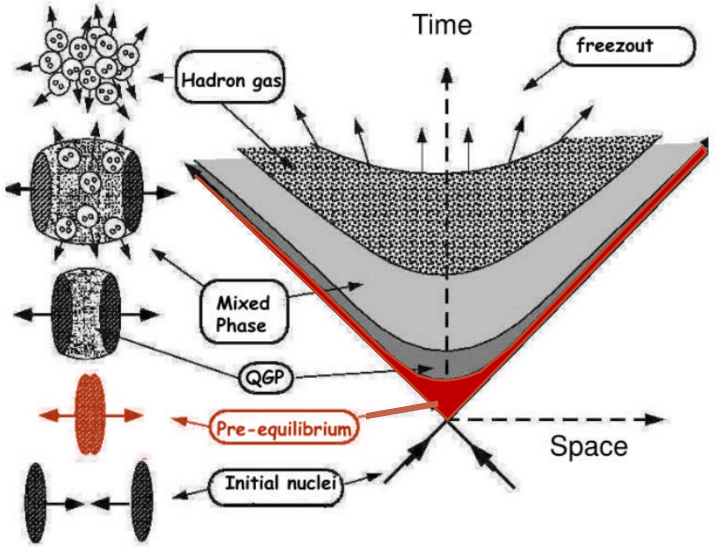
Tsinghua University

Hydrodynamization from the full Boltzmann equation



New theoretical tool :

(nonlinear) Spectral BBGKY hierarchy

--Fill the Gap in kinetic theory: Correlation, Nonlinear

Physical Insight:

How to distinguish Hydronamization from Thermalization

Xingjian Lu (lus21@mails.tsinghua.edu.cn)

arXiv: 2507.14243, 2509.23978, XJ L, Shuzhe Shi

UPC2025, 21-24 Nov., 2025, Guangzhou, China

Motivation – Nonequilibrium evolution

ultra-peripheral collision

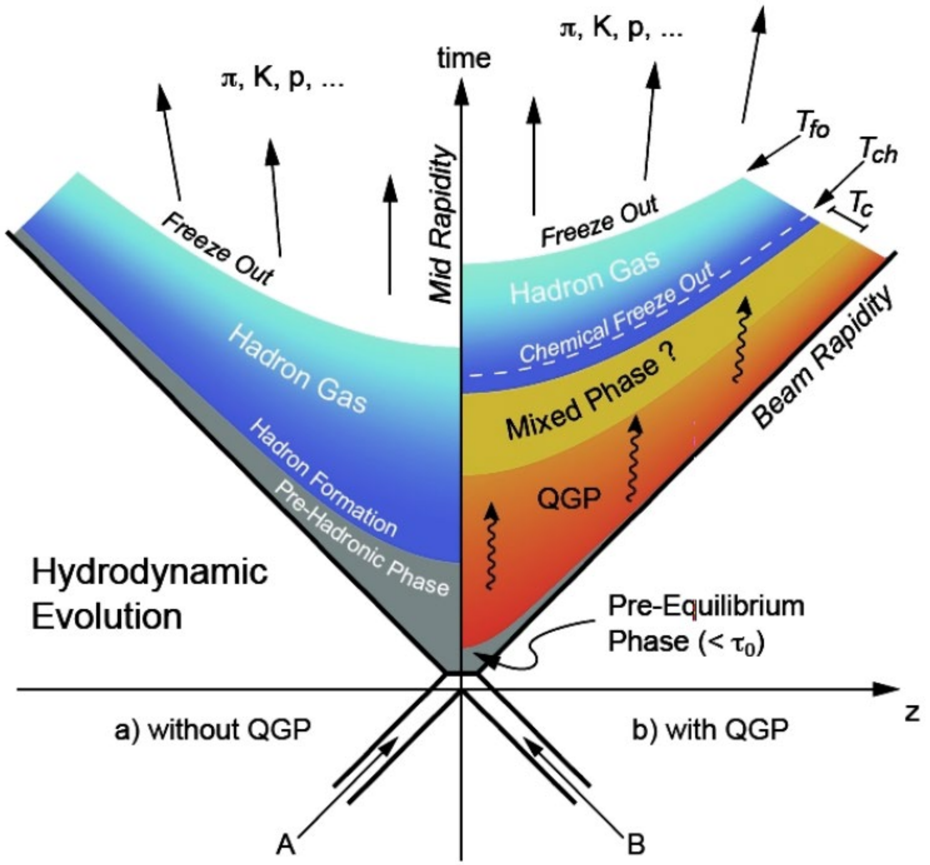
$$b \gtrsim R_A + R_B$$

- $\gamma + A \rightarrow V + A.$

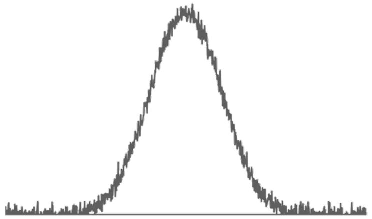
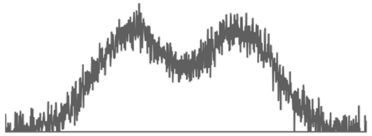
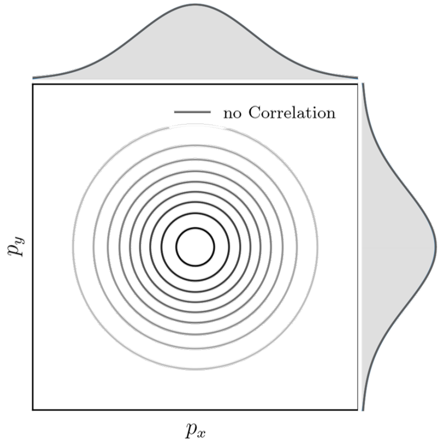
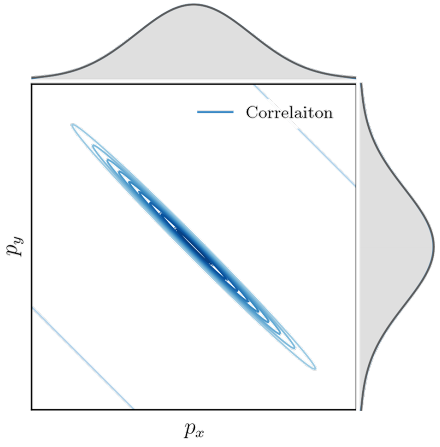
  where $V = \rho^0, \phi, J/\psi, \Upsilon,$ etc

- $\gamma + \gamma \rightarrow \ell^+ + \ell^-$

(semi-)central collision



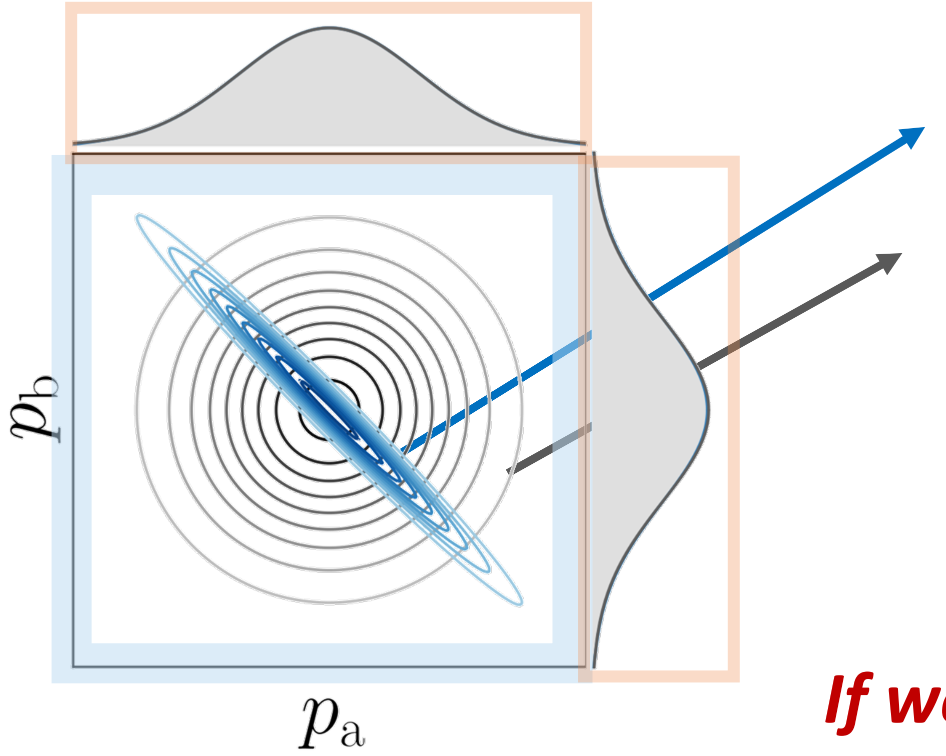
Motivation - Early Thermalization Puzzle

Conventional Method	Assumptions	Real System	Reasonable Method
Ⅰ. Linearization ✗	 close-to-equalibrium	 far-from-equalibrium	Non-linear ✓
Ⅱ. Boltzmann ✗	 no correlation	 correlation incolved	Correlation ✓

The definition of correlation in kinetic theory

The definition of pair-correlation in my talk (in kinetic theory)

$$P'_2(\phi_a, \phi_b) = P_2(\phi_a, \phi_b) - P_1(\phi_a)P_1(\phi_b)$$



Blue: with correlation

Grey: without correlation

Blue and **Grey:**

different $P_2(\phi_a, \phi_b)$

same $P_1(\phi_a)$

If we ignore correlations,
we will lose information about the system.

Map: What we do in the building of nonequilibrium statistical mechanics?

Hamilton Equation/
Schrödinger Equation

Hamilton Equation

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}, \quad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}$$

Schrödinger equation

$$\hat{H}\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

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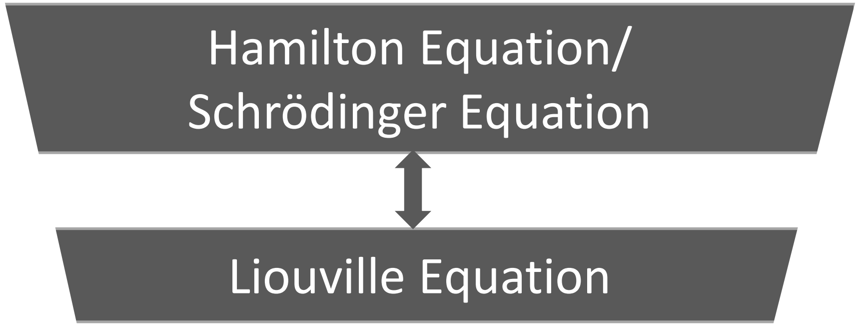


Liouville's theorem

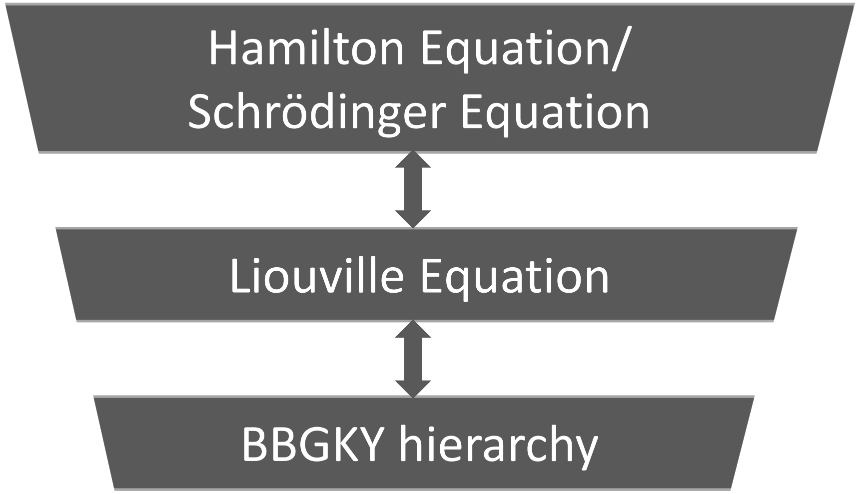
The distribution function is **constant** along any trajectory **in phase space**.

Liouville equation

$$\frac{\partial P_{(N)}}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial P_{(N)}}{\partial q_i} \dot{q}_i + \frac{\partial P_{(N)}}{\partial p_i} \dot{p}_i \right) = 0$$

Full distribution function

Map: What we do in the building of nonequilibrium statistical mechanics?



Liouville equation

$$d_t P_{(N)} = C[\int, P_{(N)}]$$

Full distribution function

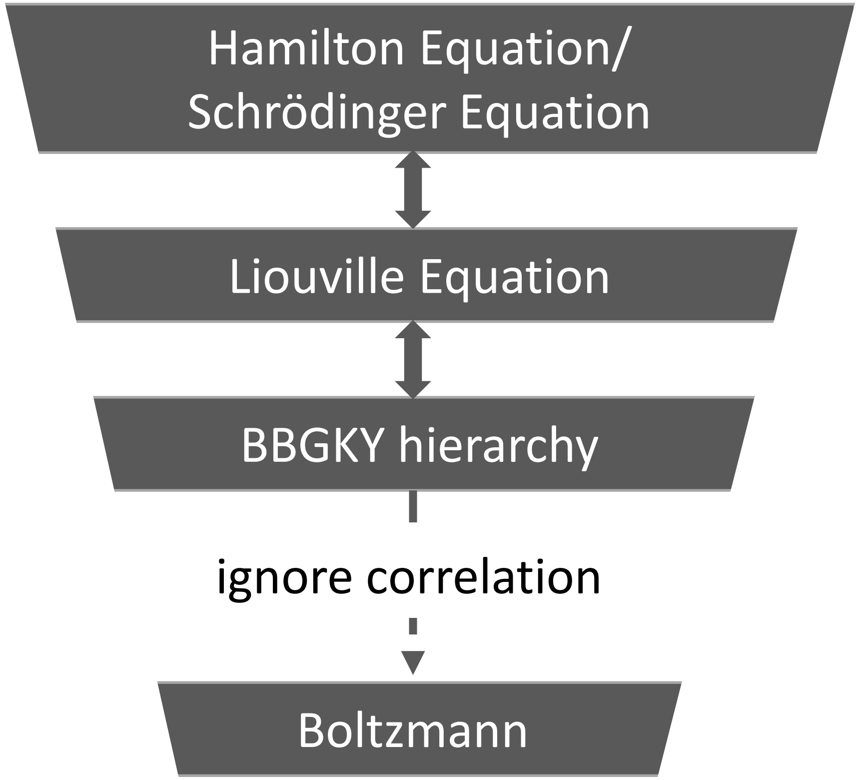
BBGKY hierarchy

(Bogoliubov–Born–Green–Kirkwood–Yvon)

$$d_t P_{(n)} = C[\int, P_{(n)}, P_{(n+1)}]$$

Reduced distribution function

Map: What we do in the building of nonequilibrium statistical mechanics?



BBGKY hierarchy

(Bogoliubov–Born–Green–Kirkwood–Yvon)

$$d_t P_{(n)} = C[\int, P_{(n)}, P_{(n+1)}]$$

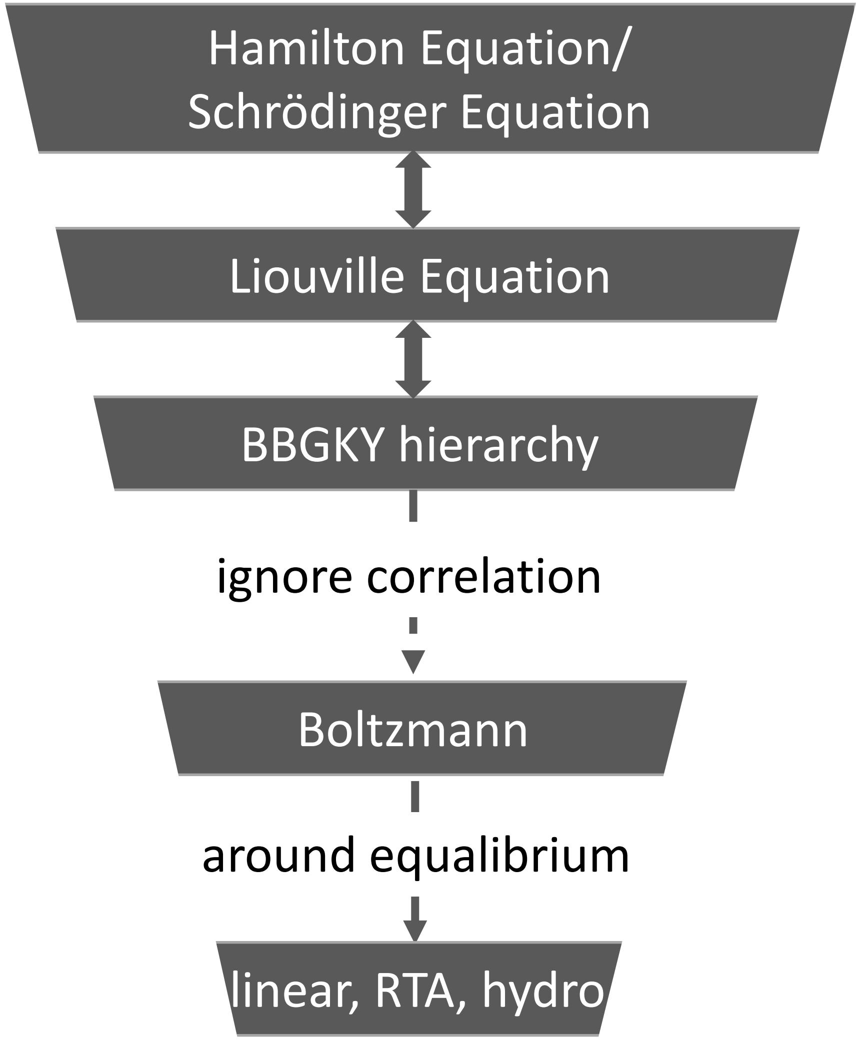
Reduced distribution function

Assume $P_2(\phi_a, \phi_b) - P_1(\phi_a) P_1(\phi_b) = 0$

Boltzmann Equation

$$d_t P_{(1)} = C[\int, P_{(1)}]$$

Map: What we do in the building of nonequilibrium statistical mechanics?



Boltzmann Equation

$$d_t f_1 \propto (f_3 f_4 - f_1 f_2)$$

Around equalibrium

Linear Boltzmann Equation

$$d_t f_1 \propto (f_3 + f_4 - f_1 - f_2)$$

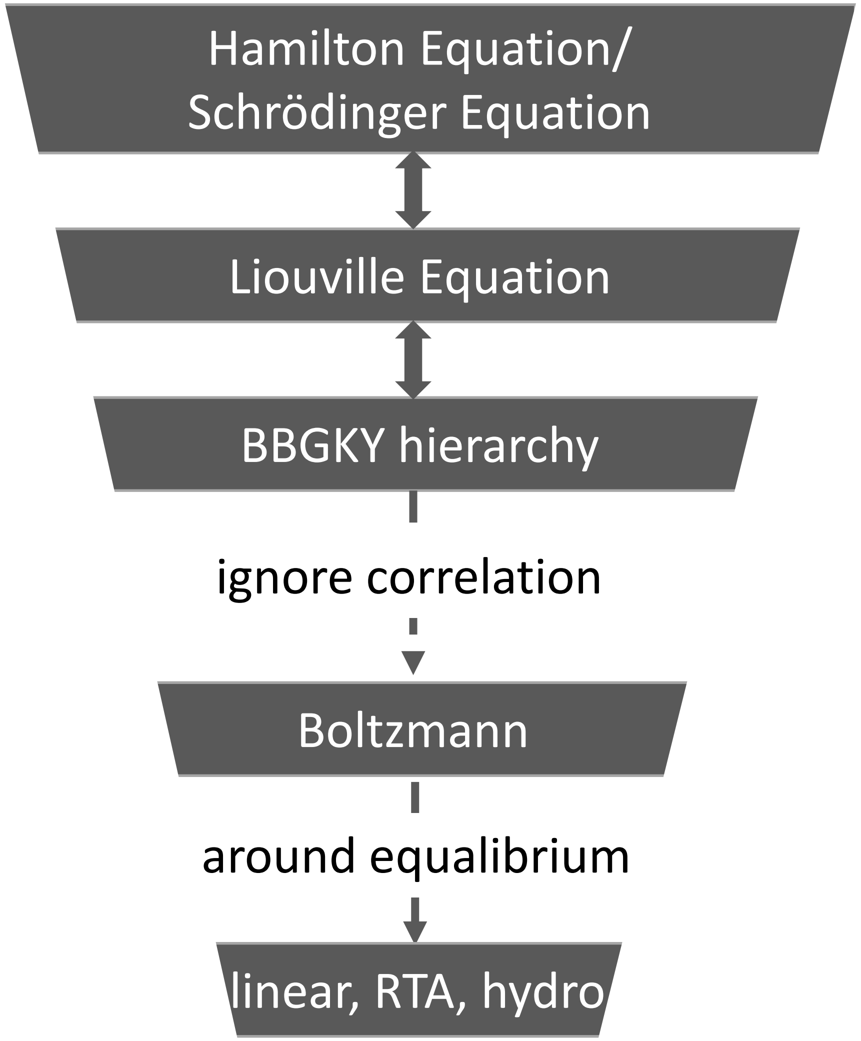
RTA Boltzmann Equation

$$d_t f_1 \propto (f_1 - f_{eq})/\tau$$

Hydrodynamics

$$\nabla_\mu T^{\mu\nu} = 0$$

Map: What we do in the building of nonequilibrium statistical mechanics?



Time-reversal symmetry, Cornerstone

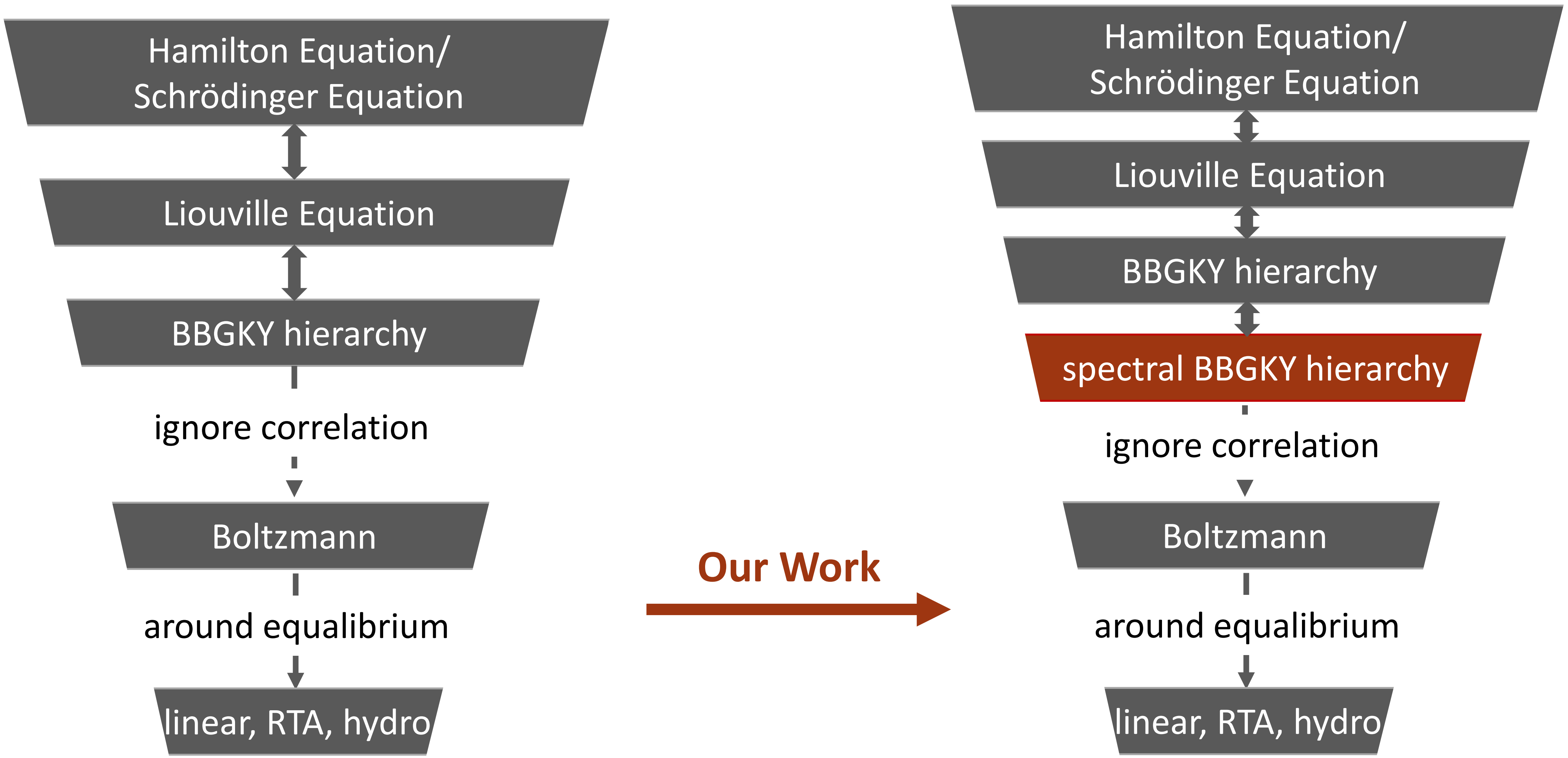
Gap: correlation

Tractable proxy

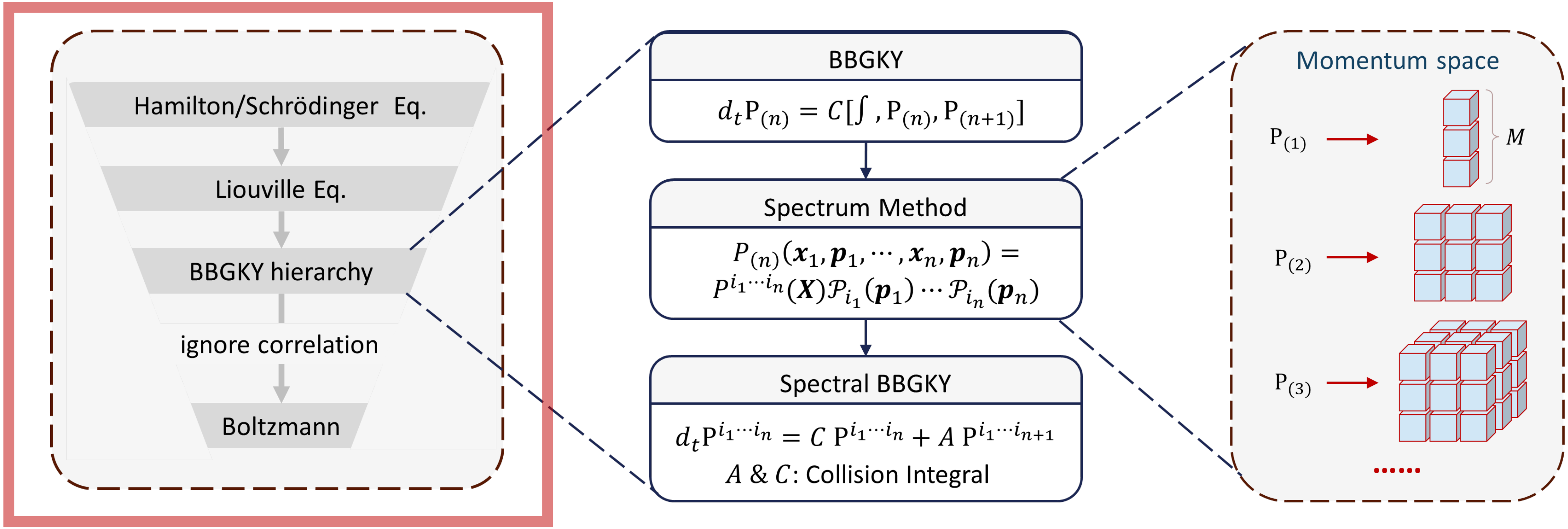
Gap: nonlinear

Workhorse

Map: What we do in the building of nonequilibrium statistical mechanics?



Spectral BBGKY Hierarchy



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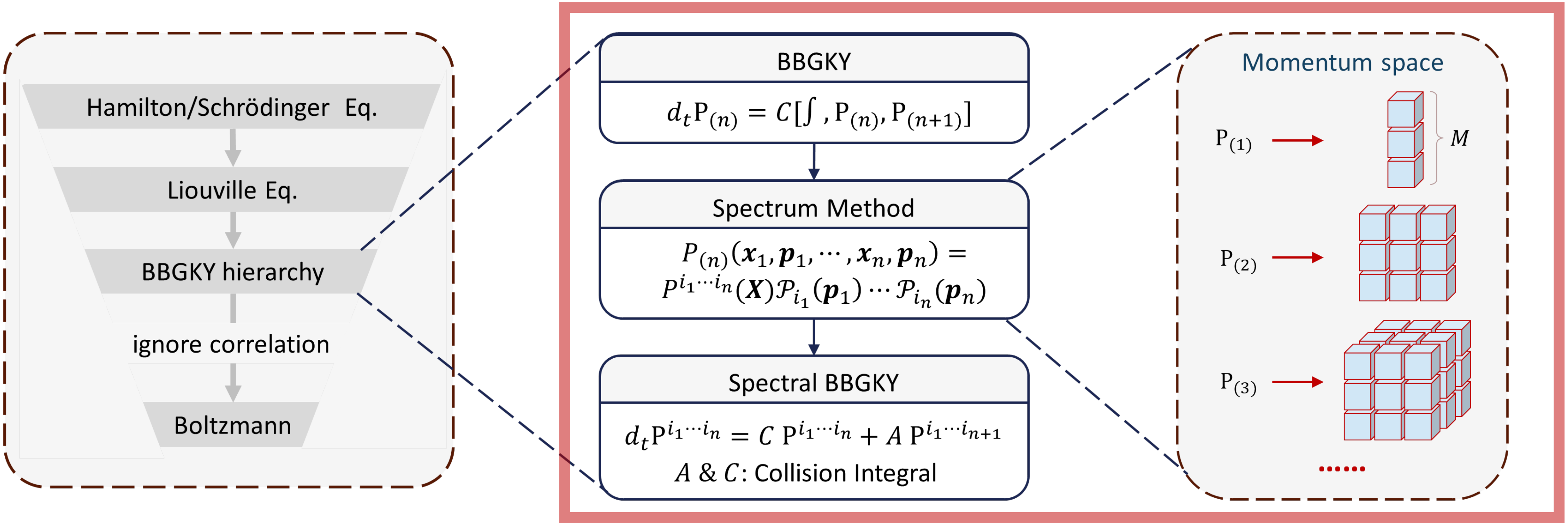


Spectral BBGKY Hierarchy

Hamilton/Schrödinger Eq.

↓

Liouville Eq.

↓

BBGKY hierarchy

↓ ignore correlation

Boltzmann

BBGKY

$$d_t \mathrm{P}_{(n)} = C[\int, \mathrm{P}_{(n)}, \mathrm{P}_{(n+1)}]$$

↓

Spectrum Method

$$P_{(n)}(\boldsymbol{x}_1, \boldsymbol{p}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{p}_n) = P^{i_1 \cdots i_n}(\boldsymbol{X}) \mathcal{P}_{i_1}(\boldsymbol{p}_1) \cdots \mathcal{P}_{i_n}(\boldsymbol{p}_n)$$

↓

Spectral BBGKY

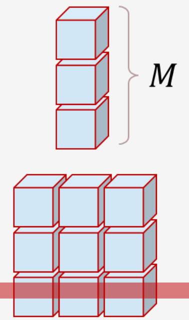
$$d_t \mathrm{P}^{i_1 \cdots i_n} = C\ \mathrm{P}^{i_1 \cdots i_n} + A\ \mathrm{P}^{i_1 \cdots i_{n+1}}$$

A & C: Collision Integral

Momentum space

$\mathrm{P}_{(1)}$ →

$\mathrm{P}_{(2)}$ →



$$n = \frac{4\sqrt{\pi}}{(2\pi)^3} T^3 f^{(0,0,0)}$$

$$\langle p^0 \rangle = \frac{12\sqrt{\pi}}{(2\pi)^3} T^4 \left(f^{(0,0,0)} - f^{(1,0,0)} \right)$$

$$\langle p^1 \rangle = \frac{16\sqrt{3\pi}}{(2\pi)^3} T^4 f^{(0,1,1)}$$

$$\langle p^2 \rangle = \frac{16\sqrt{3\pi}}{(2\pi)^3} T^4 f^{(0,1,-1)}$$

$$\langle p^3 \rangle = \frac{16\sqrt{3\pi}}{(2\pi)^3} T^4 f^{(0,1,0)}$$

Our basis

Angular Radial

$$\mathcal{P}_{n,\ell,m}(p_\mu) = e^{-p_\mu u^\mu/\Lambda} \left(\frac{p_\mu u^\mu}{\Lambda} \right)^\ell Y_{\ell,m}(\theta,\phi) L_n^{(2\ell+2)} \left(\frac{p_\mu u^\mu}{\Lambda} \right)$$

The physics

- equilibrium
- particle number, energy-momentum conservation
- nonlinear behavior

mainly depends on the first few basis.

Spectral BBGKY Hierarchy

Hamilton/Schrödinger Eq.

↓

Liouville Eq.

↓

BBGKY hierarchy

↓ ignore correlation

Boltzmann

BBGKY

$$d_t \mathrm{P}_{(n)} = C[\int, \mathrm{P}_{(n)}, \mathrm{P}_{(n+1)}]$$

↓

Spectrum Method

$$P_{(n)}(\boldsymbol{x}_1, \boldsymbol{p}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{p}_n) = P^{i_1 \cdots i_n}(\boldsymbol{X}) \mathcal{P}_{i_1}(\boldsymbol{p}_1) \cdots \mathcal{P}_{i_n}(\boldsymbol{p}_n)$$

↓

Spectral BBGKY

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Momentum space

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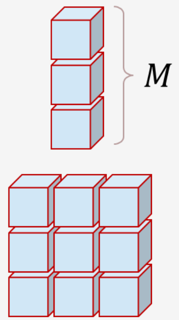

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避免了对动量空间离散化 $\mathrm{Cost}_{\text{now}} = \sqrt{\mathrm{Cost}_{\text{before}}}$

The physics

- equilibrium
- particle number, energy-momentum conservation
- nonlinear behavior

mainly depends on the first few basis.

Spectral BBGKY Hierarchy

Hamilton/Schrödinger Eq.

↓

Liouville Eq.

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$$d_t \mathrm{P}_{(n)} = C[\textstyle\int, \mathrm{P}_{(n)}, \mathrm{P}_{(n+1)}]$$

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Spectral BBGKY

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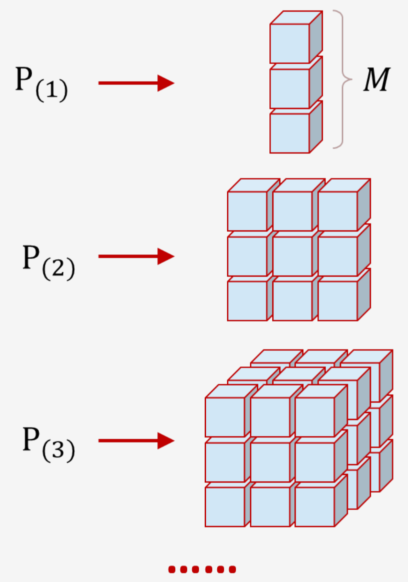
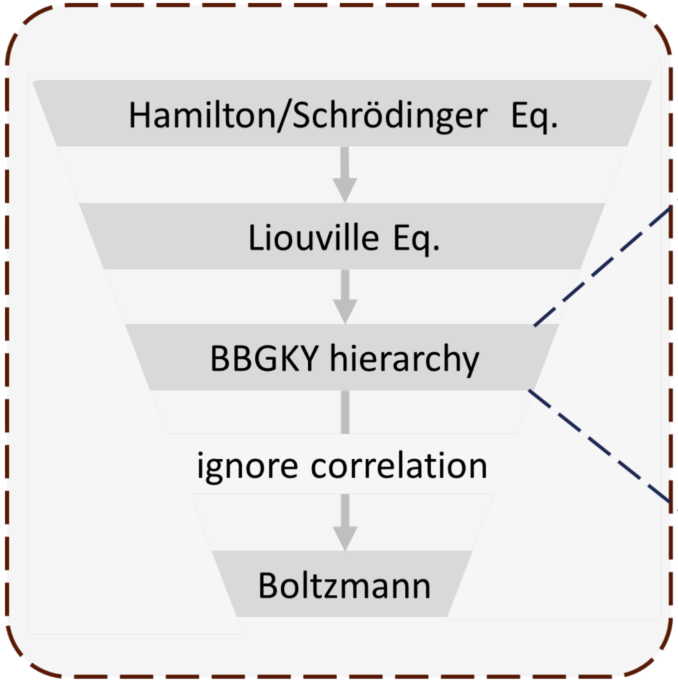
Momentum space



$$C^{\text{gain}}_{ijks} = \int_{p_{1\mu}, p_{2\mu}} \int_{\boldsymbol{p}_3, \boldsymbol{p}_4} \mathcal{Q}_i(p_{1\mu})\mathcal{Q}_j(p_{2\mu})\frac{1}{p_1^0 p_2^0}\mathcal{W}_{(\boldsymbol{p}_1,\boldsymbol{p}_2\to\boldsymbol{p}_3,\boldsymbol{p}_4)}\mathcal{P}_j(p'_{1\mu})\mathcal{P}_k(p'_{2\mu})$$

- **Simplify** — **8D Integral** → **3D Integral** —($m = 0$ case / polynomial σ)→ **Finite Sums**
- **Compact** — **Parity + rotation symmetry**

Spectral BBGKY Hierarchy



BBGKY

$$d_t P_{(n)} = C[\int, P_{(n)}, P_{(n+1)}]$$

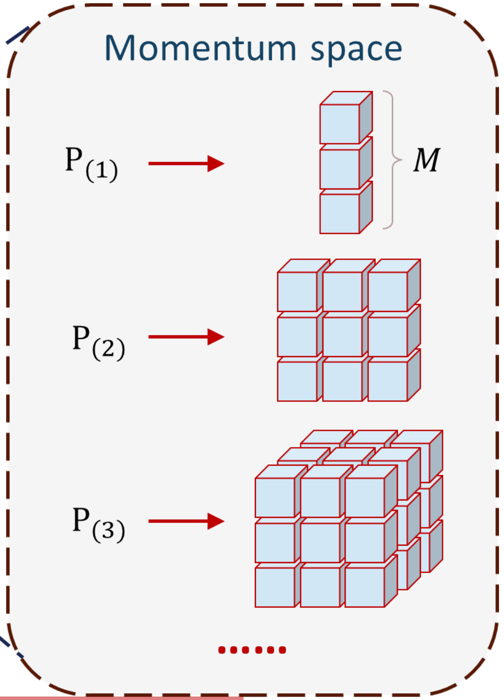
Spectrum Method

$$P_{(n)}(\boldsymbol{x}_1,\boldsymbol{p}_1,\cdots,\boldsymbol{x}_n,\boldsymbol{p}_n) = P^{i_1\cdots i_n}(\boldsymbol{X})\mathcal{P}_{i_1}(\boldsymbol{p}_1)\cdots\mathcal{P}_{i_n}(\boldsymbol{p}_n)$$

Spectral BBGKY

$$d_t P^{i_1\cdots i_n} = C\, P^{i_1\cdots i_n} + A\, P^{i_1\cdots i_{n+1}}$$

A & C: Collision Integral



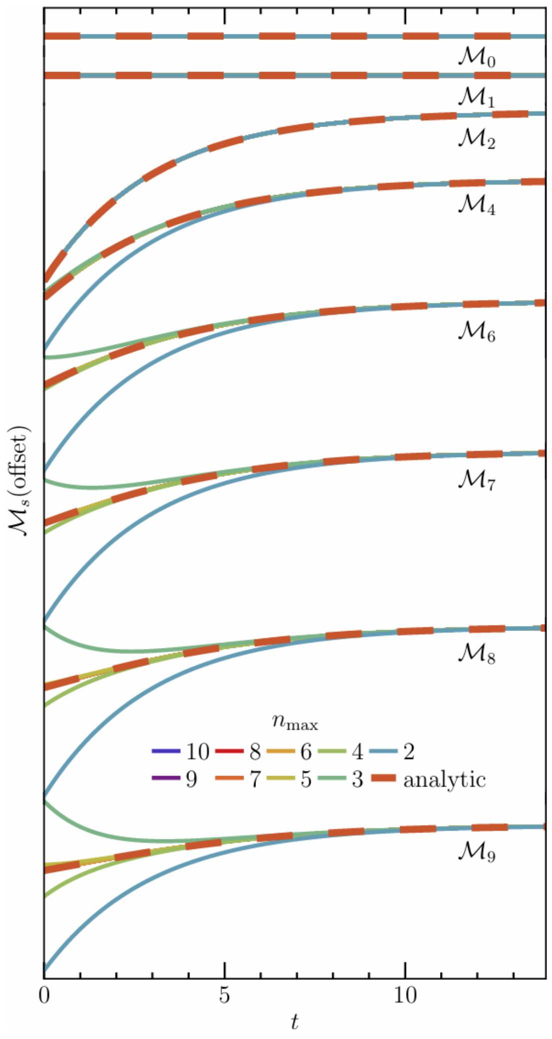
$$C^{\text{gain}}_{ijks} = \int_{p_{1\mu},p_{2\mu}} \int_{\boldsymbol{p}_3,\boldsymbol{p}_4} \mathcal{Q}_i(p_{1\mu})\mathcal{Q}_j(p_{2\mu})\frac{1}{p_1^0 p_2^0}\mathcal{W}_{(\boldsymbol{p}_1,\boldsymbol{p}_2\to\boldsymbol{p}_3,\boldsymbol{p}_4)}\mathcal{P}_j(p'_{1\mu})\mathcal{P}_k(p'_{2\mu})$$

- **Simplify**: **8D Integral** → **3D Integral** —($m = 0$ case, polynomial σ)→ **Finite Sums**
- **Compact**: **Parity + rotation symmetry**

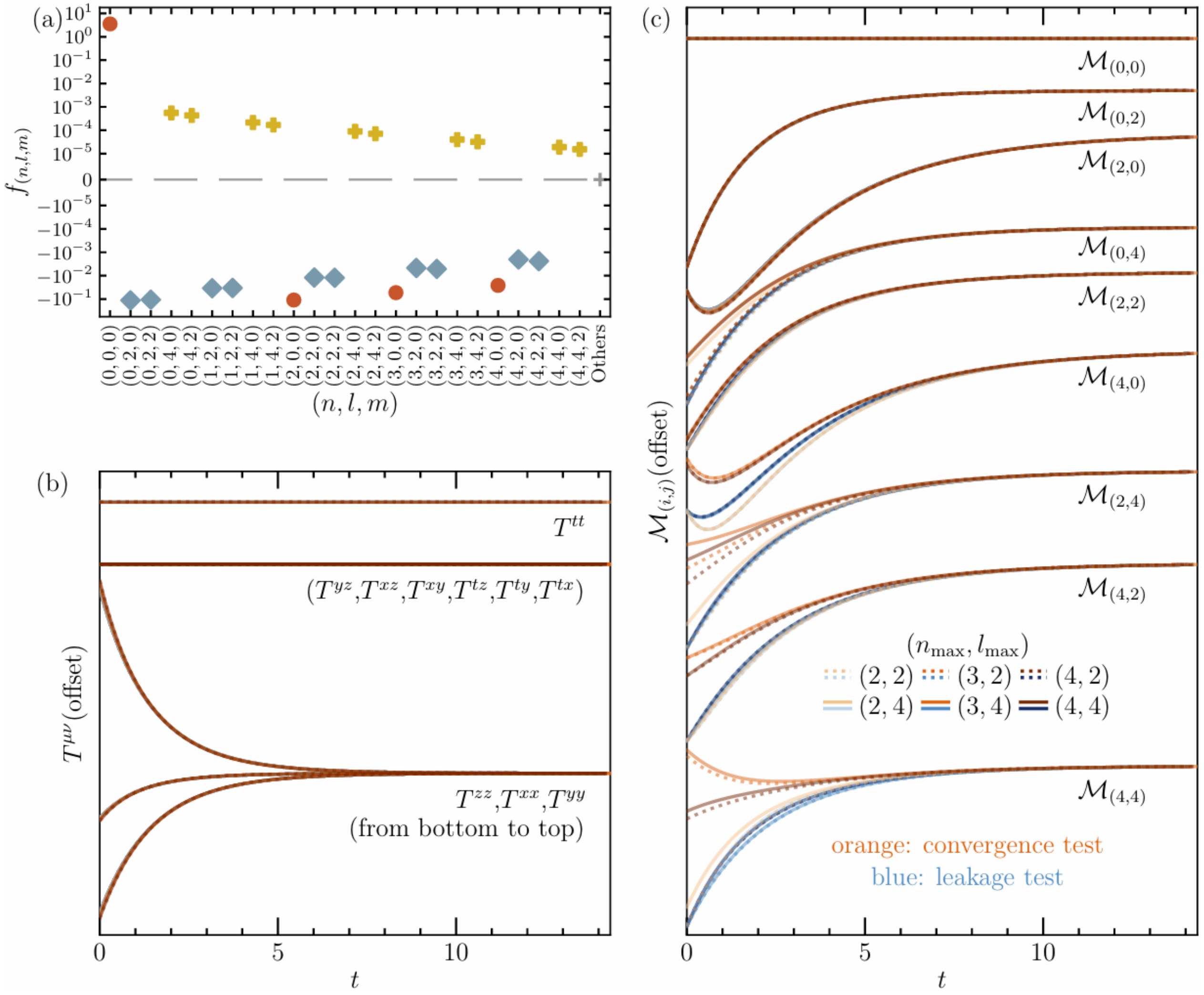
碰撞积分与含时演化解耦合, 碰撞积分可以解析预计算

Numerical validation

Compare to analytic solution



Converge test and Leakage test



Hydrodynamization from the full Boltzmann Eq.

We need:
hydrodynamization
= **near equilibrium**
People before
hydrodynamization
= **equilibrium**
Now we can determine
near equilibrium
through linearization

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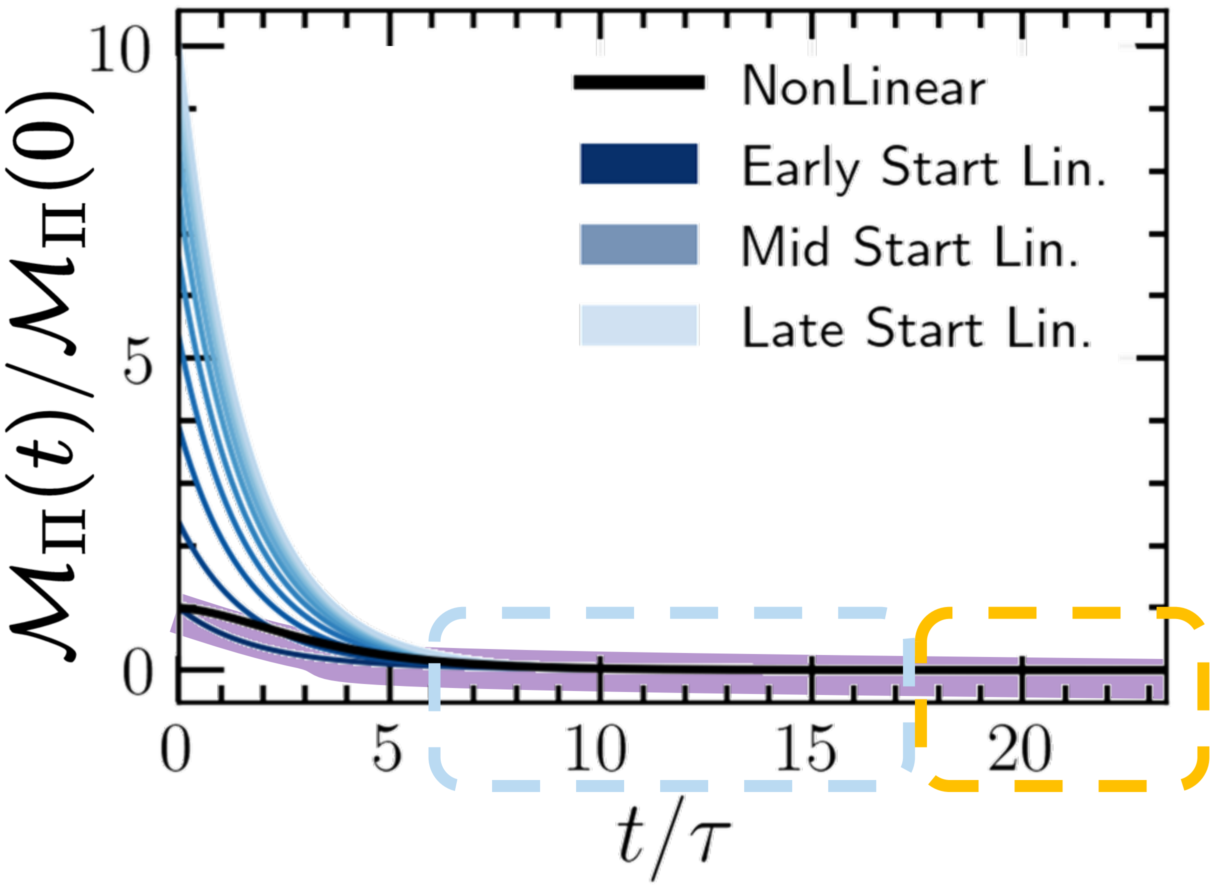
$$\mathcal{M}_{\Pi}(t) \equiv \frac{4\pi^{\frac{5}{2}}}{T^2} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 p^0} (f(t,\boldsymbol{p}) - f_{\text{eq}}(t,\boldsymbol{p}))$$

Hydrodynamization from the full Boltzmann Eq.



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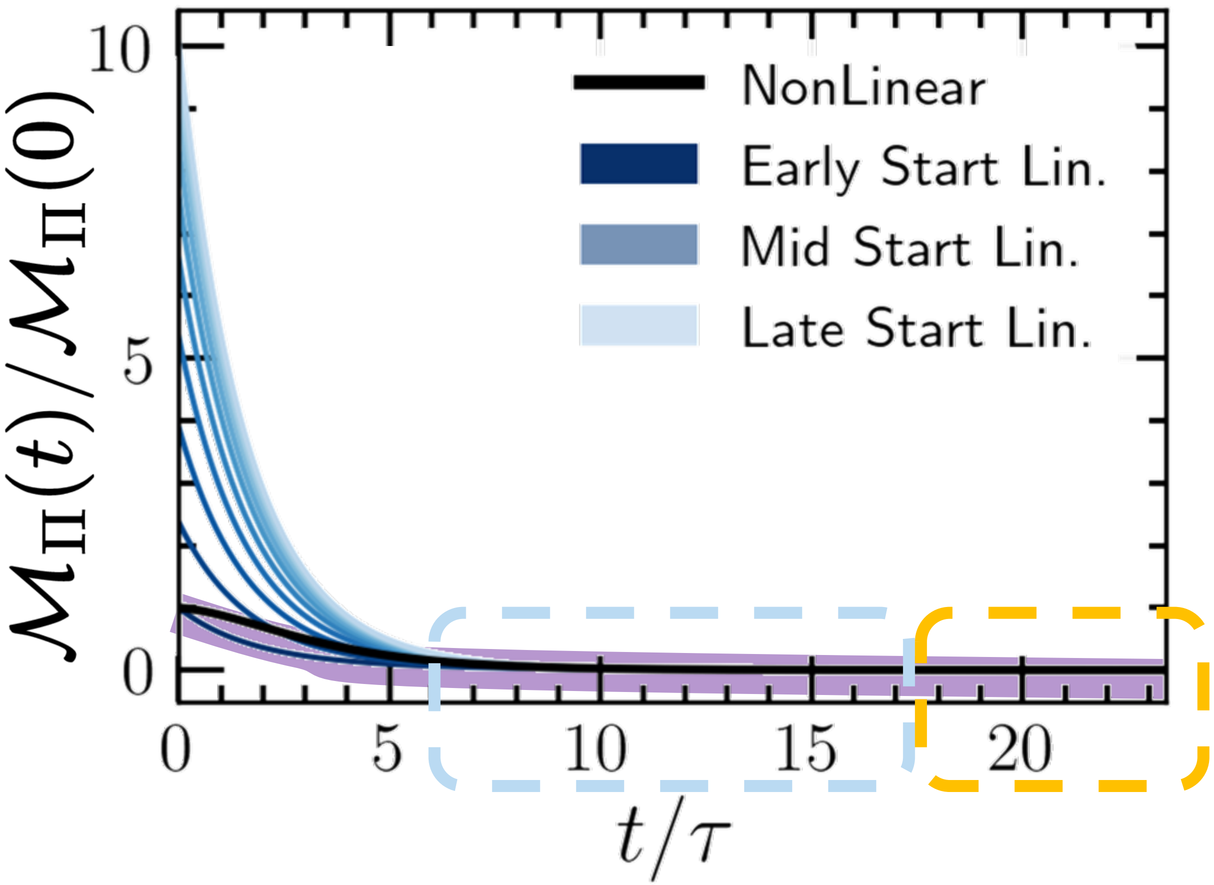
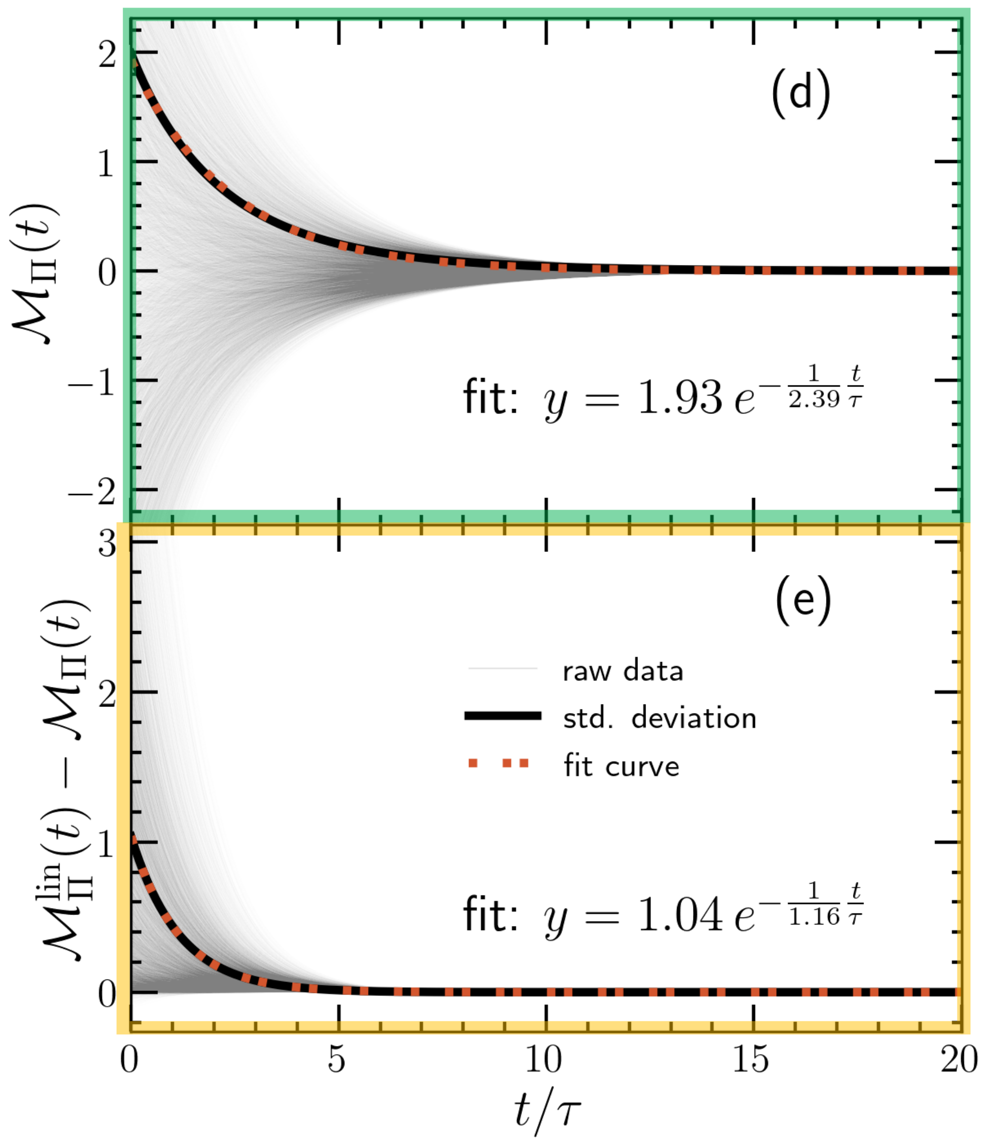


$$\mathcal{M}_\Pi(t) \equiv \frac{4\pi^{\frac{5}{2}}}{T^2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3 p^0}\left(f(t,\boldsymbol{p}) - f_{\rm eq}(t,\boldsymbol{p})\right)$$

- Under isotropic conditions, a nonlinear system already enters a linear regime before it approaches thermal equilibrium

$$\tau_{\rm lin}^{[O]} < \tau_{\rm therm}^{[O]}$$

Hydrodynamization from the full Boltzmann Eq.



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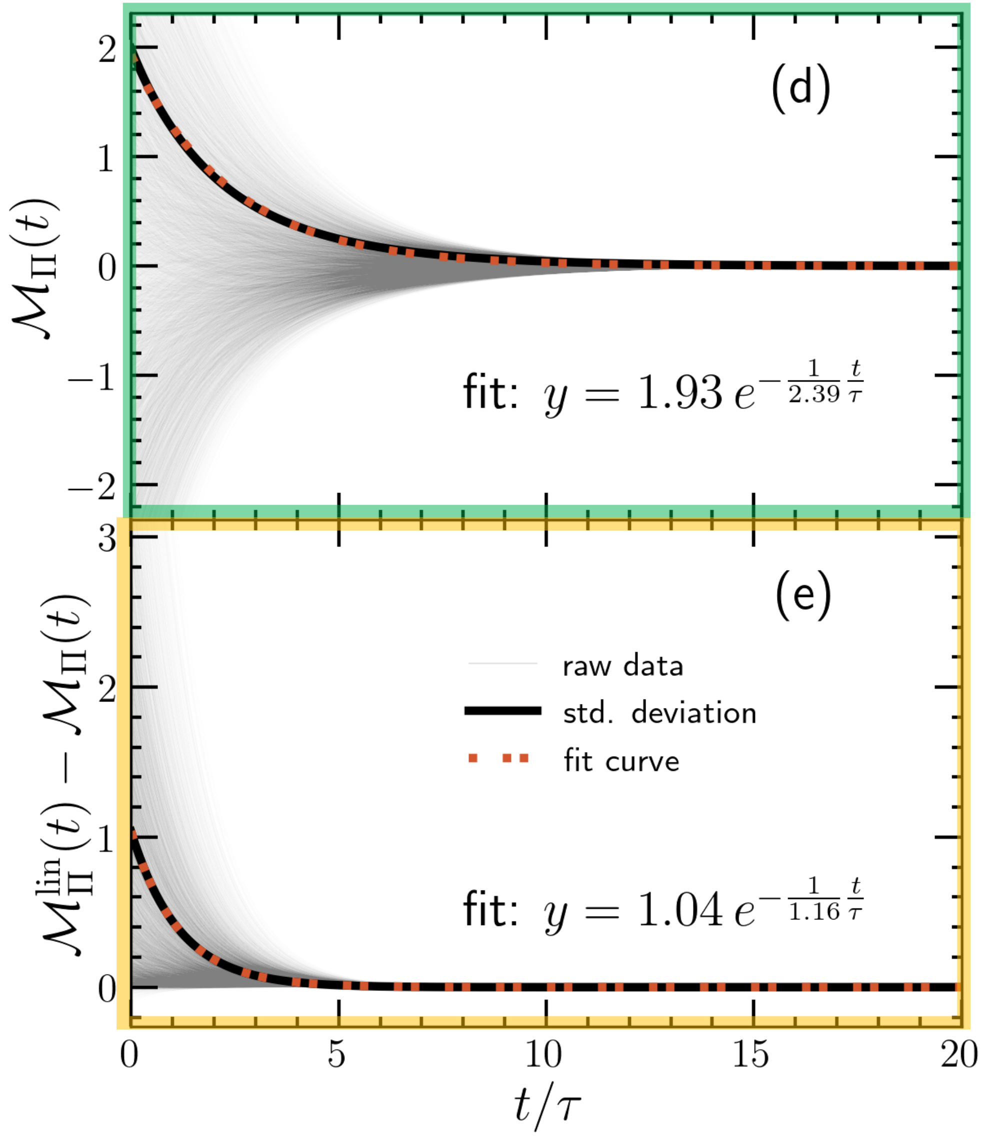

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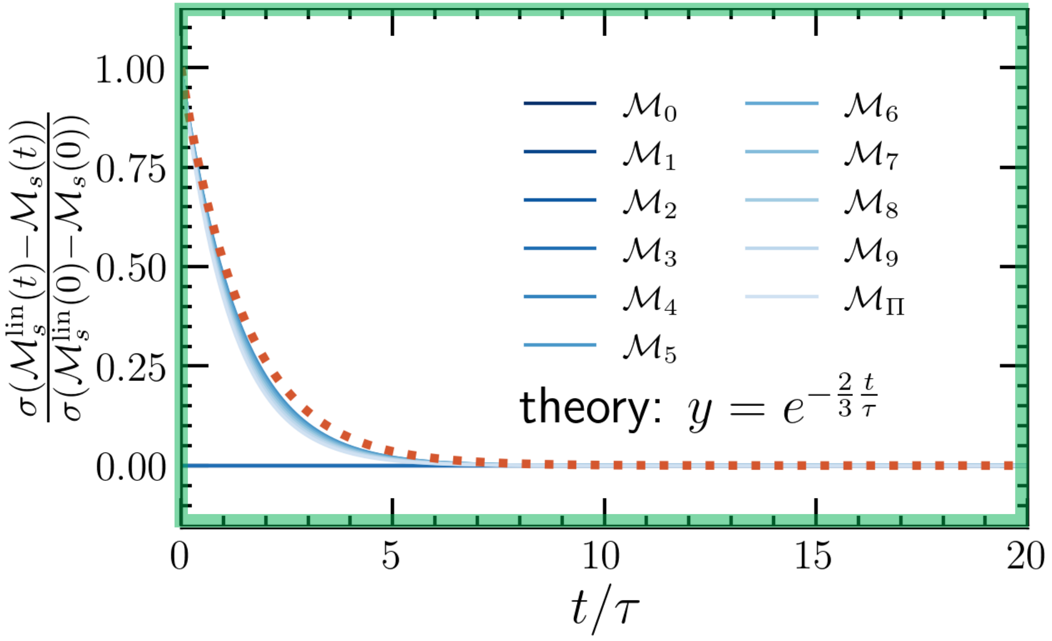
"distribution ensemble"

Hydrodynamization from the full Boltzmann Eq.



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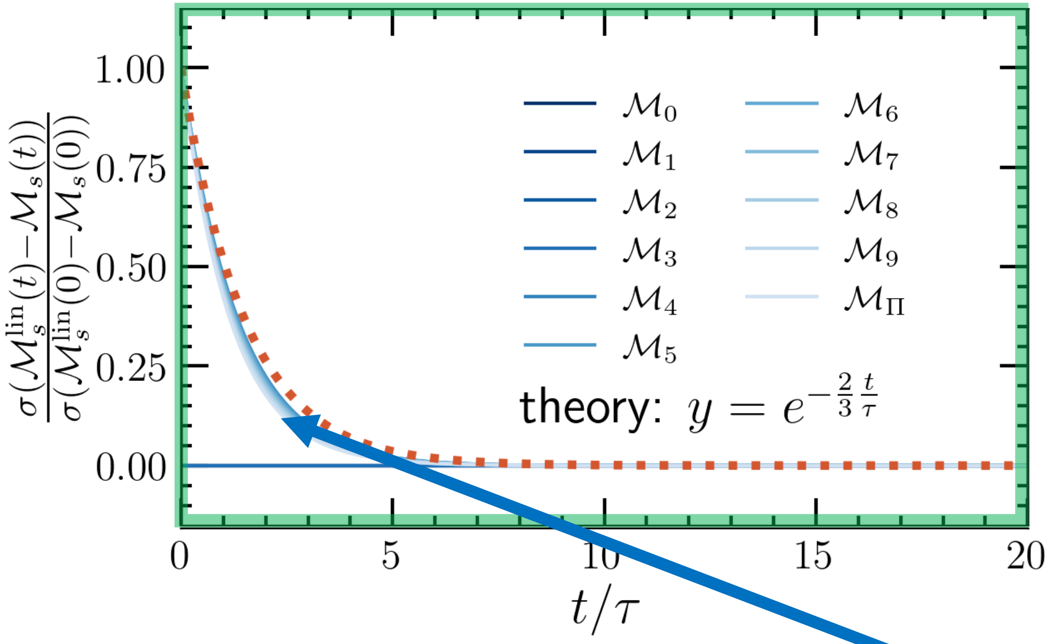

- Observables

$$\mathcal{M}_\Pi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{E} f(t,\boldsymbol{p})$$

$$\mathcal{M}_i = \int \frac{d^3p}{(2\pi)^3} E^i f(t,\boldsymbol{p}) \quad i = 0,1,\cdots,10$$

look at the blue bands

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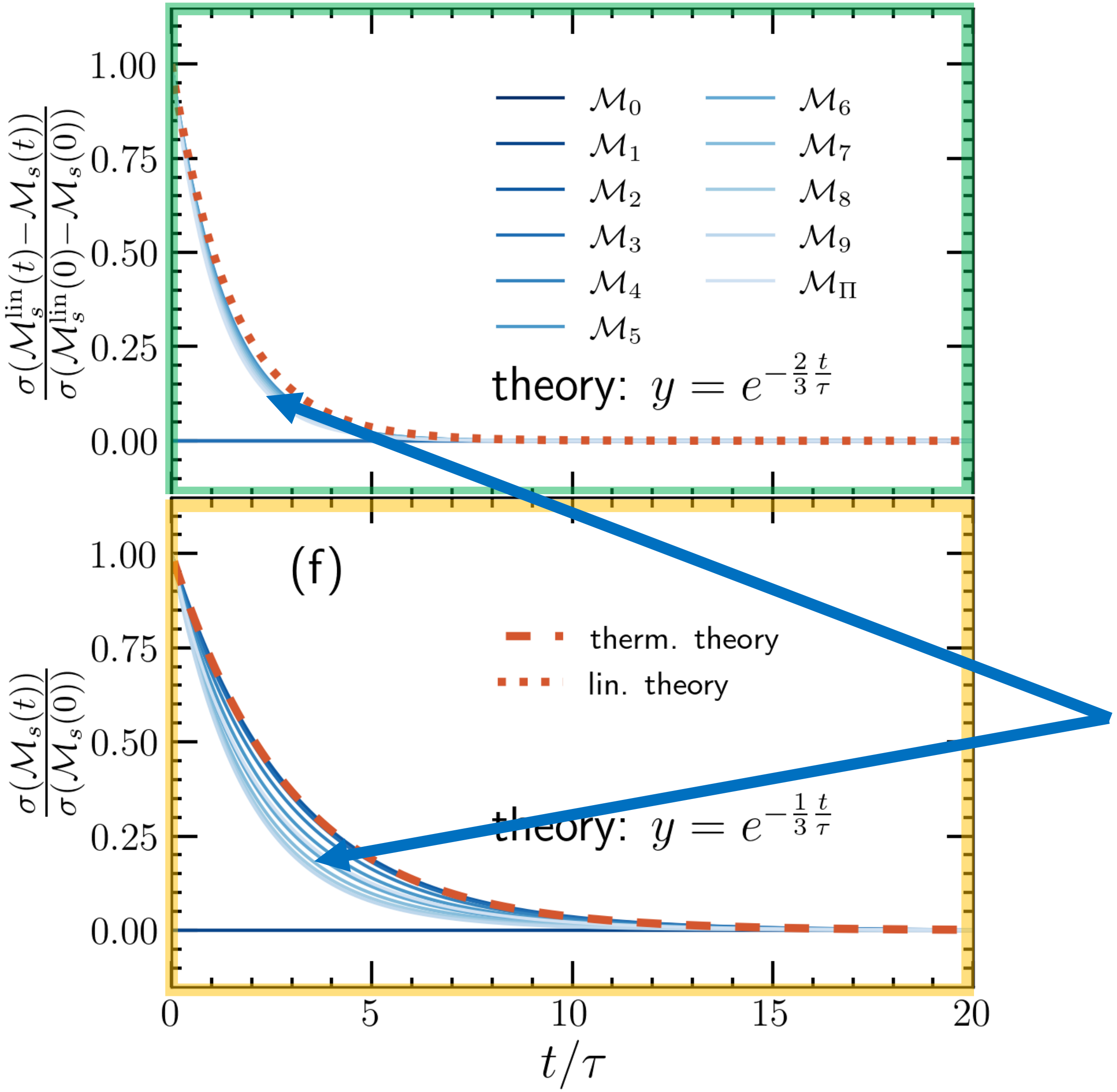


- Observables

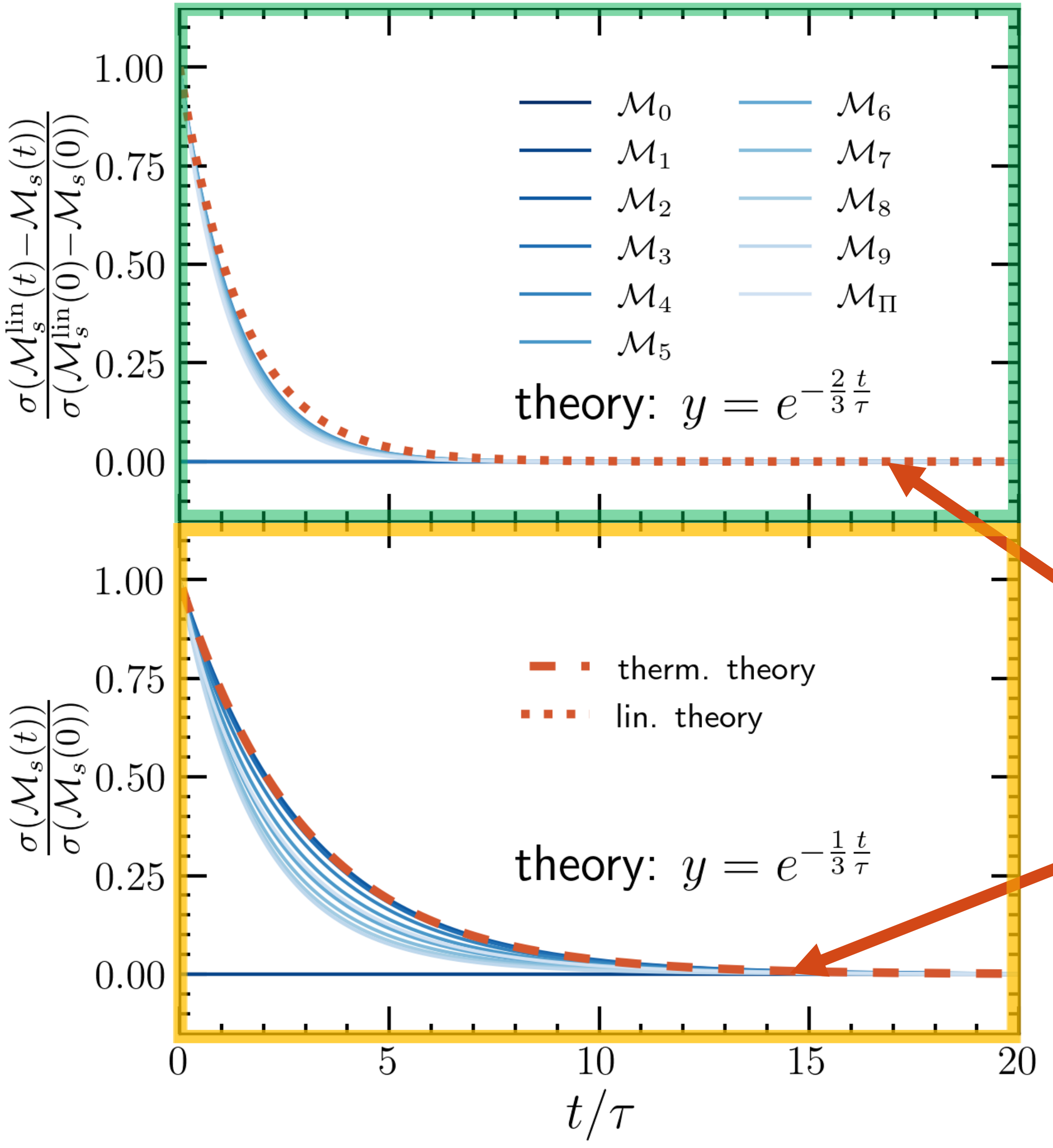
$$\mathcal{M}_\Pi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{E} f(t,\boldsymbol{p})$$

$$\mathcal{M}_i = \int \frac{d^3p}{(2\pi)^3} E^i f(t,\boldsymbol{p}) \quad i = 0,1,\cdots,10$$

- Different observables share **similar** thermalization and linearization times

$$\tau_{\rm lin}^{[\sigma(\mathcal{M}_\Pi)]} \approx \tau_{\rm lin}^{[\sigma(\mathcal{M}_s)]}, \quad \tau_{\rm therm}^{[\sigma(\mathcal{M}_\Pi)]} \approx \tau_{\rm therm}^{[\sigma(\mathcal{M}_s)]}$$

Hydrodynamization from the full Boltzmann Equation



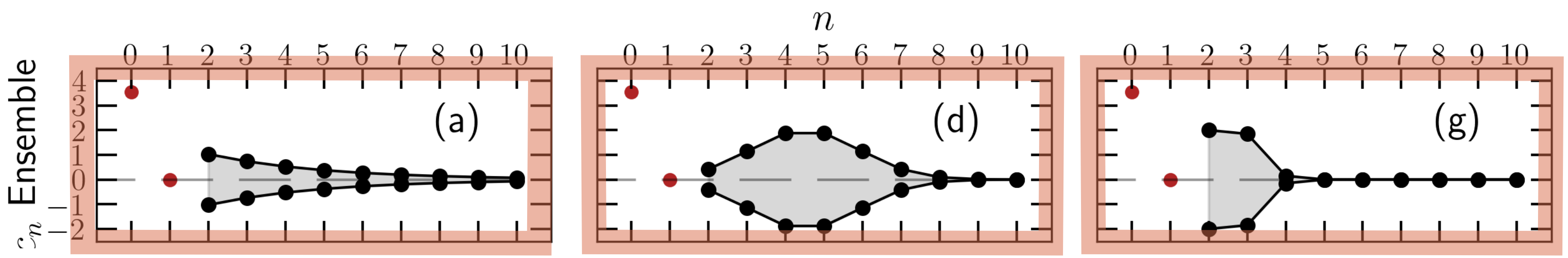
Both thermalization and linearization have slowest

theoretical limits,

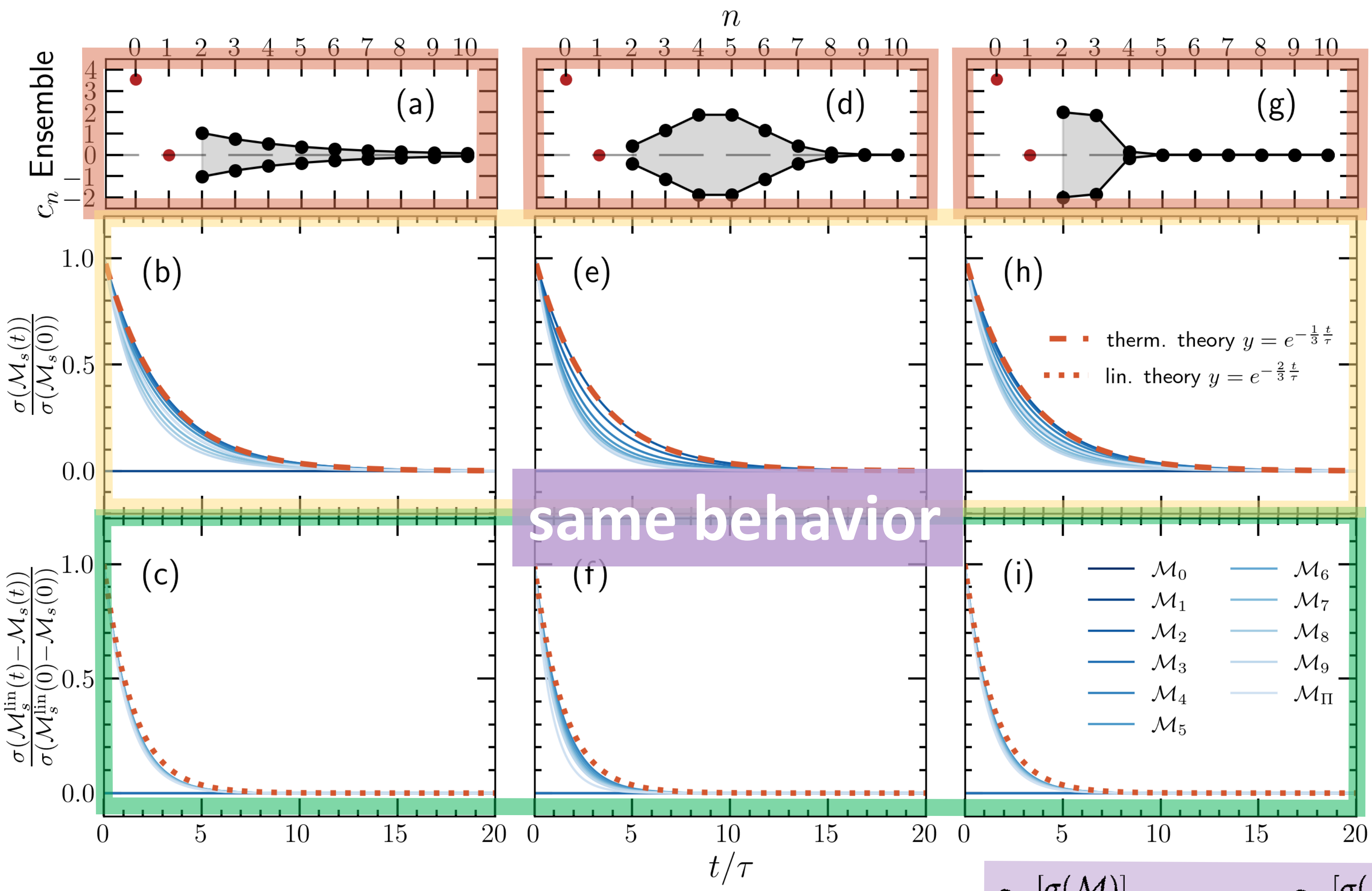
upper limit $\tau_{\rm lin}^{[\sigma(\mathcal{M}_s)]} = 1/2$ upper limit $\tau_{\rm therm}^{[\sigma(\mathcal{M}_s)]}$

look at the orange lines

Hydrodynamization from the full Boltzmann Equation



Hydrodynamization from the full Boltzmann Equation

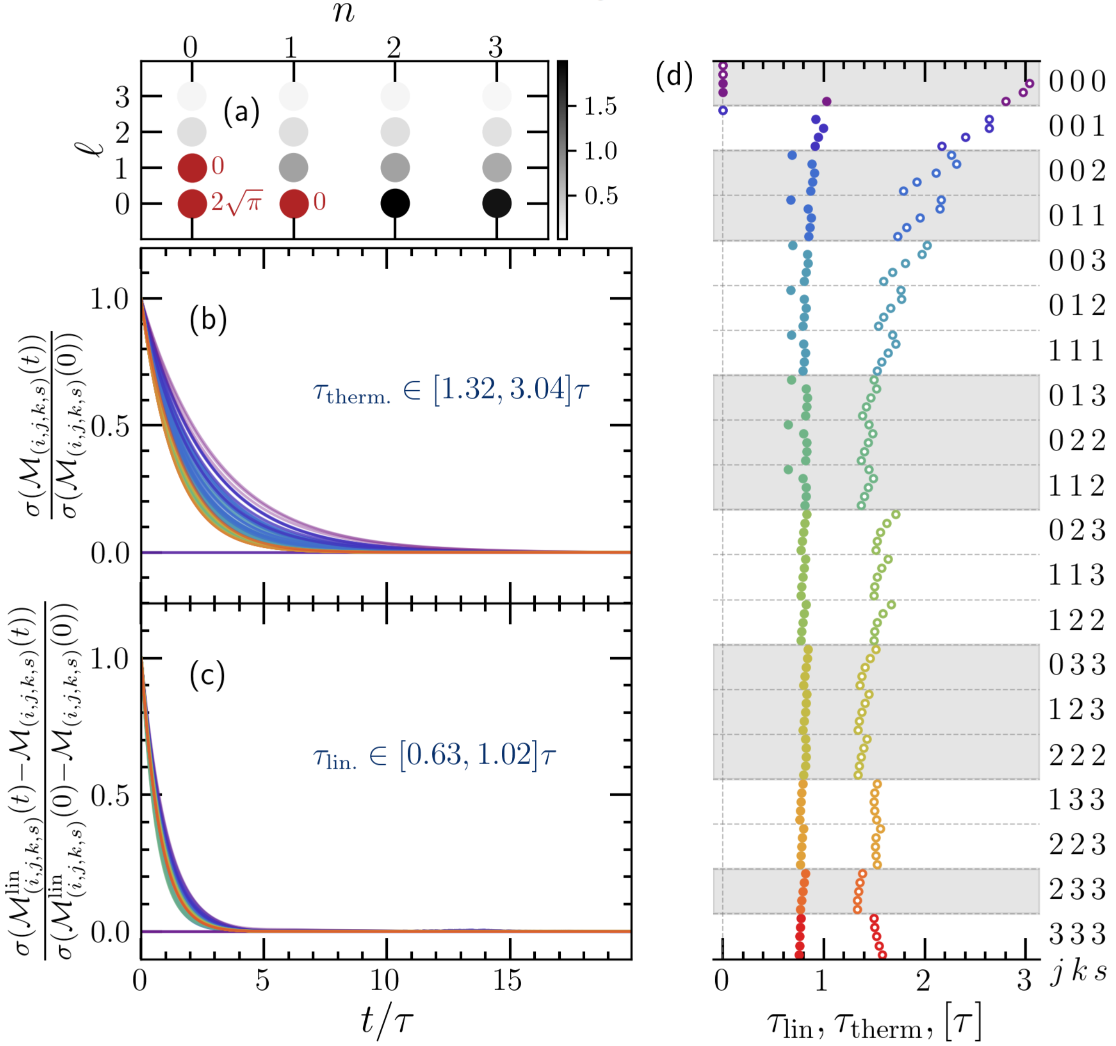


Independent of the initial distribution ensembles

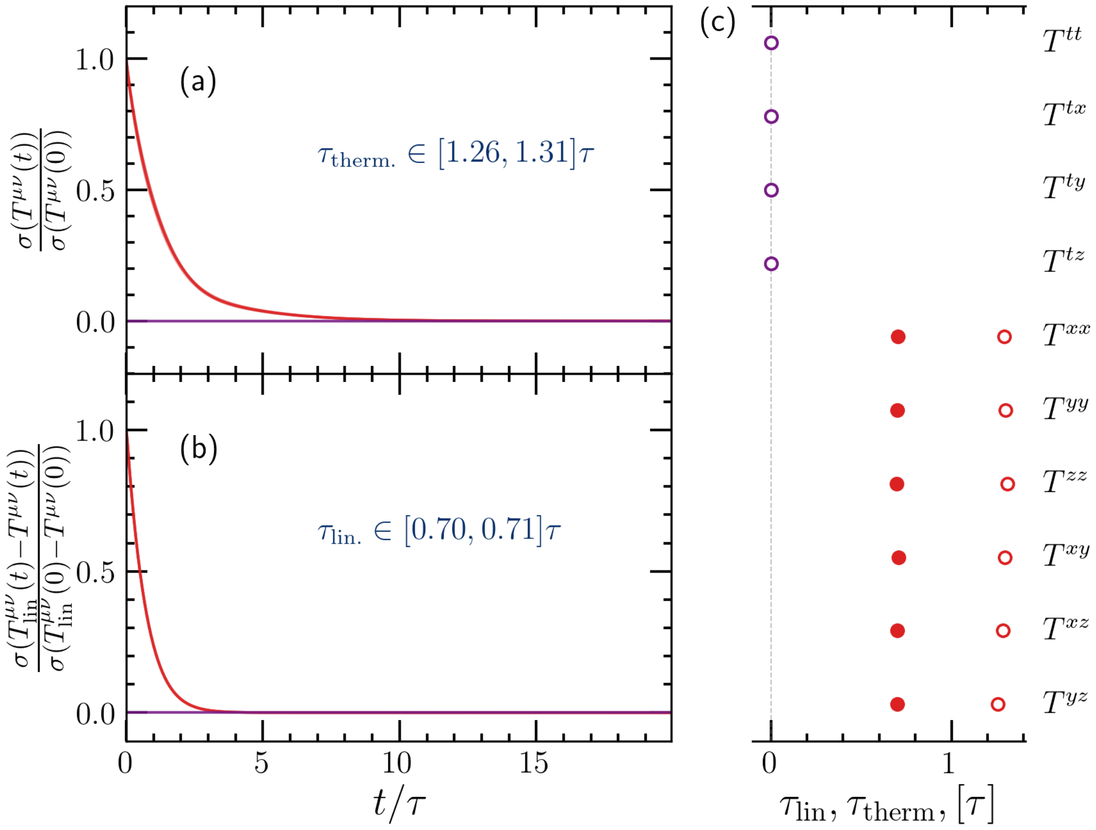
$$\frac{\delta\tau_{\rm therm}^{[\sigma(\mathcal{M})]}}{\delta f_{\rm init}} = 0, \quad \frac{\delta\tau_{\rm lin}^{[\sigma(\mathcal{M})]}}{\delta f_{\rm init}} = 0$$

Hydrodynamization from the full Boltzmann equation

Moment $\mathcal{M}_{i,j,k,s}(t) = \int d^3\boldsymbol{p}\, f(t,\boldsymbol{p})\,(E_{\boldsymbol{p}})^i (p^x)^j (p^y)^k (p^z)^s$



$T^{\mu\nu}$ $T^{\mu\nu} = \int_{\boldsymbol{p}} p^\mu p^\nu f(t,\boldsymbol{x},\boldsymbol{p})$

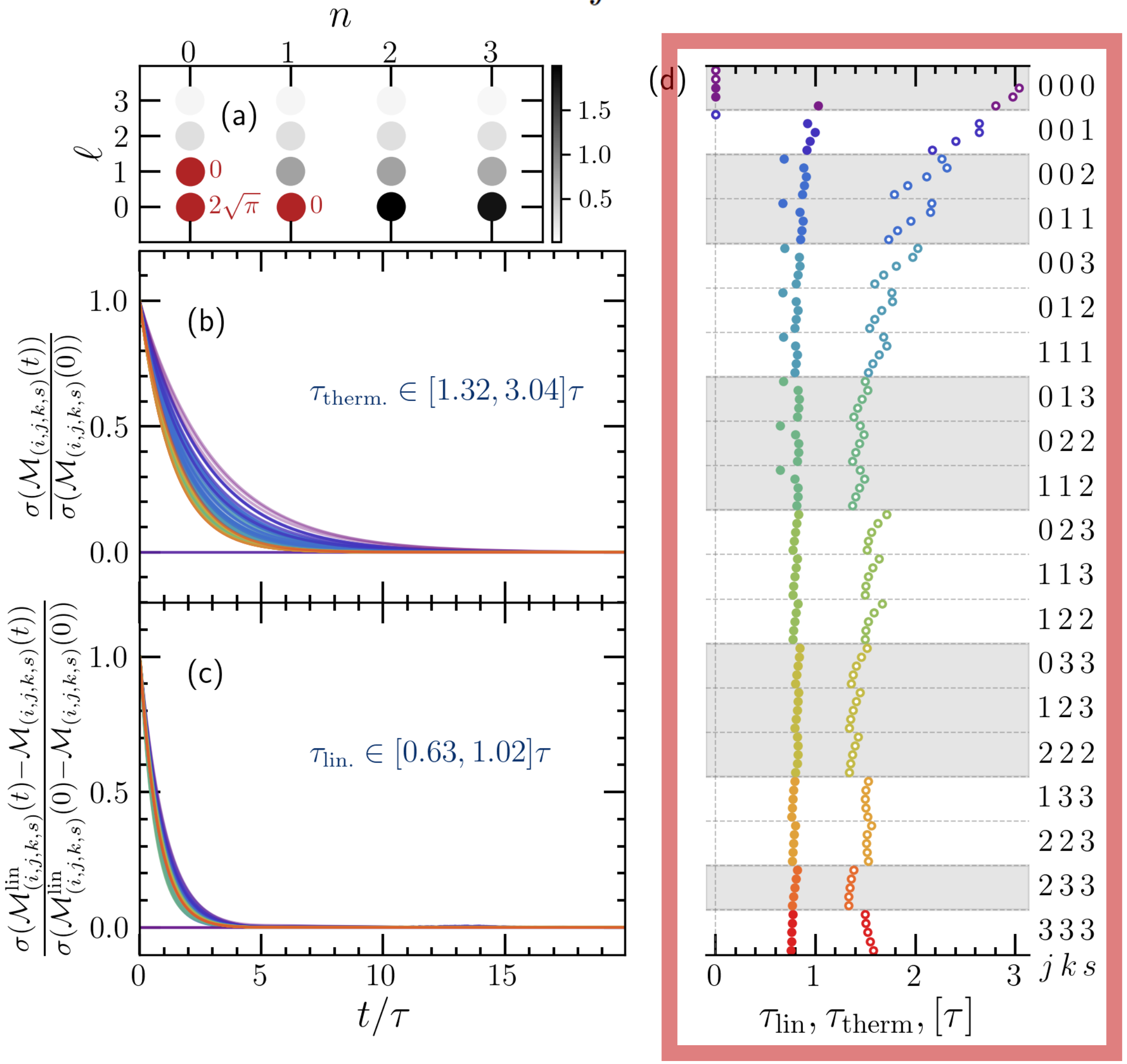


Consistent: $\tau_{lin} \approx 0.5\, \tau_{therm}$

Hydrodynamization from the full Boltzmann equation

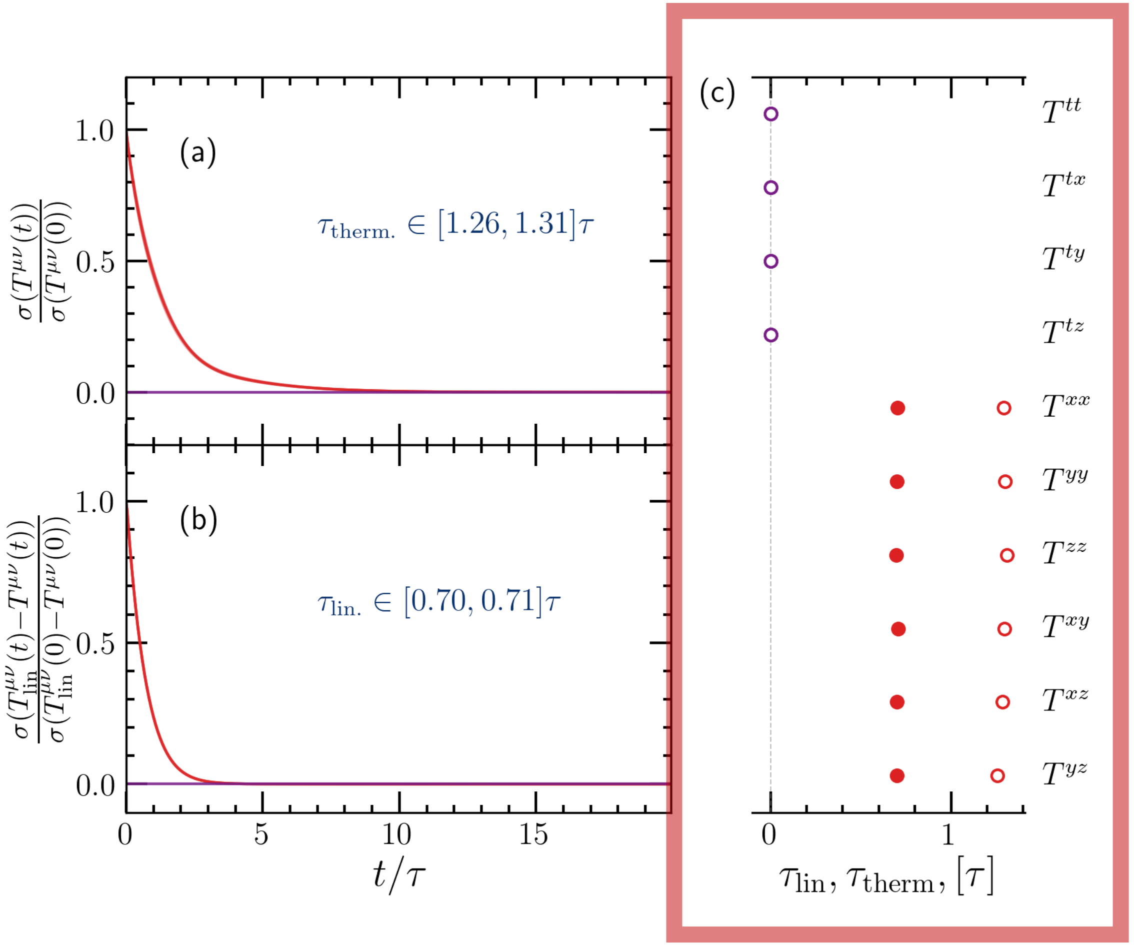
Moment

$$\mathcal{M}_{i,j,k,s}(t) = \int d^3\boldsymbol{p}\, f(t,\boldsymbol{p})\, (E_{\boldsymbol{p}})^i (p^x)^j (p^y)^k (p^z)^s$$



$T^{\mu\nu}$

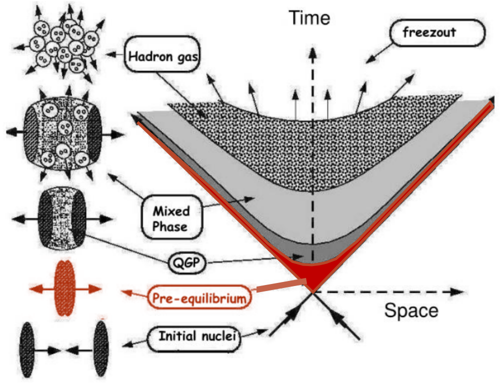
$$T^{\mu\nu} = \int_{\boldsymbol{p}} p^\mu p^\nu f(t,\boldsymbol{x},\boldsymbol{p})$$



Consistent: $\tau_{lin} \approx 0.5\, \tau_{therm}$

Summary

- New theoretical tool: **Spectral BBGKY Hierarchy**(arXiv:25(
    - **Fill the Gap: Correlation, Nonlinear**



- Physical Insight: Decoupling hydrodynamization from thermalization via **nonlinear** Boltzmann equation(arXiv:2509.23978)

$$\tau_{lin} \approx \tau_{therm}/2$$

Outlook

- The thermalization time and linearization time in an expanding system
- The correlation contribution to thermalization process
- $gg \leftrightarrow ggg$
- Bose Enhancement, Pauli Blocking

From Framework to Computation: How We Actually Solve It

momentum expansion

evaluation of the collision integral

BBGKY

Bogoliubov, Born, Green, Kirkwood, Yvon

Liouville Equation

$$\frac{\partial P_{(N)}(\{\boldsymbol{\phi}\})}{\partial t} + \sum_{i=1}^{N} \frac{\boldsymbol{p}_i}{p_i^0} \cdot \frac{\partial P_{(N)}(\{\boldsymbol{\phi}\})}{\partial \boldsymbol{x}_i} = \int d^{6N}\{\boldsymbol{\phi}'\} \Gamma_{\{\boldsymbol{\phi}\}\to\{\boldsymbol{\phi}'\}} \Big(P_{(N)}(\{\boldsymbol{\phi}'\}) - P_{(N)}(\{\boldsymbol{\phi}\}) \Big)$$

Interaction

$$\Gamma_{\{\boldsymbol{\phi}\}\to\{\boldsymbol{\phi}'\}} = \sum_{\substack{i,j=1 \\ i<j}}^{N} \mathcal{W}_{\boldsymbol{p}_i,\boldsymbol{p}_j\to\boldsymbol{p}'_i,\boldsymbol{p}'_j} \delta^{(3)}(\boldsymbol{x}_i - \boldsymbol{x}_j)\delta^{(3)}(\boldsymbol{x}'_i - \boldsymbol{x}'_j)\delta^{(3)}(\boldsymbol{x}_i - \boldsymbol{x}'_i) \left(\prod_{k\neq i,j} \delta^{(6)}(\boldsymbol{\phi}_k - \boldsymbol{\phi}'_k) \right)$$

BBGKY

$$\frac{\partial P_{(n)}(\{\boldsymbol{\phi}\})}{\partial t} + \sum_{i=1}^{n} \frac{\boldsymbol{p}_i}{p_i^0} \cdot \frac{\partial P_{(n)}(\{\boldsymbol{\phi}\})}{\partial \boldsymbol{x}_i}$$

$$= \sum_{\substack{i,j=1 \\ i<j}}^{n} \delta^{(3)}(\boldsymbol{x}_i - \boldsymbol{x}_j) \frac{1}{p_i^0 p_j^0} \int_{\boldsymbol{p}'_i,\boldsymbol{p}'_j} \mathcal{W}_{\boldsymbol{p}_i,\boldsymbol{p}_j\to\boldsymbol{p}'_i,\boldsymbol{p}'_j} \left(P_{(n)}(\boldsymbol{p}'_i, \boldsymbol{p}'_j, \cdots) - P_{(n)}(\boldsymbol{p}_i, \boldsymbol{p}_j, \cdots) \right)$$

$$+ (N-n) \sum_{i=1}^{n} \frac{1}{p_i^0} \int_{\boldsymbol{p}'_i,\boldsymbol{p}'_j,\boldsymbol{p}_j} \mathcal{W}_{\boldsymbol{p}_i,\boldsymbol{p}_j\to\boldsymbol{p}'_i,\boldsymbol{p}'_j} \left(P_{(n+1)}(\boldsymbol{p}'_i, \cdots; \boldsymbol{p}'_j) - P_{(n+1)}(\boldsymbol{p}_i, \cdots; \boldsymbol{p}_j) \right)$$

Spectral BBGKY

Spectral BBGKY

$$\frac{\partial}{\partial t}\boldsymbol{P}^{t_1 t_2 \cdots t_n}(t, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n)$$
$$+ \sum_{j=1}^{n} \boldsymbol{B}_{t_j h_j} \cdot \frac{\partial}{\partial \boldsymbol{x}_j} \boldsymbol{P}^{t_1 t_2 \cdots h_j \cdots t_n}(t, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n)$$
$$= \sum_{\substack{i,j=1 \\ i<j}}^{n} \delta^{(3)}(\boldsymbol{x}_i - \boldsymbol{x}_j) C_{t_i t_j h_i h_j} \boldsymbol{P}^{t_1 t_2 \cdots h_i \cdots h_j \cdots t_n}(t, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n)$$
$$+ (N-n) \sum_{i=1}^{n} A_{t_i h_i h_j} \boldsymbol{P}^{t_1 t_2 \cdots h_i \cdots t_n h_j}(t, \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_i, \cdots, \boldsymbol{x}_n, \boldsymbol{x}_i)$$

$$\boldsymbol{B}_{ij} = \int_{p_\mu} \mathcal{Q}_i(p_\mu)\mathcal{P}_j(p_\mu)\frac{\boldsymbol{p}}{p^0}$$

$$A_{ijk} = A^{\text{gain}}_{ijk} - A^{\text{loss}}_{ijk}\,,$$
$$C_{ijks} = C^{\text{gain}}_{ijks} - C^{\text{loss}}_{ijks}\,,$$

$$A^{\text{gain}}_{ijk} = \int_{p_{1\mu}} \int_{\boldsymbol{p}_2,\boldsymbol{p}_3,\boldsymbol{p}_4} \mathcal{Q}_i(p_{1\mu}) \frac{1}{p_1^0} \mathcal{W}_{(\boldsymbol{p}_1,\boldsymbol{p}_2 \to \boldsymbol{p}_3,\boldsymbol{p}_4)} \mathcal{P}_j(p'_{1\mu}) \mathcal{P}_k(p'_{2\mu})\,,$$
$$A^{\text{loss}}_{ijk} = \int_{p_{1\mu}} \int_{\boldsymbol{p}_2,\boldsymbol{p}_3,\boldsymbol{p}_4} \mathcal{Q}_i(p_{1\mu}) \frac{1}{p_1^0} \mathcal{W}_{(\boldsymbol{p}_1,\boldsymbol{p}_2 \to \boldsymbol{p}_3,\boldsymbol{p}_4)} \mathcal{P}_j(p_{1\mu}) \mathcal{P}_k(p_{2\mu})\,,$$
$$C^{\text{gain}}_{ijks} = \int_{p_{1\mu},p_{2\mu}} \int_{\boldsymbol{p}_3,\boldsymbol{p}_4} \mathcal{Q}_i(p_{1\mu}) \mathcal{Q}_j(p_{2\mu}) \frac{1}{p_1^0 p_2^0} \mathcal{W}_{(\boldsymbol{p}_1,\boldsymbol{p}_2 \to \boldsymbol{p}_3,\boldsymbol{p}_4)} \mathcal{P}_j(p'_{1\mu}) \mathcal{P}_k(p'_{2\mu})$$
$$C^{\text{loss}}_{ijks} = \int_{p_{1\mu},p_{2\mu}} \int_{\boldsymbol{p}_3,\boldsymbol{p}_4} \mathcal{Q}_i(p_{1\mu}) \mathcal{Q}_j(p_{2\mu}) \frac{1}{p_1^0 p_2^0} \mathcal{W}_{(\boldsymbol{p}_1,\boldsymbol{p}_2 \to \boldsymbol{p}_3,\boldsymbol{p}_4)} \mathcal{P}_j(p_{1\mu}) \mathcal{P}_k(p_{2\mu})$$

$$\mathcal{P}_{n,\ell,m}(p_\mu) = e^{-p_\mu u^\mu/\Lambda} \left(\frac{p_\mu u^\mu}{\Lambda}\right)^{\ell} Y_{\ell,m}(\theta,\phi) L_n^{(2\ell+2)}\left(\frac{p_\mu u^\mu}{\Lambda}\right)$$
$$\mathcal{Q}_{n,\ell,m}(p_\mu) = \frac{n!}{\Gamma(2l+n+3)} \left(\frac{p_\mu u^\mu}{\Lambda}\right)^{\ell+2} Y_{\ell,m}(\theta,\phi) L_n^{(2l+2)}\left(\frac{p_\mu u^\mu}{\Lambda}\right)$$

Minimizing Collision Integral Evaluations A_{ijk}, C_{ijks}

- Reduction relationship

$$C_{i0\langle ks\rangle} = C_{0i\langle ks\rangle} = \frac{(2\pi)^3}{\sqrt{4\pi}}\frac{1}{T^3}A_{i\langle ks\rangle}$$

- Parity symmetry

$$\mathrm{P}\,Y_\ell^m(\theta,\varphi)\,\mathrm{P} \;=\; (-1)^\ell Y_\ell^m(\theta,\varphi) \longrightarrow C_{ijks} = 0, \quad \text{if } \ell_i + \ell_j + \ell_k + \ell_s \text{ is odd}$$

- Rotational invariance

$$Y_\ell^m(\theta',\varphi') = \sum_{m'=-\ell}^{\ell}\left[D_{mm'}^{(\ell)}(\mathcal{R})\right]^* Y_\ell^{m'}(\theta,\varphi) \longrightarrow 0 = (\mathcal{D}_{ijks,i'j'k's'} - I)C_{i'j'k's'}$$

$$Y_\ell^m(\theta,\varphi+\Delta\varphi) = e^{im\Delta\varphi}Y_\ell^m(\theta,\varphi) \longrightarrow C_{ijks} = 0, \quad \text{if } m_i + m_j \neq m_k + m_s$$

$$\boldsymbol{n} \in \{0,1,2\}, \boldsymbol{l} \in \{0,1,2,3\} \quad 2\times 48^4 = 1.06\times 10^7 \xrightarrow{0.2\%} 26244$$

Integral Form of the Collision Kernel

Gernal Cases

8 Fold

↓

3 Fold

$$
\begin{aligned}
&A_{n\ell m,n_1\ell_1 m_1,n_2\ell_2 m_2,n_3\ell_3 m_3,n_4\ell_4 m_4}\\
&= \frac{n!}{\Gamma(2\ell+n+3)} \frac{S^{*(m)}_{(\mu_1\cdots\mu_\ell)} S^{(m_1)}_{(\alpha_1\cdots\alpha_{\ell_1})} S^{(m_2)}_{(\varepsilon_1\cdots\varepsilon_{\ell_2})} S^{(m_3)}_{(\omega_1\cdots\omega_{\ell_3})} S^{(m_4)}_{(\lambda_1\cdots\lambda_{\ell_4})}}{(2\pi)^5\Lambda^{\ell+\ell_1+\ell_2+\ell_3+\ell_4+5}}\\
&\quad\times \int d|\boldsymbol{P}||\boldsymbol{P}|^2 \int_{|\boldsymbol{P}|}^{\infty} dP^0 P^2 e^{-2P^\mu u_\mu/\Lambda} \frac{\sqrt{-Q^2}}{2P} \frac{\sqrt{-Q'^2}}{2P} \sum_{g=0}^{\infty} \sigma^{(g)}\left(\frac{P}{\Lambda}\right) \frac{4\pi}{2g+1} \sum_{h=-g}^{g}\\
&\quad\times \Big[\int d\Omega_P \Big(\prod_{i=1}^{\ell} \mathcal{M}^{\mu^i}_{\nu_i}\Big)\Big(\prod_{i=1}^{\ell_1} \mathcal{M}^{\alpha^i}_{\beta_i}\Big)\Big(\prod_{i=1}^{\ell_2} \mathcal{M}^{\varepsilon^i}_{\zeta_i}\Big)\Big(\prod_{i=1}^{\ell_3} \mathcal{M}^{\omega^i}_{\kappa_i}\Big)\Big(\prod_{i=1}^{\ell_4} \mathcal{M}^{\lambda^i}_{\rho_i}\Big)\Big]\\
&\quad\times \int_0^{\pi} \sin\theta\, d\theta \left(\frac{p_{1\mu}u^\mu}{|\boldsymbol{p}_1|}\right)^{\ell+\ell_1+1} \left(\frac{p_{2\mu}u^\mu}{|\boldsymbol{p}_2|}\right)^{\ell_2} L_n^{(2\ell+2)}\left(\frac{p_{1\mu}u^\mu}{\Lambda}\right) L_{n_1}^{(2\ell_1+2)}\left(\frac{p_{1\mu}u^\mu}{\Lambda}\right) L_{n_2}^{(2\ell_2+2)}\left(\frac{p_{2\mu}u^\mu}{\Lambda}\right)\\
&\quad\times \int_0^{\pi} \sin\theta'\, d\theta' \left(\frac{p_{3\mu}u^\mu}{|\boldsymbol{p}_3|}\right)^{\ell_3} \left(\frac{p_{4\mu}u^\mu}{|\boldsymbol{p}_4|}\right)^{\ell_4} L_{n_3}^{(2\ell_3+2)}\left(\frac{p_{3\mu}u^\mu}{\Lambda}\right) L_{n_4}^{(2\ell_4+2)}\left(\frac{p_{4\mu}u^\mu}{\Lambda}\right)\\
&\quad\times \Big[\int_0^{2\pi} d\phi \Big(\prod_{i=1}^{\ell} p_1^{\nu_k}\Big)\Big(\prod_{i=1}^{\ell_1} p_1^{\beta_i}\Big)\Big(\prod_{i=1}^{\ell_2} p_2^{\zeta_i}\Big) Y_g^{*h}(\theta,\phi)\Big]\Big[\int_0^{2\pi} d\phi' \Big(\prod_{i=1}^{\ell_3} p_3^{\kappa_i}\Big)\Big(\prod_{i=1}^{\ell_4} p_4^{\rho_i}\Big) Y_g^{h}(\theta',\phi')\Big].
\end{aligned}
$$

Integral Form of the Collision Kernel

Gernal Cases

8 Fold

↓

3 Fold

$$
\begin{aligned}
& C_{n_a l_a m_a, n_b l_b m_b, n_1 \ell_1 m_1, n_2 \ell_2 m_2, n_3 \ell_3 m_3, n_4 \ell_4 m_4} \\
&= \frac{n_a!}{\Gamma(2\ell_a + n_a + 3)} \frac{n_b!}{\Gamma(2\ell_b + n_b + 3)} \frac{S^{*(m_a)}_{(\mu_1 \cdots \mu_{\ell_a})} S^{*(m_b)}_{(\eta_1 \cdots \eta_{\ell_b})} S^{(m_1)}_{(\alpha_1 \cdots \alpha_{\ell_1})} S^{(m_2)}_{(\varepsilon_1 \cdots \varepsilon_{\ell_2})} S^{(m_3)}_{(\omega_1 \cdots \omega_{\ell_3})} S^{(m_4)}_{(\lambda_1 \cdots \lambda_{\ell_4})}}{(2\pi)^2 \, \Lambda^{\ell_a + \ell_b + \ell_1 + \ell_2 + \ell_3 + \ell_4 + 8}} \\
& \times \int d|\boldsymbol{P}| |\boldsymbol{P}|^2 \int_{|\boldsymbol{P}|}^{\infty} dP^0 P^2 e^{-2P^\mu u_\mu / \Lambda} \frac{\sqrt{-Q^2}}{2P} \frac{\sqrt{-Q'^2}}{2P} \sum_{g=0}^{\infty} \sigma^{(g)} \left(\frac{P}{\Lambda} \right) \frac{4\pi}{2g+1} \sum_{h=-g}^{g} \\
& \times \left[\int d\Omega_P \left(\prod_{k=1}^{\ell_a} \mathcal{M}^{\mu^k}_{\nu_k} \right) \left(\prod_{k=1}^{\ell_b} \mathcal{M}^{\eta^k}_{\chi_k} \right) \left(\prod_{k=1}^{\ell_1} \mathcal{M}^{\alpha^k}_{\beta_k} \right) \left(\prod_{k=1}^{\ell_2} \mathcal{M}^{\varepsilon^k}_{\zeta_k} \right) \left(\prod_{k=1}^{\ell_3} \mathcal{M}^{\omega^k}_{\kappa_k} \right) \left(\prod_{k=1}^{\ell_4} \mathcal{M}^{\lambda^k}_{\rho_k} \right) \right] \\
& \times \int_0^\pi \sin\theta \, d\theta \left(\frac{p_{1\mu} u^\mu}{|\boldsymbol{p}_1|} \right)^{\ell_a + \ell_1 + 1} \left(\frac{p_{2\mu} u^\mu}{|\boldsymbol{p}_2|} \right)^{\ell_b + \ell_2 + 1} \\
& \quad \times L^{(2\ell_a + 2)}_{n_a} \left(\frac{p_{1\mu} u^\mu}{\Lambda} \right) L^{(2\ell_b + 2)}_{n_b} \left(\frac{p_{2\mu} u^\mu}{\Lambda} \right) L^{(2\ell_1 + 2)}_{n_1} \left(\frac{p_{1\mu} u^\mu}{\Lambda} \right) L^{(2\ell_2 + 2)}_{n_2} \left(\frac{p_{2\mu} u^\mu}{\Lambda} \right) \\
& \times \int_0^\pi \sin\theta' \, d\theta' \left(\frac{p_{3\mu} u^\mu}{|\boldsymbol{p}_3|} \right)^{\ell_3} \left(\frac{p_{4\mu} u^\mu}{|\boldsymbol{p}_4|} \right)^{\ell_4} L^{(2\ell_3 + 2)}_{n_3} \left(\frac{p_{3\mu} u^\mu}{\Lambda} \right) L^{(2\ell_4 + 2)}_{n_4} \left(\frac{p_{4\mu} u^\mu}{\Lambda} \right) \\
& \times \left[\int_0^{2\pi} d\phi \left(\prod_{k=1}^{\ell_i} p_1^{\nu_k} \right) \left(\prod_{k=1}^{\ell_j} p_2^{\chi_k} \right) \left(\prod_{i=1}^{\ell_1} p_1^{\beta_i} \right) \left(\prod_{i=1}^{\ell_2} p_2^{\zeta_i} \right) Y^{*h}_g(\theta, \phi) \right] \\
& \times \left[\int_0^{2\pi} d\phi' \left(\prod_{i=1}^{\ell_3} p_3^{\kappa_i} \right) \left(\prod_{i=1}^{\ell_4} p_4^{\rho_i} \right) Y^h_g(\theta', \phi') \right].
\end{aligned}
$$

Summation Form of the Collision Kernel

$$
\begin{aligned}
&A_{n_a\ell_a0,n_1\ell_10,n_2\ell_20,n_3\ell_30,n_4\ell_40}\\
&= \frac{n_a!}{\Gamma(2l_a+n_a+3)} \prod_{i\in\{a,1,2,3,4\}} \sqrt{2\ell_i+1}\\
&\sum_{w'_1=0}^{\lfloor\frac{\ell_a}{2}\rfloor+\lfloor\frac{\ell_1}{2}\rfloor} \sum_{\eta'_1=0}^{w'_1} \sum_{\epsilon'_1=0}^{w'_1-\eta'_1} \sum_{f'_1=0}^{n_a+n_1} \Big[\prod_{i\in\{2,3,4\}} \sum_{w_i=0}^{\lfloor\frac{\ell_i}{2}\rfloor} \sum_{\eta_i=0}^{w_i} \sum_{\epsilon_i=0}^{w_i-\eta_i} \sum_{f_i=0}^{n_i} \Big] \sum_{t'_{12}=0}^{f'_1+f_2} \sum_{t_{34}=0}^{f_3+f_4}\\
&\sum_{x'_{i,12},x'_{j,12},x'_{k,12}=0}^{\substack{x'_{i,12}+x'_{j,12}+x'_{k,12}\le\\ 2\eta'_{12}}} \sum_{y'_{i,12},y'_{j,12},y'_{k,12}=0}^{\substack{y'_{i,12}+y'_{j,12}+y'_{k,12}\le\\ 2\epsilon'_{12}}} \sum_{z'_{i,12},z'_{k,12}=0}^{\substack{z'_{i,12}+z'_{k,12}\le\\ \ell'_{12}-2\eta'_{12}-2\epsilon'_{12}}}\\
&\sum_{x_{i,34},x_{j,34},x_{k,34}=0}^{\substack{x_{i,34}+x_{j,34}+x_{k,34}\le\\ 2\eta_{34}}} \sum_{y_{i,34},y_{j,34},y_{k,34}=0}^{\substack{y_{i,34}+y_{j,34}+y_{k,34}\le\\ 2\epsilon_{34}}} \sum_{z_{i,34},z_{k,34}=0}^{\substack{z_{i,34}+z_{k,34}\le\\ \ell_{34}-2\eta_{34}-2\epsilon_{34}}}\\
&\times \mathcal{I}_\alpha(w'_1,\eta'_1,\epsilon'_1,0,\ell_a,0,\ell_1,0) \prod_{i\in\{2,3,4\}} \binom{2\ell_i-2w_i}{\ell_i-w_i,\ell_i-2w_i,\eta_i,\epsilon_i,w_i-\eta_i-\epsilon_i}\\
&\times \mathcal{I}_\beta(x'_{i,12},x'_{j,12},x'_{k,12},2\eta'_1,2\eta_2)\\
&\times \mathcal{I}_\beta(y'_{i,12},y'_{j,12},y'_{k,12},2\epsilon'_1,2\epsilon_2)\\
&\times \mathcal{I}_\beta(z'_{i,12},0,z'_{k,12},\ell'_1-2\eta'_1-2\epsilon'_1,\ell_2-2\eta_2-2\epsilon_2)\\
&\times \mathcal{I}_\beta(x_{i,34},x_{j,34},x_{k,34},2\eta_3,2\eta_4)\\
&\times \mathcal{I}_\beta(y_{i,34},y_{j,34},y_{k,34},2\epsilon_3,2\epsilon_4)\\
&\times \mathcal{I}_\beta(z_{i,34},0,z_{k,34},\ell_3-2\eta_3-2\epsilon_3,\ell_4-2\eta_4-2\epsilon_4)\\
&\times \mathcal{I}_\gamma(f'_1,n_a,\ell_a,n_1,\ell_1)\Big[\prod_{i\in\{2,3,4\}}\binom{n_i+2\ell_i+2}{n_i-f_i}\frac{1}{f_i!}\Big]\\
&\times \mathcal{I}_\beta(t'_{12},0,0,f'_1,f_2)\mathcal{I}_\beta(t_{34},0,0,f_3,f_4)\\
&\times \mathcal{S}_A(t'_{12},r'_{k,12},r'_{i,12},r'_{j,12},t_{34},r_{k,34},r_{i,34},r_{j,34},\ell_s)\\
&\times (-1)^{w_s-x_{i,s}-y_{i,s}-y_{j,s}+f_s}\frac{1}{2^{\ell_s+10}}\frac{1}{\pi^4\sqrt{\pi}}\Lambda^1\sigma^{(0)}4^{-6-\ell_s-f_s}\\
&\times \frac{\left(t'_{12}+r'_{k,12}-1\right)!!\left(r'_{i,12}-1\right)!!\left(r'_{j,12}-1\right)!!}{\left(t'_{12}+r'_{i,12}+r'_{j,12}+r'_{k,12}+1\right)!!}\frac{(t_{34}+r_{k,34}-1)!!\,(r_{i,34}-1)!!\,(r_{j,34}-1)!!}{(t_{34}+r_{i,34}+r_{j,34}+r_{k,34}+1)!!}\\
&\times (5+\ell_s+f_s)!\frac{(r_{i,s}+r_{j,s}+2)!!\,(1-r_{i,s}-r_{j,s}-r_{k,s}+\ell_s+t_s)!!}{(5-r_{k,s}+\ell_s+t_s)!!}\\
&\times \frac{(z_{i,s}+2\eta_s-x_{i,s}-x_{j,s}+2\epsilon_s-y_{i,s}-y_{j,s})!!\,(x_{i,s}+y_{i,s}+\ell_s-2\epsilon_s-2\eta_s-z_{i,s}-1)!!}{(1+\ell_s-x_{j,s}-y_{j,s})!!}\\
&\times \frac{(x_{j,s}+2\epsilon_s-y_{j,s}-1)!!\,(y_{j,s}+2\eta_s-x_{j,s}-1)!!}{(2\epsilon_s+2\eta_s)!!},
\end{aligned}
$$

(C.8)

A: 34 Fold

Finite sum

← **massless particles** →

$$
\begin{aligned}
&C_{n_al_am_a,n_bl_bm_b,n_1\ell_1m_1,n_2\ell_2m_2,n_3\ell_3m_3,n_4\ell_4m_4}\\
&= \prod_{i\in\{a,b\}}\frac{n_i!}{\Gamma(2l_i+n_i+3)}\prod_{i\in\{a,b,1,2,3,4\}}\sqrt{2\ell_i+1}\left[\frac{(\ell_i-|m_i|)!}{(\ell_i+|m_i|)!}\right]^{1/2}\mathcal{F}_\alpha(m_i)\\
&\sum_{w'_1=0}^{\lfloor\frac{\ell_a-|m_a|}{2}\rfloor+\lfloor\frac{\ell_1-|m_1|}{2}\rfloor}\sum_{p'_1=0}^{|m_a|+|m_1|}\sum_{w'_2=0}^{\lfloor\frac{\ell_b-|m_b|}{2}\rfloor+\lfloor\frac{\ell_2-|m_2|}{2}\rfloor}\sum_{p'_2=0}^{|m_b|+|m_2|}\sum_{f'_1=0}^{n_a+n_1}\sum_{f'_2=0}^{n_b+n_2}\Big[\prod_{i\in\{1,2\}}\sum_{\eta'_i=0}^{w'_i}\sum_{\epsilon'_i=0}^{w'_i-\eta'_i}\Big]\sum_{t''_{12}=0}^{f'_1+f'_2}\\
&\Big[\prod_{i\in\{3,4\}}\sum_{w_i=0}^{\lfloor\frac{\ell_i-|m_i|}{2}\rfloor}\sum_{p_i=0}^{|m_i|}\sum_{\eta_i=0}^{w_i}\sum_{\epsilon_i=0}^{w_i-\eta_i}\sum_{f_i=0}^{n_i}\Big]\sum_{t_{34}=0}^{f_3+f_4}\\
&\sum_{x''_{i,12},x''_{j,12},x''_{k,12}=0}^{\substack{x''_{i,12}+x''_{j,12}+x''_{k,12}\le\\ 2\eta''_{12}+p''_{12}}}\sum_{y''_{i,12},y''_{j,12},y''_{k,12}=0}^{\substack{y''_{i,12}+y''_{j,12}+y''_{k,12}\le\\ 2\epsilon''_{12}+|m|''_{12}-p''_{12}}}\sum_{z''_{i,12},z''_{k,12}=0}^{\substack{z''_{i,12}+z''_{k,12}\le\\ \ell''_{12}-|m|''_{12}-2\eta''_{12}-2\epsilon''_{12}}}\\
&\sum_{x_{i,34},x_{j,34},x_{k,34}=0}^{\substack{x_{i,34}+x_{j,34}+x_{k,34}\le\\ 2\eta_{34}+p_{34}}}\sum_{y_{i,34},y_{j,34},y_{k,34}=0}^{\substack{y_{i,34}+y_{j,34}+y_{k,34}\le\\ 2\epsilon_{34}+|m|_{34}-p_{34}}}\sum_{z_{i,34},z_{k,34}=0}^{\substack{z_{i,34}+z_{k,34}\le\\ \ell_{34}-|m|_{34}-2\eta_{34}-2\epsilon_{34}}}\\
&\times\mathcal{I}_\alpha(w'_1,\eta'_1,\epsilon'_1,p'_1,\ell_a,m_a,\ell_1,m_1)\mathcal{I}_\alpha(w'_2,\eta'_2,\epsilon'_2,p'_2,\ell_b,m_b,\ell_2,m_2)\\
&\prod_{i\in\{3,4\}}\binom{2\ell_i-2w_i}{\ell_i-w_i,\ell_i-2w_i-|m_i|,p_i,|m_i|-p_i,\eta_i,\epsilon_i,w_i-\eta_i-\epsilon_i}|m_i|!e^{\mathcal{F}_\beta(m_i)\times\mathrm{i}\left((|m_i|-p_i)\frac{\pi}{2}\right)}\\
&\times\mathcal{I}_\beta(x''_{i,12},x''_{j,12},x''_{k,12},2\eta'_1+p'_1,2\eta'_2+p'_2)\\
&\times\mathcal{I}_\beta(y''_{i,12},y''_{j,12},y''_{k,12},2\epsilon'_1+|m|'_1-p'_1,2\epsilon'_2+|m|'_2-p'_2)\\
&\times\mathcal{I}_\beta(z''_{i,12},0,z''_{k,12},\ell'_1-|m|'_1-2\eta'_1-2\epsilon'_1,\ell'_2-|m|'_2-2\eta'_2-2\epsilon'_2)\\
&\times\mathcal{I}_\beta(x_{i,34},x_{j,34},x_{k,34},2\eta_3+p_3,2\eta_4+p_4)\\
&\times\mathcal{I}_\beta(y_{i,34},y_{j,34},y_{k,34},2\epsilon_3+|m_3|-p_3,2\epsilon_4+|m_4|-p_4)\\
&\times\mathcal{I}_\beta(z_{i,34},0,z_{k,34},\ell_3-|m_3|-2\eta_3-2\epsilon_3,\ell_4-|m_4|-2\eta_4-2\epsilon_4)\\
&\times\mathcal{I}_\gamma(f'_1,n_a,\ell_a,n_1,\ell_1)\mathcal{I}_\gamma(f'_2,n_b,\ell_b,n_2,\ell_2)\Big[\prod_{i\in\{3,4\}}\binom{n_i+2\ell_i+2}{n_i-f_i}\frac{1}{f_i!}\Big]\\
&\times\mathcal{I}_\beta(t''_{12},0,0,f'_1,f'_2)\mathcal{I}_\beta(t_{34},0,0,f_3,f_4)\\
&\times\mathcal{S}_C(t''_{12},r''_{k,12},r''_{i,12},r''_{j,12},t_{34},r_{k,34},r_{i,34},r_{j,34},p_s,|m|_s,\ell_s)\\
&\times(-1)^{w_s-x_{i,s}-y_{i,s}-y_{j,s}+f_s}\frac{1}{2^{\ell_s+9}}\frac{1}{\pi^2}\frac{1}{\Lambda^2}\sigma^{(0)}4^{-6-\ell_s-f_s}\\
&\times\frac{\left(t''_{12}+r''_{k,12}-1\right)!!\left(r''_{i,12}-1\right)!!\left(r''_{j,12}-1\right)!!}{\left(t''_{12}+r''_{i,12}+r''_{j,12}+r''_{k,12}+1\right)!!}\frac{(t_{34}+r_{k,34}-1)!!\,(r_{i,34}-1)!!\,(r_{j,34}-1)!!}{(t_{34}+r_{i,34}+r_{j,34}+r_{k,34}+1)!!}\\
&\times(5+\ell_s+f_s)!\frac{(r_{i,s}+r_{j,s}+2)!!\,(1-r_{i,s}-r_{j,s}-r_{k,s}+\ell_s+t_s)!!}{(5-r_{k,s}+\ell_s+t_s)!!}\\
&\times\frac{(z_{i,s}+2\eta_s-x_{i,s}-x_{j,s}+2\epsilon_s+|m|_s-y_{i,s}-y_{j,s})!!\,(x_{i,s}+y_{i,s}+\ell_s-2\epsilon_s-|m|_s-2\eta_s-z_{i,s}-1)!!}{(1+\ell_s-x_{j,s}-y_{j,s})!!}\\
&\times\frac{(x_{j,s}+(2\epsilon_s+|m|_s-p_s)-y_{j,s}-1)!!\,(y_{j,s}+(2\eta_s+p_s)-x_{j,s}-1)!!}{((2\epsilon_s+|m|_s-p_s)+(2\eta_s+p_s))!!},
\end{aligned}
$$

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(C.31)

C: 38 Fold

Finite sum