Jacobi polynomial

覃立言

1 integral equation

Sacherer 积分方程，注意求和在积分里

$$
\begin{aligned}
&[\Omega-\omega_s(H)l]R_l(H)\\
&=-\kappa\omega_s(H)\psi_0'(H)\int_0^\infty dH'\frac{1}{\omega_s(H')}\sum_m R_m(H')G_{l,m}(H,H')
\end{aligned}
\tag{1}
$$

其中

$$
\kappa=\frac{2Ne^2c}{E_0T_0}
\tag{2}
$$

$$
G_{l,m}(J,J')=l\ \mathrm{Im}\int_0^\infty d\omega\frac{Z(\omega)}{\omega}h_l(J,\omega)h_m^*(J',\omega)
\tag{3}
$$

$$
\begin{aligned}
&[\Omega-\omega_s(H)l]\frac{R_l(H)}{\omega_s(H)}\\
&=-\kappa\psi_0'(H)\int_0^\infty dH'\sum_m\frac{R_m(H')}{\omega_s(H')}G_{l,m}(H,H')
\end{aligned}
\tag{4}
$$

令 $S_l(H)=\frac{R_l(H)}{\omega_s(H)}$

$$
\begin{aligned}
&[\Omega-\omega_s(H)l]S_l(H)\\
&=-\kappa\psi_0'(H)\int_0^\infty dH'\sum_m S_m(H')G_{l,m}(H,H')
\end{aligned}
\tag{5}
$$

令 $K=H-H_{\min}$，截断哈密顿量为 $\hat{K}$

$$
\begin{aligned}
&[\Omega-\omega_s(K)l]S_l(H)\\
&=-\kappa\psi_0'(K)\int_0^\infty dK'\sum_m S_m(H')G_{l,m}(K,K')
\end{aligned}
\tag{6}
$$

选取 Jacobi 正交基

$$S_l(K) = \sum_{\alpha,w} C_l^{\alpha,w} f_\alpha^{w,l}(K) \tag{7}$$

$$f_\alpha^{w,l}(x(K)) = 2^w \sqrt{\frac{2\alpha+w+l+1}{2^{1+w+l}} \frac{\alpha!\Gamma(l+w+\alpha+1)}{\Gamma(\alpha+l+1)\Gamma(\alpha+w+1)}} (\frac{K}{\hat{K}})^w [\frac{2(\hat{K}-K)}{\hat{K}}]^{l/2} P_\alpha^{w,l}(x) \tag{8}$$

$$= 2^w (\frac{K}{\hat{K}})^w r_\alpha^{w,l}(x(K)) \tag{9}$$

$$r_\alpha^{w,l}(x(K)) = \sqrt{\frac{2\alpha+w+l+1}{2^{1+w+l}} \frac{\alpha!\Gamma(l+w+\alpha+1)}{\Gamma(\alpha+l+1)\Gamma(\alpha+w+1)}} [\frac{2(\hat{K}-K)}{\hat{K}}]^{l/2} P_\alpha^{w,l}(x) \tag{10}$$

$$= 2^{-w} (\frac{K}{\hat{K}})^{-w} f_\alpha^{w,l}(x(K)) \tag{11}$$

其中 $x = 1 - 2K/\hat{K}$，$P_\alpha^{w,l}(x)$ 为 Jacobi 多项式。满足正交条件

$$\int_{-1}^{1} f_\alpha^{w,l}(x) r_\beta^{w,l}(x) dx = \delta_{\alpha\beta} \tag{12}$$

$$-\frac{2}{\hat{K}} \int_0^{\hat{K}} f_\alpha^{w,l}(K) r_\beta^{w,l}(K) dK = \delta_{\alpha\beta} \tag{13}$$

代入得到

$$[\Omega - \omega_s(K)l] \sum_{\alpha,w} C_l^\alpha f_\alpha^{w,l}(K)$$
$$= -\kappa \psi_0'(K) \int_0^\infty dK' \sum_m \sum_{\beta,w'} C_m^\beta f_\beta^{w',m}(K') G_{l,m}(K,K') \tag{14}$$

对 $\psi_0'(K)$ 做级数展开

$$\psi_0'(K) = -\frac{A}{\sqrt{2\pi}\sigma_\delta^3\eta} \exp[-\frac{H_{\min}}{\eta\sigma_\delta^2}] \exp[-\frac{K}{\eta\sigma_\delta^2}] \tag{15}$$

$$= -\frac{A}{\sqrt{2\pi}\sigma_\delta^3\eta} \exp[-\frac{H_{\min}}{\eta\sigma_\delta^2}] \sum_w (\frac{K}{\hat{K}})^w (\frac{-\hat{K}}{\eta\sigma_\delta^2})^w (\frac{1}{w!}) \tag{16}$$

$$= \sum_w c_w^{\exp} (\frac{K}{\hat{K}})^w \tag{17}$$

$$c_w^{\exp} = -\frac{A}{\sqrt{2\pi}\sigma_\delta^3\eta} \exp[-\frac{K_{\min}}{\eta\sigma_\delta^2}] (\frac{-\hat{K}}{\eta\sigma_\delta^2})^w (\frac{1}{w!}) \tag{18}$$

代入得到

$$
\begin{aligned}
&[\Omega-\omega_s(K)l]\sum_{\alpha,w}C^{\alpha}_{w,l}f^{w,l}_{\alpha}(K)\\
&=-\kappa\sum_{w}c^{\exp}_{w}(\frac{K}{\hat{K}})^{w}\int_0^{\infty}dK'\sum_m\sum_{\beta,w'}C^{\beta}_{m}f^{w',m}_{\beta}(K')G_{l,m}(K,K')
\end{aligned}
\tag{19}
$$

两边同时乘以 $r^{w,l}_{\alpha'}(K)$，做正交积分

$$
\begin{aligned}
&\int dK[\Omega-\omega_s(K)l]\sum_{\alpha,w}C^{\alpha}_{w,l}f^{w,l}_{\alpha}(K)r^{w,l}_{\alpha'}(K)\\
&=-\int dK\kappa\sum_{w}c^{\exp}_{w}(\frac{K}{\hat{K}})^{w}\int_0^{\infty}dK'\sum_m\sum_{\beta,w'}C^{\beta}_{m}f^{w',m}_{\beta}(K')G_{l,m}(K,K')r^{w,l}_{\alpha'}(K)
\end{aligned}
\tag{20}
$$

等号左边

$$
\begin{aligned}
&\int dK[\Omega-\omega_s(K)l]\sum_{\alpha,w}C^{\alpha}_{w,l}f^{w,l}_{\alpha}(K)r^{w,l}_{\alpha'}(K)\\
&=-\sum_{w}\frac{2}{\hat{K}}\Omega C^{\alpha}_{w,l}\delta_{\alpha\alpha'}-l\int dK\omega_s(K)\sum_{\alpha,w}f^{w,l}_{\alpha}(K)r^{w,l}_{\alpha'}(K)C^{\alpha}_{w,l}
\end{aligned}
\tag{21}
$$

等号右边

$$
-l\kappa\int_0^{\infty}dK\sum_{w}c^{\exp}_{w}(\frac{K}{\hat{K}})^{w}\int_0^{\infty}dK'\sum_m\sum_{\beta,w'}C^{\beta}_{w',m}f^{w',m}_{\beta}(K')\ \mathrm{Im}\int_0^{\infty}d\omega\frac{Z(\omega)}{\omega}h_l(K,\omega)h^*_m(K',\omega)r^{w,l}_{\alpha'}(K)
\tag{22}
$$

$$
=-l\kappa\ \mathrm{Im}\int_0^{\infty}d\omega\frac{Z(\omega)}{\omega}\int_0^{\infty}dK\sum_{w}c^{\exp}_{w}(\frac{K}{\hat{K}})^{w}r^{w,l}_{\alpha'}(K)h_l(K,\omega)\int_0^{\infty}dK'\sum_m\sum_{\beta,w'}C^{\beta}_{w',m}f^{w',m}_{\beta}(K')h^*_m(K',\omega)
\tag{23}
$$

$$
=-l\kappa\ \mathrm{Im}\int_0^{\infty}d\omega\frac{Z(\omega)}{\omega}\int_0^{\infty}dK\sum_{w}c^{\exp}_{w}2^{-w}f^{w,l}_{\alpha'}(K)h_l(K,\omega)\int_0^{\infty}dK'\sum_m\sum_{\beta,w'}C^{\beta}_{w',m}f^{w',m}_{\beta}(K')h^*_m(K',\omega)
\tag{24}
$$

令

$$
g^{\alpha}_{w,l}(\omega)=\int_0^{\infty}dKf^{w,l}_{\alpha}(K)h_l(K,\omega)
\tag{25}
$$

得到

$$
-l\kappa\ \mathrm{Im}\int_0^{\infty}d\omega\frac{Z(\omega)}{\omega}\sum_{w}c^{\exp}_{w}2^{-w}g^{\alpha'}_{w,l}(\omega)\sum_m\sum_{\beta,w'}g^{*\beta}_{w',m}(\omega)C^{\beta}_{w',m}
\tag{26}
$$

$$
=-l\kappa\sum_{w}\sum_m\sum_{\beta,w'}\ \mathrm{Im}\int_0^{\infty}d\omega\frac{Z(\omega)}{\omega}c^{\exp}_{w}2^{-w}g^{\alpha'}_{w,l}(\omega)g^{*\beta}_{w',m}(\omega)C^{\beta}_{w',m}
\tag{27}
$$

最终等式化为

$$
\begin{aligned}
&-\sum_w \frac{2}{\hat{K}}\Omega C^{\alpha}_{w,l}\delta_{\alpha\alpha'} - l\int dK\omega_s(K)\sum_{\alpha,w} f^{w,l}_{\alpha}(K) r^{w,l}_{\alpha'}(K) C^{\alpha}_{w,l} && (28)\\
&= -l\kappa \sum_w \sum_m \sum_{\beta,w'} \operatorname{Im}\int_0^{\infty} d\omega \frac{Z(\omega)}{\omega} c^{\exp}_w 2^{-w} g^{\alpha'}_{w,l}(\omega) g^{*\beta}_{w',m}(\omega) C^{\beta}_{w,m} && (29)
\end{aligned}
$$

两边同时去掉对 w 的求和，$\beta \to \alpha$， $\alpha' \to \beta$ 合并为

$$
\begin{aligned}
&-\frac{2}{\hat{K}}\Omega C^{\alpha}_{w,l}\delta_{\alpha\beta} && (30)\\
&= l\int dK\omega_s(K)\sum_{\alpha} f^{w,l}_{\alpha}(K) r^{w,l}_{\beta}(K) C^{\alpha}_{w,l} - l\kappa \sum_m \sum_{\alpha,w'} \operatorname{Im}\int_0^{\infty} d\omega \frac{Z(\omega)}{\omega} c^{\exp}_w 2^{-w} g^{\beta}_{w,l}(\omega) g^{*\alpha}_{w',m}(\omega) C^{\alpha}_{w,m} && (31)
\end{aligned}
$$

统一求和号

$$
-\frac{2}{\hat{K}}\Omega C^{\beta}_{w,l} = l\sum_{\alpha,w',m}\left[\int dK\omega_s(K) f^{w,l}_{\alpha}(K) r^{w,l}_{\beta}(K)\delta_{lm}\delta_{ww'} - \kappa a_w \operatorname{Im}\int_0^{\infty} d\omega \frac{Z(\omega)}{\omega} g^{\beta}_{w,l}(\omega) g^{*\alpha}_{w',m}(\omega)\right] C^{\alpha}_{w',m} \tag{32}
$$

$$
a_w = c^{\exp}_w 2^{-w} \tag{33}
$$

Taylor 展开约 15 阶对 exp 函数拟合可以接受