Hidden Structures in Scattering Amplitudes

From colored scalars to pions, gluons, and more

Qu Cao (Zhejiang University)

Based on 2312.16282, 2401.00041, 2401.05483, 2403.08855, 2406.03838,2408.11891,2412.19629, 2503.15860, 2504.21676…

W/~ Nima Arkani-Hamed, Jin Dong, Carolina Figueiredo, Song He, Canxi Shi, Fanky Zhu.

Contents

Motivation

Question: How to **understand** scattering amplitudes in the QFT at the weak coupling?

$$\mathcal{S} = \langle out \,|\, in \rangle$$

Answer 1: Feynman diagram expansion. ——————————Not good

Too many diagrams. Not manifest symmetry (gauge invariance) in each diagram.

Answer 2: Global description (Amplitu-hedron/integral)——better

One global object. Manifest symmetry.

Motivation

Question: How to **understand** scattering amplitudes in the QFT at the weak coupling?

$$\mathcal{S} = \langle out \,|\, in \rangle$$

Toy model: when QFT is the $\mathrm{Tr}(\phi^3)$ theory

$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3)$$

Answer 1: $\mathcal{A}_n(1,\ldots,n) = \sum_{g\in\mathcal{T}} \text{Feynman diagrams} = \sum_{g\in\mathcal{T}} \prod_{e\in\mathcal{E}(g)} \frac{1}{P_e^2}$

Motivation

Question: How to **understand** scattering amplitudes in the QFT at the weak coupling?

$$\mathcal{S} = \langle out \,|\, in \rangle$$

Toy model: when QFT is the $\mathrm{Tr}(\phi^3)$ theory

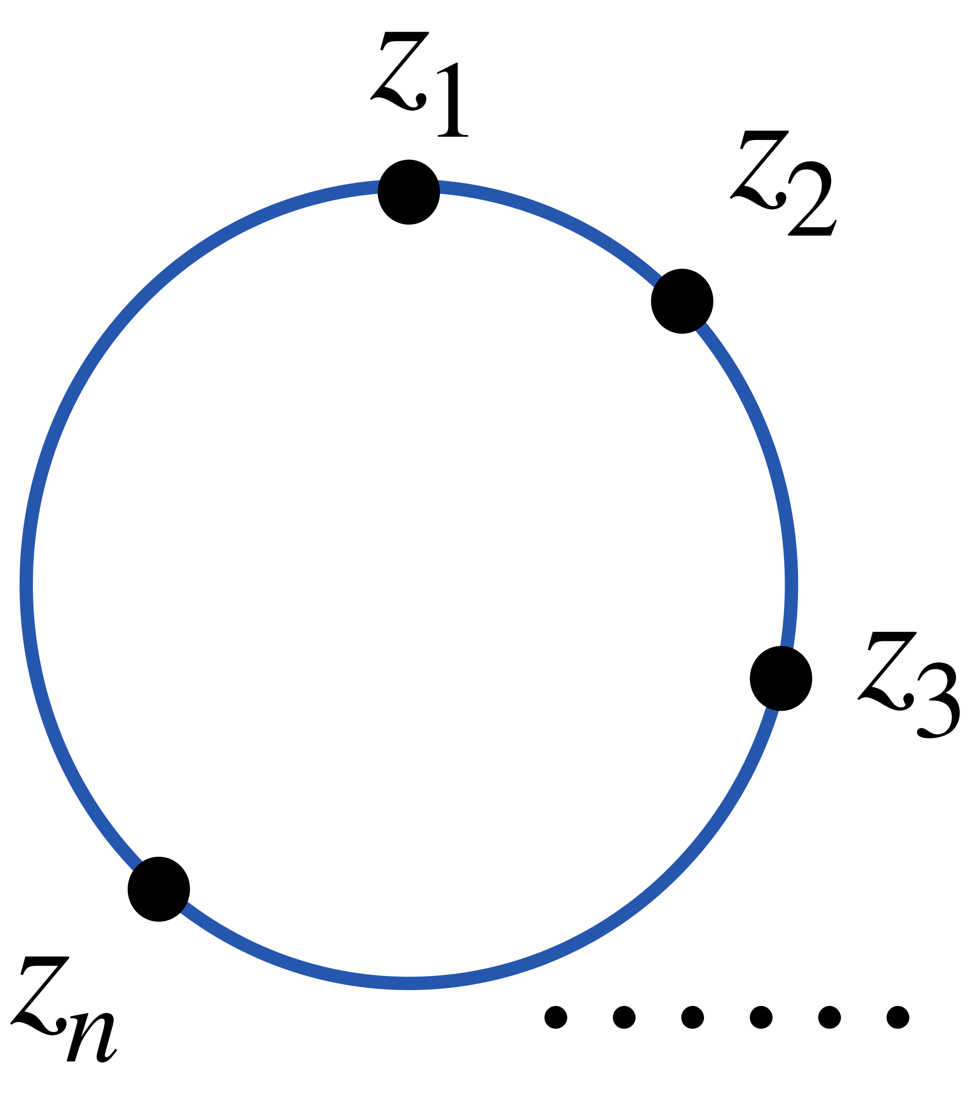
$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3)$$

Stringy integral

Answer 2: $\mathcal{A}_n(1,\ldots,n) \leftrightarrow \lim_{\alpha'\to 0} \int_{D(1\ldots n)} \frac{\mathrm{d}z_1\ldots\ \mathrm{d}z_n}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{R})} \frac{1}{z_{1,2}z_{2,3}\cdots z_{n,1}} \times \prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}$

Stringy integral

$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{D(1\ldots n)} \frac{dz_1\ldots\ dz_n}{\text{vol SL}(2,\mathbb{R})} \frac{1}{z_{1,2}z_{2,3}\cdots z_{n,1}} \times \prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}$$



$D(1\ldots n) := z_1 < \ldots < z_n$ (with three of them fixed)

$z_{i,j} := z_i - z_j$

$\frac{1}{z_{1,2}z_{2,3}\cdots z_{n,1}}$ Parke-Taylor factor

$\prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}$ Koba-Nielsen factor

$p_i^2 = 0$ Massless

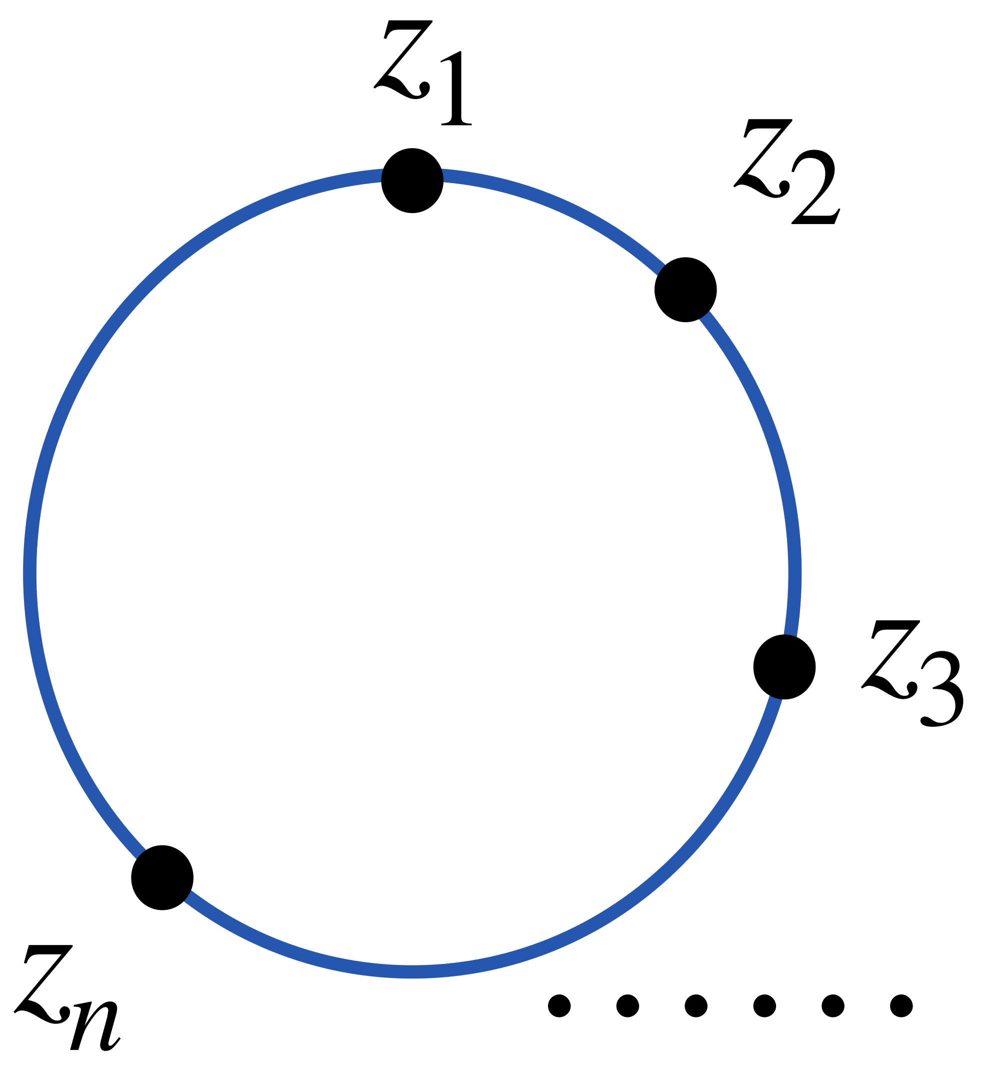
$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \langle J_1 e^{ip_1X(z_1)}\ldots J_n e^{ip_nX(z_n)}\rangle$$

Correlation function on open-string worldsheet

(Tree-level open-string amplitude)

Stringy integral

$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{D(1\ldots n)} \frac{dz_1\ldots\ dz_n}{\text{vol SL}(2,\mathbb{R})} \frac{1}{z_{1,2}z_{2,3}\ldots z_{n,1}} \times \prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}$$

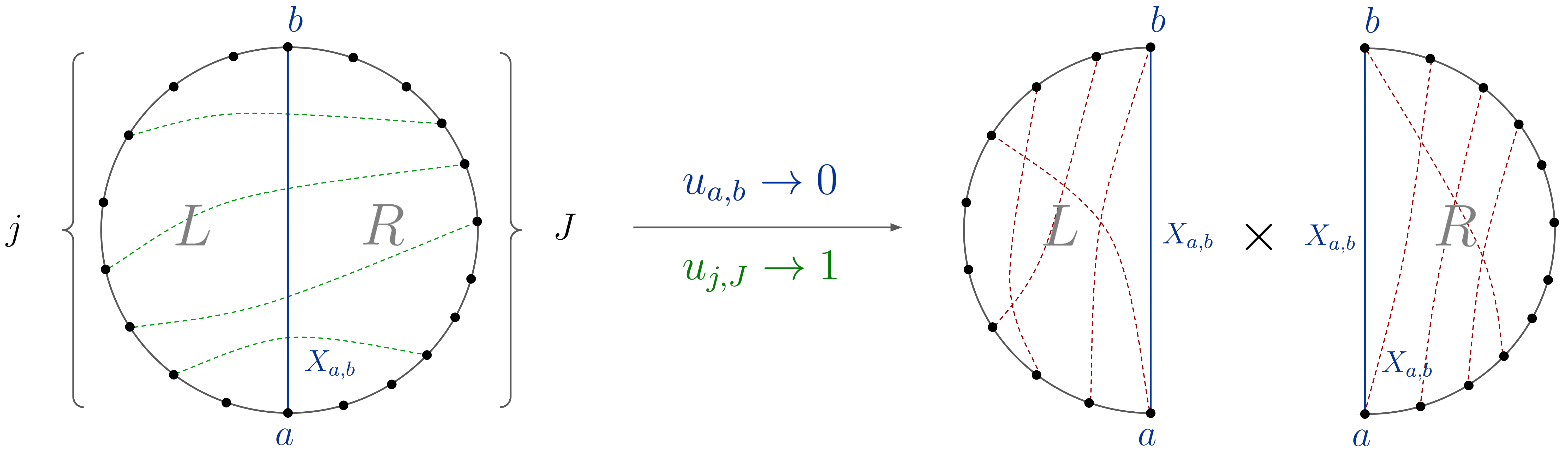


$$u_{i,j} = \frac{z_{i-1,j}z_{i,j-1}}{z_{i,j}z_{i-1,j-1}} \qquad X_{i,j} = (p_i + p_{i+1} + \ldots + p_{j-1})^2$$

$$\prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j} = \prod_{i<j} u_{i,j}^{\alpha' X_{i,j}}$$

$$u_{i,j} + \prod_{(k,l)\text{ cross }(i,j)} u_{k,l} = 1$$

u-equations



$$\prod_X u_X^X \longrightarrow \prod_{X_L} u_{X_L}^{X_L} \times \prod_{X_R} u_{X_R}^{X_R}$$

Stringy integral

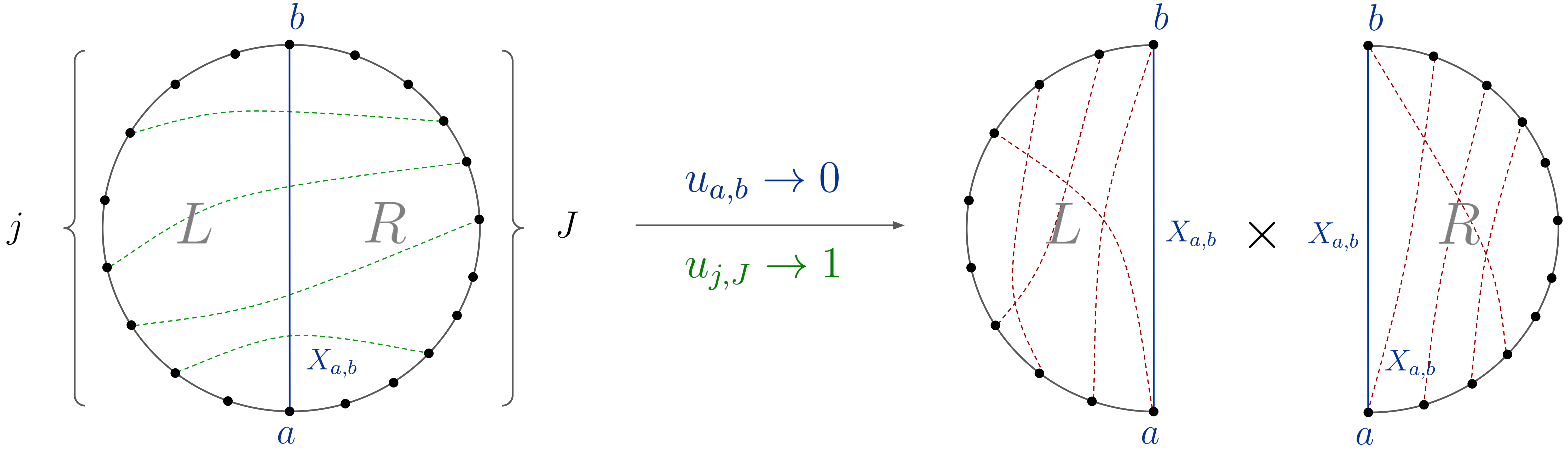
$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{U_n^+} \Omega\left(U_n^+\right) \prod_{i<j} u_{i,j}^{\alpha' X_{i,j}}$$

Stringy integral in u-variables
(Global description)

$$u_{i,j} = \frac{z_{i-1,j} z_{i,j-1}}{z_{i,j} z_{i-1,j-1}} \qquad X_{i,j} = (p_i + p_{i+1} + \ldots + p_{j-1})^2$$

$$\prod_{i<j} z_{i,j}^{2\alpha' p_i \cdot p_j} = \prod_{i<j} u_{i,j}^{\alpha' X_{i,}}$$

$$u_{i,j} + \prod_{(k,l)\ \text{cross}\ (i,j)} u_{k,l} = 1$$



$$\prod_X u_X^X \longrightarrow \prod_{X_L} u_{X_L}^{X_L} \times \prod_{X_R} u_{X_R}^{X_R}$$

Stringy integral

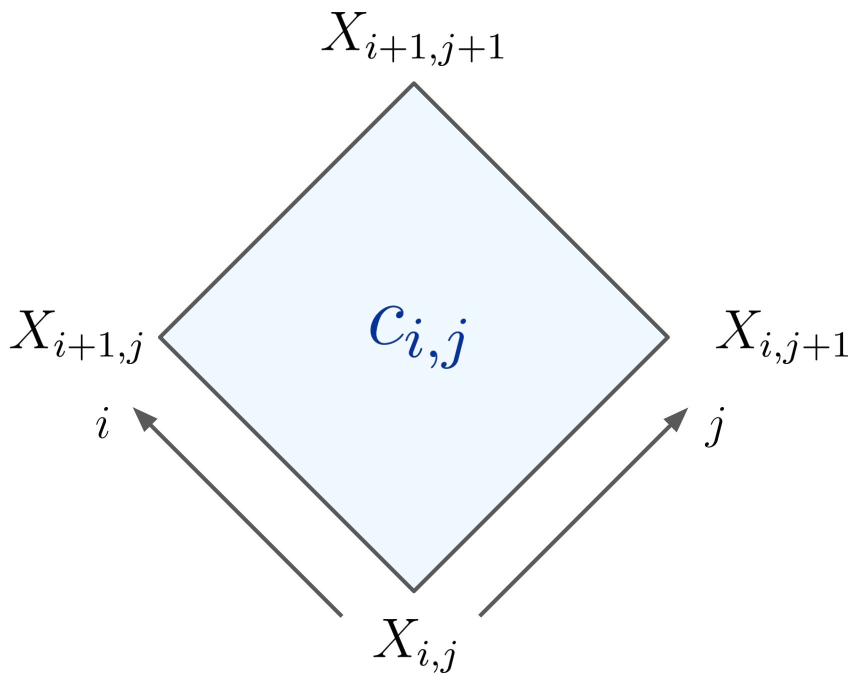
$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{U_n^+} \Omega\left(U_n^+\right) \prod_{i<j} u_{i,j}^{\alpha' X_{i,j}}$$

Stringy integral in u-variables (Global description)

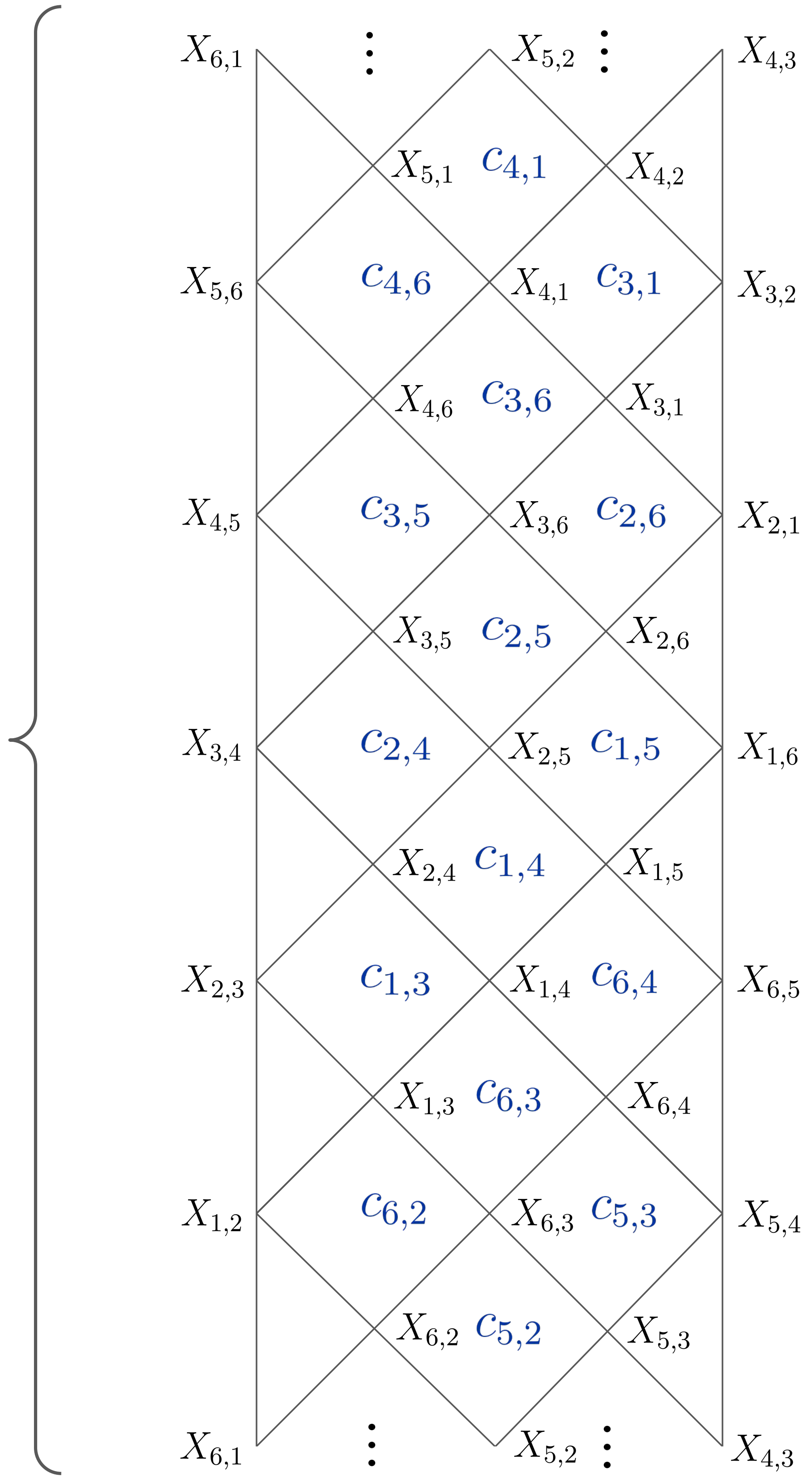
$$u_{i,j} = \frac{z_{i-1,j} z_{i,j-1}}{z_{i,j} z_{i-1,j-1}} \qquad X_{i,j} = (p_i + p_{i+1} + \ldots + p_{j-1})^2$$

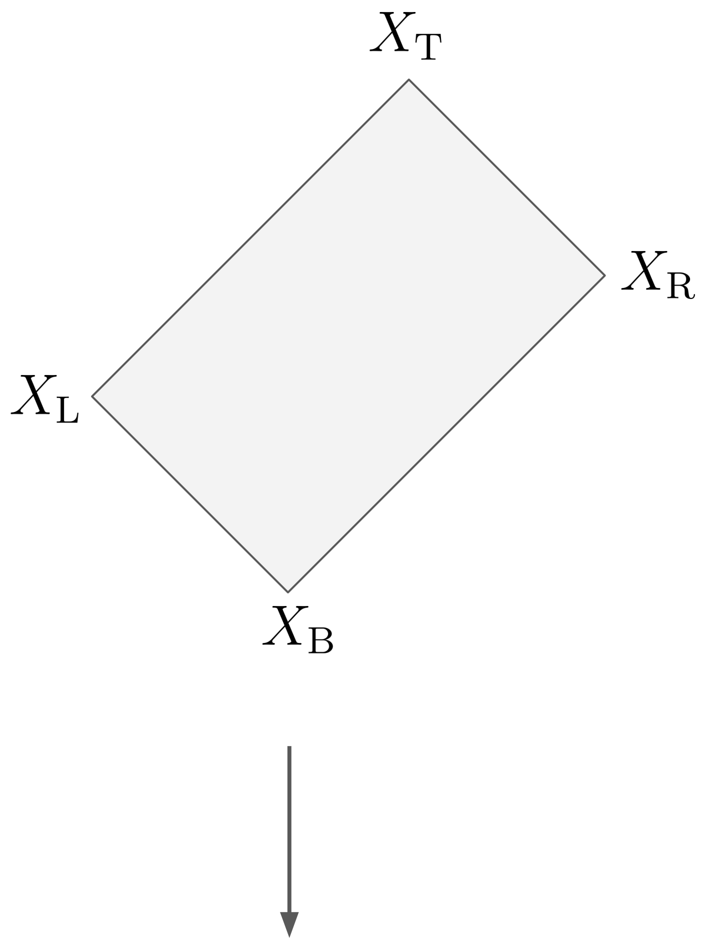
$$X_{i,j} : \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

$$c_{i,j} := -2p_i \cdot p_j = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j}$$



Kinematic mesh





$$X_{\mathrm{B}} + X_{\mathrm{T}} - X_{\mathrm{L}} - X_{\mathrm{R}} = \sum_{c_{i,j} \in \diamond} c_{i,j}$$

Stringy integral

$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{U_n^+} \Omega\left(U_n^+\right) \prod_{i<j} u_{i,j}^{\alpha' X_{i,j}}$$

Stringy integral in u-variables (Global description)

$$u_{i,j} = \frac{z_{i-1,j} z_{i,j-1}}{z_{i,j} z_{i-1,j-1}} \qquad X_{i,j} = (p_i + p_{i+1} + \ldots + p_{j-1})^2$$

$$X_{i,j} : \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

All $\frac{n(n-3)}{2}$ $X_{i,j}$'s can form a basis.

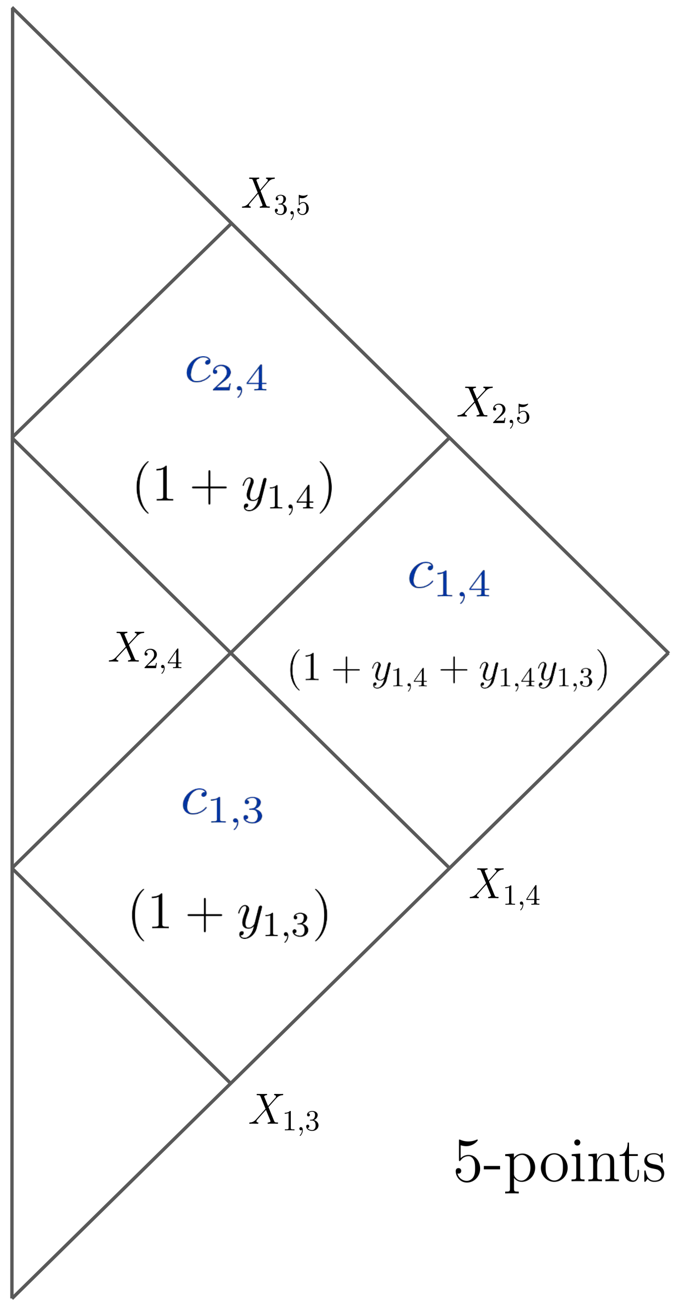
One can also use $(n-3)$ $X_{i,j}$'s of a triangulation and some $c_{i,j}$'s form a basis.

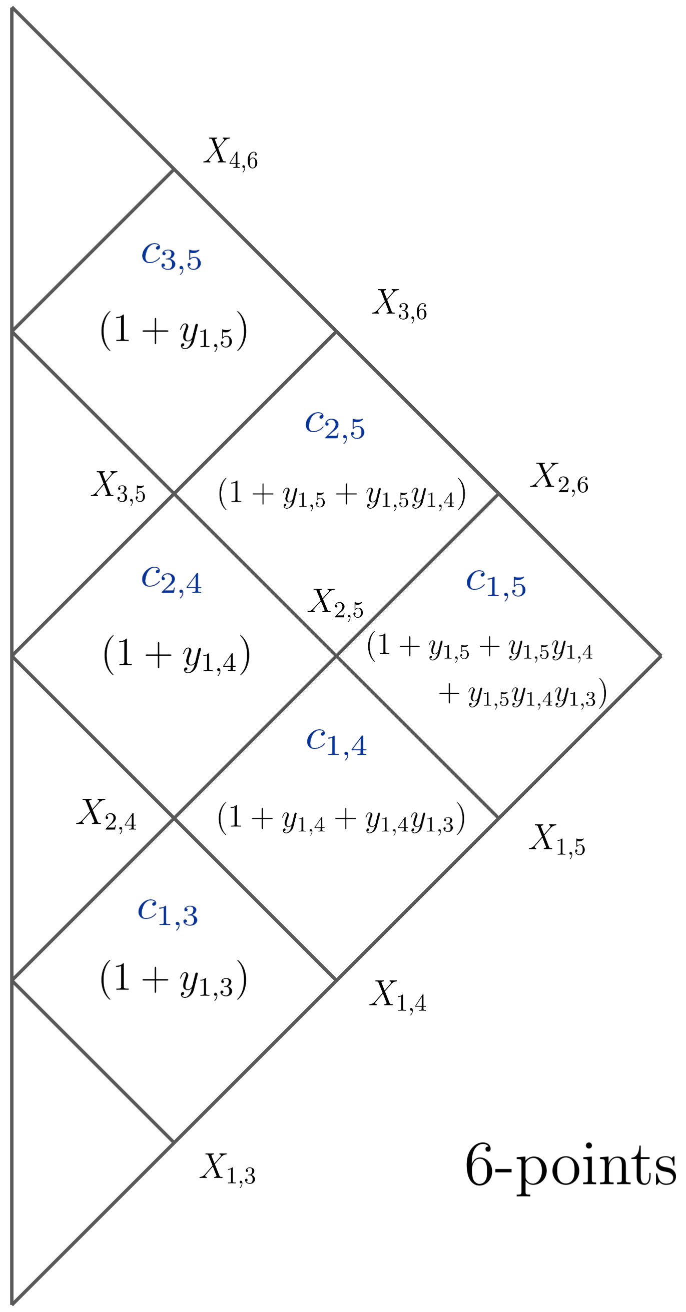
$$c_{i,j} := -2p_i \cdot p_j = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j}$$

Stringy integral

$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{U_n^+} \Omega\left(U_n^+\right) \prod_{i<j} u_{i,j}^{\alpha' X_{i,j}}$$

Stringy integral in u-variables
(Global description)





Basis 1 : $\{X_{1,3}, X_{1,4}, X_{1,5}, c_{1,3}, c_{1,4}, c_{1,5}, c_{2,4}, c_{2,5}, c_{3,5}\}$

$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{\mathbb{R}_{>0}^{n-3}} \prod_{i=3}^{n-1} \frac{\mathrm{d}y_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}} \prod_{1\leq i,j<n} F_{i,j}(y)^{-\alpha' c_{i,j}}$$

$$F_{i,j} = 1 + y_{1,j} + y_{1,j}y_{1,j-1} + \cdots + y_{1,j}\ldots y_{1,i+2}$$

Stringy integral in y-variables
(Local description in ray-like T)

Zeros and factorizations

$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{\mathbb{R}_{>0}^{n-3}} \prod_{i=3}^{n-1} \frac{\mathrm{d}y_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}} \prod_{1\leq i,j<n} F_{i,j}(y)^{-\alpha' c_{i,j}}$$

$$F_{i,j} = 1 + y_{1,j} + y_{1,j}y_{1,j-1} + \cdots + y_{1,j}\ldots y_{1,i+2}$$

<u>Manifest zeros and a new kind of factorization near the zeros.</u>

$$\mathcal{I}_4^{tr(\phi^3)} = \int_{\mathbb{R}_{>0}} \frac{\mathrm{d}y_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}} (1+y_{1,3})^{-\alpha' c_{13}} = \frac{\Gamma[\alpha' X_{1,3}]\Gamma[\alpha'(c_{1,3}-X_{1,3})]}{\Gamma[\alpha' c_{1,3}]} = \frac{\Gamma[\alpha' X_{1,3}]\Gamma[\alpha' X_{2,4}]}{\Gamma[\alpha' c_{1,3}]}$$

In the field-theory limit ($\alpha' \to 0$), we get $\mathcal{I}_4^{tr(\phi^3)} \to \dfrac{c_{1,3}}{\alpha' X_{1,3} X_{2,4}} = \dfrac{1}{\alpha' X_{1,3}} + \dfrac{1}{\alpha' X_{2,4}}$, which vanishes when $c_{1,3} = 0$.

Zeros and factorizations

$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{\mathbb{R}_{>0}^{n-3}} \prod_{i=3}^{n-1} \frac{\mathrm{d}y_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}} \prod_{1\leq i,j<n} F_{i,j}(y)^{-\alpha' c_{i,j}}$$

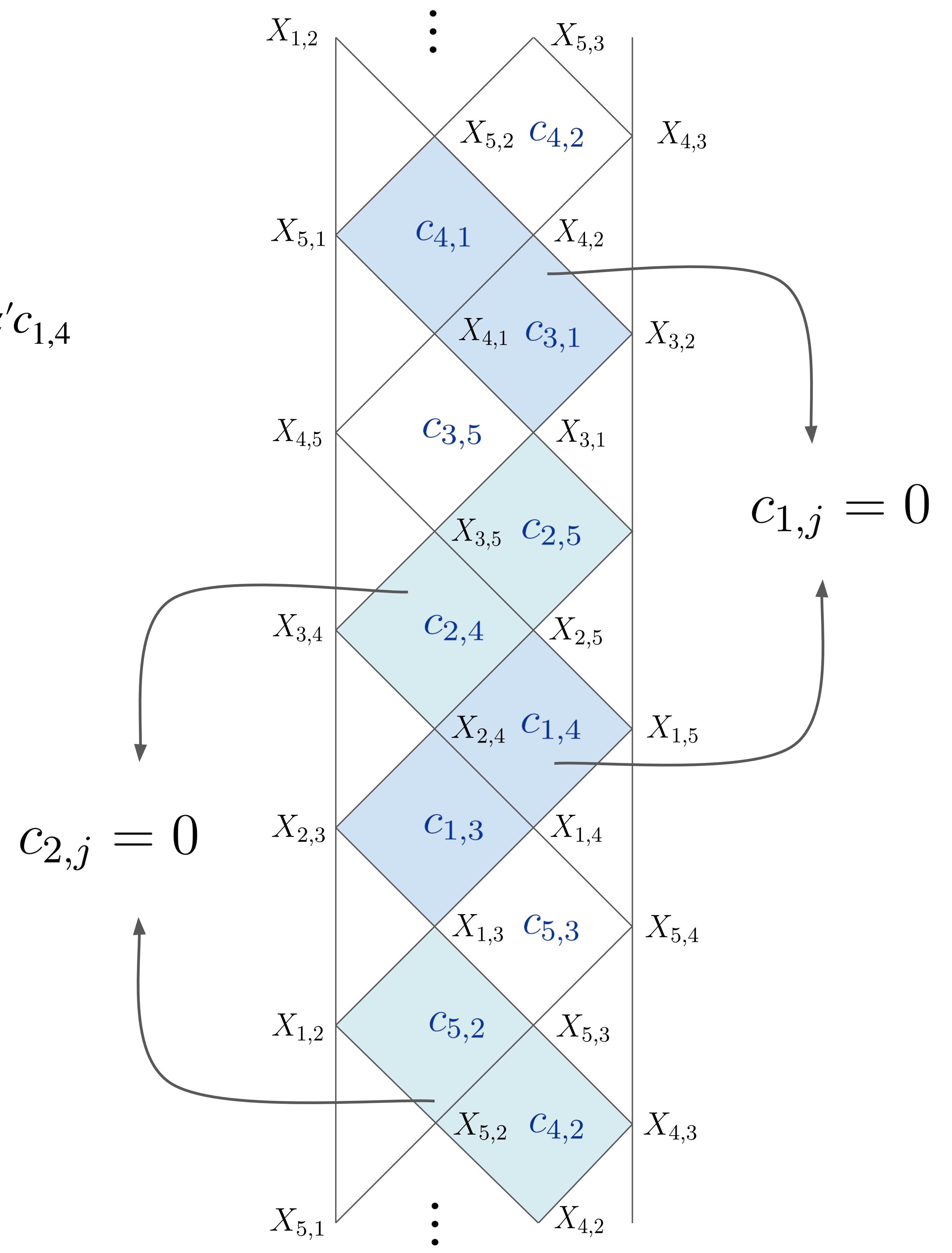
$$F_{i,j} = 1 + y_{1,j} + y_{1,j}y_{1,j-1} + \cdots + y_{1,j}\ldots y_{1,i+2}$$

$$\mathcal{I}_5^{tr(\phi^3)} = \int_0^\infty \frac{\mathrm{d}y_{1,3}}{y_{1,3}}\frac{\mathrm{d}y_{1,4}}{y_{1,4}} y_{1,3}^{\alpha' X_{1,3}} y_{1,4}^{\alpha' X_{1,4}} \left(1+y_{1,3}\right)^{-\alpha' c_{1,3}} \left(1+y_{1,4}\right)^{-\alpha' c_{2,4}} \left(1+y_{1,4}+y_{1,3}y_{1,4}\right)^{-\alpha' c_{1,4}}$$

Setting $c_{1,3} = c_{1,4} = 0$,

$$\mathcal{I}_5^{tr(\phi^3)} \to \left(\int_0^\infty \frac{\mathrm{d}y_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}}\right) \int_0^\infty \frac{\mathrm{d}y_{1,4}}{y_{1,4}} y_{1,4}^{\alpha' X_{1,4}} \left(1+y_{1,4}\right)^{-\alpha' c_{2,4}} = 0$$

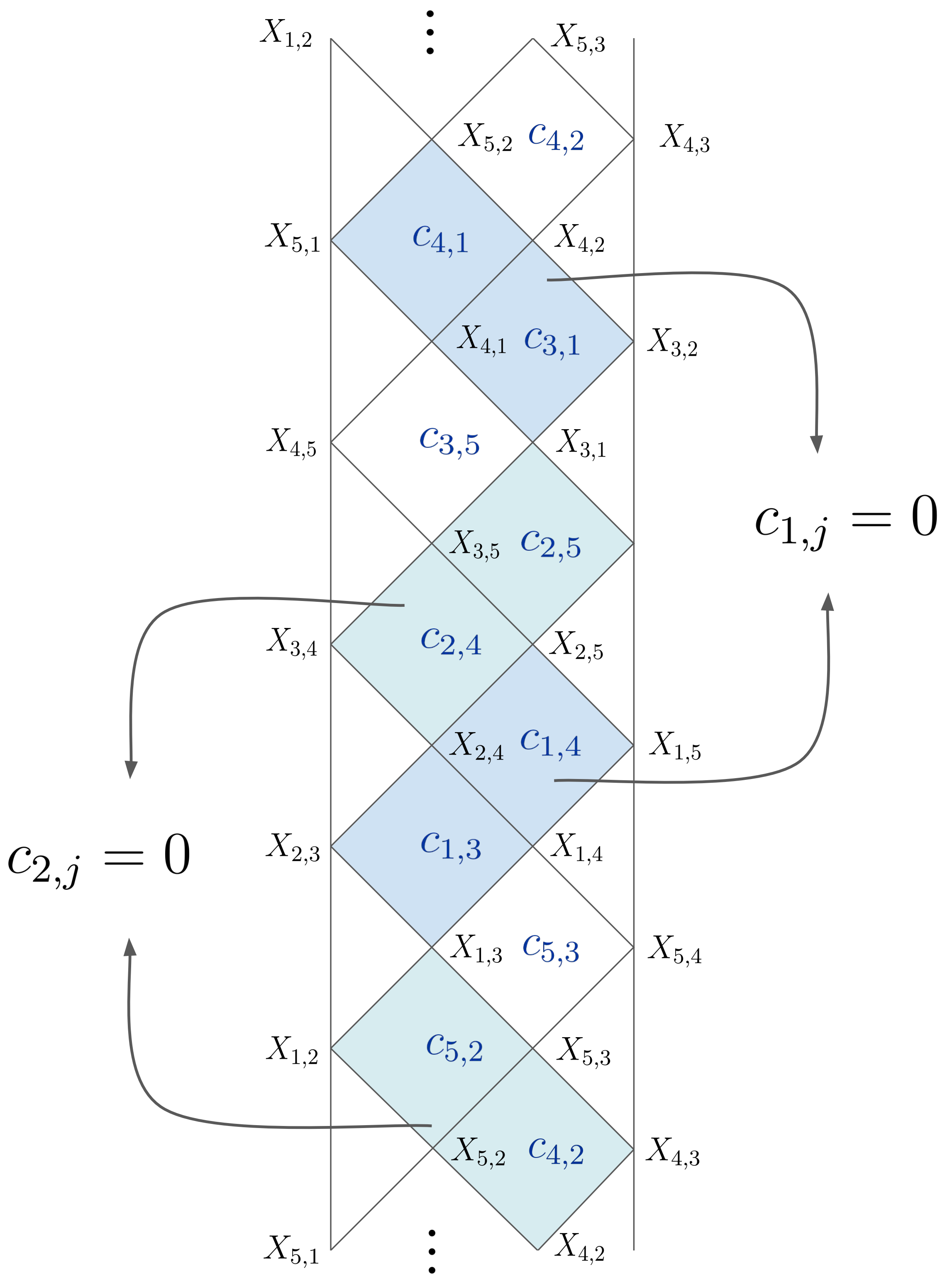
$$\int_{\mathbb{R}_{>0}} \frac{dy}{y} y^{\alpha' X} = 0$$

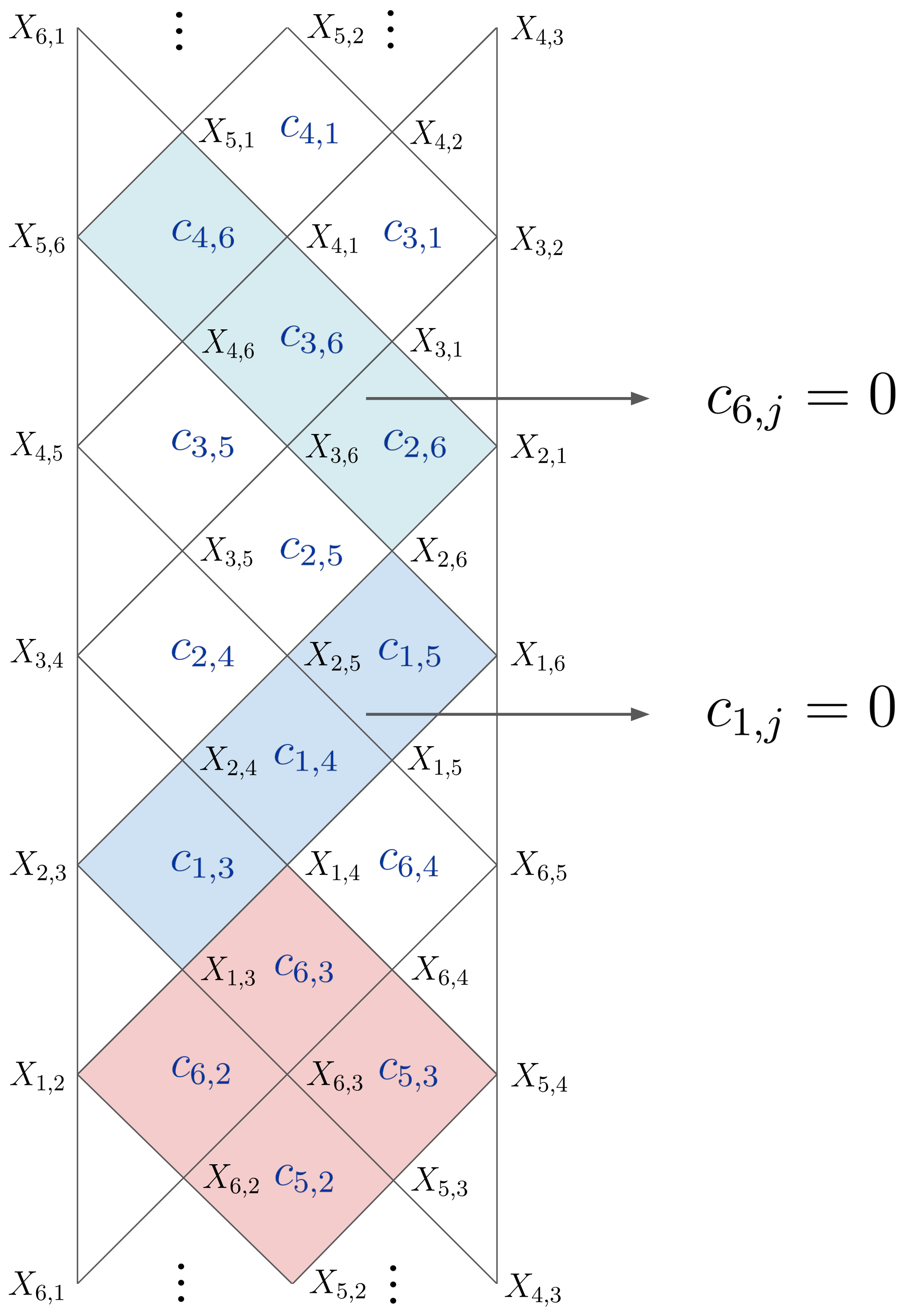


Zeros and factorizations

$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{\mathbb{R}_{>0}^{n-3}} \prod_{i=3}^{n-1} \frac{\mathrm{d}y_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}} \prod_{1\le i,j<n} F_{i,j}(y)^{-\alpha' c_{i,j}}$$

$$F_{i,j} = 1 + y_{1,j} + y_{1,j}y_{1,j-1} + \cdots + y_{1,j}\ldots y_{1,i+2}$$





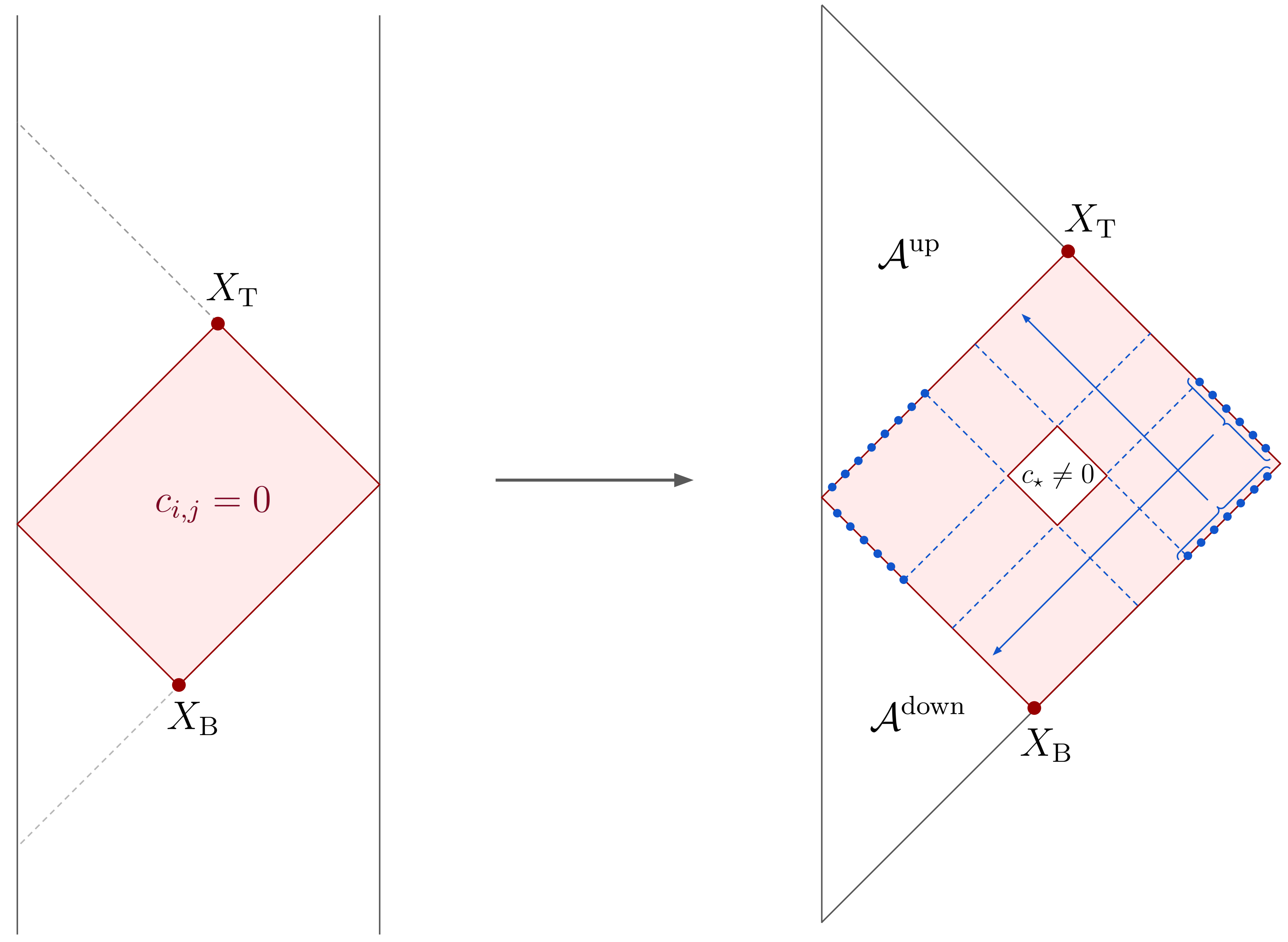
Zero:

$$\alpha' c_{ab} = -n_{a,b}\,, \quad \text{for} \quad 1 \le a \le i-2\,, \quad i \le b \le n-1, \quad n_{a,b} \in \mathbb{N}_0$$

Zeros and factorizations

$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{\mathbb{R}^{n-3}_{>0}} \prod_{i=3}^{n-1} \frac{\mathrm{d}y_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}} \prod_{1\le i,j<n} F_{i,j}(y)^{-\alpha' c_{i,j}}$$

$$F_{i,j} = 1 + y_{1,j} + y_{1,j}y_{1,j-1} + \cdots + y_{1,j}\ldots y_{1,i+2}$$



Zero:

$$\alpha' c_{ab} = -n_{a,b}, \quad \text{for} \quad 1 \le a \le i-2, \quad i \le b \le n-1, \quad n_{a,b} \in \mathbb{N}_0$$

Factorization near zeros:

$$\mathcal{I}_n^{tr(\phi^3)} \to \mathcal{I}_i^{\mathrm{down,tr}(\phi^3)} \times \mathcal{I}_{n-i+2}^{\mathrm{up,tr}(\phi^3)} \times \mathcal{I}_4^{\mathrm{tr}(\phi^3)}(\alpha' X_{1,i}, \alpha'(c_{km} - X_{1,i}))$$

Zeros and factorizations

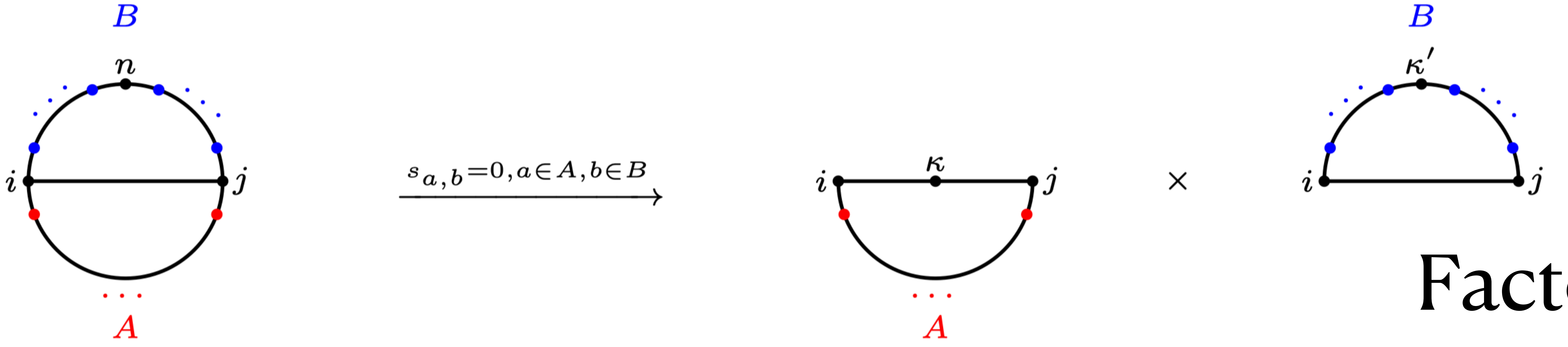
$$\mathcal{I}_n^{tr(\phi^3)}(1,2,\ldots,n) = \int_{\mathbb{R}^{n-3}_{>0}} \prod_{i=3}^{n-1} \frac{\mathrm{d}y_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}} \prod_{1\le i,j<n} F_{i,j}(y)^{-\alpha' c_{i,j}}$$

$$F_{i,j} = 1 + y_{1,j} + y_{1,j}y_{1,j-1} + \cdots + y_{1,j}\ldots y_{1,i+2}$$

Zero

$$\alpha' c_{ab} = -n_{a,b}\,, \quad \text{for} \quad 1 \le a \le i-2\,, \quad i \le b \le n-1, \quad n_{a,b} \in \mathbb{N}_0$$

2-splits



$$\mathcal{I}_n^{tr(\phi^3)} \to \mathcal{I}_i^{\text{down},\text{tr}(\phi^3)} \times \mathcal{I}_{n-i+3}^{\text{up},\text{tr}(\phi^3)}$$

Factorization near zeros:

$$\mathcal{I}_n^{tr(\phi^3)} \to \mathcal{I}_i^{\text{down},\text{tr}(\phi^3)} \times \mathcal{I}_{n-i+2}^{\text{up},\text{tr}(\phi^3)} \times \mathcal{I}_4^{\text{tr}(\phi^3)}(\alpha' X_{1,i}, \alpha'(c_{km} - X_{1,i}))$$

δ-eformation stringy integral

$$\mathcal{I}_{2n}^{tr(\phi^3)} = \int_{\mathbb{R}_{>0}^{2n-3}} \prod_{I=1}^{2n-3} \frac{\mathrm{dy_I}}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}}$$

If we want to keep all the zeros and factorization near zeros, there is a natural generalization.

↓ δ-eformation

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \prod_{I=1}^{2n-3} \frac{\mathrm{dy_I}}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}} \left(\frac{\prod_{(e,e)} u_{e,e}}{\prod_{(o,o)} u_{o,o}} \right)^{\alpha' \delta}$$

e: Even

o: Odd

$$\mathcal{I}_{2n}^{\delta} = \mathcal{I}_{2n}^{tr(\phi^3)}[\alpha' X_{e,e} \to \alpha'(X_{e,e} + \delta), \alpha' X_{o,o} \to \alpha'(X_{o,o} - \delta)]$$

δ-eformation stringy integral

$$\mathcal{I}^{\delta}_{2n} = \int_{\mathbb{R}^{2n-3}_{>0}} \prod_{I=1}^{2n-3} \frac{\mathrm{dy}_I}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}} \left(\frac{\prod_{(e,e)} u_{e,e}}{\prod_{(o,o)} u_{o,o}} \right)^{\alpha'\delta}$$

Stringy integral

Field-theory limit

Lagrangian

$\alpha' \to 0$

$$\mathcal{I}^{\delta}_{2n} = \begin{cases} \mathcal{I}^{\mathrm{tr}(\phi^3)}_{2n} & \alpha'\delta = 0 \\ \mathcal{I}^{NLSM}_{2n} & \alpha'\delta = \mathrm{non-integer} \\ \mathcal{I}^{YMS}_{2n} & \alpha'\delta = 1, \end{cases}$$

→

$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3)$$

$$\mathcal{L}_{\mathrm{NLSM}} = \frac{1}{8\lambda^2}\mathrm{Tr}\left(\partial_\mu \mathrm{U}^\dagger \partial^\mu \mathrm{U}\right)$$

$$\mathcal{L}_{\mathrm{YMS}} = -\mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^\mu\phi^I D_\mu\phi^I - \frac{g^2_{\mathrm{YM}}}{4}\sum_{I\neq J}\left[\phi^I,\phi^J\right]^2\right).$$

↓

$$Res_{y_s=0,X_s=0}\mathcal{I}^{YMS}_{2n} = \mathcal{I}^{YM}_{n}$$

Yang-Mills stringy integral can be obtained by taking the scaffolding residue.

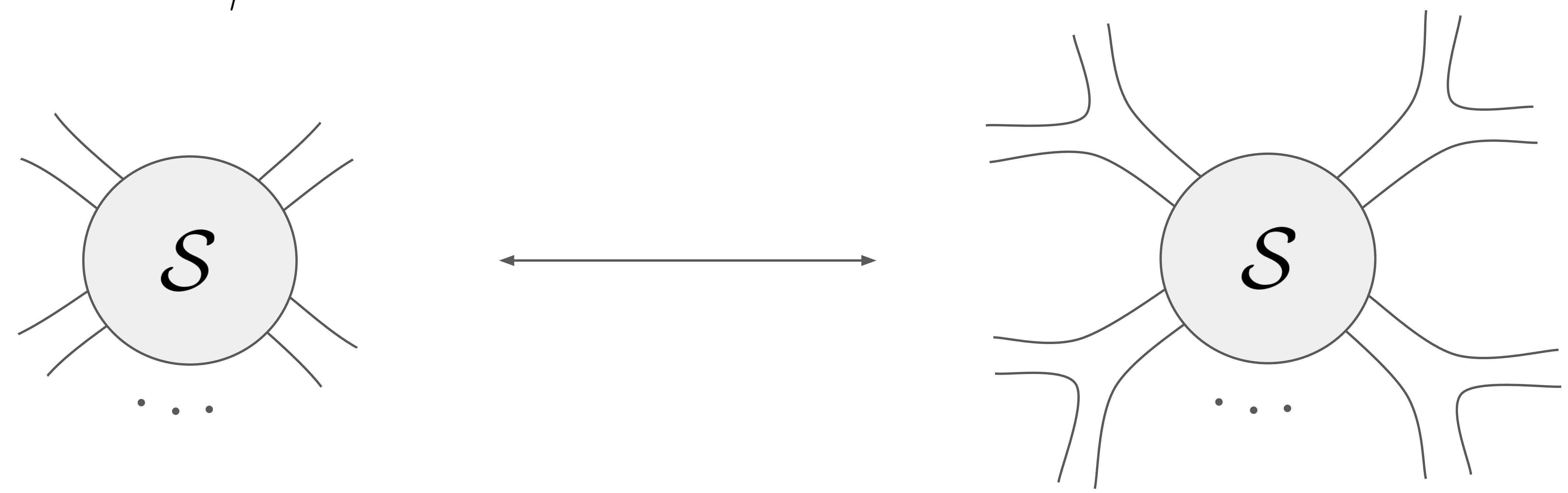
δ-eformation stringy integral

$$\mathcal{I}^{\delta}_{2n} = \int_{\mathbb{R}^{2n-3}_{>0}} \prod_{I=1}^{2n-3} \frac{\mathrm{d}y_I}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}} \left(\frac{\prod_{(e,e)} u_{e,e}}{\prod_{(o,o)} u_{o,o}} \right)^{\alpha'\delta}$$

$$\alpha'\delta = 1$$

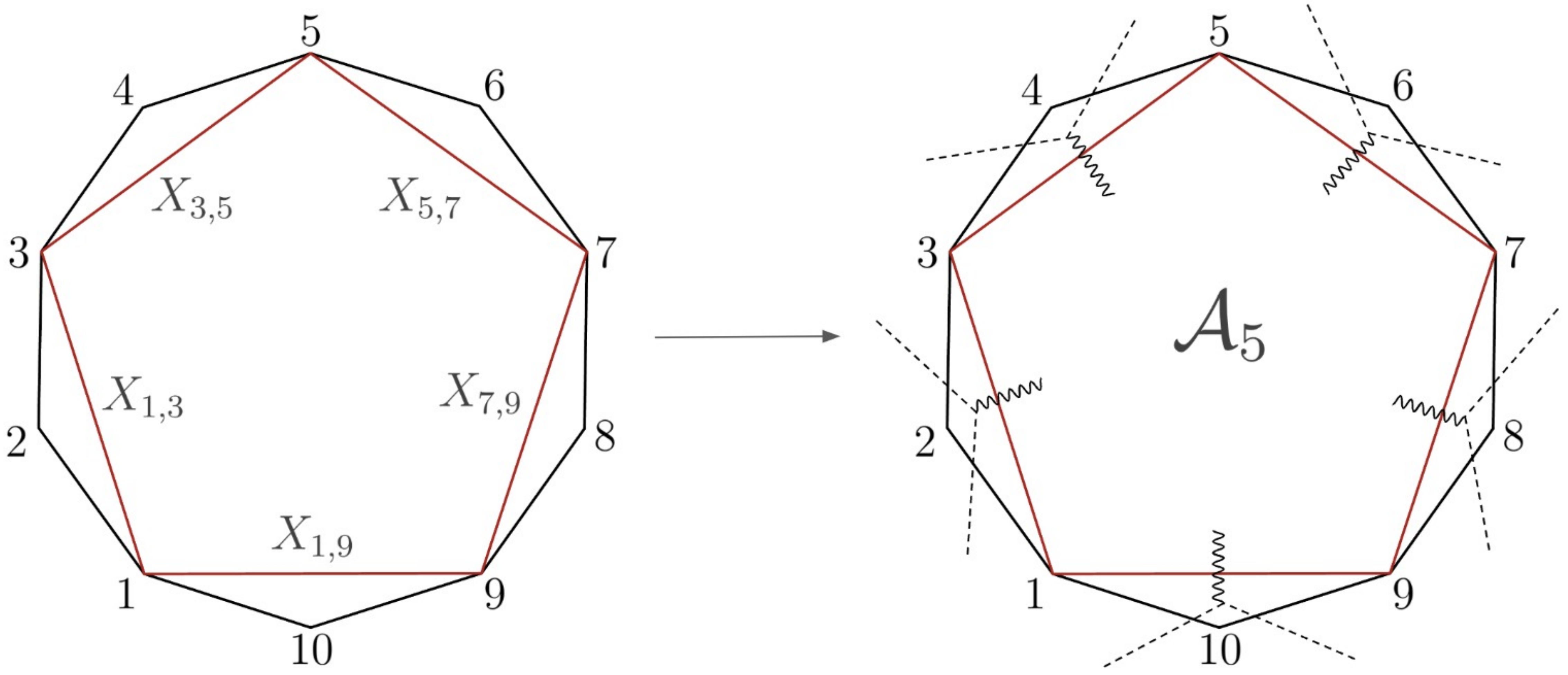
$$Res_{y_s=0, X_s=0} \mathcal{I}^{YMS}_{2n} = \mathcal{I}^{YM}_{n}$$

Tr ϕ^3 | YM



$$\int_0^\infty \prod \frac{\mathrm{d}y}{y} \prod u_X^{\alpha' X}$$

$$\int_0^\infty \prod \frac{\mathrm{d}y}{y^2} \prod u_X^{\alpha' X}$$

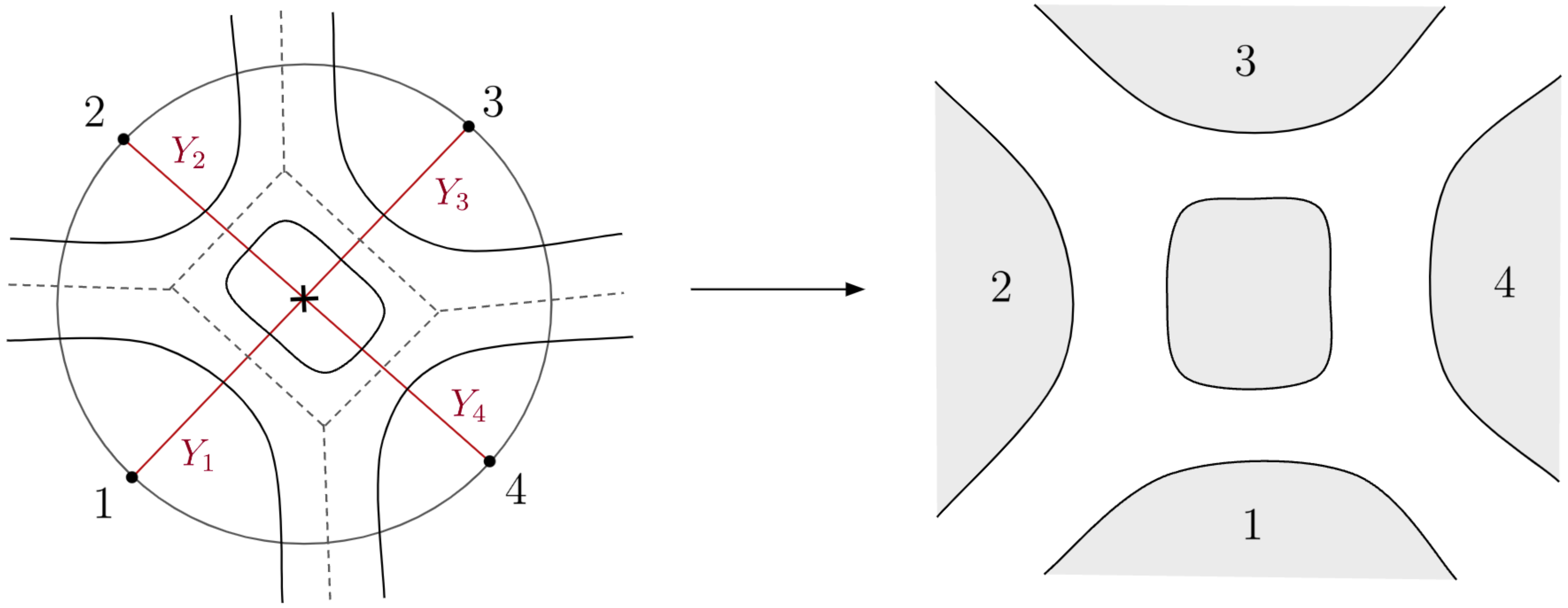


Loop integrands

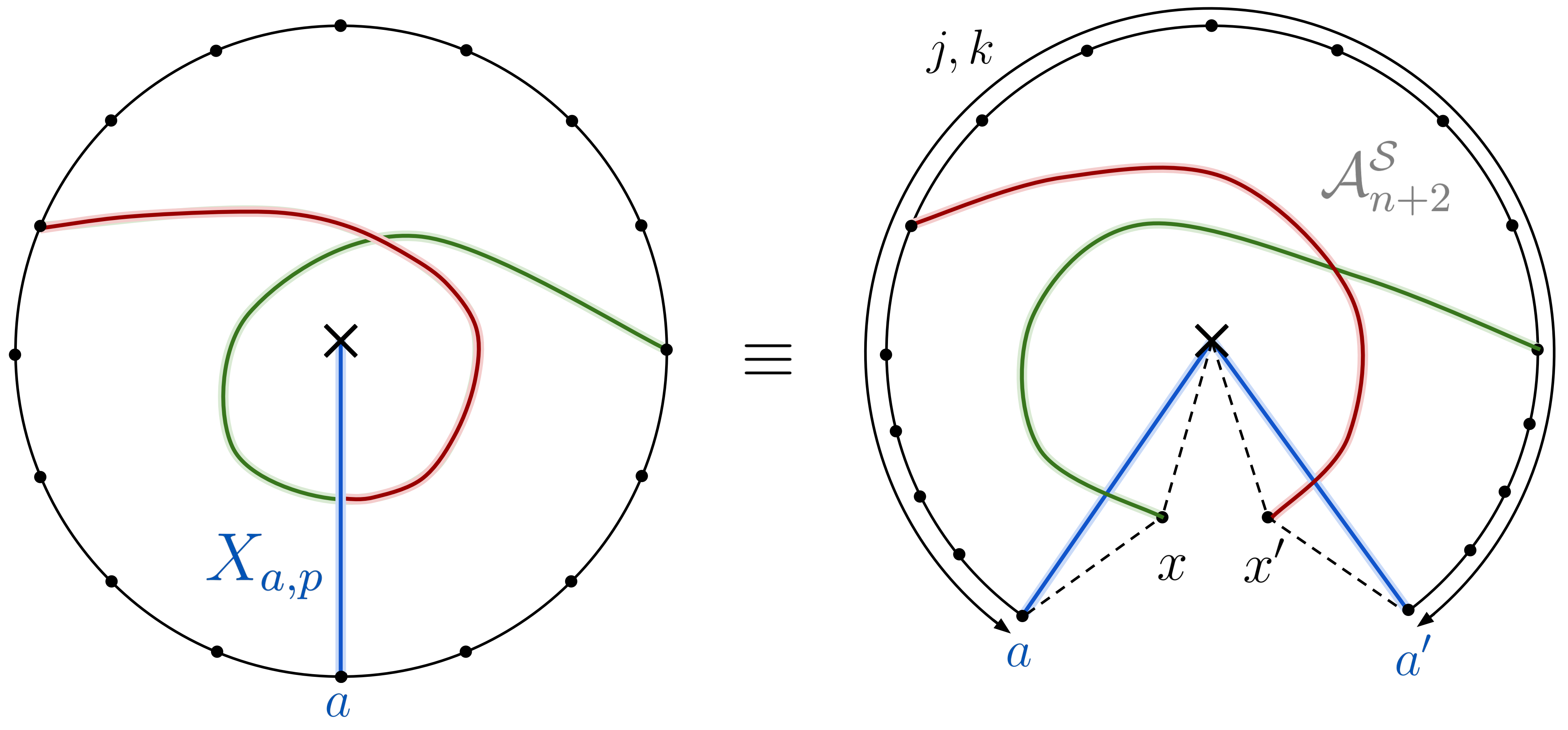
$$\mathcal{I}_{\mathcal{S}}^{\mathrm{tr}(\phi^3)} = \int_{\mathbb{R}^d_>} \prod_{I=1}^{d} \frac{\mathrm{d}y_I}{y_I} \prod u_\Gamma(y)^{\alpha' X_\Gamma}$$

Γ: Curves on the surface.

$\mathcal{S}$: Surface——the punctured disk with n marked points on the boundary.



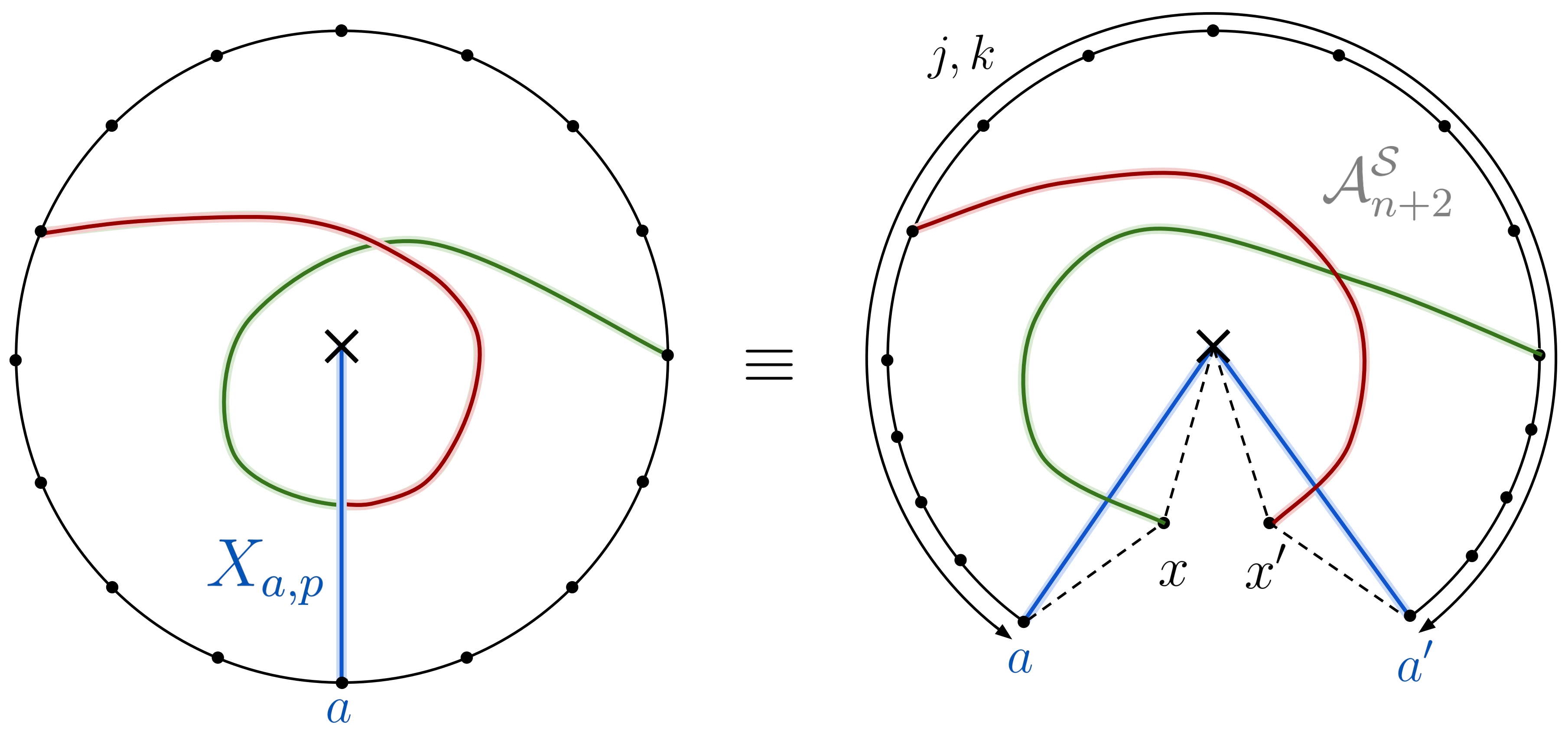
1-loop 4-point surface

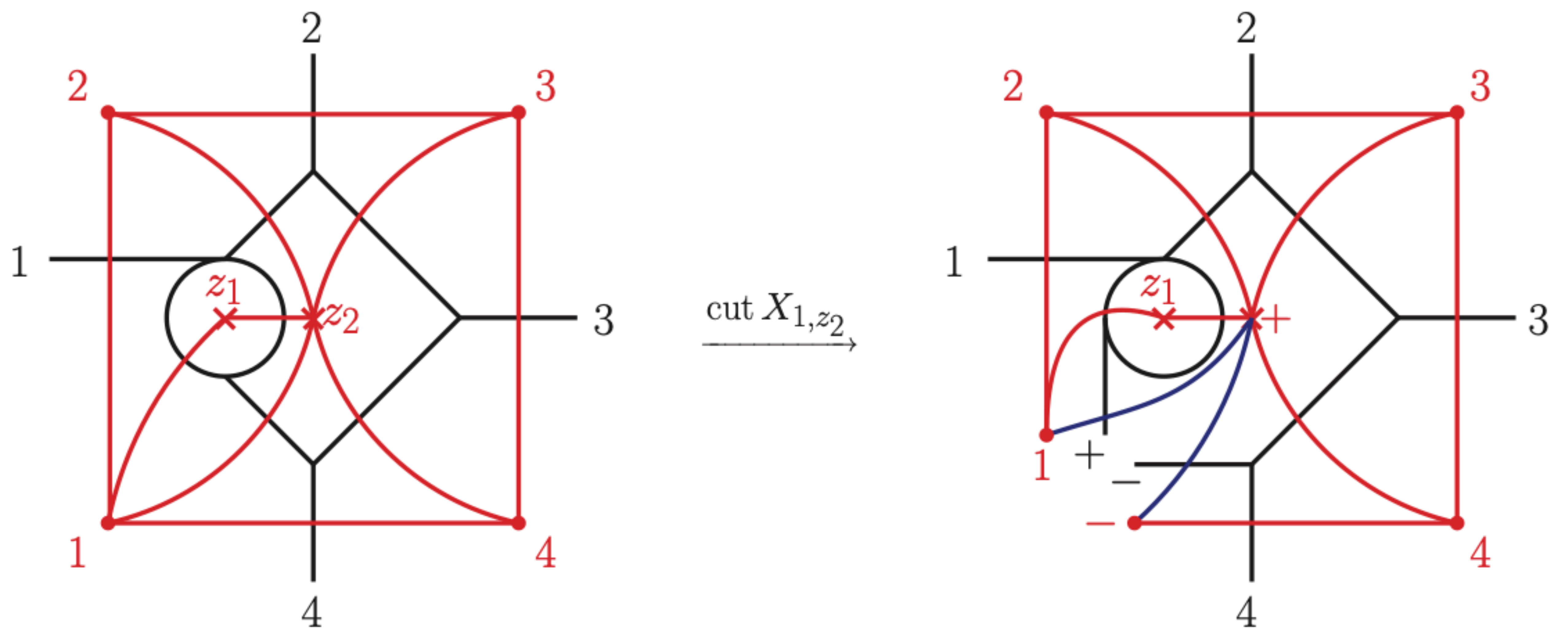


Loop-cut to reconstruct the integrand

Loop integrands

- All-loop canonical Yang-Mills integrands from the surface [Arkani-Hamed, QC, Dong, Figueiredo & He (2024)].
- At one-loop, the single cut reconstruction can be generalized to general gauge theory, and obtain the universal expansion [QC, Dong, He &Zhu(2024)], even for one-loop gravity integrands [QC, He, Zhang &Zhu (2025)].
- At all-loop, the cut equations work for scalar theories [Arkani-Hamed, Frost &Salvatori (2024)], and can be generalized to Yang-Mills all-loop integrands from the cut equations [QC & Zhu (2025)].





Remarks

- The origins of hidden zeros: ABHY associahedron, String amplitudes, Enhanced BCFW scales [Rodina(2024)], Double copy [Bartsch, Brown, Kampf, Oktem, Paranjape &Trnka(2024)][Li, Roest & Veldhuis(2024)], Feyman diagrams [Zhou(2024)]…
- Zeros determine the scattering amplitudes at tree-level & one-loop integrands [Rodina(2024)][Backus&Rodina(2025)].
- Zeros and splits constrain the Wilson coefficients in S-matrix bootstrap [Berman, Elvang & Figueiredo(2025)].
- All-loop splits in surfaceology and some mathematical contexts [Arkani-Hamed & Figueiredo(2024)][Umbert& Sturmfels(2025)][Early(2025)].
- Inspiration for more new factorizations and recursions for scattering amplitudes [Guevara&Zhang(2024)][Zhang(2024)][Backus(2025)].
- Applications for colored Yukawa theory[De, Pokraka, Skowronek, Spradlin& Volovich(2024)] and Cosmohedra[Arkani-Hamed, Figueiredo&Vazao(2024)].

Ongoing projects

- Form factors from surfaceolgy.
- Surfaceology == String amplitudes at higher-genus surface.
- Dual resonance and massive factorizations.

Summary

- Stringy integrals for scalar/pion/gluon———depend on surfaceology.
- Hidden zeros/factorization for scattering amplitudes———from the 2-splits.
- All-loop integrands from surface——————— recursion from the single-cut.

Thank you for attention!