

Multi-loop Feynman Integrals new developments

New Frontier on Quantum Fields and Gravity

Peking University, Jan. 12. 2026

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Outline

Introduction

Analytic Feynman integral frontier

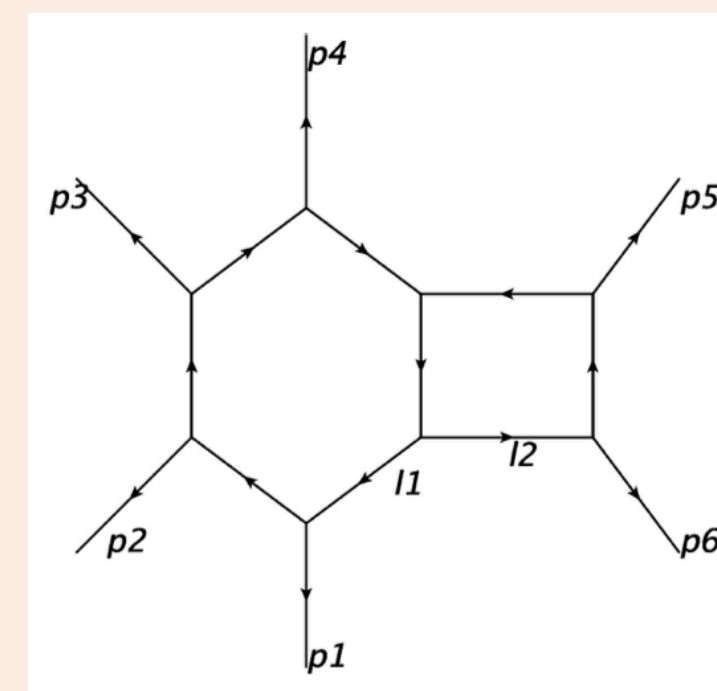
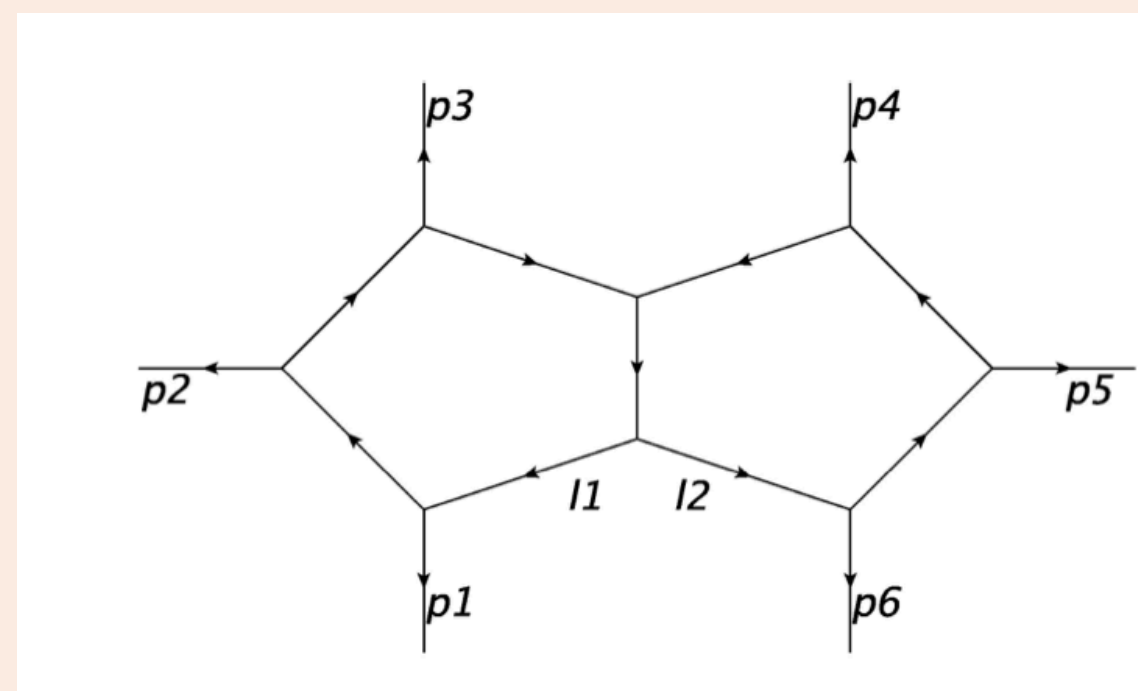
Case 1: 2loop 6point Feynman integrals

Case 2: 3loop 5point Feynman integrals

Summary



Based on Feynman integral Integral evaluation



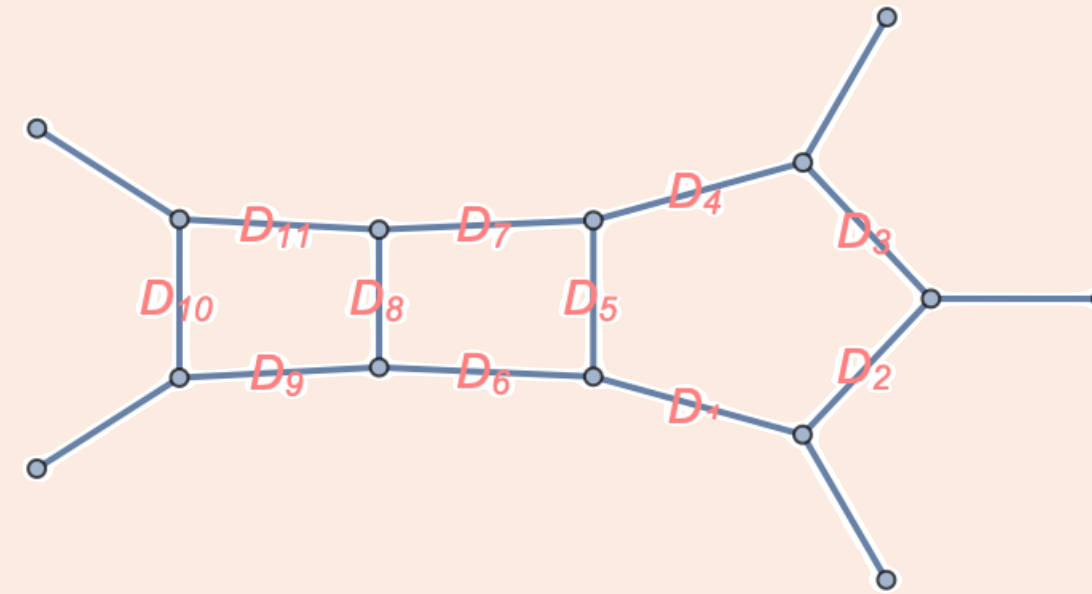
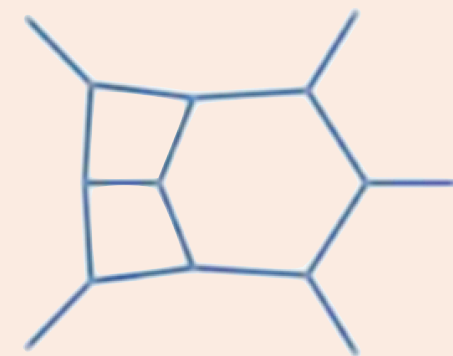
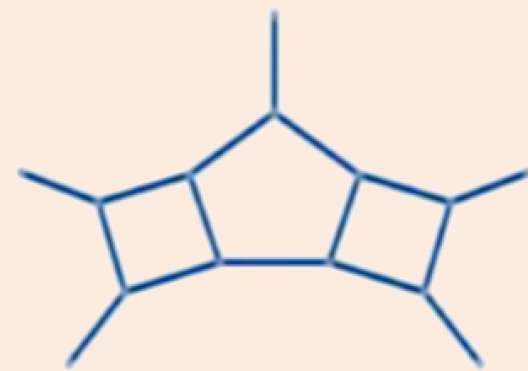
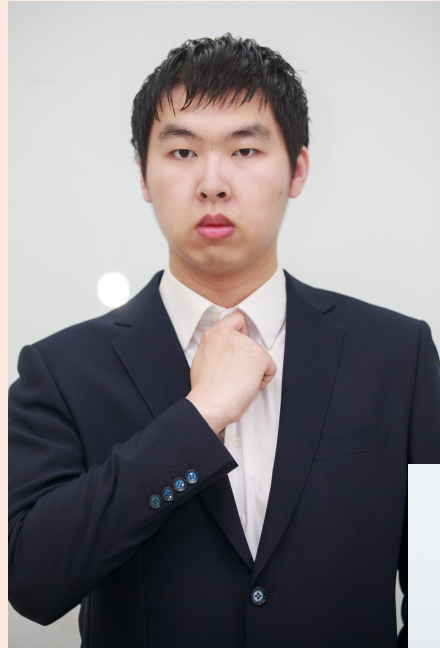
analytic computation of all **2loop 6point**
massless planar integrals



Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027

Based on Feynman integral Integral evaluation



analytic computation of all
3loop 5point Feynman integral family



Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021 (Editors' Suggestion)

Chicherin, Wu, Wu, Xu, Zhang, YZ, arXiv: 2512.17330

Based on package development

“**NeatIBP 1.0**, a package generating small-size integration-by-parts relations for Feynman integrals”
Wu, Boehm, Ma, Xu, YZ, Comput. Phys. Commun. 295 (2024), 108999

“Performing integration-by-parts reductions using **NeatIBP 1.1 + Kira**”
Wu, Boehm, Ma, Usovitsch, Xu, YZ, Comput.Phys.Commun. 316 (2025) 109798

NeatIBP collaboration

pfd-parallel, *a Singular/GPI-Space package for massively parallel multivariate partial fractioning*
Bendle, Boehm, Heymann Ma, Rahn, Ristau, Wittmann, Wu, Xu, YZ, Comput.Phys.Commun. 294 (2024) 108942

pfd collaboration

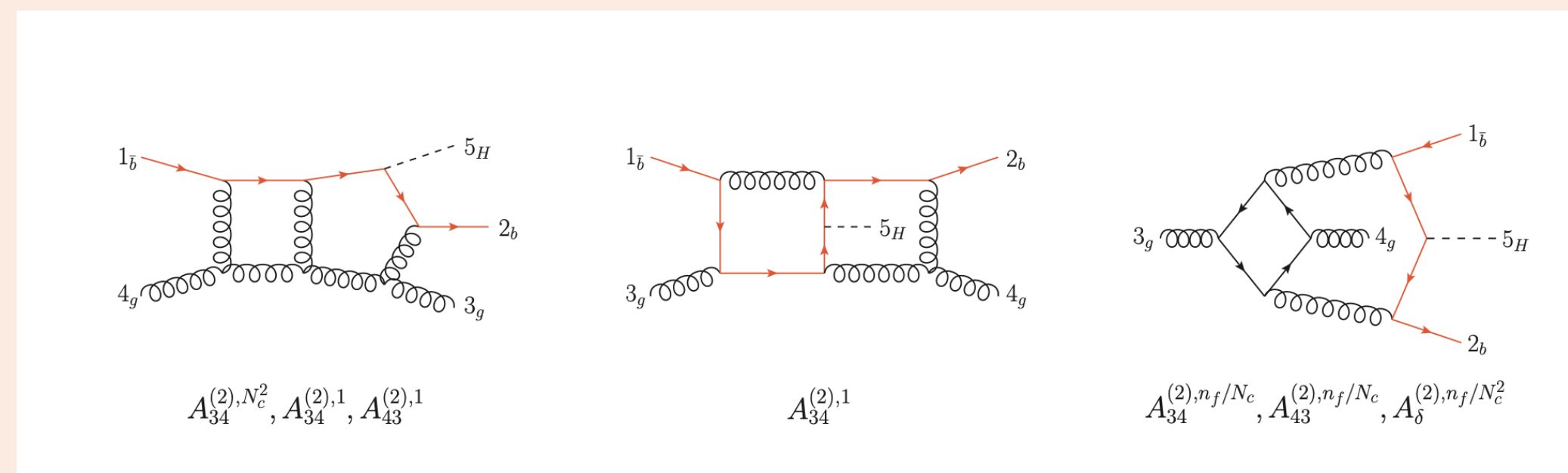
Based on perturbative QCD computations

“Two-loop amplitudes for $O(\alpha_s^2)$ corrections to $W\gamma\gamma$ production at the LHC”

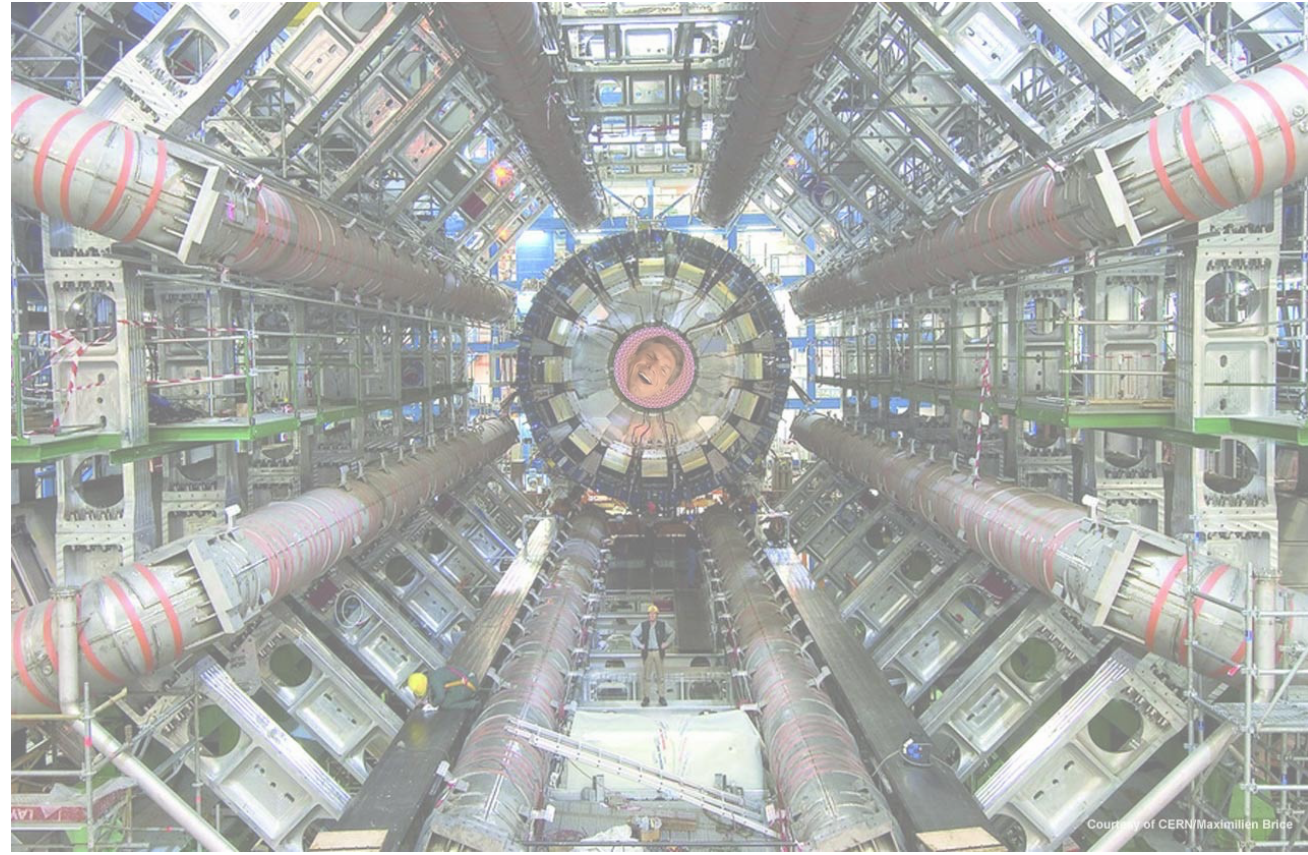
Badger, Hartanto, Wu, YZ, Zoia, JHEP12(2024) 221

“Full-colour double-virtual amplitudes for associated production of a Higgs boson with a bottom-quark pair at the LHC”

Badger, Hartanto, Poncelet, Wu, YZ, Zoia, JHEP03(2025) 066

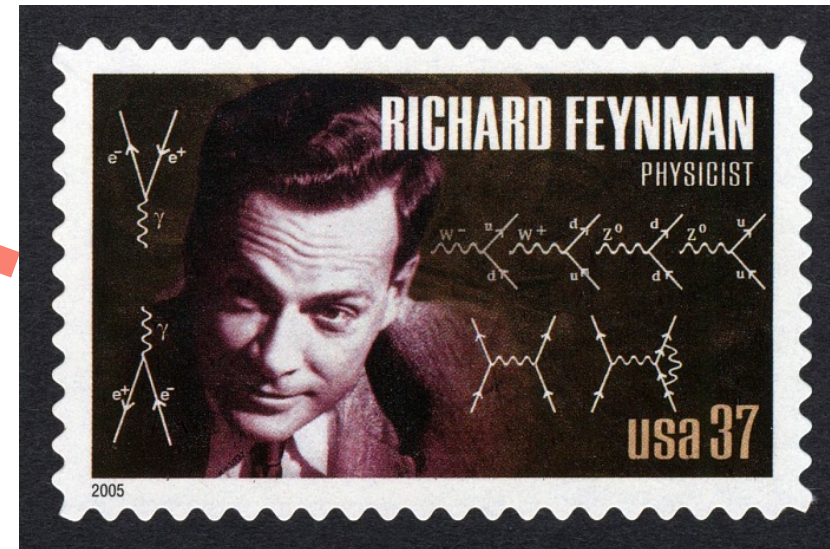


Introduction



Precision physics

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \sigma^{NNLO}$$



Feynman integrals



(CERN art image for supergravity)

Formal theories

N=8 supergravity UV finiteness



Gravitational wave template computations

for instance,
Driesse, Jakobsen, Klemm, Mogull, Nega, Plefka, Sauer, Usovitsch
Nature 641 (2025) 8063, 603-607

Why *analytic* multi-loop Feynman integrals/amplitudes?

- Analytic results can be very fast for numeric results
- Analytic method can be a source for developing new numeric methods
- Theoretical aspects of quantum field theory

for examples: 2loop N=4 SYM theory **spacelike** splitting amplitude

Henn, Ma, Xu, Yan, YZ, Zhu, Phys.Rev.D 112 (2025) 7, 076003

- Quantum field theory computation of gravitational wave
analytic continuation/ Fourier transform is needed
- **Function space bootstrap** for physical observable

Carrôlo, Chicherin, Henn, Yang, YZ, JHEP 07 (2025) 214

Analytic Feynman integral computation and the methodology

Current status of analytic Feynman integral computation

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results	?	?	?
Three-loop	some results	?	?	?	?
Four-loop	some results	?	?	?	?

with dimensional regulation

Current status of analytic Feynman integral computation

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results	?	?	?
Three-loop	some results	?	?	?	?
Four-loop	some results	?	?	?	?

with dimensional regulation

Frontier

Current status of analytic Feynman integral computation

	4-point	5-point	6-point	7-point	8-point
One-loop	known	known	known	known	known
Two-loop	most known	some results	?	?	?
Three-loop	some results	?	?	?	?
Four-loop	some results	?	?	?	?

with dimensional regulation

Frontier

Liu, Matijasic, Miczajka, Xu, Xu, YZ,
Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion
Chicherin, Wu, Wu, Xu, Zhang, YZ,
arXiv

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, PRL. 135, 031601
Henn, Matijasic, Miczajka, Peraro, Xu, YZ, JHEP 08(2024) 027
Henn, Peraro, Xu, YZ, JHEP 03 (2022) 056
Abreu, Monni, Page, Usovitsch JHEP 06 (2025) 112

Canonical Differential Equation

Uniformly transcendental (UT) basis determination
Canonical differential equation

$$\frac{\partial}{\partial x_i} I(x, \epsilon) = \epsilon A_i(x) I(x, \epsilon)$$

Henn 2013



Solving differential equation
with boundary value

$$\int d \log(W_{i_1}) \circ \dots \circ d \log(W_{i_k})$$

analytic result

→ polylogarithm functions or one-fold integration

Canonical Differential Equation, new insights

ϵ -factorized differential equation algorithm ϵ -collaboration 2506.09124

Better Integration-by-parts (IBP) reduction **Blade**, Guan, Liu, Ma, Wu 2024
Comput.Phys.Commun. 310 (2025) 109538
NeatIBP, Wu, Boehm, Ma, Xu, YZ 2023
Comput.Phys.Commun. 295 (2024) 108999

Alphabet searching **Effortless**, Matijasic, Miczajka to appear
<https://github.com/antonela-matijasic/Effortless>
BaikovLetter, Jiang, Liu, Xu, Yang,
Phys.Lett.B 864 (2025) 139443
SOFIA Correia, Giroux, Mizera 2503.16601

Solving differential equation Novel representation of one-fold integration
Liu, Matijasic, Miczajka, Xu, Xu, YZ,
Phys.Rev.D 112 (2025) 1, 016021

*for Next-to-Next-to-leading order
production of 4 jets, 4 photons ...*

2loop 6point Feynman integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett.* 135 (2025) 3, 031601

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *JHEP* 08(2024) 027

Abreu, Monni, Page, Usovitsch *JHEP* 06 (2025) 112

2loop Feynman integral: Scale frontier

2loop 5point massless

Gehrmann, Henn, Lo Presti 2015

Chicherin, Gehrmann, Henn, Wasser, YZ, Zoia 2019

5 scales

2loop 5point one-mass

Papadopoulos, Tommasini, Wever 2019

Abreu, Ita, Moriello, Page, Tschernow, Zeng 2020

Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia 2023

Jiang, Liu, Xu, Yang 2024

Badger, Becchetti, Giraudo, Zoia 2024

6 scales

2loop 5point two-mass

Cordero, Figueiredo, Kraus, Page and Reina 2023

for leading-Color $pp \rightarrow ttH$ amplitudes with a light-quark loop

7 scales

2loop 6point massless

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, 2025

for NNLO 4 jets production, 2 jets+ 2 photons

8 scales!

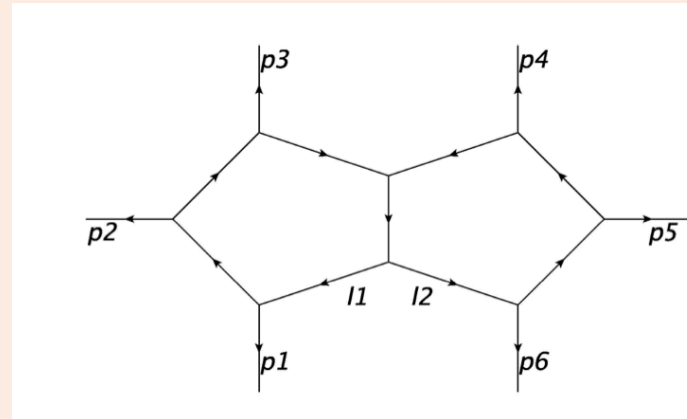
$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{345}$

All planar 2loop 6point integrals

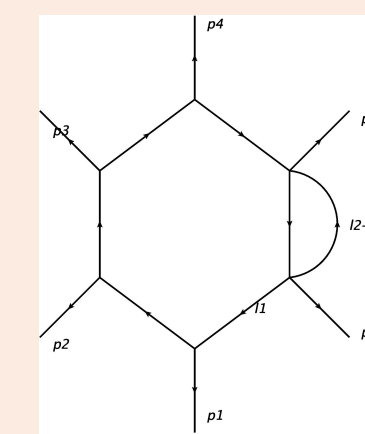
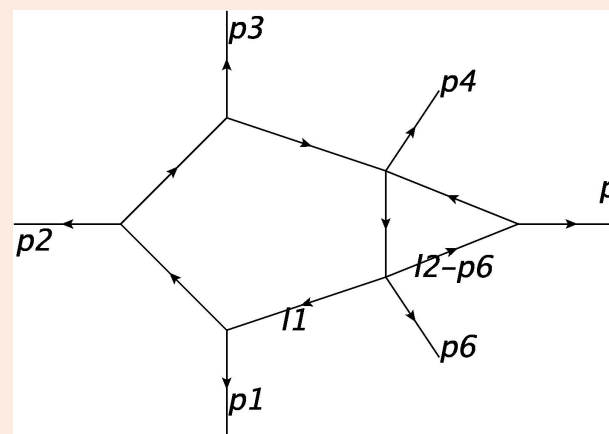
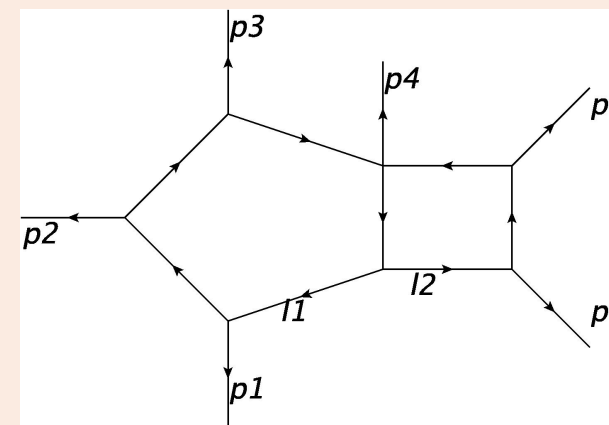
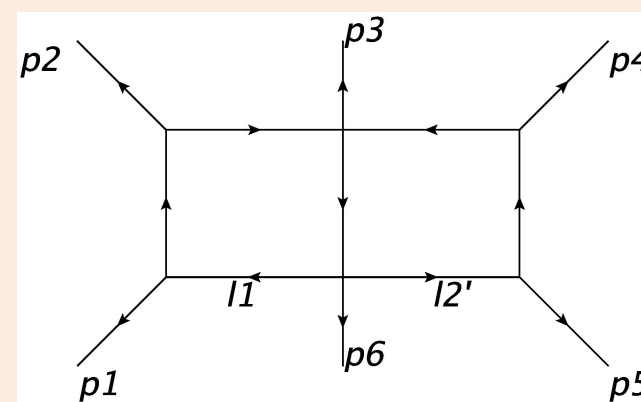
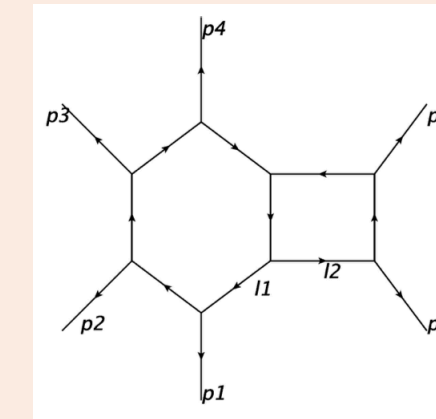
Henn, Matijasic, Miczajka, Peraro, Xu, YZ, *Phys. Rev. Lett.* 135 (2025) 3, 031601

Abreu, Monni, Page, Usovitsch, *JHEP* 06 (2025) 112

267
UT integrals



202
UT integrals



Momentum Twistor

external momenta

$d=4$

$$p_i = x_{i+1} - x_i$$

dual coordinates

$$\epsilon_{\dot{\beta}\dot{\alpha}} x^{\dot{\alpha}\gamma} \lambda_{A,\gamma} = \mu_{A,\dot{\beta}}$$

$$\epsilon_{\dot{\beta}\dot{\alpha}} x^{\dot{\alpha}\gamma} \lambda_{B,\gamma} = \mu_{B,\dot{\beta}}$$

$$(Z_A, Z_B) \rightarrow x$$

$$(Z_i, Z_{i+1}) \rightarrow x_{i+1}$$

$$Z_i = \begin{pmatrix} \lambda_{i,\alpha} \\ \mu_{i,\dot{\beta}} \end{pmatrix}, \quad i = 1, \dots, 6$$

momentum twistor

$\langle Z_A Z_B Z_C Z_D \rangle$ is dual conformally invariant

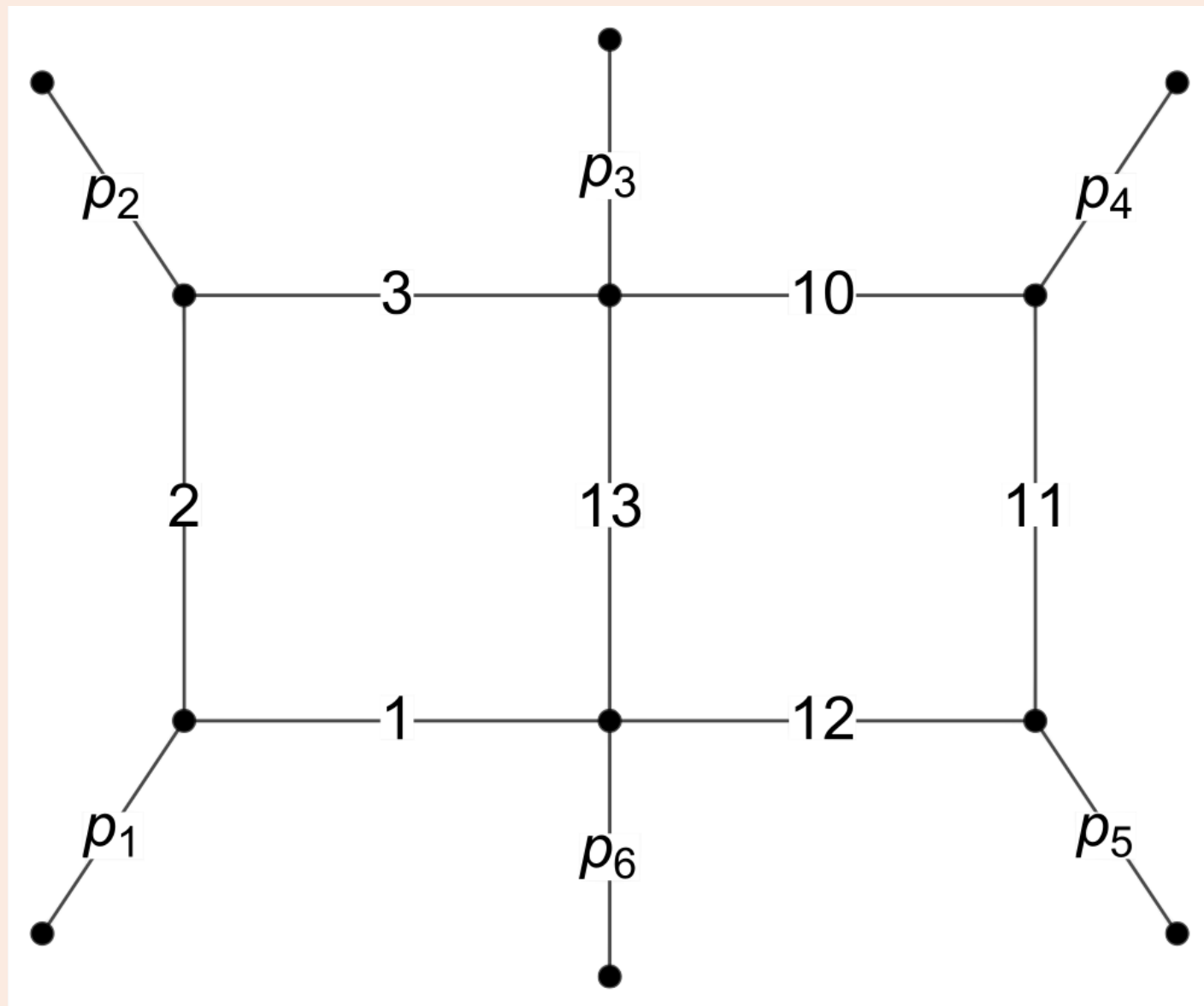
$$\tilde{\lambda}_i = \frac{\langle i, i+1 \rangle \mu_{i-1} + \langle i+1, i-1 \rangle \mu_i + \langle i-1, i \rangle \mu_{i+1}}{\langle i, i+1 \rangle \langle i-1, i \rangle},$$

$$(\lambda_{i,\alpha}, \tilde{\lambda}_{i,\dot{\alpha}})$$

spinor helicity

Uniformly transcendental (UT) basis determination

key step



Chiral numerator

(Arkani-Hamed, Bourjaily, Cachazo, Trnka 2011)

/ Gram determinant

correspondence

$$\Delta_6 = \langle 12 \rangle [23] \langle 34 \rangle [45] \langle 56 \rangle [61] - \langle 23 \rangle [34] \langle 45 \rangle [56] \langle 61 \rangle [12] .$$

$$I_{\text{db},i} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_i}{D_1 D_2 D_3 D_{10} D_{11} D_{12} D_{13}} , \quad i = 1, \dots, 7$$

$$N_1 = -s_{12} s_{45} s_{156} ,$$

$$N_2 = -s_{12} s_{45} (l_1 + p_5 + p_6)^2 ,$$

$$N_3 = \frac{s_{45}}{\epsilon_{5126}} G \left(\begin{array}{cccc} l_1 & p_1 & p_2 & p_5 + p_6 \\ p_1 & p_2 & p_5 & p_6 \end{array} \right) ,$$

$$N_4 = \frac{s_{12}}{\epsilon_{1543}} G \left(\begin{array}{cccc} l_2 - p_6 & p_5 & p_4 & p_1 + p_6 \\ p_1 & p_5 & p_4 & p_3 \end{array} \right) ,$$

$$N_5 = -\frac{1}{4} \frac{\epsilon_{1245}}{G(1, 2, 5, 6)} G \left(\begin{array}{ccccc} l_1 & p_1 & p_2 & p_5 & p_6 \\ l_2 & p_1 & p_2 & p_5 & p_6 \end{array} \right) ,$$

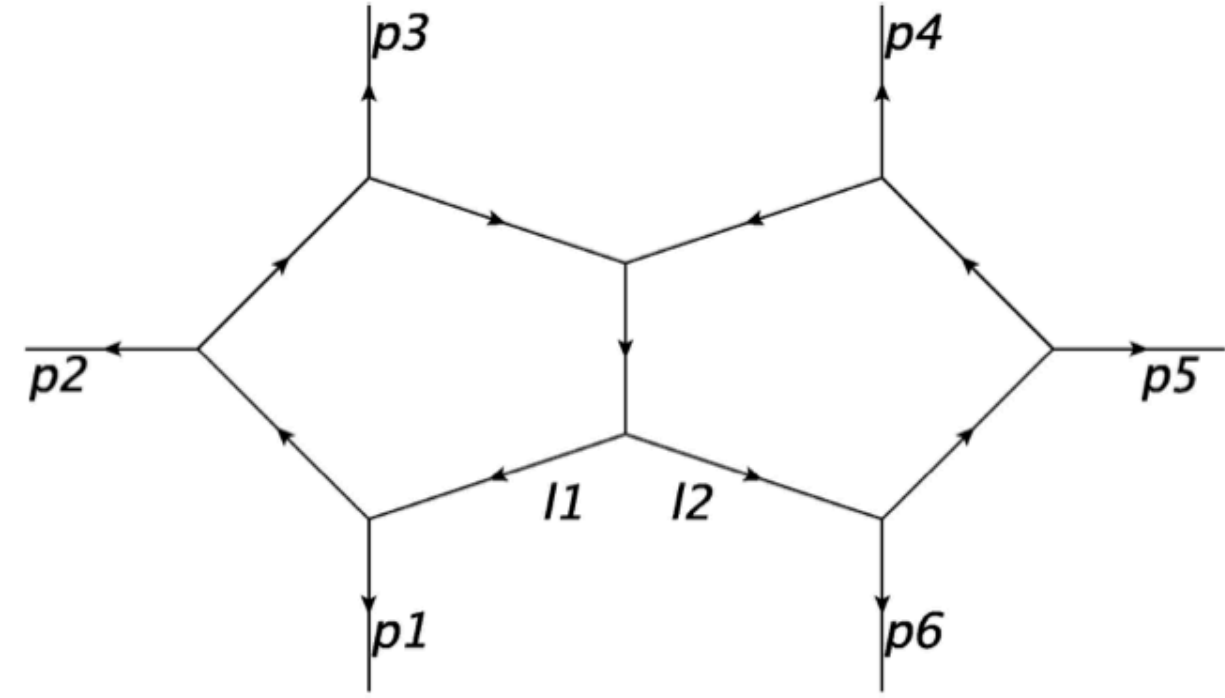
$$N_6 = \frac{1}{8} G \left(\begin{array}{ccc} l_1 & p_1 & p_2 \\ l_2 - p_6 & p_4 & p_5 \end{array} \right) + \frac{D_2 D_{11} (s_{123} + s_{126})}{8} ,$$

$$N_7 = -\frac{1}{2\epsilon} \frac{\Delta_6}{G(1, 2, 4, 5) D_{13}} G \left(\begin{array}{ccccc} l_1 & p_1 & p_2 & p_4 & p_5 \\ l_2 & p_1 & p_2 & p_4 & p_5 \end{array} \right) .$$

2loop 6point top sector, UT integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

UT integrals list



5 MIs (this sector)
267 MIs (whole family)

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

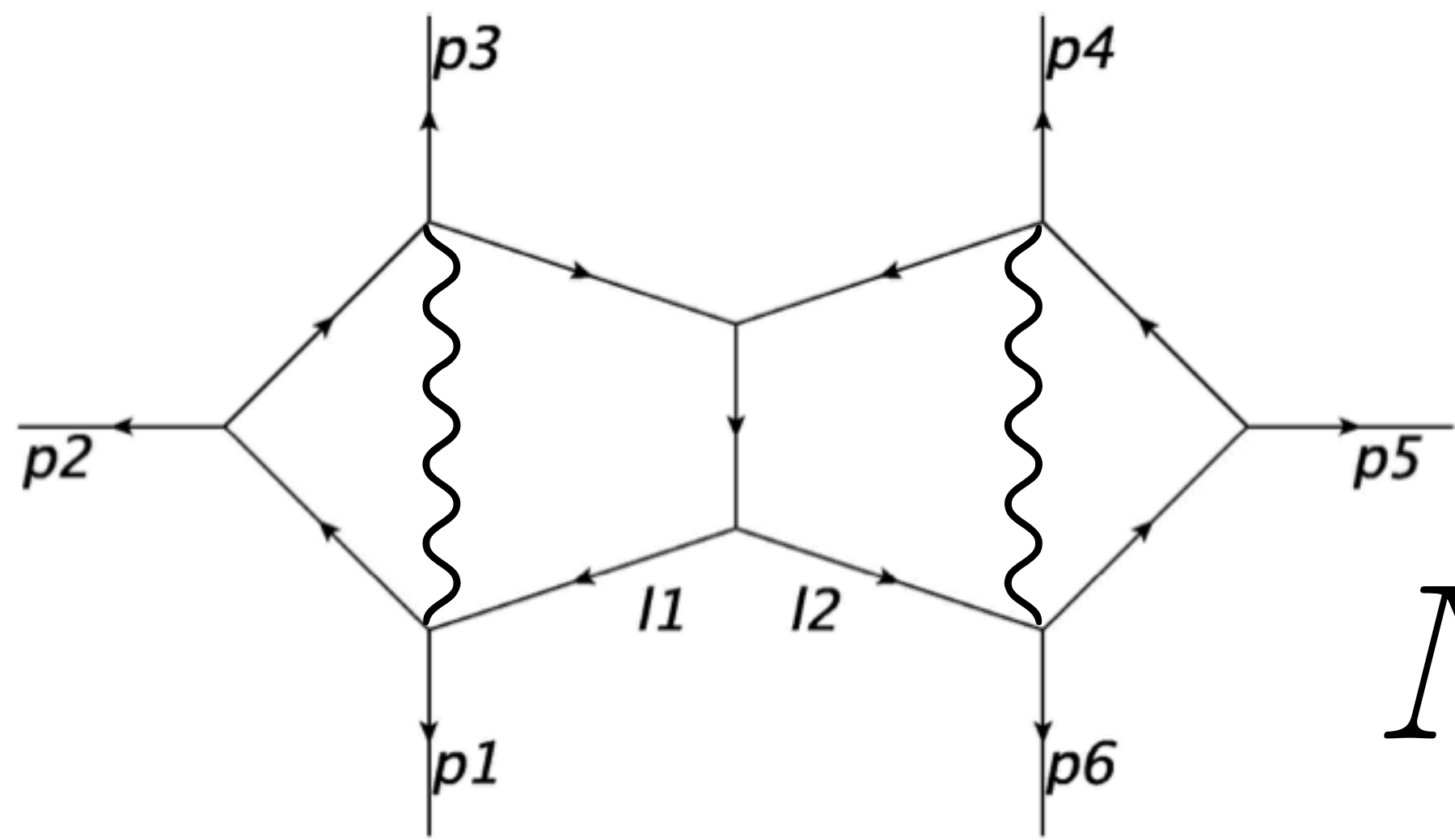
$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \longrightarrow \text{evanescent, 6D weight-6 integral}$$

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5 \mu_{12}}{D_1 \dots D_9}$$

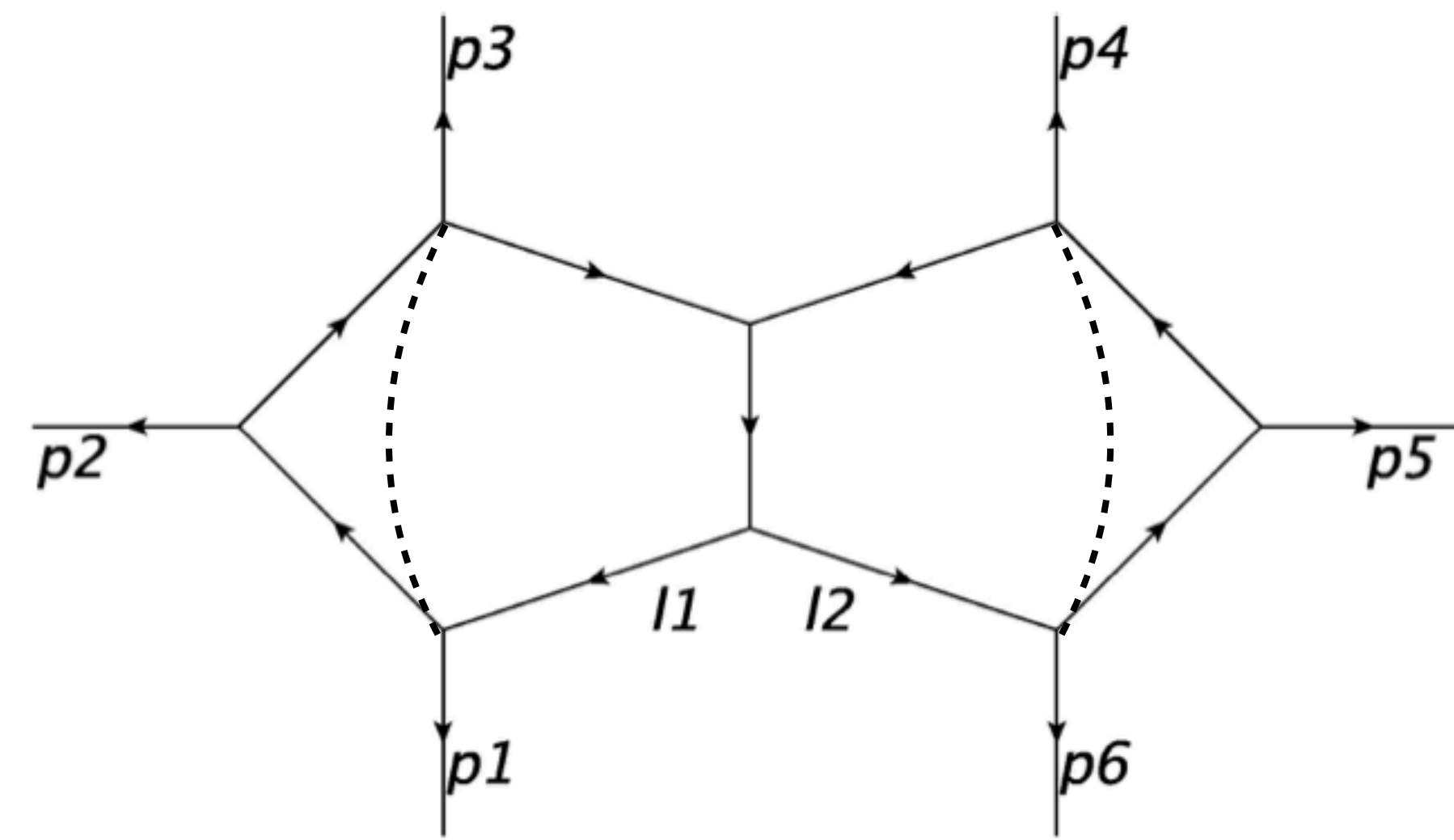
N_1, N_2, N_3 and N_4 are chiral numerators

Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010



N_1

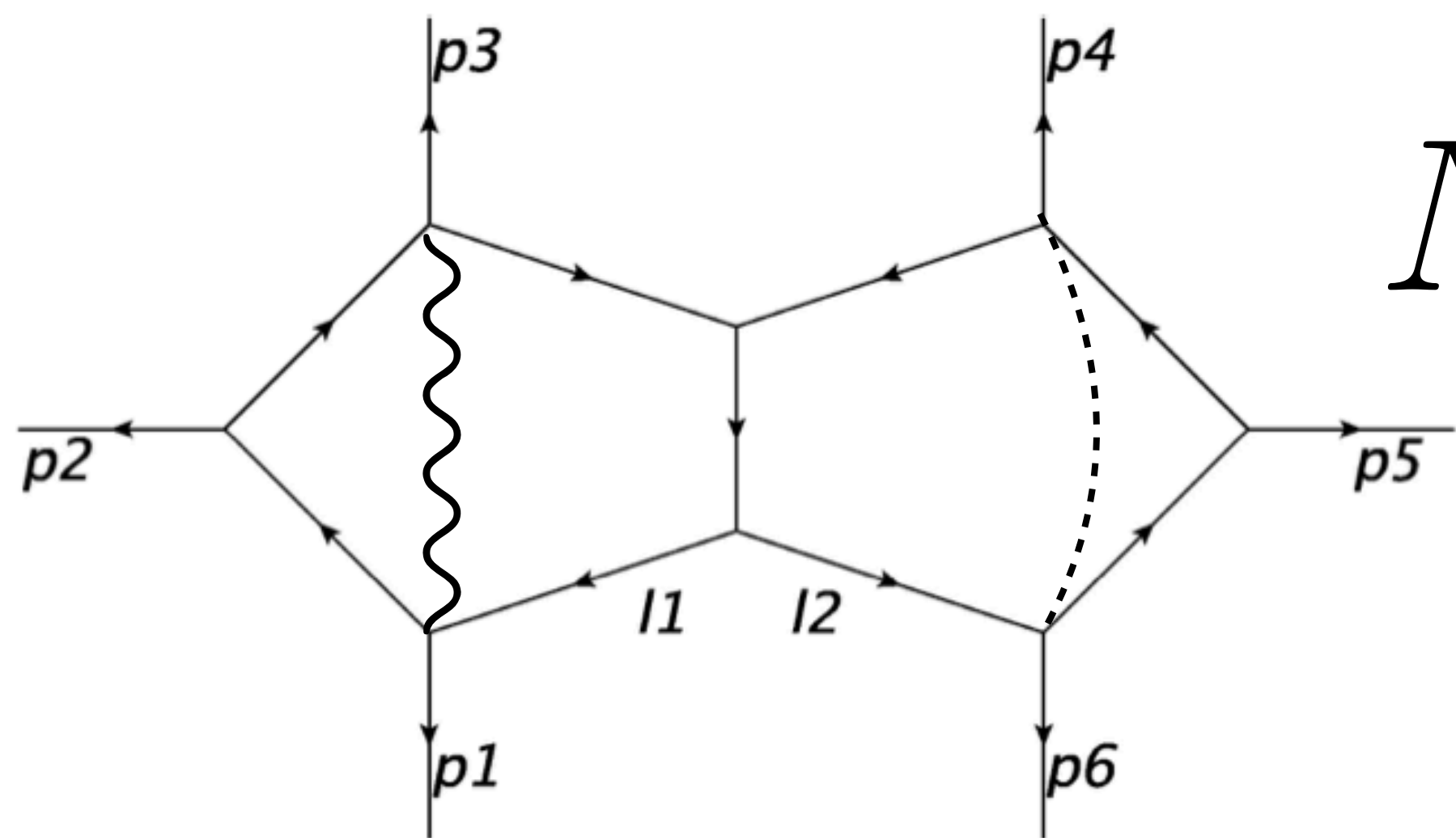
N_4



$$ut_2 = I[N_2 - N_3] = -2\tilde{\Omega}_{\text{odd}} + O(\epsilon),$$

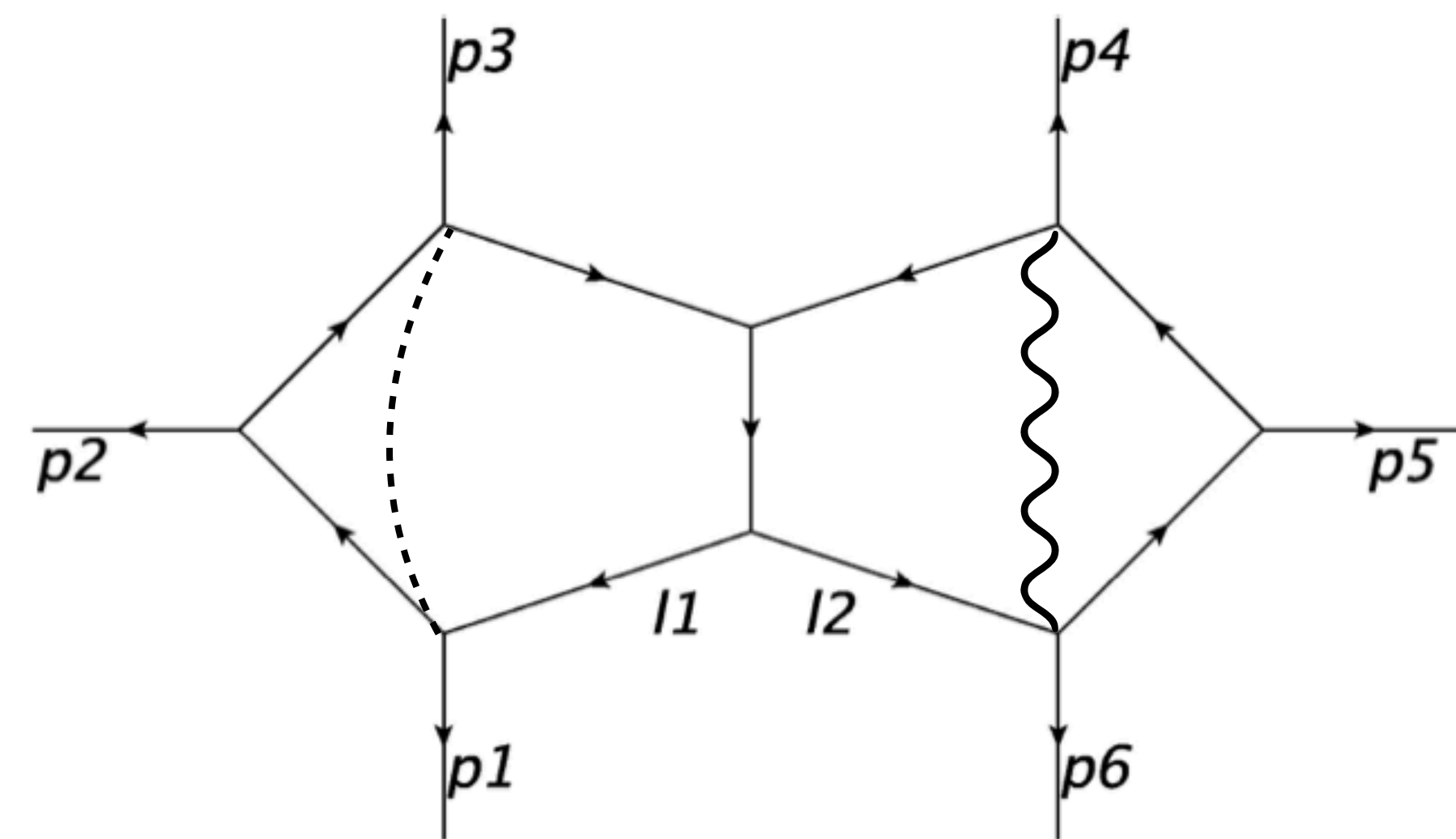
$$ut_5 = I[N_1 + N_4] = 2\Omega_{\text{even}} + O(\epsilon)$$

chiral-numerator integrals are finite and
calculated to weight-4, Dixon, Drummond, Henn 2011



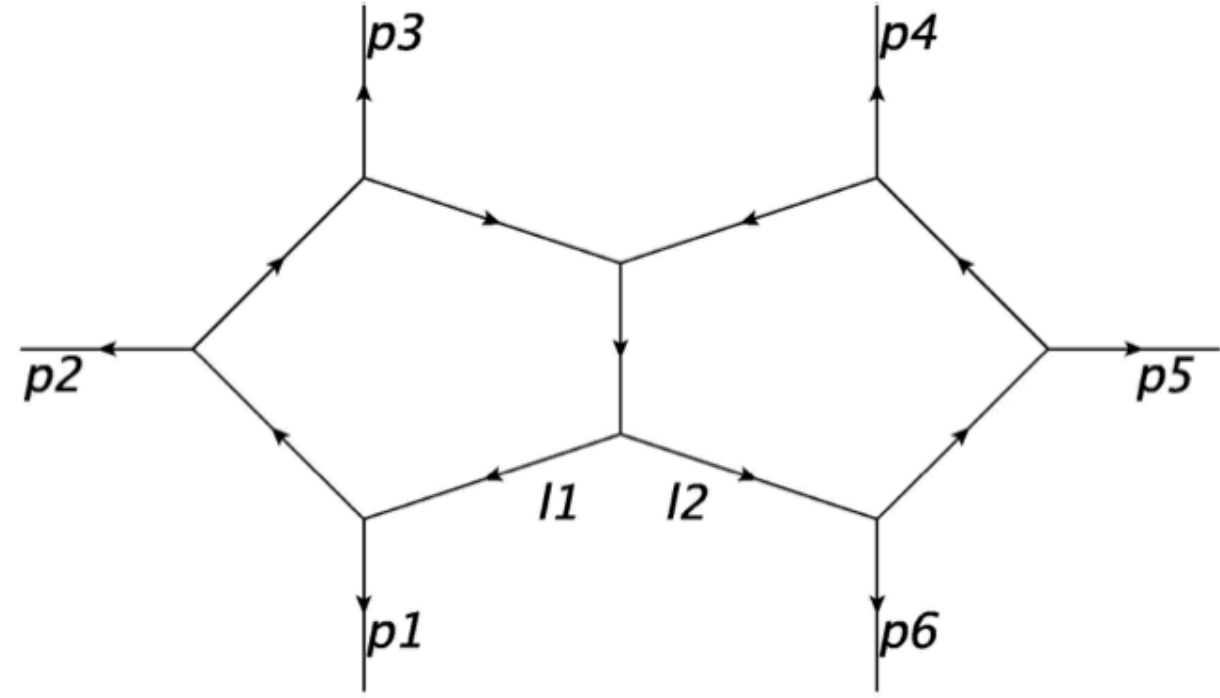
N_2

N_3



2loop 6point top sector, UT integrals

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601



UT integrals list

$$I_1^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} - N_4^{\text{DP-a}}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_2^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_2^{\text{DP-a}} - N_3^{\text{DP-a}}}{D_1 \dots D_9}$$

$$I_3^{\text{DP-a}} = F_3 \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{\mu_{12}}{D_1 \dots D_9} \longrightarrow \text{evanescent}$$

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \longrightarrow \text{evanescent, 6D weight-6 integral}$$

$$I_5^{\text{DP-a}} = \int \frac{d^{4-2\epsilon} l_1}{i\pi^{2-\epsilon}} \frac{d^{4-2\epsilon} l_2}{i\pi^{2-\epsilon}} \frac{N_1^{\text{DP-a}} + N_4^{\text{DP-a}} + F_5 \mu_{12}}{D_1 \dots D_9}$$

5 MIs (this sector)
267 MIs (whole family)

Those integrals provide **no independent** weight-4 functions ...
not needed for two-loop amplitudes to the $O(\epsilon^0)$ order

Complete canonical differential equation for 2l6p planar integrals

Use **momentum twistor**
Variables

267×267 for double pentagon
 202×202 for hexagon box

$$\frac{\partial}{\partial x_i} I(x, \epsilon) = \epsilon A_i(x) I(x, \epsilon)$$

$$A_i = \frac{\partial}{\partial x_i} \tilde{A}, \quad \tilde{A} = \sum_k \tilde{a}_k \log(W_k)$$

use alphabet to fit the canonical differential equation

Even letter, Odd letter and the more complicated ...

Even letter $F(s)$ a polynomial in Mandelstam variables
or homogeneously linear in square roots

a Feynman integral's even letters are all from Landau singularity

Odd letter $\frac{P(s) - \sqrt{Q(s)}}{P(s) + \sqrt{Q(s)}}$ $\log(W) \mapsto -\log(W)$ under the sign change of the square root

“square roots”: $\epsilon_{ijkl}, \Delta_6, \sqrt{\lambda(s_{12}, s_{34}, s_{56})}$

pseudo
scalar

leading
singularity
hexagon

Källin function
from massive triangle
diagrams

More
complicated
letter

$$\frac{P(s) - \sqrt{Q_1(s)}\sqrt{Q_2(s)}}{P(s) + \sqrt{Q_1(s)}\sqrt{Q_2(s)}}$$

Even letter, Odd letter and the more complicated ...

156 Even letters

$$\begin{aligned}
 & s_{12}, \quad s_{123} \\
 & s_{12} - s_{123} \\
 & \dots \\
 & -s_{12}s_{45} + s_{123}s_{345} \\
 & \dots \\
 & \sqrt{\lambda(s_{12}, s_{34}, s_{56})}, \quad \epsilon_{ijkl}
 \end{aligned}$$

245 letters up to weight four

79 Odd letters

$$\begin{aligned}
 & \frac{s_{12} + s_{34} - s_{56} - \sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{s_{12} + s_{34} - s_{56} + \sqrt{\lambda(s_{12}, s_{34}, s_{56})}} \\
 & \dots \\
 & \frac{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) - \epsilon_{1234}}{s_{12}s_{23} - s_{23}s_{34} + s_{23}s_{56} + s_{34}s_{123} - s_{234}(s_{12} + s_{123}) + \epsilon_{1234}}, \\
 & \dots \quad \dots \\
 & \frac{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) - \Delta_6}{-s_{12}s_{45}s_{234} + s_{34}s_{61}s_{123} + s_{345}(-s_{23}s_{56} + s_{123}s_{234}) + \Delta_6}
 \end{aligned}$$

10 More complicated letters

$$\frac{P - \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}}{P + \epsilon_{1234}\sqrt{\lambda(s_{12}, s_{34}, s_{56})}} \quad \dots$$

more letters from the 6D double pentagon, which will appear on weight 6 ...

Then the canonical DE is derived analytically after ~50 times of numeric IBP running

Boundary Values

Numeric boundary values

It is fine to use the package AMFlow to get ~ 100 digits as the boundary value for double-box, pentagon-triangle, hexagon-bubble diagrams

Liu, Wang, Ma, 2018
Liu, Ma 2022

Analytic boundary values

It is still possible to get *fully analytic* boundary values

$$X_0 : \{s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}\} \rightarrow \{-1, -1, -1, -1, -1, -1, -1, -1, -1\}$$

Solve the canonical DE on a curve starting with X_0 and require the finite solution
Some known integrals20 boundary values

} analytic boundary value

boundary value for a point in the **physical region** also obtained

Boundary Values

Analytic boundary values

Boundary values at the initial point, are combination of poly-logarithm of roots of unity

$$\epsilon^4 I_{\text{db},1}(X_0) = 1 + \frac{\pi^2}{6} \epsilon^2 + \frac{38}{3} \zeta_3 \epsilon^3 + \left(\frac{49\pi^4}{216} + \frac{32}{3} \text{Im} [\text{Li}_2(\rho)]^2 \right) \epsilon^4,$$

$$\epsilon^4 I_{\text{db},2}(X_0) = 1 + \frac{\pi^2}{6} \epsilon^2 + \frac{34}{3} \zeta_3 \epsilon^3 + \left(\frac{71\pi^4}{360} + 20 \text{Im} [\text{Li}_2(\rho)]^2 \right) \epsilon^4,$$

$$I_{\text{db},3}(X_0) = I_{\text{db},4}(X_0) = I_{\text{db},5}(X_0) = 0,$$

$$\epsilon^4 I_{\text{db},6}(X_0) = - \left(\frac{\pi^4}{540} + \frac{4}{3} \text{Im} [\text{Li}_2(\rho)]^2 \right) \epsilon^4,$$

$$\epsilon^4 I_{\text{db},7}(X_0) = 0.$$

from the ordinary differential equation
spurious pole asymptotic analysis

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Solution of canonical DE

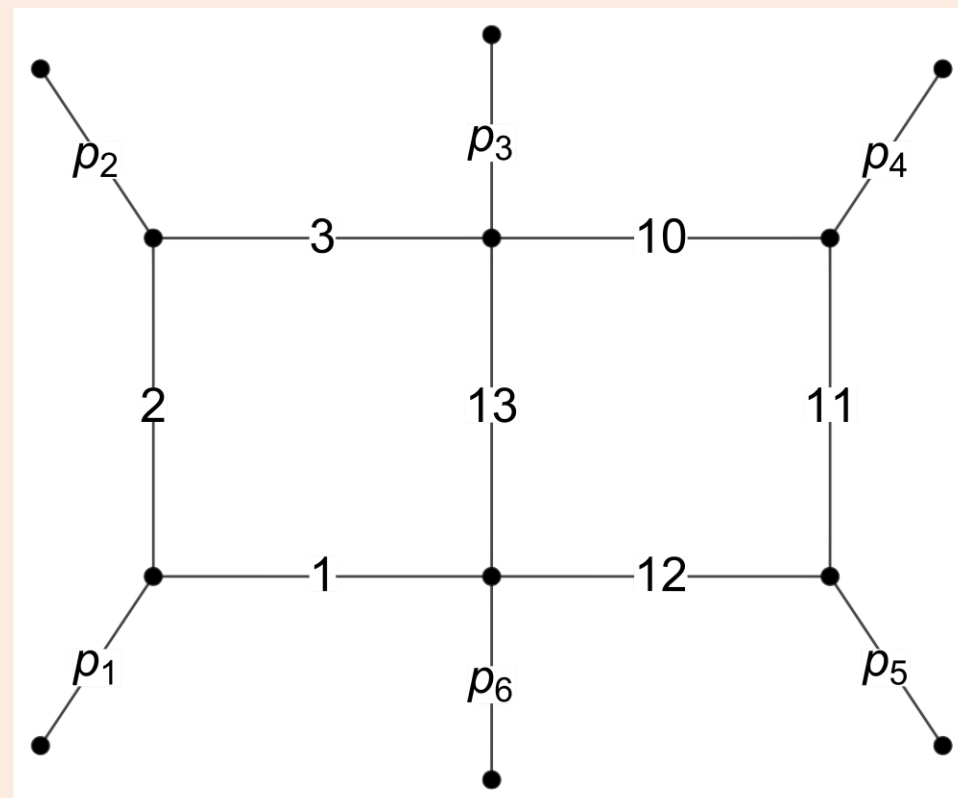
$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4} \left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots \right)$$

$$I^{(n)} = I_{X_0}^{(n)} + \int_{\gamma} (d\tilde{A}) I^{(n-1)}$$

weight-1, weight-2

All in logarithm and classical poly-logarithm



$$I_{\text{db},1}^{(2)} =$$

$$\begin{aligned} & -\log(-v_1)\log(-v_2) - \log(-v_1)\log(-v_3) + \log(-v_1)\log(-v_4) - \log(-v_1)\log(-v_5) - \\ & \log(-v_1)\log(-v_6) + 4\log(-v_1)\log(-v_8) + \frac{1}{2}\log^2(-v_1) + \log(-v_2)\log(-v_3) - \\ & \log(-v_2)\log(-v_4) - \text{Li}_2\left(1 - \frac{v_2v_5}{v_7v_8}\right) + \log(-v_2)\log(-v_6) + \log(-v_2)\log(-v_7) - \\ & 2\text{Li}_2\left(1 - \frac{v_2}{v_8}\right) - \log(-v_2)\log(-v_8) - \log^2(-v_2) - \log(-v_3)\log(-v_4) + \log(-v_3)\log(-v_5) - \\ & \text{Li}_2\left(1 - \frac{v_3v_6}{v_8v_9}\right) - 2\text{Li}_2\left(1 - \frac{v_3}{v_8}\right) - \log(-v_3)\log(-v_8) + \log(-v_3)\log(-v_9) - \\ & \log^2(-v_3) - \log(-v_4)\log(-v_5) - \log(-v_4)\log(-v_6) + 4\log(-v_4)\log(-v_8) + \\ & \frac{1}{2}\log^2(-v_4) + \log(-v_5)\log(-v_6) + \log(-v_5)\log(-v_7) - 2\text{Li}_2\left(1 - \frac{v_5}{v_8}\right) - \log(-v_5)\log(-v_8) - \\ & \log^2(-v_5) - 2\text{Li}_2\left(1 - \frac{v_6}{v_8}\right) - \log(-v_6)\log(-v_8) + \log(-v_6)\log(-v_9) - \log^2(-v_6) - \\ & \log(-v_7)\log(-v_8) - \frac{1}{2}\log^2(-v_7) - \log(-v_8)\log(-v_9) + 3\log^2(-v_8) - \frac{1}{2}\log^2(-v_9) + \\ & \frac{\pi^2}{6} \end{aligned}$$

Solution of canonical DE

$$dI = \epsilon(d\tilde{A})I$$

$$I = \frac{1}{\epsilon^4} \left(I^{(0)} + \epsilon I^{(1)} + \epsilon^2 I^{(2)} + \epsilon^3 I^{(3)} + \epsilon^4 I^{(4)} + \dots \right) \quad I^{(n)} = I_{X_0}^{(n)} + \int_{\gamma} (d\tilde{A}) I^{(n-1)}$$

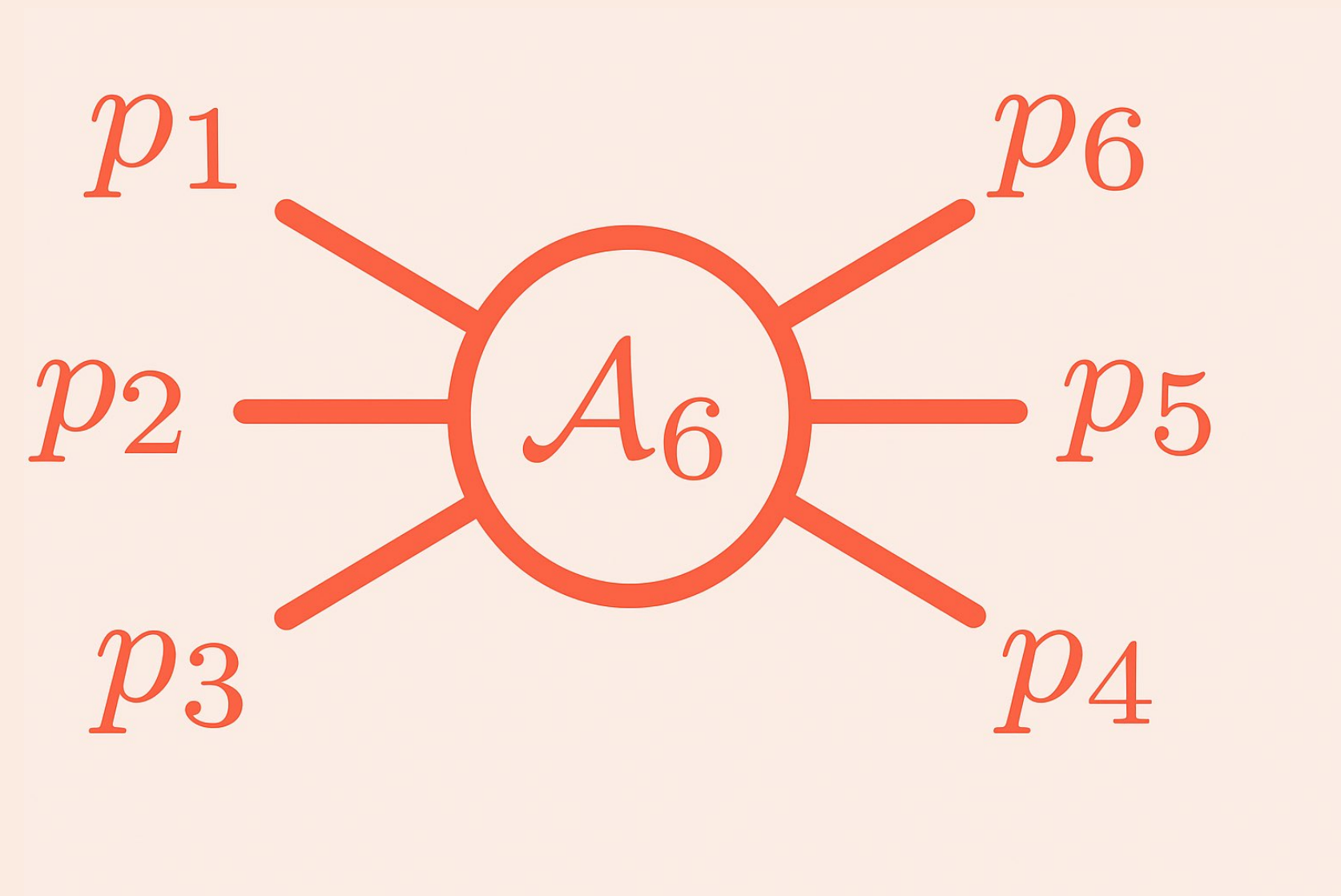
weight-3, weight-4

$$\begin{aligned} \vec{I}^{(4)} &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 dt \frac{d\tilde{A}}{dt} \vec{I}^{(3)}(\vec{x}_0) + \int_0^1 dt_1 \int_0^{t_1} dt_2 \frac{d\tilde{A}}{dt_1} \frac{d\tilde{A}}{dt_2} \vec{f}^{(2)}(t_2) \\ &= \vec{I}^{(4)}(\vec{x}_0) + \int_0^1 dt \left(\frac{d\tilde{A}}{dt} \vec{I}^{(3)}(\vec{x}_0) + \left(\tilde{A}(1) - \tilde{A}(t) \right) \frac{d\tilde{A}}{dt} \vec{f}^{(2)}(t) \right). \end{aligned} \quad \text{one-fold integration}$$

It takes **minutes on a laptop** to get 14 digits for all 2loop 6point integrals
from our solution in both Euclidean and Physical regions

Refer to Johannes Henn's talk this week

for the application of those integrals, **bootstrap**



“Bootstrapping Six-Gluon QCD Amplitudes”

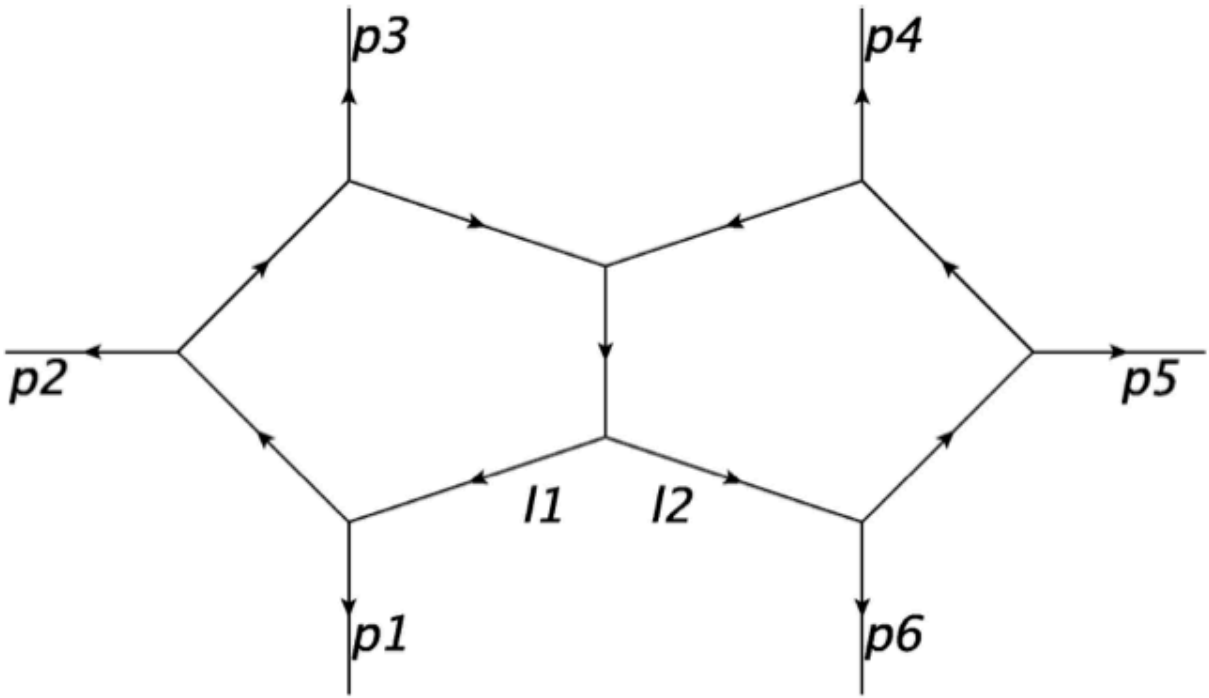
*for Next-to-Next-to-Next-to-leading order
production of 4 jets, 4 photons ...*

What about the higher weight (>4) for
two-loop six-point integrals?

Liu, Matijasic, Peraro, Yang, Xu, YZ
to appear

Double pentagon sector

Henn, Peraro, Xu, YZ, *JHEP* 03 (2022) 056

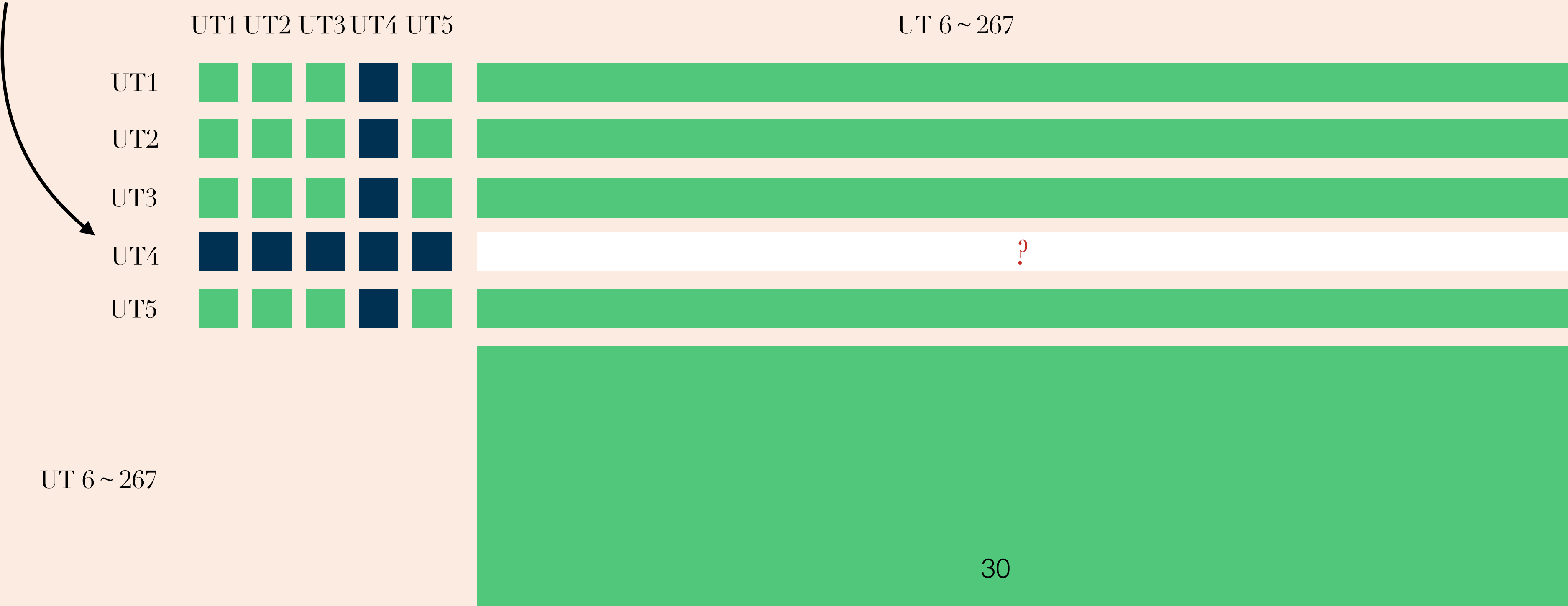


6D double pentagon, starts with weight six

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \sim O(\epsilon^2)$$

$$F_4 = \frac{1}{8} \left(G(1, 2, 3, 6) s_{45}^2 + G(1, 2, 3, 5) s_{46}^2 + G(1, 2, 3, 4) s_{56}^2 \right. \\ \left. - 2\epsilon_{1235} \epsilon_{1236} s_{45} s_{46} - 2\epsilon_{1234} \epsilon_{1236} s_{45} s_{56} - 2\epsilon_{1234} \epsilon_{1235} s_{46} s_{56} \right)^{1/2},$$

a square root
not rationalised
by momentum twistor

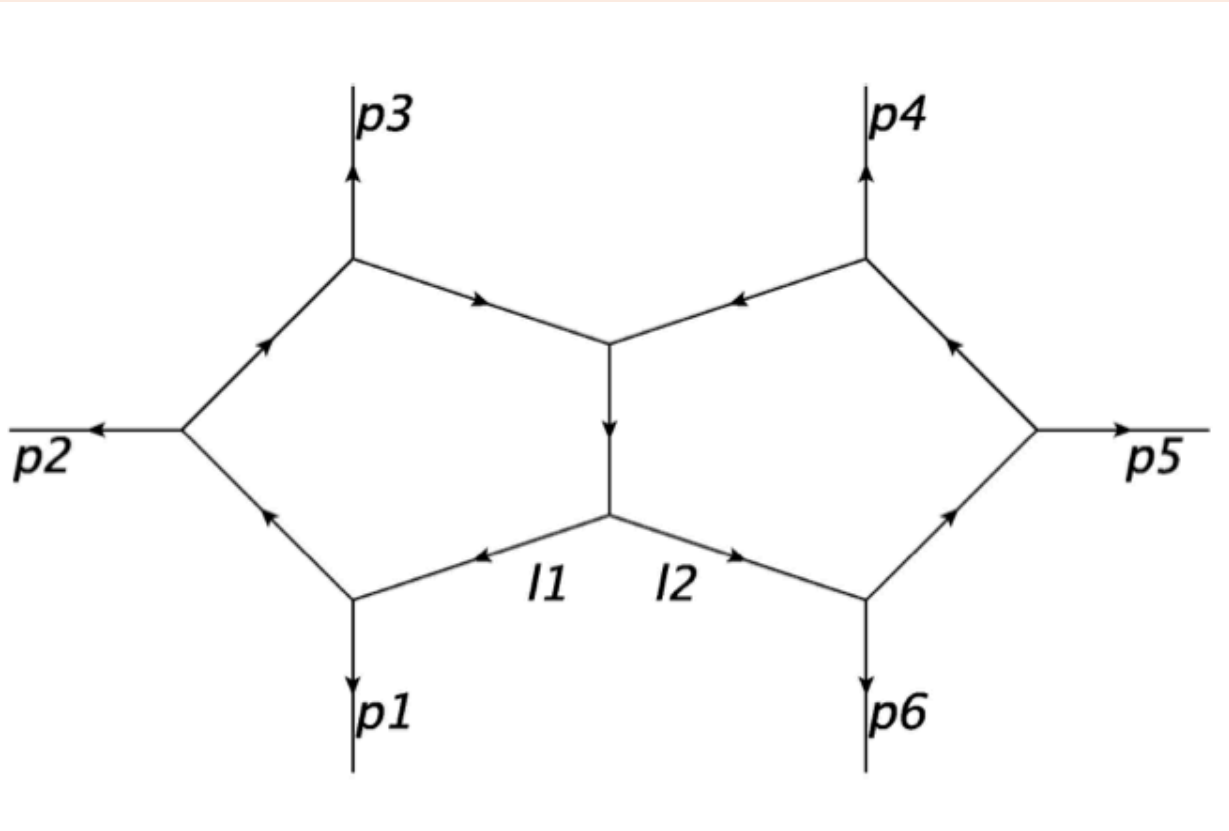


$$\frac{F_4 + R_i}{F_4 - R_i}, \quad i = 1, \dots, 5$$

18 more letters
from the diagonal block

Double pentagon sector

Henn, Peraro, Xu, YZ, *JHEP* 03 (2022) 056

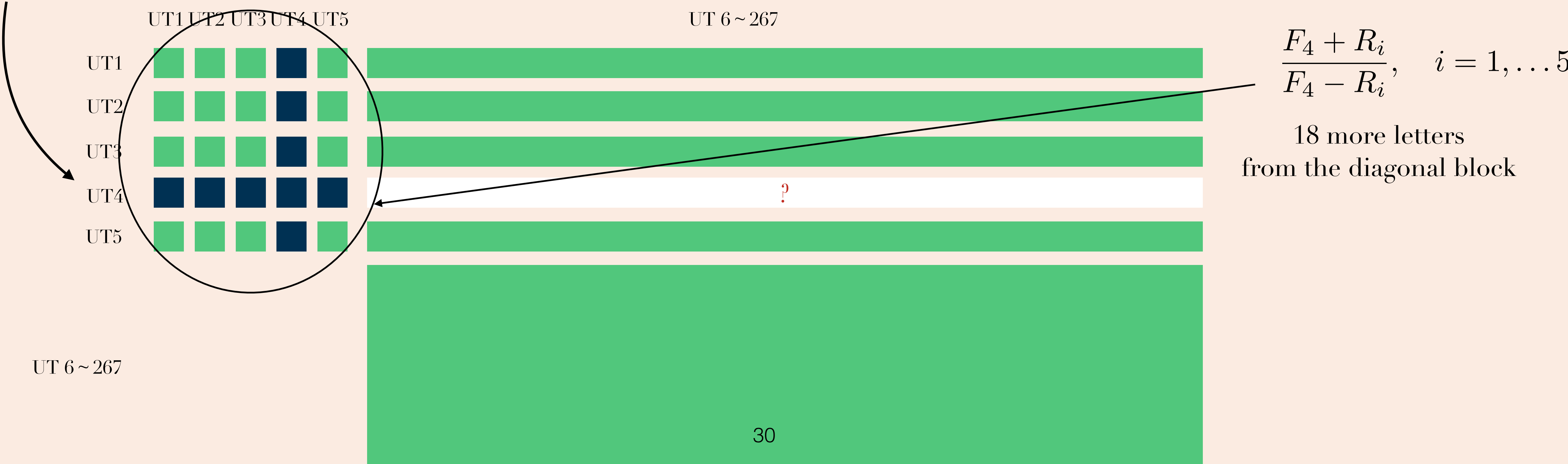


6D double pentagon, starts with weight six

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \sim O(\epsilon^2)$$

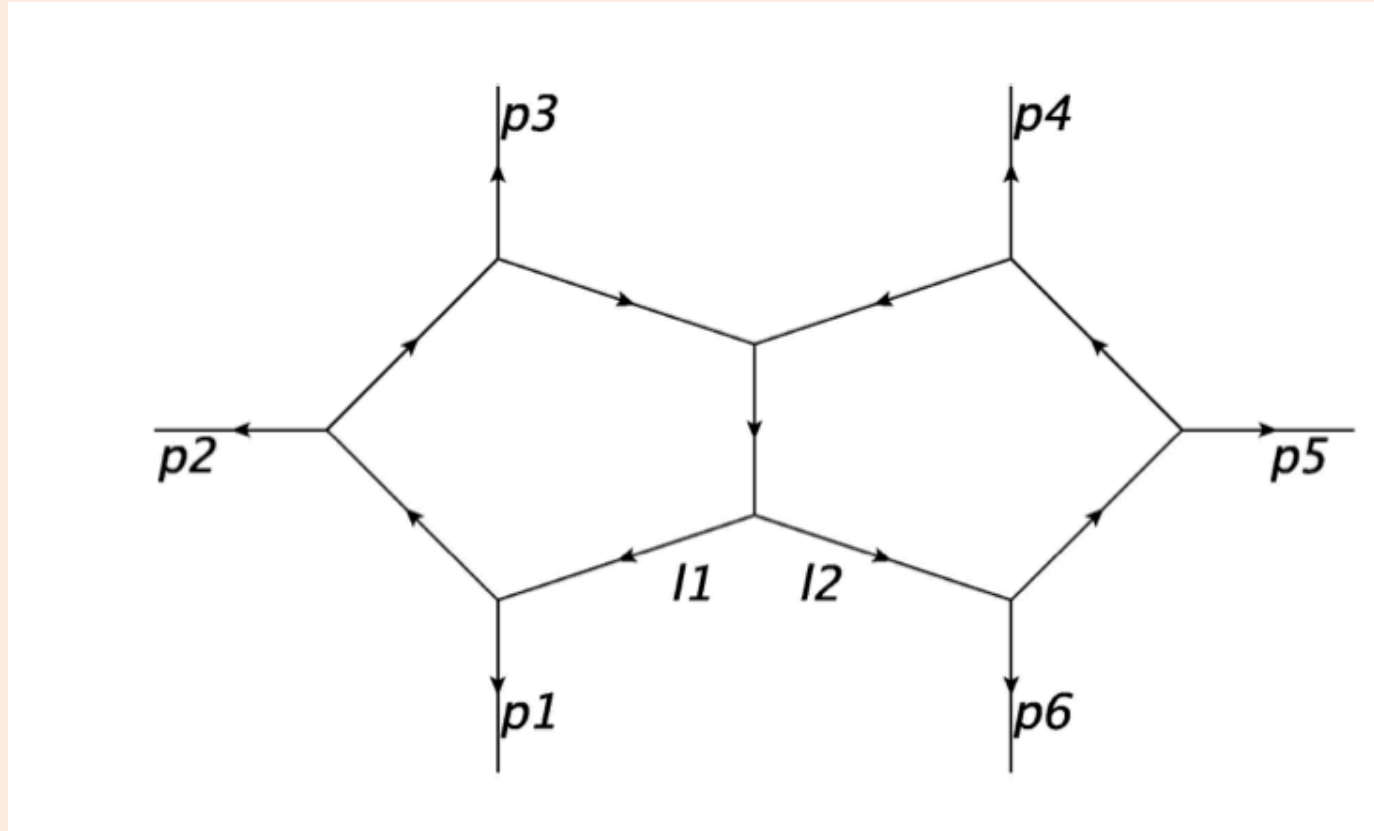
$$F_4 = \frac{1}{8} \left(G(1, 2, 3, 6) s_{45}^2 + G(1, 2, 3, 5) s_{46}^2 + G(1, 2, 3, 4) s_{56}^2 \right. \\ \left. - 2\epsilon_{1235} \epsilon_{1236} s_{45} s_{46} - 2\epsilon_{1234} \epsilon_{1236} s_{45} s_{56} - 2\epsilon_{1234} \epsilon_{1235} s_{46} s_{56} \right)^{1/2},$$

a square root
not rationalised
by momentum twistor



Double pentagon sector

Liu, Matijasic, Peraro, Yang, Xu, YZ, to appear

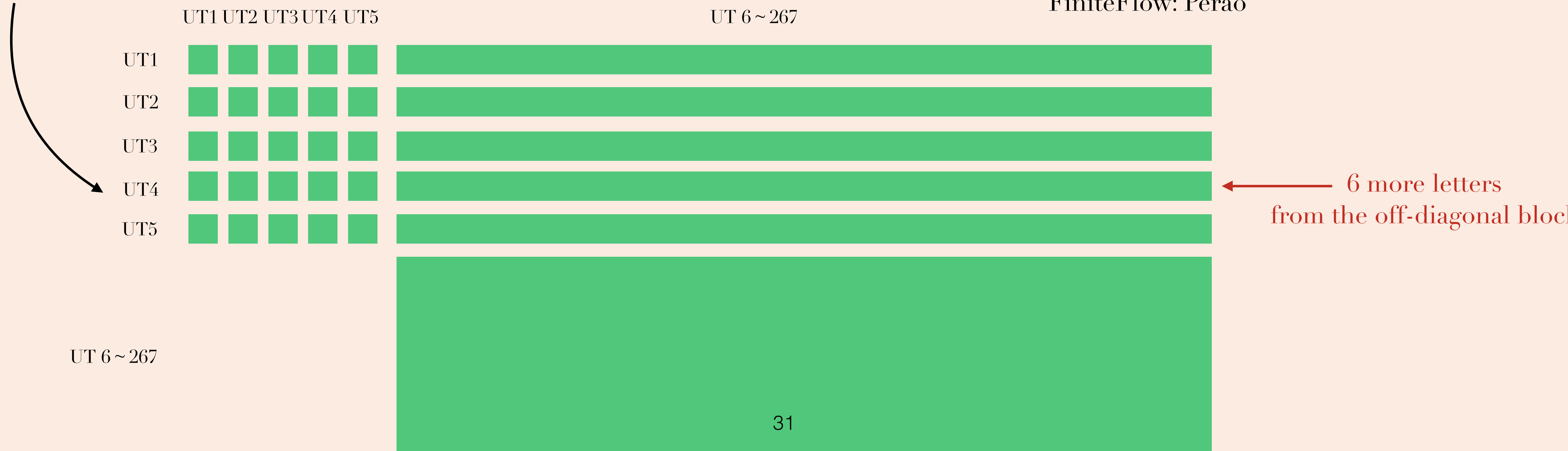


6D double pentagon, starts with weight six

$$I_4^{\text{DP-a}} = F_4 \epsilon^2 \int \frac{d^{6-2\epsilon} l_1}{i\pi^{3-\epsilon}} \frac{d^{6-2\epsilon} l_2}{i\pi^{3-\epsilon}} \frac{1}{D_1 \dots D_9} \sim O(\epsilon^2)$$

Calico: Bertolini, Fontana, Peraro

FiniteFlow: Perao



Two-loop Six-point integrals to all orders of ϵ

Henn, Matijasic, Miczajka, Peraro, Xu, YZ, Phys. Rev. Lett. 135 (2025) 3, 031601

Liu, Matijasic, Peraro, Yang, Xu, YZ, to appear

closed under the
dihedral group action

$245 + 18 + 6 = 269$ letters in total

up to weight-4

2loop 6point massless planar integrals
calculated to **weight-six**

Henn, Peraro, Xu, YZ, *JHEP* 03 (2022) 056

*for Next-to-Next-to-Next-to-leading order
production of 3 jets, 3 photons ...*

from the request of John Ellis

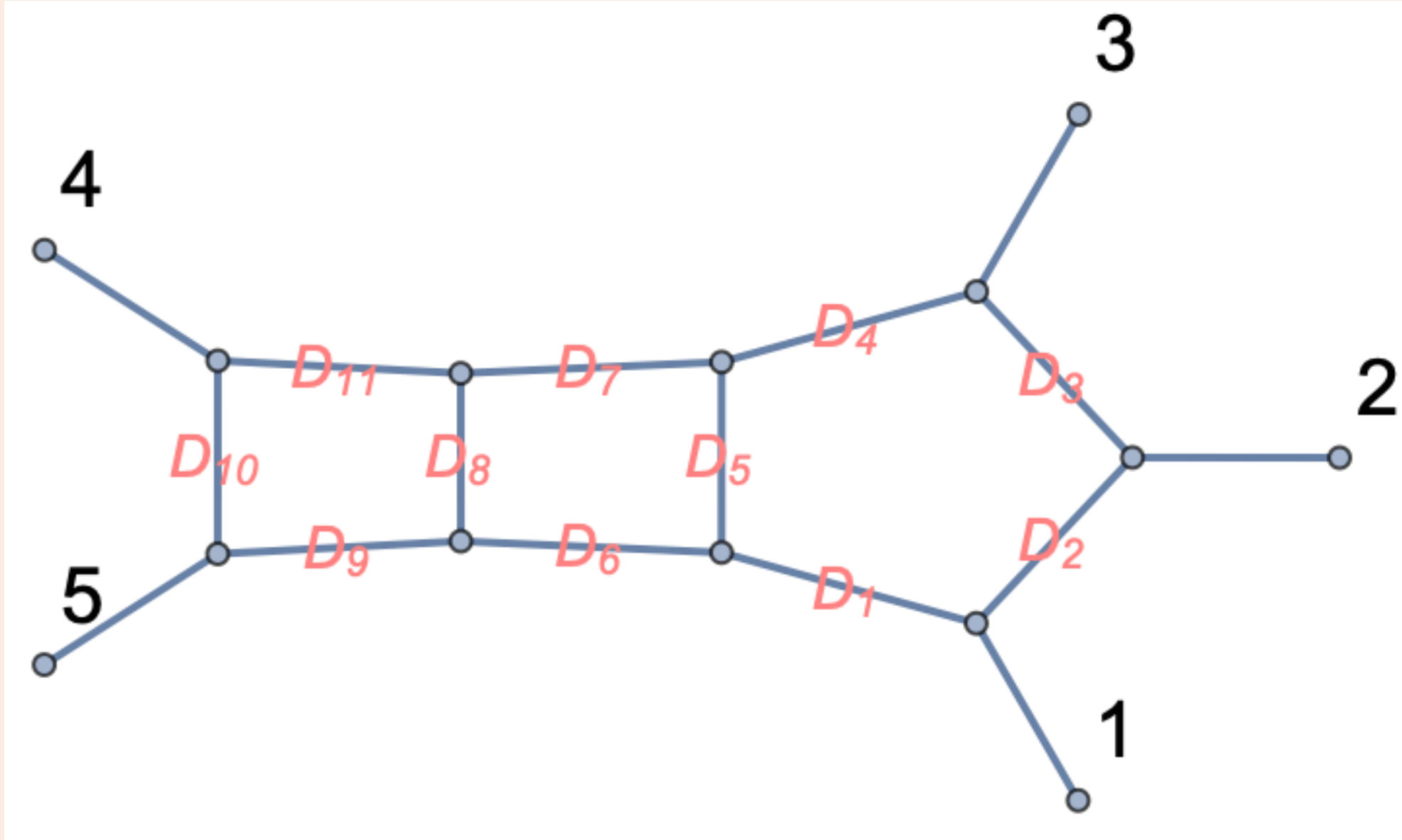
3loop 5point Feynman integrals

Liu, Matijasic, Miczajka, Xu, Xu, YZ,
Phys.Rev.D 112 (2025) 1, 016021, editors' suggestion

Chicherin, Wu, Wu, Xu, Zhang, YZ,
arXiv 2512.17330

3loop 5point planar family

Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021



5 scales

$s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals



UT basis found!

Baikov analysis
Gram determinant

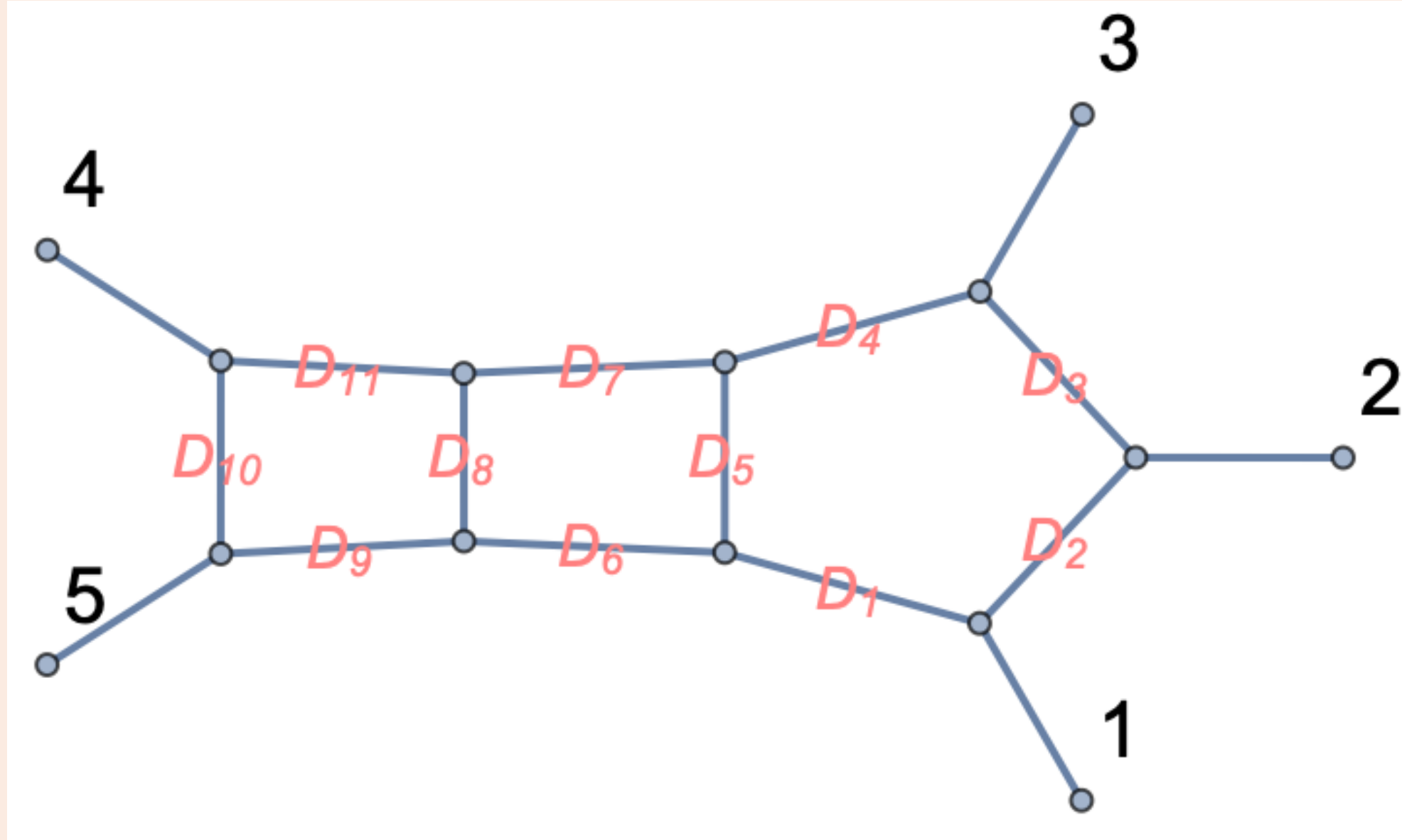
Canonical
differential
equation
complicated ?

We use **NeatIBP 1.0** to
derive the differential equation
 ~ 100 million IBPs $\rightarrow$ 85000 IBPs

hard to integrate
to weight-6?

A novel one-fold
representation

First family of 3loop 5point



5 scales $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$

316 Master Integrals

✓ UT basis found!

✓ Canonical differential equation found with NeatIBP

31 letters ...

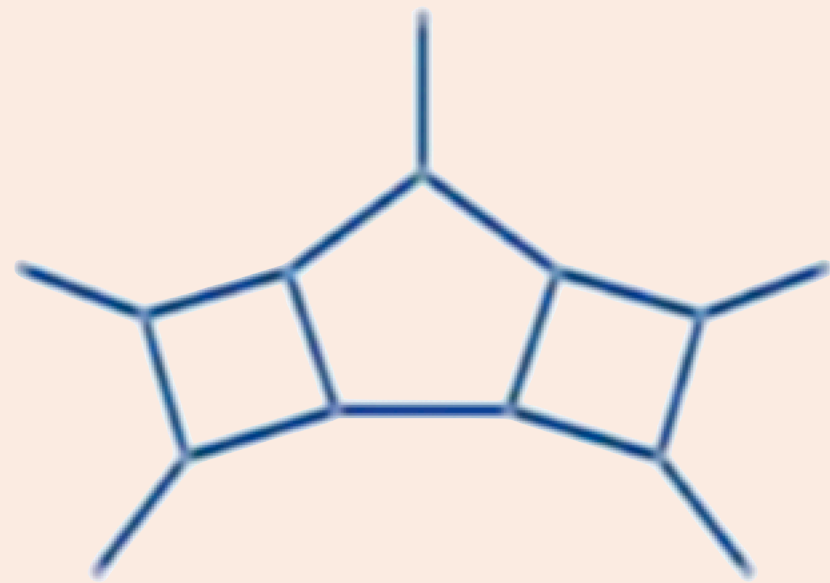
All boundary values up to weight-6 are obtained by spurious pole analysis

✓ weight-1,2,3 classical polylogarithm, weight-4,5,6 one-fold integration
It takes 2 minutes on a laptop to get 10 digits from our analytic solution

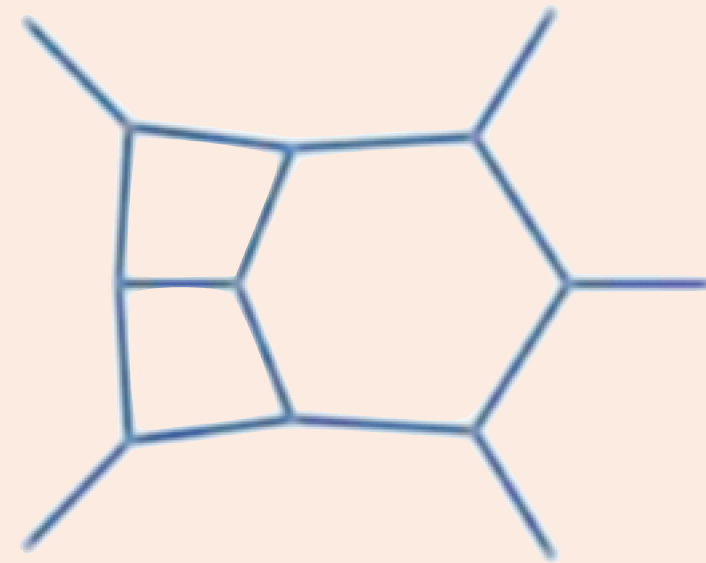
Liu, Matijasic, Miczajka, Xu, Xu, YZ, Phys.Rev.D 112 (2025) 1, 016021 editors' suggestion

Complete result of all 3loop 5point integrals

Chicherin, Wu, Wu, Xu, Zhang, YZ,
arXiv 2512.17330



367 MIs



431 MIs



734 MIs

IBP reduction is extremely complicated, we have to use the latest **NeatIBP 1.1** version

NeatIBP 1.1: Syzygy + Spanning Cuts

Most complicated IBP system with cut, consists of $\sim 10^5$ equations

Complete result of all 3loop 5point integrals

Chicherin, Wu, Wu, Xu, Zhang, YZ,
arXiv 2512.17330

pure basis determination



top sector has 19 MIs

loop-by-loop dlog integrand generation, dimensional shift
module lift method

$$\mathcal{N} = \sum_I c_I ISP^{\alpha_I}$$

$\sum_I c_I \text{residue}_j(ISP^{\alpha_I}) = r_j$,
 c_I **polynomial** in Mandelstam variables, r_j **rational numbers**

$$\mathcal{N}_4 = \frac{\varepsilon(s_{12} - s_{45})}{(1 + 2\varepsilon)\epsilon_5} G(\ell_2, \ell_3, k_1, k_2, k_3, k_4),$$

$$\mathcal{N}_{13} = s_{12}s_{34}s_{45}(\ell_2 - k_1)^2(\ell_3 - k_{123})^2,$$

$$\mathcal{N}_{17} = s_{12}s_{45}(\ell_3 + k_5)^2 \left((\ell_2 - k_{123})^2(\ell_3 - k_{12})^2 - (\ell_3 - k_{123})^2(\ell_2 - k_{12})^2 \right),$$

$$\mathcal{N}_{19} = s_{12}s_{15}s_{45} \left((\ell_2 - k_{123})^2(\ell_3 - k_{12})^2 - (\ell_3 - k_{123})^2(\ell_2 - k_{12})^2 \right).$$

We have not used chiral numerator for the pure integral construction, $\text{tr}(\gamma_5 \dots)$, ...

Complete result of all 3loop 5point integrals

Chicherin, Wu, Wu, Xu, Zhang,YZ,
arXiv 2512.17330

From the canonical differential equation

We confirmed the 3loop 5point letters from Chicherin, Henn, Trnka, Zhang *JHEP* 04 (2025) 022

56 Letters

$$\mathbb{A}_{\text{P}}^{3\text{-loop}} = \mathbb{A}_{\text{P}}^{2\text{-loop}} \cup \{\widehat{W}_i\}_{i=1}^5 \cup \{\widetilde{W}_i\}_{i=16}^{20} \cup \{\widetilde{W}_i\}_{i=41}^{55} \cup \{\widetilde{W}_i\}_{i=76}^{80}$$

26 Letters

Symbols

weight	1	2	3	4	5	6
3loop 5point Symbols	5	20	76	285	1000	2220

$$\begin{aligned}\widehat{W}_1 &= s_{23}s_{34} - s_{34}s_{45} + s_{45}s_{15} \,, \\ \widetilde{W}_{16} &= \Delta_4^{(1)} \,, \\ \widetilde{W}_i &= \frac{q_i - \sqrt{\Delta_4^{(1)}}}{q_i + \sqrt{\Delta_4^{(1)}}}, \; i \in \{41, 46, 51\} \,, \\ \widetilde{W}_{76} &= \frac{q_{76} - \epsilon_5 \sqrt{\Delta_4^{(1)}}}{q_{76} + \epsilon_5 \sqrt{\Delta_4^{(1)}}} \,,\end{aligned}$$

Three-loop pentagonal Wilson loop with Lagrangian insertion
Chicherin, Henn, Xu, Zhang,YZ arxiv: 2512.17881

small number
good news for bootstrap

Summary and Outlook

Analytic computation of all 2loop 6point planar massless integrals,
all 3loop 5point integrals are done;

Cutting-edge multi-loop amplitude computations: full color $W\gamma\gamma$, $Hb\bar{b}$

NeatIBP, a powerful package for cutting-edge IBP reduction for precision physics

Multi-loop multi-leg Feynman integrals are no longer that difficult!