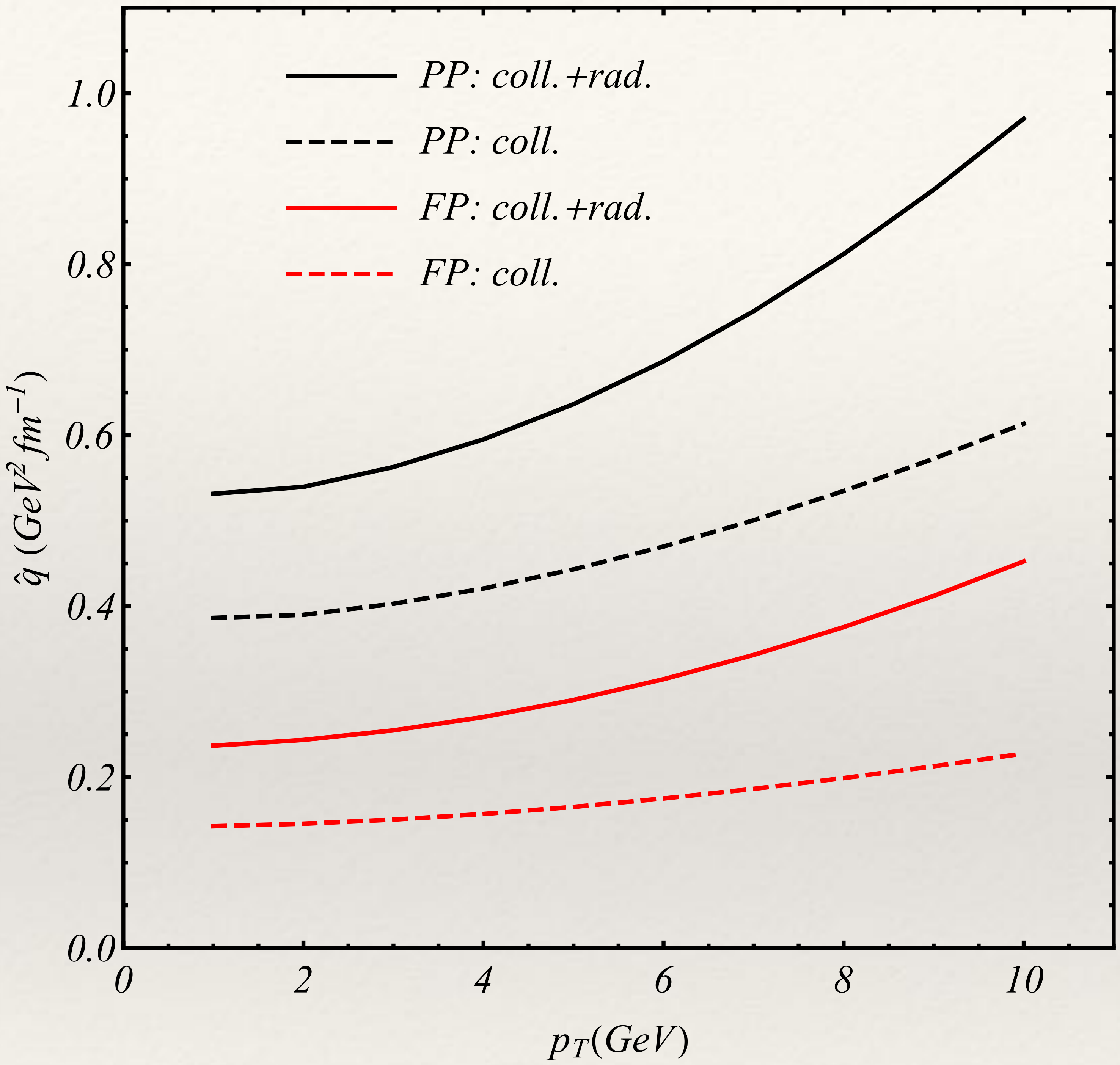
T. Bhattacharyya, E. Megías, and A. Deppman, Phys. Lett. B 856 , 138907 (2024)
T. Bhattacharyya, Physica A 624, 128910 1-12 (2023)

A. Deppman, A. K. Golmankhaneh, E. Megias, R. Pasechnik, Phys. Lett. B 839, 137752 (2023)

Plastino-Plastino transport coefficients

$$B_{ij,q} = \int \text{dynamics} \otimes \text{phase space("interplay")} \times (p_i - p_i')(p_j - p_j')$$

Jet-quenching parameter



$$\hat{q} \sim \frac{4E}{p_L} B_\perp$$

T. Bhattacharyya, E. Megías, and A. Deppman, Phys. Lett. B 856 , 138907 (2024)

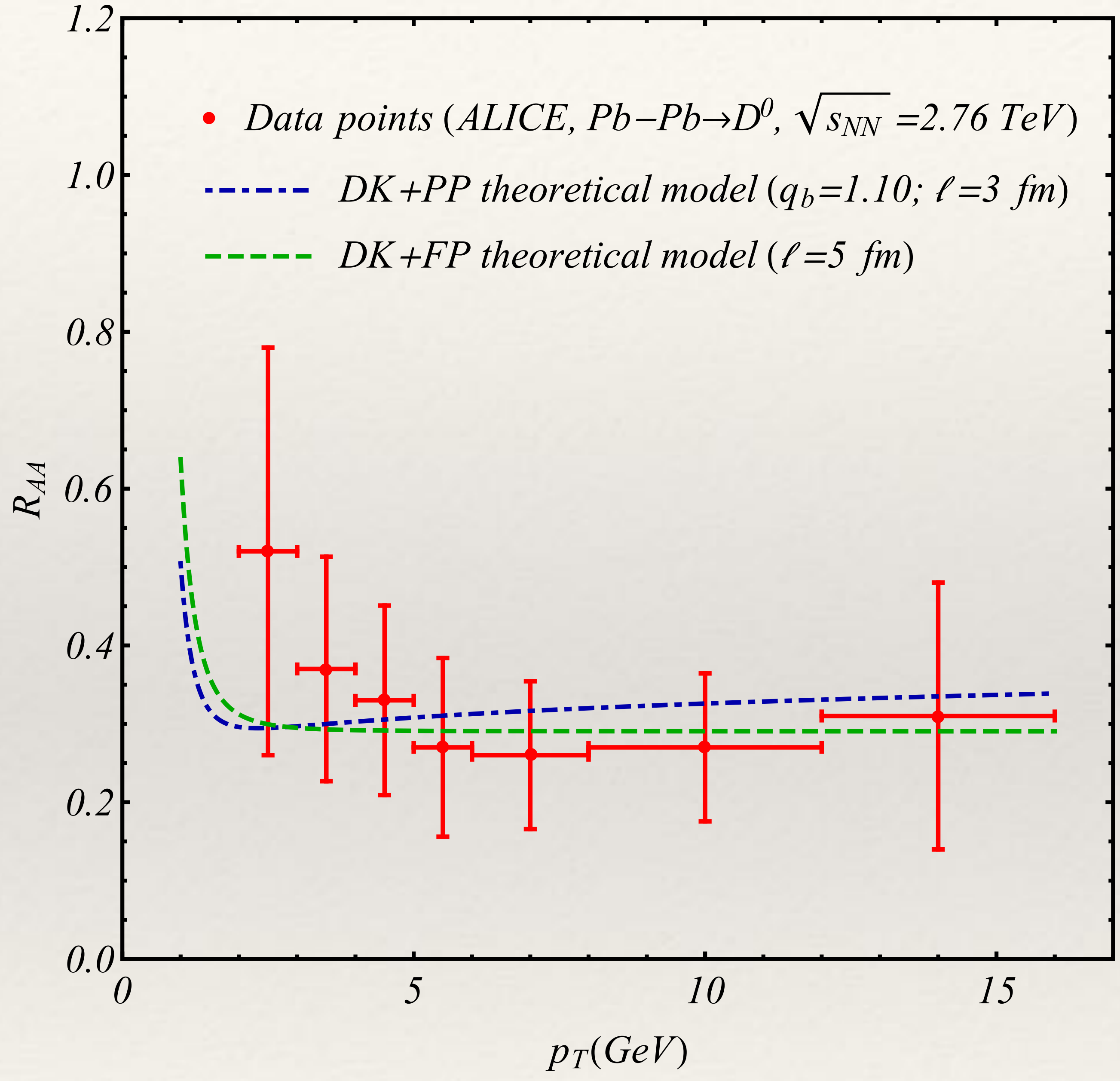
Differential energy loss

$$-\left(\frac{dE}{dx}\right)_q \equiv \left\langle\left\langle \frac{E-E'}{v} \right\rangle\right\rangle_q = \left\langle\left\langle \frac{E^2-E\,E'}{p} \right\rangle\right\rangle_q$$

A modified form of E. Braaten and M.H. Thoma, Phys. Rev. D 44, R2625 (1991) formula

$$R^{\mathrm{h}}_{\mathrm{AA}}(P_{\mathrm{T}},\ell) \simeq \exp\left[-\frac{2\alpha_s C_F}{\sqrt{\pi}}\ell\sqrt{\frac{\hat{q}\mathcal{L}_h^{\mathrm{abs}}}{P_{\mathrm{T}}}} + \frac{16\alpha_s C_F}{9\sqrt{3}}\ell\left(\frac{\hat{q}m^2}{m^2+P_{\mathrm{T}}^2}\right)^{1/3}\right].$$

Y. Dokshitzer and D. Kharzeev, Phys. Lett. B **519** , 199 (2001)


Data points (ALICE, Pb−Pb→D⁰, √s_NN =2.76 TeV)
DK+PP theoretical model (q_b=1.10; ℓ=3 fm)
DK+FP theoretical model (ℓ=5 fm)
R_AA
p_T(GeV)
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0
5
10
15

Summary and conclusions

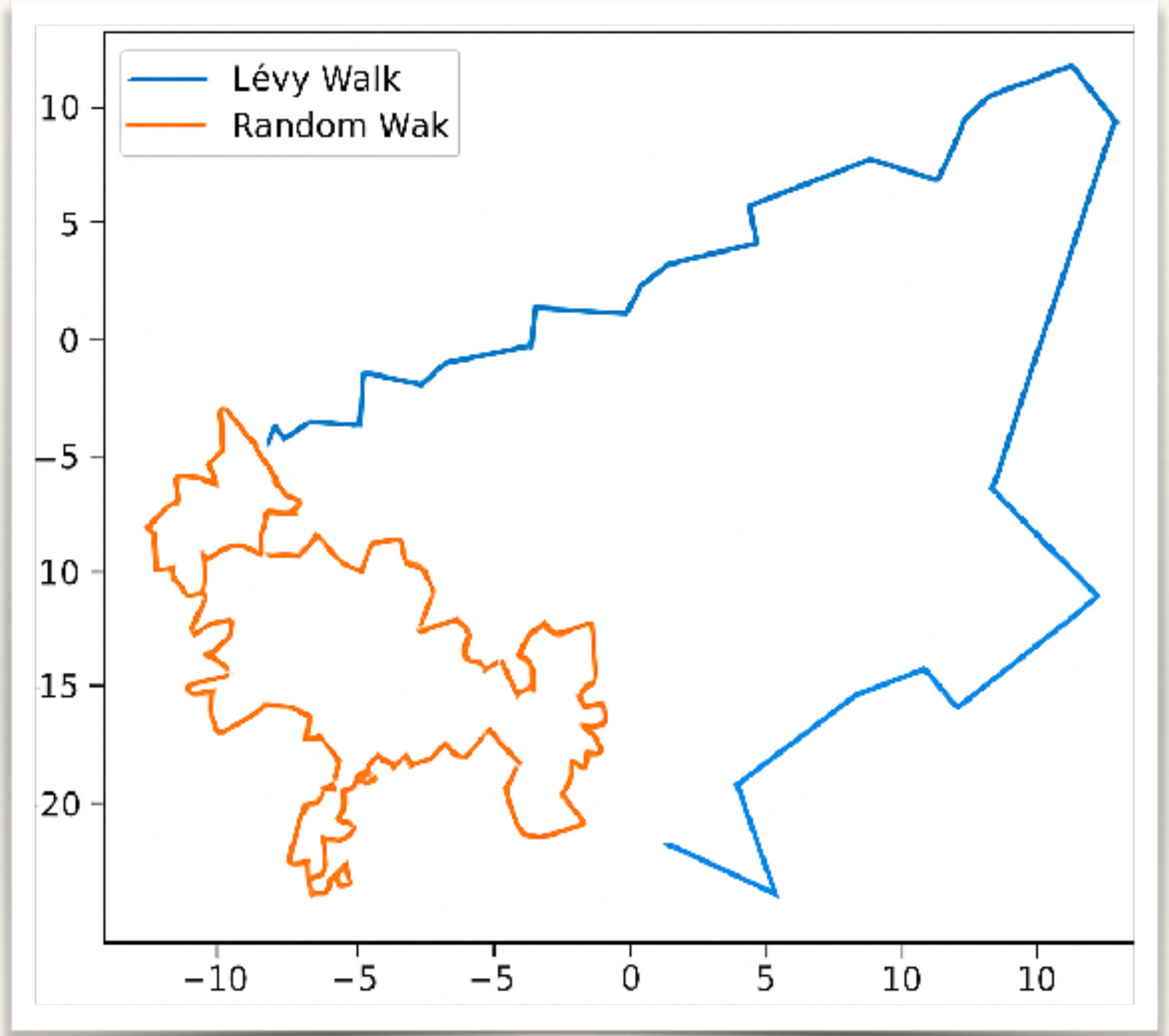
- We find that the power-law stationary distributions arise in the studies of energetic particles passing through the quark-gluon plasma.
- We motivate the use of a generalized transport equation that yields such a stationary state.
- We discuss methods of computing transport coefficients (drag, diffusion, jet quenching parameter) and differential energy loss.
- We investigated a connection with an experimental observable (nuclear suppression factor).
- Numerical solutions with the input of transport coefficients, collectivity

Acknowledgements

All collaborators, organizers

Funded by
the European Union

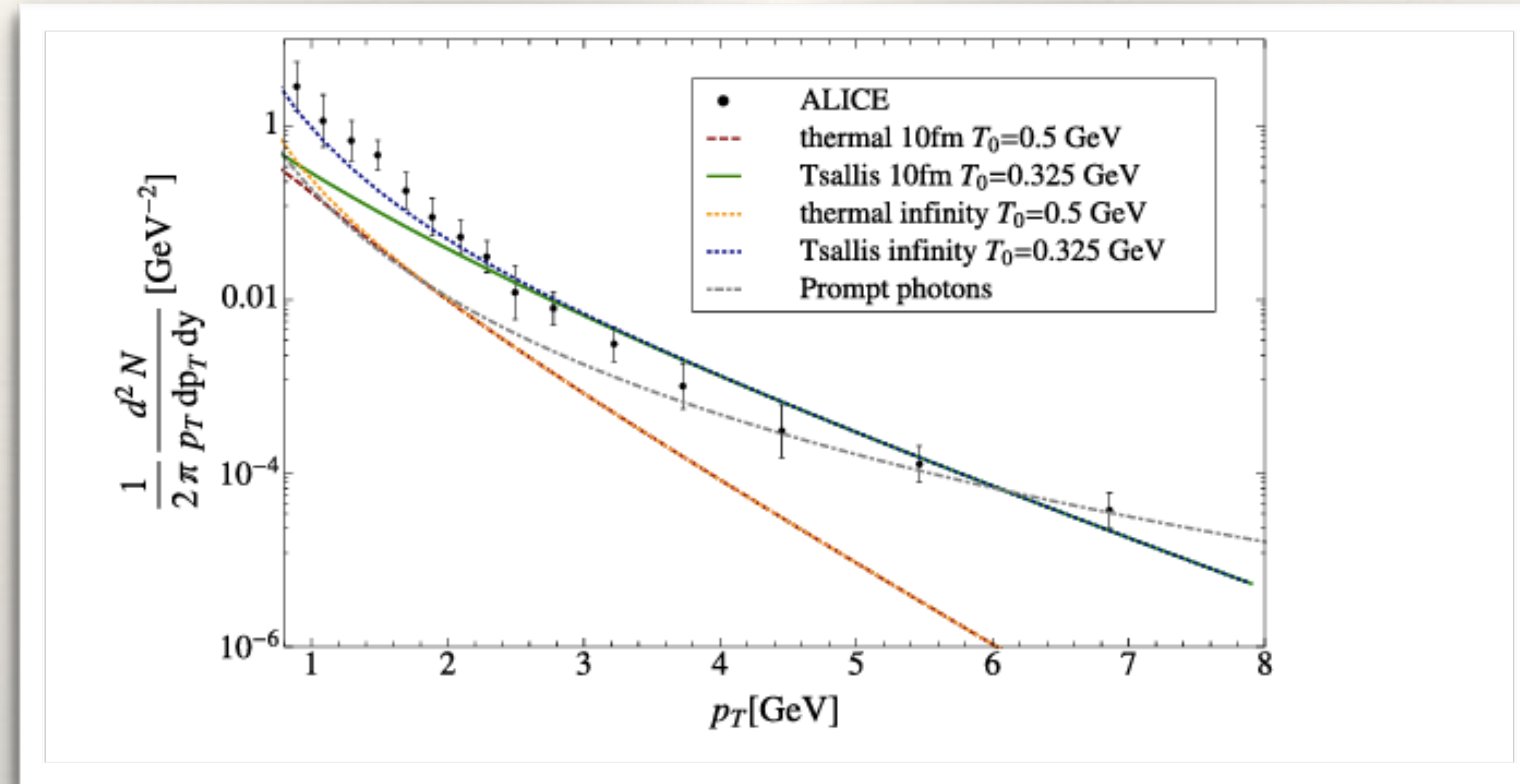
Thank you !!


Lévy Walk
Random Wak
10
5
0
−5
10
15
20
−10
−5
−5
0
5
10
10

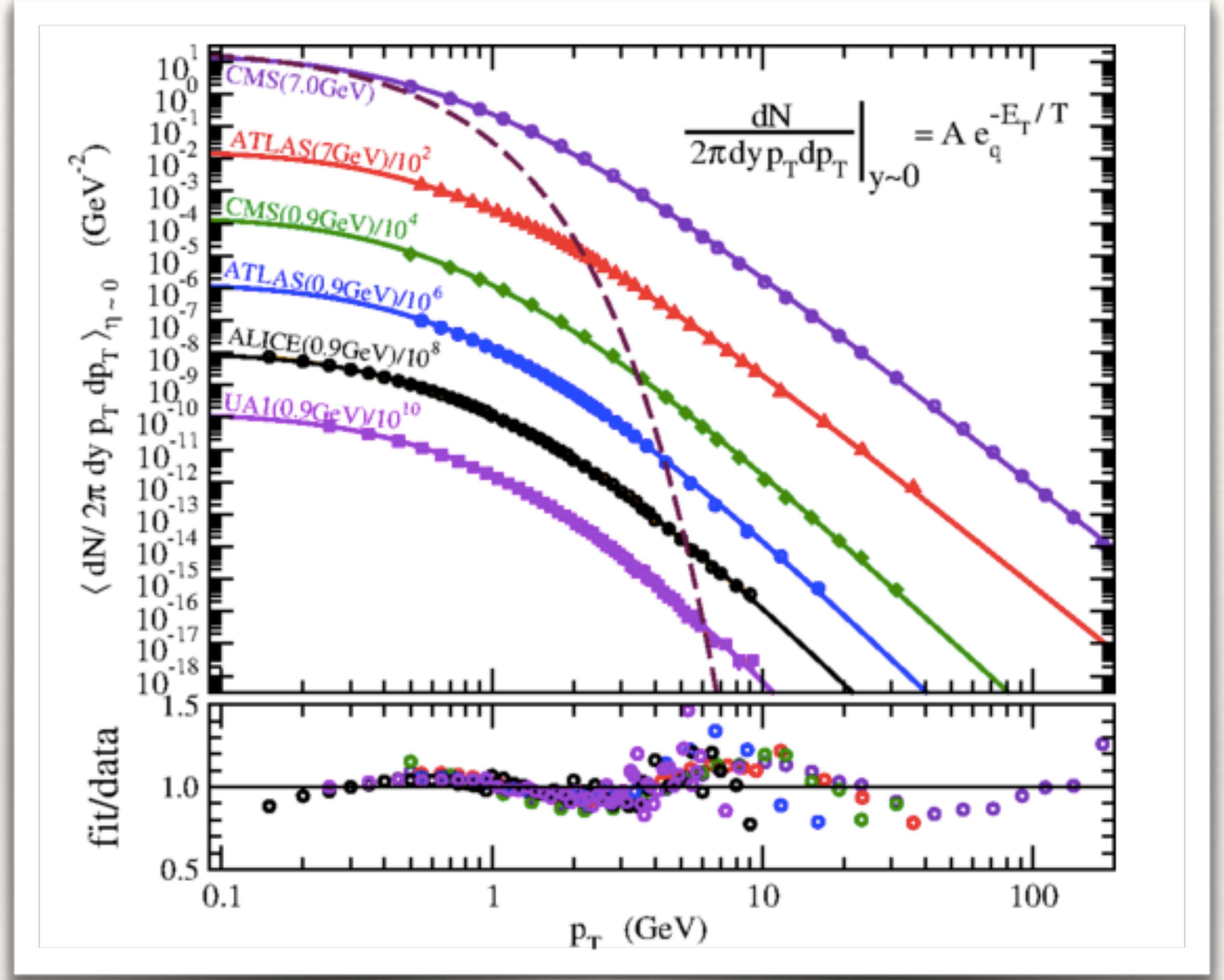
"Complex systems, by their nature, pose challenges to our understanding. They typically involve many components intricately connected through networks. Neither the parts nor the networks are static. Instead, they mutually influence each other as they co-evolve"

From the webpage of **Complexity Science Hub, Vienna**

$$
\begin{aligned}
-\frac{dE}{dr} &= \left\langle\!\!\left\langle \frac{E^2 - EE'}{P} \right\rangle\!\!\right\rangle^{PP} = \left\langle\!\!\left\langle \frac{P^2 + m^2 - EE' - \vec{P}\cdot\vec{P}' + \vec{P}\cdot\vec{P}'}{P} \right\rangle\!\!\right\rangle^{PP} \\
&= \left\langle \frac{1}{P}\left[P^2 - \vec{P}.\vec{P}' + \frac{1}{2}\left\{E^2 - P^2 + E'^2 - P'^2 - 2EE' + 2\vec{P}\cdot\vec{P}'\right\}\right]\right\rangle^{PP} \\
&= \left\langle\!\!\left\langle \left[\frac{P_i K_i}{P} + \frac{1}{2P}\left\{(E-E')^2 - (\vec{P}-\vec{P}')^2\right\}\right]\right\rangle\!\!\right\rangle^{PP} \\
&= \frac{P_i}{P}\langle\langle K_i\rangle\rangle^{PP} + \frac{1}{2P}\left\langle\!\left\langle (E-E')^2 \right\rangle\!\right\rangle^{PP} - \frac{1}{2P}\left\langle\!\!\left\langle (\vec{P}-\vec{P}')^2 \right\rangle\!\!\right\rangle^{PP}
\end{aligned}
$$



L. McLerran and B. Schenke, Nucl. Phys. A ***946,*** *158 (2016)*



*C.-Y. Wong, G. Wilk, L.J.L. Cirto, C. Tsallis, Phys. Rev. D **91** , 114027 (2015)*

PHYSICAL REVIEW D **101,** 034019 (2020)

Fractals, nonextensive statistics, and QCD

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[1]*Instituto de Física, Rua do Matão 1371-Butantã, São Paulo-SP, CEP 05508-090 Brazil*
[2]*Departamento de Física Atómica, Molecular y Nuclear and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, E-18071 Granada, Spain*
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In this work, we analyze how scaling properties of Yang-Mills field theory manifest as self-similarity of truncated n-point functions by scale evolution. The presence of such structures, which actually behave as fractals, allows for recurrent nonperturbative calculation of any vertex. Some general properties are indeed independent of the perturbative order, what simplifies the nonperturbative calculations. We show that for sufficiently high perturbative orders a statistical approach can be used, the nonextensive statistics is obtained, and the Tsallis index, q, is deduced in terms of the field theory parameters. The results are applied to QCD in the one-loop approximation, where q can be calculated, resulting in a good agreement with the value obtained experimentally. We discuss how this approach allows us to understand some intriguing experimental findings in high energy collisions, as the behavior of multiplicity against collision energy, long-tail distributions, and the fractal dimension observed in intermittency analysis.

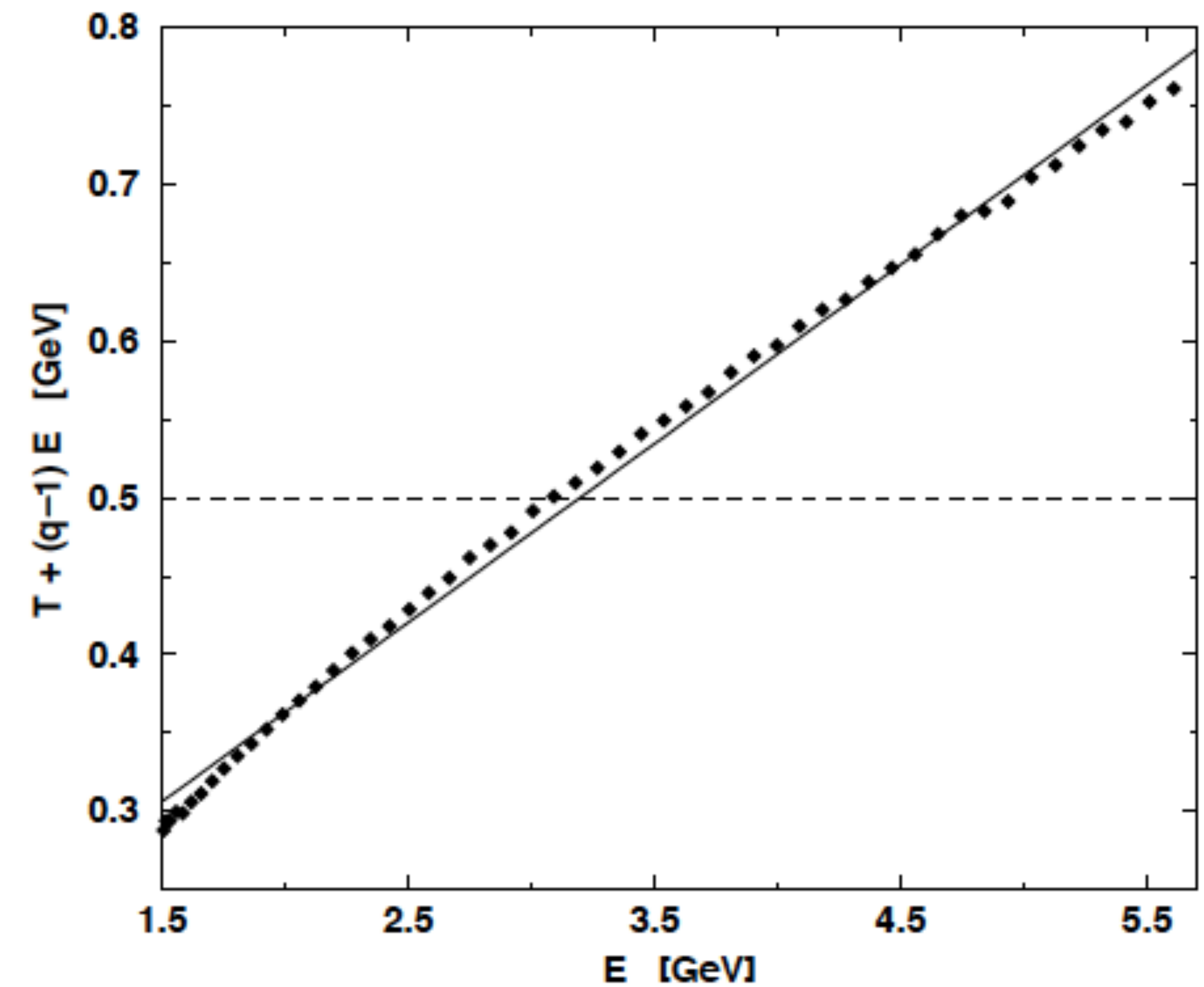
DOI: 10.1103/PhysRevD.101.034019

$$A(p,T) = \frac{1}{p}\frac{d\Phi}{dp}B_{\parallel}(p,T) - \frac{1}{p}\frac{dB_{\parallel}}{dp} - \frac{n-1}{p^2}[B_{\parallel}(p,T) - B_{\perp}(p,T)],$$

n=3

$$\Phi_{\mathrm{Ts}} = \frac{1}{q-1}\ln\left(1 + (q-1)\frac{E(p)}{T}\right)$$

$$T + (q-1)E = \frac{dE}{dp}\frac{B_{\parallel}}{pA + \frac{dB_{\parallel}}{dp} + \frac{n-1}{p}(B_{\parallel} - B_{\perp})}.$$



D.B. Walton and J. Rafelski Phys. Rev. Lett. 84, 31 (2000)