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Lineshape of $a_1(1260)$ from heavy-lepton decay $\tau \rightarrow \upsilon 3\pi$

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Outline

Motivation

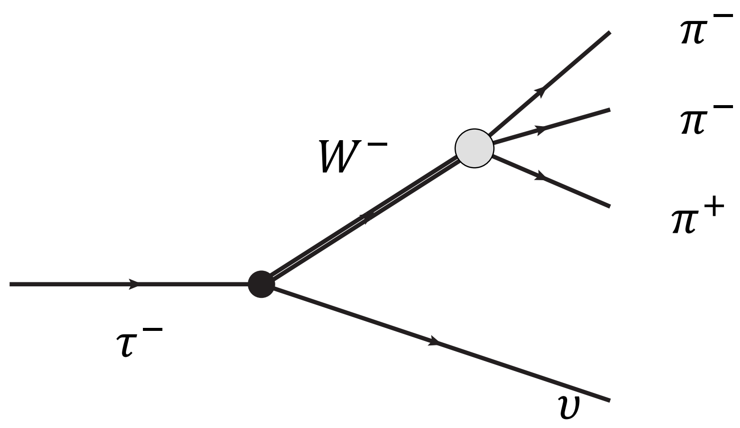
- ALEPH and CLEO data on $\tau \to \upsilon 3\pi$
- $\tau \to \upsilon 3\pi$ offers a clean view of $a_1(1260)$ production
- $a_1(1420)$ triangle singularity in $a_1(1260) \to \bar{K}K^* \to f_0\pi \to 3\pi$

Dispersive framework for three-pion decays

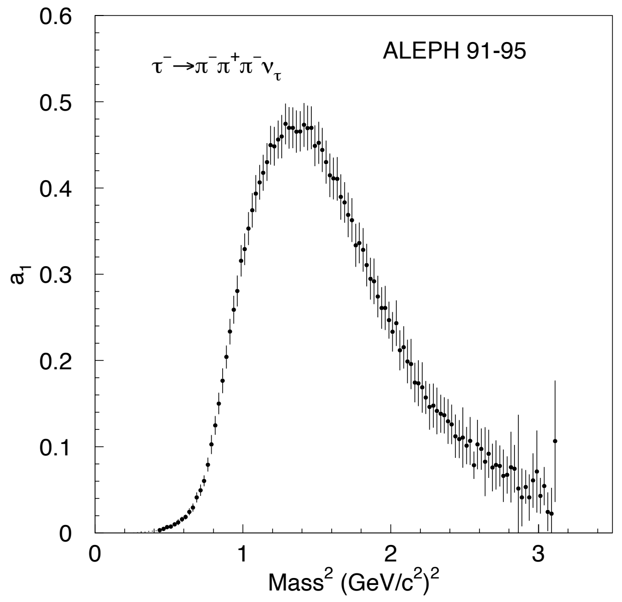
- construction by unitarity and analyticity
- phase-shifts with good accuracy available
- integral equations for $a_1 \to 3\pi$

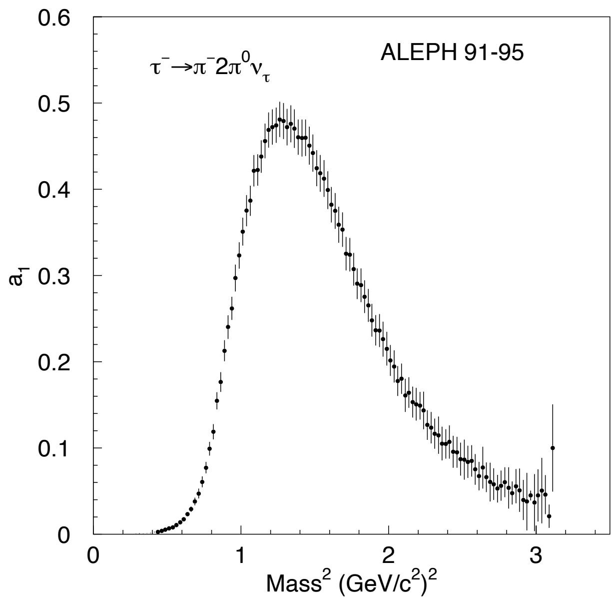
Numerical calculation

$\tau \to \upsilon 3\pi$ decay



ALEPH data on $\tau \to \upsilon 3\pi$





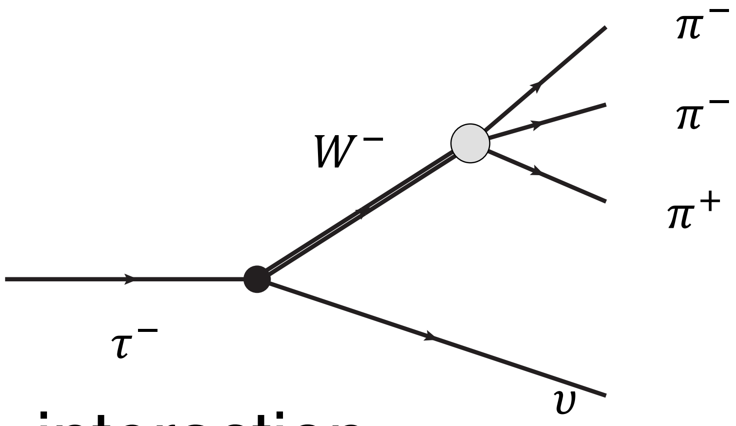
- accurate measurements of the mass distribution
- component $J^{PC} = 0^{-+}$ is suppressed by the PCAC
- the $J^{PC} = 1^{++}$ is dominant

The spectral function for the 3π final state

ALEPH Collaboration, Phys. Rept. 421, 191(2005).

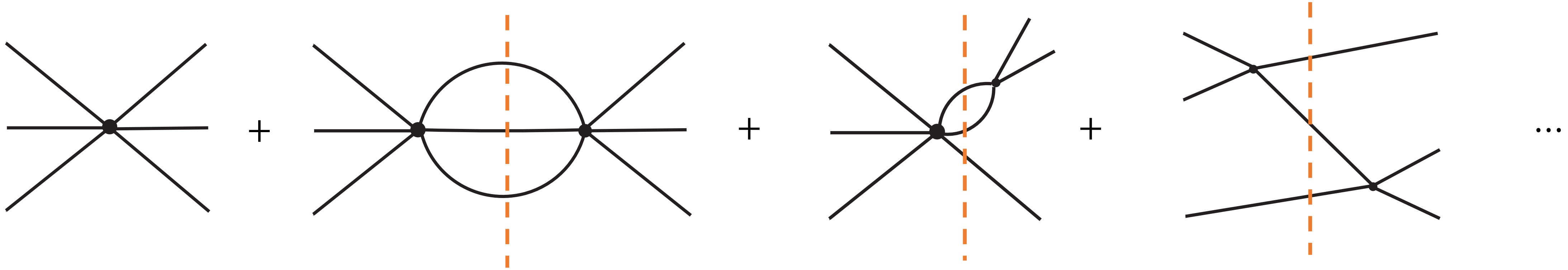
J. H. Kuhn and E. Mirkes, Z. Phys. C 56, 661 (1992)

$\tau \rightarrow \upsilon 3\pi$ decay



Final-state interactions in hadronic three-body decays play essential role

- three-body unitary framework used to describe 3π interaction



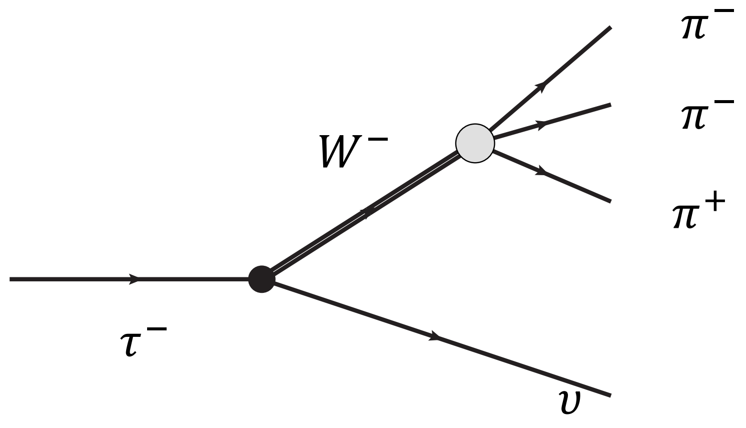
Daniel Sadasivan et al., Phys. Rev. D 105, 054020 (2022)

- $a_1(1260)$ generated from 3π interaction

- approximate three-body unitary

JPAC Collaboration, Phys. Rev. D 98, 096021 (2018)

$\tau \to \upsilon 3\pi$ decay

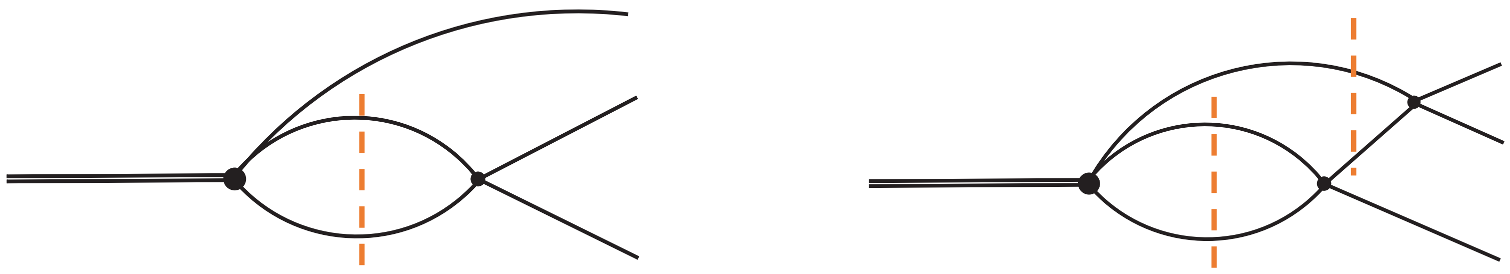


- model of this work $\tau \to \upsilon 3\pi$ decay

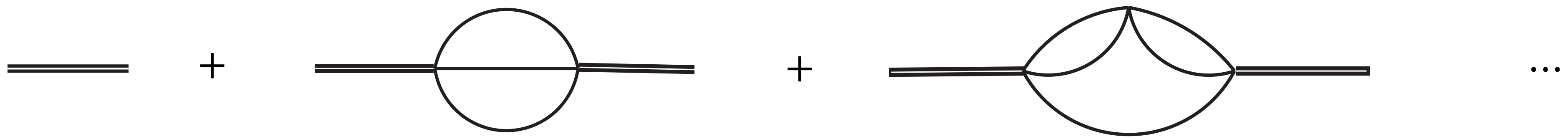
$$\frac{d\Gamma}{dS} = \text{cons.}\,(m_\tau^2 - S)^2 \sum_\lambda \left| \frac{A_\lambda(S)}{S - m_{a_1}^2 + \Sigma(S)} \right|^2 d\Phi^3$$

- unitarity relation of $a_1 \to 3\pi$

N. Khuri, S. Treiman, Phys. Rev. 119, 1115 (1960)

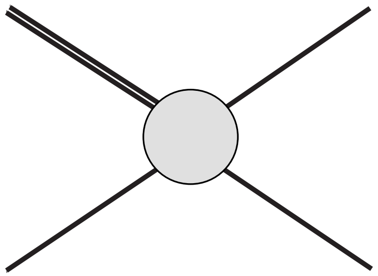


- self-energy $\Sigma(S)$



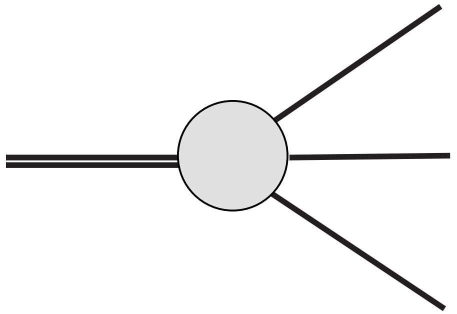
Three-body decay $a_1 \to 3\pi$

scattering region



$M < 3m_\pi$
$s \in [\,(M+m_\pi)^2\,,\infty]$

continuation in
M and *s*

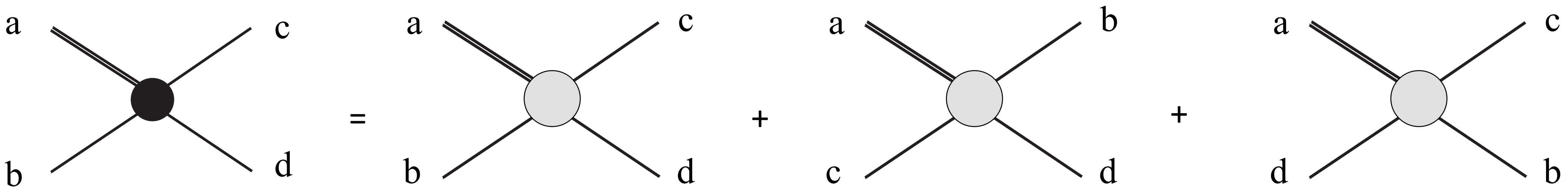


decay region

$M > 3m_\pi$
$s \in [\,4m_\pi^2\,,(M-m_\pi)^2]$

dispersive treatment

Cartesian isospin indices



s-channel

t-channel

u-channel

amplitude can be decomposed as

$$A_\lambda^{(s)abcd}(s,t,u) = \bar{A}_\lambda^{(s)abcd}(s,t,u) + \Sigma_{\lambda'}(-1)^{\lambda'}\, d^J_{\lambda'\lambda}(\omega_t)\bar{A}_\lambda^{(t)acbd}(t,s,u) + \Sigma_{\lambda'}(-1)^{\lambda'}\, d^J_{\lambda'\lambda}(\omega_u)\bar{A}_\lambda^{(u)adbc}(u,t,s)$$

From unitarity to integral equations

$$\bar{A}_\lambda(s,t,u) = \Sigma_j\,(2j+1)d^j_{\lambda 0}(\theta_s)a_{j\lambda}(s)$$

$$A^{(I)}_\lambda(s,t,u) = \frac{1}{(2I+1)}\Sigma_{a,b,c,d}P^I_{a,b,c,d}\,A^{(s)abcd}_\lambda(s,t,u)$$

$$T^{(I)}(s,\cos\theta_s) = 32\pi\sum_{l=0}^{\infty}(2l+1)\,P_l(\cos\theta_s)\tau^{(I)}_l(s)$$

- unitarity relation

$$\text{Disc}\,A^{(I)}_\lambda(s,t,u) = \frac{\rho(s)}{64\pi^2}\int d\Omega'\;T^{*(I)}(s,t'',u'')\times A^{(I)}_\lambda(s,t',u')$$

$$\text{Disc}\,a^{(I)}_{j\lambda}(s) = \rho(s)\tau^{(I)}_j(s)\left\{\underbrace{a^{(I)}_{j\lambda}(s)}_{\text{right-hand cut}} + \underbrace{\bar{\bar{a}}^{(I)}_{j\lambda}(s)}_{\text{left-hand cut}}\right\}$$

JPAC, Phys. Rev. D 101, 054018 (2020)

- kinematic singularities arising from the kinematics (spinor)

kinematic singularities free (KSF) amplitude

$$a^{(I)}_{j\lambda}(s) = \frac{(\lambda_X^{1/2}(s))^{j-1}(\lambda_\pi^{1/2}(s))^j}{s^{|\lambda|/2}}\,\hat{a}^{(I)}_{j\lambda}(s)$$

Källén or triangle function

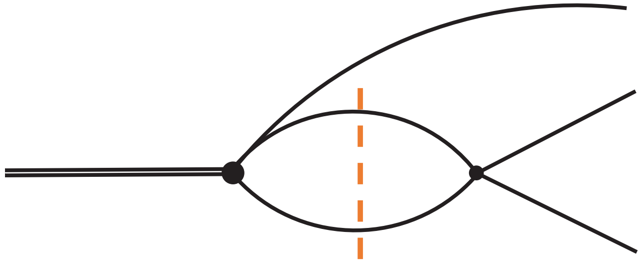
$$\lambda_M^{1/2}(s) = \lambda(s,M^2,m_\pi^2)$$

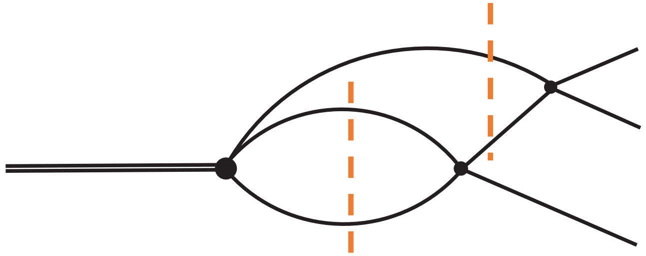
$$\lambda_\pi^{1/2}(s) = \lambda(s,m_\pi^2,m_\pi^2)$$

From unitarity to integral equations

- unitarity relation

$$\text{Disc}\, \hat{a}_{j\lambda}^{(I)}(s) = \rho(s)\tau_j^{(I)}(s)\left\{\underbrace{\hat{a}_{j\lambda}^{(I)}(s)}_{\text{right-hand cut}} + \underbrace{\tilde{a}_{j\lambda}^{(I)}(s)}_{\text{left-hand cut}}\right\}$$





- homogeneity contribution

$$\hat{a}_{j\lambda}^{(I)}(s) = P(s)\Omega(s) = P(s)\exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\delta_j^I(s')}{s'(s'-s-i\varepsilon)}\,\mathrm{d}s'\right]$$

R. Omnès, Nuovo Cim. 8, 316 (1958)

Integral equations

A. V. Anisovich and H. Leutwyler, Phys. Lett. B 375, 335 (1996)

- unitarity relation

Disc $\hat{a}_{j\lambda}^{(I)}(s) = \rho(s)\tau_j^{(I)}(s)\left\{\hat{a}_{j\lambda}^{(I)}(s) + \tilde{a}_{j\lambda}^{(I)}(s)\right\}$

$$\hat{a}_{j\lambda}^{(I)}(s) = P(s) + \frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_j^I(s')\, e^{-i\delta_j^I(s')}}{(s'-s-i\varepsilon)}\left\{\hat{a}_{j\lambda}^{(I)}(s') + \tilde{a}_{j\lambda}^{(I)}(s')\right\}$$

$$\hat{a}_{j\lambda}^{(I)}(s) = \Omega_j^I(s)\left\{P(s) + \frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_j^I(s')}{|\Omega_j^I|(s'-s-i\varepsilon)}\left\{\hat{a}_{j\lambda}^{(I)}(s') + \tilde{a}_{j\lambda}^{(I)}(s')\right\}\right\}$$

Analytic continuation

- inhomogeneities $\tilde{a}^{(I)}_{j\lambda}(s)$

$$C_{II'} = \frac{1}{(2I\ +\ 1)} \Sigma_{a,b,c,c} P^{I}_{a,b,c,d} P^{I'}_{a,b,c,d}$$

$$d^{j}_{\lambda 0}(\theta) = (\sin\theta)^{|\lambda|} \hat{d}^{j}_{\lambda\,0}(\theta)$$

$$\tilde{a}^{(I)}_{j\lambda}(s) = (-1)^{\lambda} \sum_{I',\lambda',j'} (2\,j'\ +\ 1)\, C_{II'} \int d\cos\theta'\ \hat{C}^{jj'}_{II'}(s,\theta')\ \hat{d}^{j}_{\lambda\,0}(\theta')\hat{d}^{j'}_{\lambda' 0}(\theta'_t)\hat{a}^{(I')}_{j'\lambda'}(t')$$

$\cos\theta' \to s'$

$$\tilde{a}^{(I)}_{j\lambda}(s) = (-1)^{\lambda} \sum_{I',\lambda',j'} (2\,j'\ +\ 1)\, C_{II'}\ \underbrace{\frac{2s}{\lambda_M^{n/2}(s)\lambda_\pi^{n/2}(s)}}_{\text{singularities}} \int d\,s'\ K(s,s')\ \hat{a}^{(I')}_{j'\lambda'}(s')$$

$n = 1,3,5$

singularities

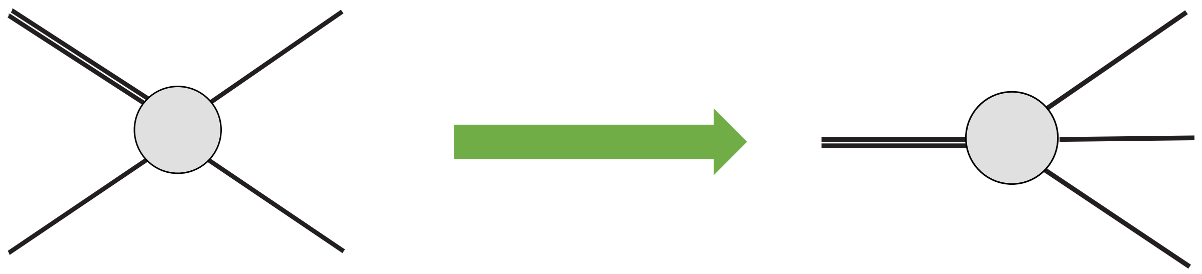
$$\hat{C}^{jj'}_{II'}(s,\theta) = (\sin^2\theta)^{|\lambda|} K^{-1}_{j\lambda}(s,\theta)\ d^{1}_{\lambda'\,\lambda}(\omega_t\,(s,\theta)) K_{j'\lambda'}(t(s,\theta),\theta_t(s,\theta))$$

$$K_{j\lambda}(s,\theta) = (s^{-1/2}\sin\theta)^{|\lambda|}\ (\lambda_X^{1/2}(s))^{|j-1|}(\lambda_\pi^{1/2}(s))^{j}$$

$$s' = (M^2 + 3m_\pi^2 - s)/2\ +\ \lambda_M^{1/2}(s)\lambda_\pi^{1/2}(s)\,/(2s)\ \cos\theta'$$

Analytic continuation

- continuation into the decay region

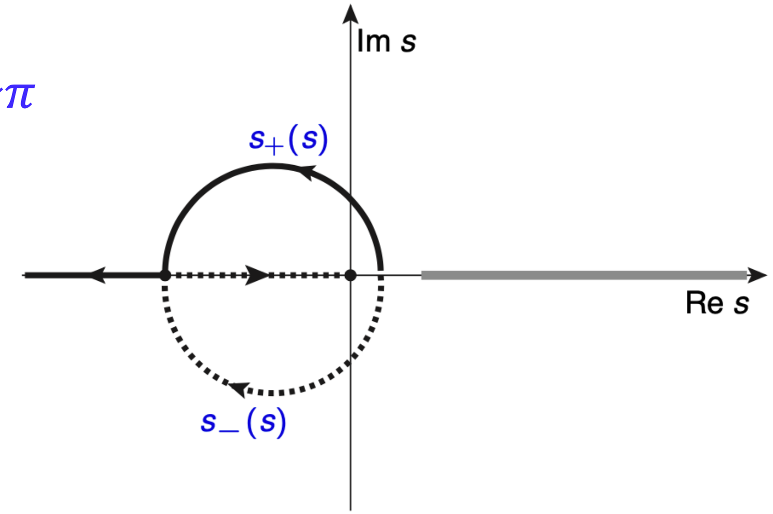


- $M \to M + i\,\delta$

J. B. Bronzan and C. Kacser, Phys. Rev. 132, 2703 (1963)

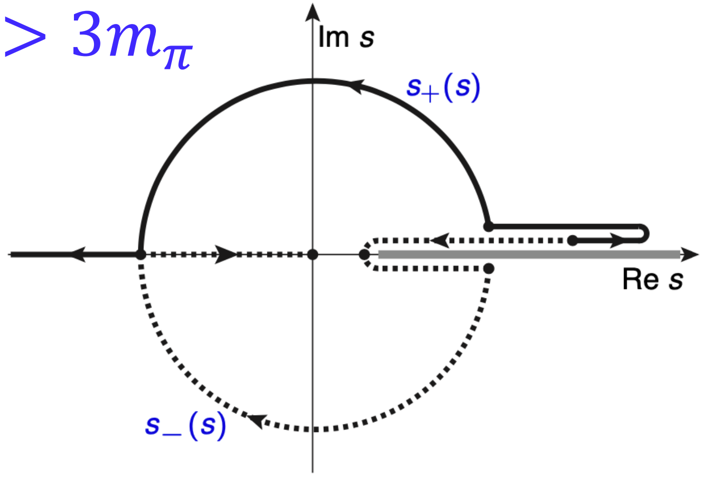
- path deformation needed

$M < 3m_\pi$



Integration does not run over the cut

$M > 3m_\pi$



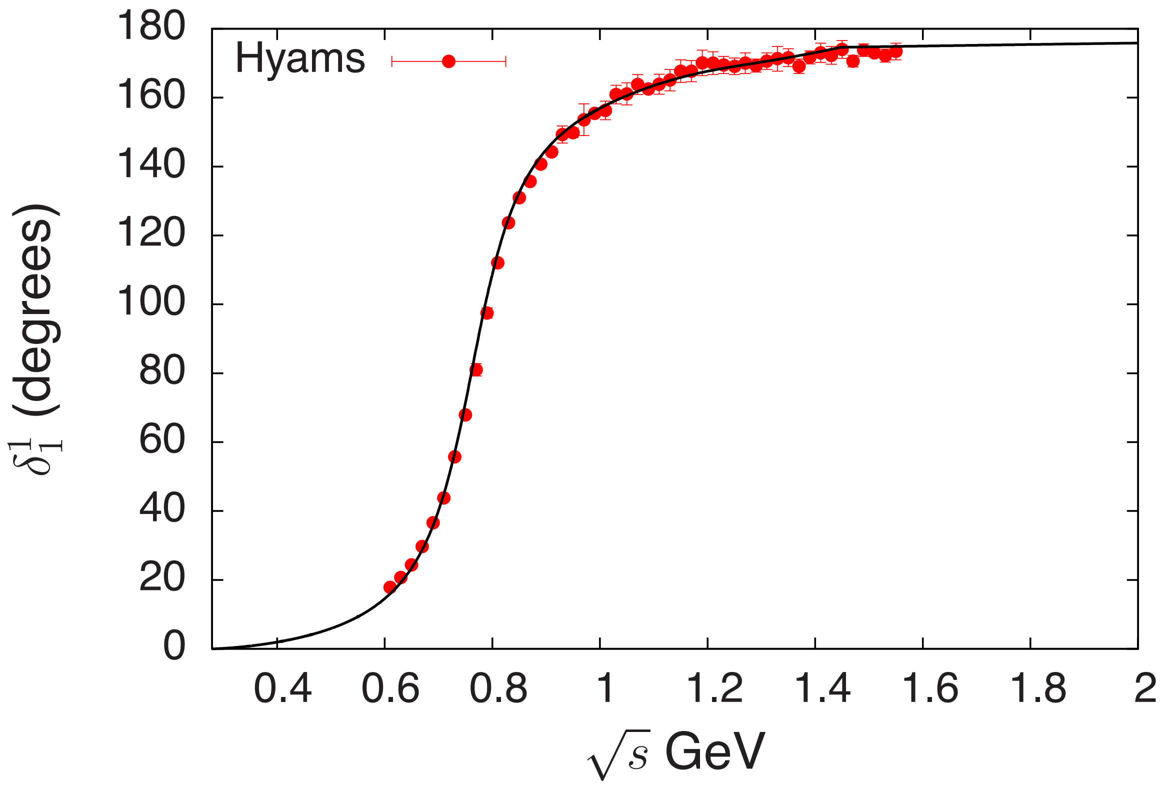
Integration does run over the cut

Phase-shift input

- P-wave, $I = 1$

R. García-Martín et al., Phys. Rev. D 83, 074004 (2011)

$\pi\pi$ scattering constrained by analyticity and unitarity



Roy equations = partial-wave dispersion relations + crossing symmetry + unitarity

B. Hyams et al., Nucl.Phys. B64, 134 (1973)

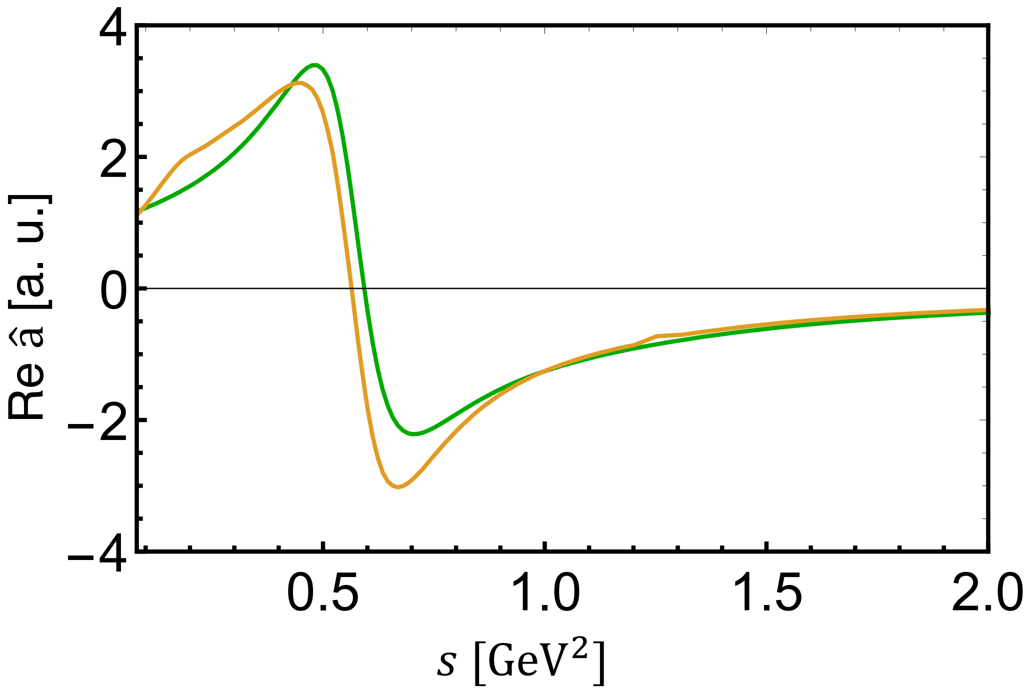
Numerical results

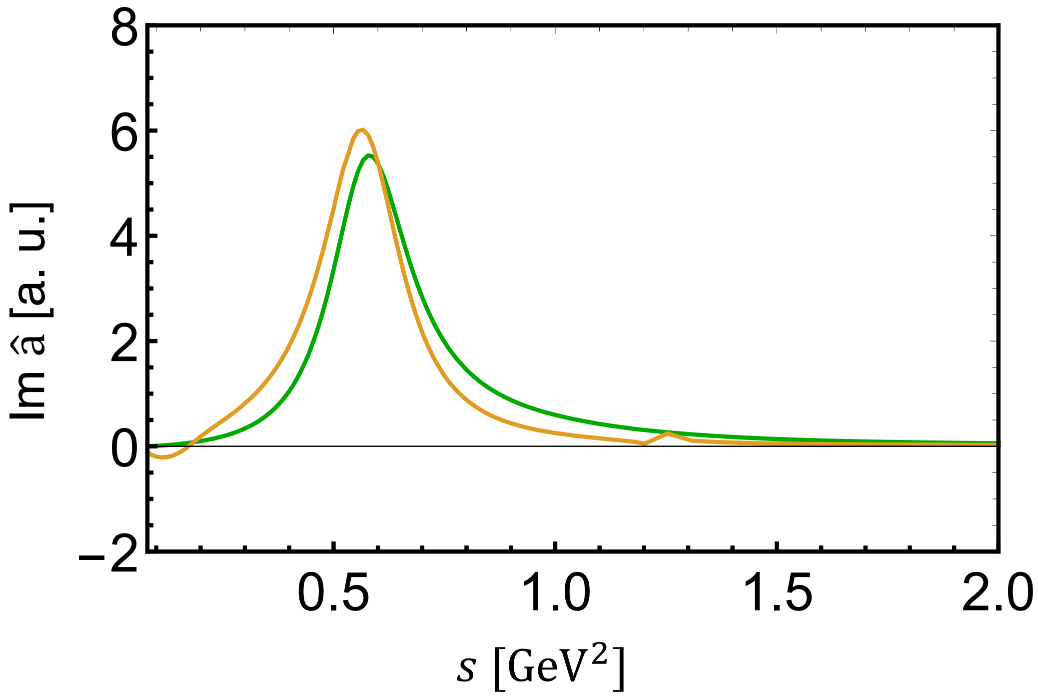
- **KSF amplitude $\hat{a}_{j\lambda}^{(I)}(s)$**

$I = 1,\ j = 1,\ \lambda = 0$

Only P-wave $\pi\pi$ scattering included

$M = 1.25$ GeV





- use Omnès function as input
- crossed-channel rescattering effects play a role
- rescattering effects phase-space dependent

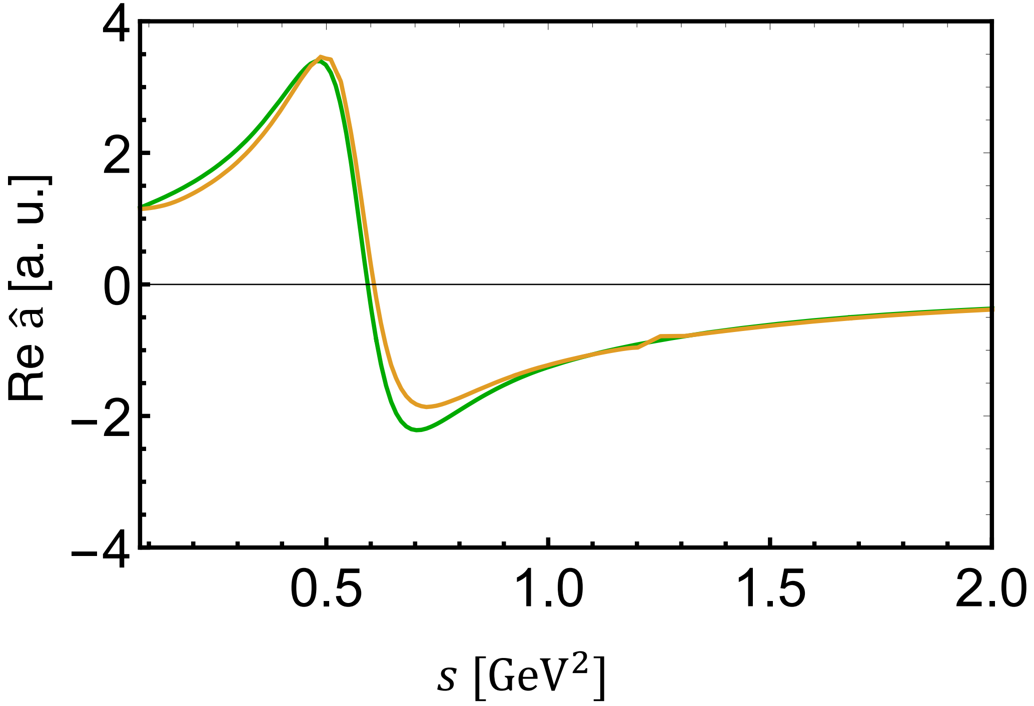
Once-subtracted solution
Green lines : $\tilde{a}_{j\lambda}^{(I)}(s) = 0$
Orange lines : Full result

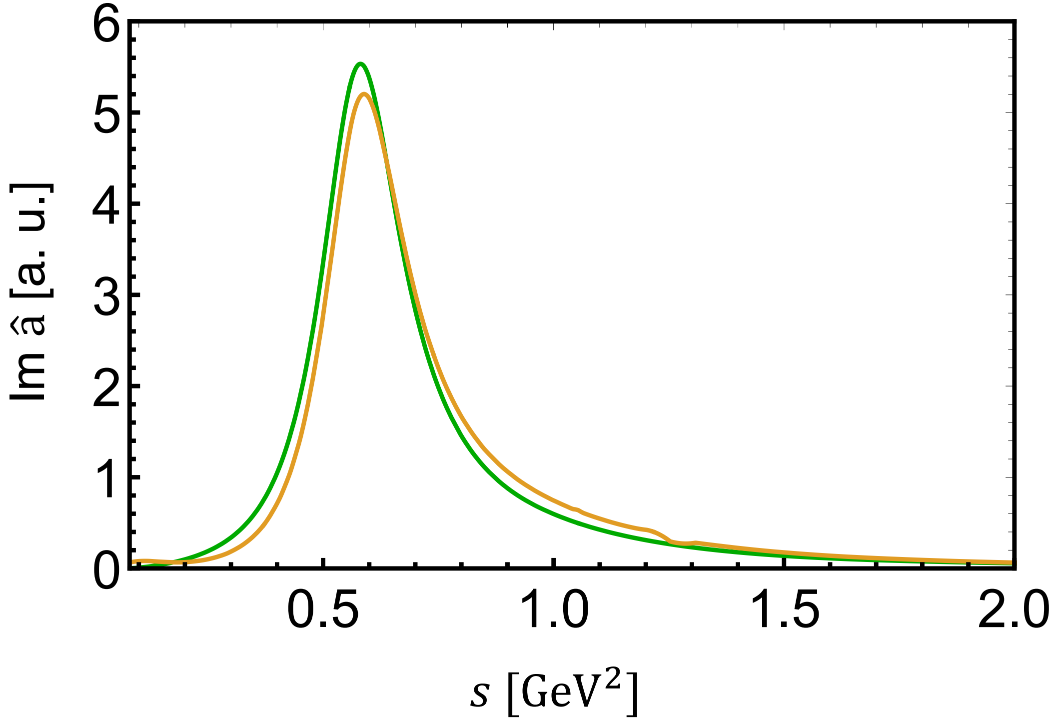
Numerical results

- **KSF amplitude $\hat{a}_{j\lambda}^{(I)}(s)$**

Only P-wave $\pi\pi$ scattering included

$I = 1,\ j = 1,\ \lambda = +1$





$$\frac{2s}{\lambda_\pi^{3/2}(s)}\int d\,s'\ K(s,s')\hat{a}_{j'\lambda'}^{(I')}(s') \sim a_P + b_p(s_{III}-s)^{\frac{1}{2}} + c_p(s_{III}-s)^{3/2} \ldots$$

$$s_{III} = (M - m_\pi)^2$$

Improve the stability of numerical integration

Green lines : $\tilde{a}_{j\lambda}^{(I)}(s) = 0$

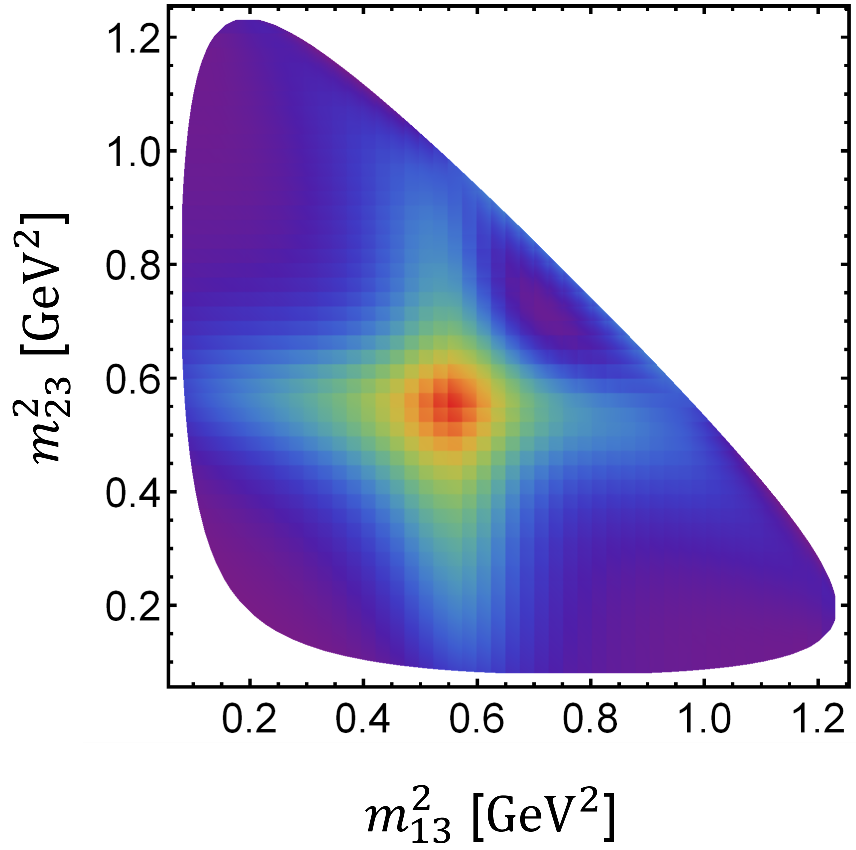
Orange lines : Full result

Dominik Stamen et al., Eur. Phys. J. C 83, 510 (2023)

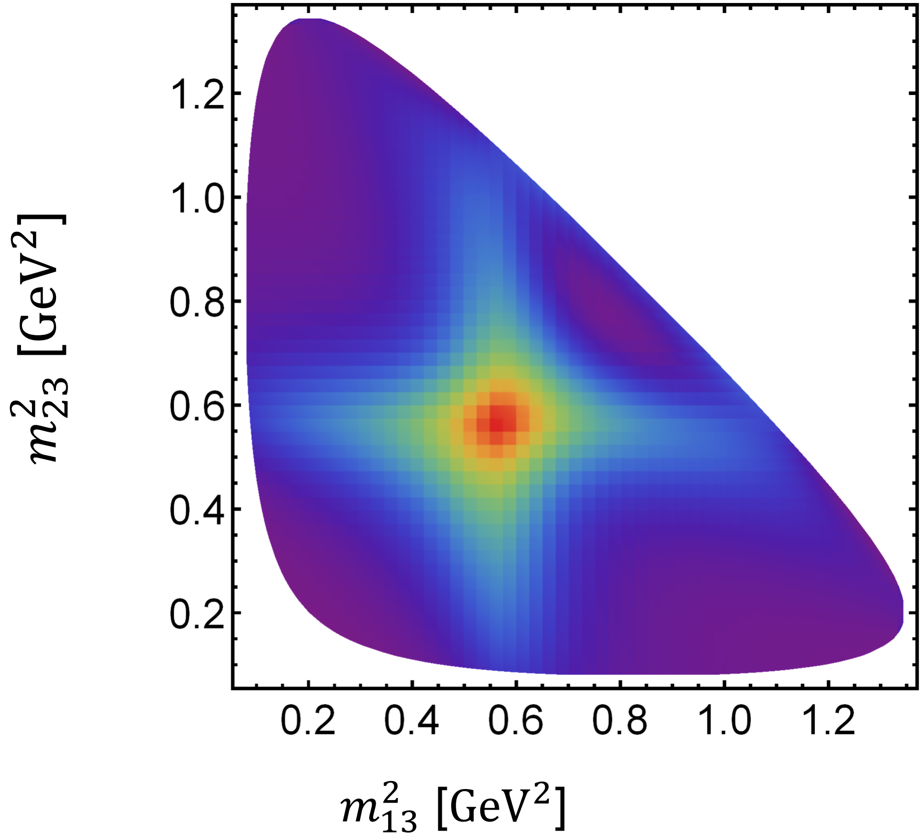
Numerical results

Dalitz plots

- $a_1^- \to \pi^-(p_1)\ \pi^-(p_2)\ \pi^+(p_3)$



$M = 1.25$ GeV



$M = 1.3$ GeV

Dalitz plot shows ρ-resonance bands

Summary

- only P-wave $\pi\pi$ scattering is included
- the 3π mass distribution needs to be fitted
- $a_1(1420)$ triangle singularity in $a_1(1260) \to \bar{K}K^* \to f_0\pi \to 3\pi$ will be studied

Thank you !