Problem Sheet for Integrability

1 The $O(N)$ Invariant S-matrix

Please find the $O(N)$ invariant S-matrix missed by the original paper [1].
[Hint: In the original paper, they missed it because they assumed implicitly that the coefficient in front of identity is nonzero.]
[Refs: [1–3]]

2 The Coordinate Bethe Ansatz

Derive the two-particle Bethe wave function of the Heisenberg chain with the periodic boundary condition. And then try to see that there is a natural generalization for multi-particle states given that the Hamiltonian is nearest neighbor. Note that the Hamiltonian is given by

$$H_{\text{Heisenberg}} \propto \sum_{j=1}^{L} \left(I_{j,j+1} - P_{j,j+1} \right), \tag{2.1}$$

where I is the identity operator and P is the permutation operator. For the periodic boundary condition, the lower index $L+1 \equiv 1$.
[Hint: Write the ansatz and require it to be the eigenstate of the Hamiltonian.]

3 Three-point Functions in $\mathcal{N}=4$ SYM

Consider three-point functions of the following operators in the $SU(2)$ sector of $\mathcal{N}=4$ SYM:

$$\mathcal{O}_1 = (ZYZZY\cdots) + \cdots, \quad \mathcal{O}_2 = \text{tr}(\bar{Z}^{L_2}), \quad \mathcal{O}_3 = \text{tr}\left((Z+\bar{Z}+Y-\bar{Y})^{L_3}\right),$$

where $Z = \Phi_1 + i\Phi_2$ and $Y = \Phi_3 + i\Phi_4$. The precise form of $\mathcal{O}_1$ is determined by the condition that $\mathcal{O}_1$ is the eigenvector of the dilatation operator:

$$D \cdot \mathcal{O}_1 = \Delta_1 \mathcal{O}_1.$$

The leading order $\mathcal{O}_1^{(0)}$ of the operator $\mathcal{O}_1$ in its expansion with respect to the coupling constant is determined by the equation

$$D^{(1)} \cdot \mathcal{O}_1^{(0)} = \Delta_1^{(1)} \mathcal{O}_1^{(0)},$$

where $D^{(1)}$ is the one-loop dilatation operator and can be identified with the Heisenberg Hamiltonian (2.1) upon the map

$$Z \mapsto \uparrow, \quad Y \mapsto \downarrow . \tag{3.1}$$

Please calculate the structure constant c_{123} at tree level when the operator $\mathcal{O}_1^{(0)}$ corresponds to the single-magnon state and the two-magnon state of the integrable spin chain. And then try to generalize it to the multi-magnon case.
[Refs: [4]]

References

[1] Alexander B. Zamolodchikov and Alexei B. Zamolodchikov. Factorized s Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models. *Annals Phys.*, 120:253–291, 1979.

[2] Nikolay Gromov, Vladimir Kazakov, Kazuhiro Sakai, and Pedro Vieira. Strings as multi-particle states of quantum sigma-models. *Nucl. Phys. B*, 764:15–61, 2007.

[3] Lucía Córdova and Pedro Vieira. Adding flavour to the S-matrix bootstrap. *JHEP*, 12:063, 2018.

[4] Shota Komatsu. Three-point functions in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. 10 2017.