

# Generating functions for classical two-body observables

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施灿欣 / Canxin Shi

Institute of Theoretical Physics, Chinese Academy of Sciences  
中国科学院理论物理研究所

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Based on work  
with Gonzo: PRD [2304.06066], PRL [2405.09687];  
with Gonzo & Alessio: PRD [2506.03249]

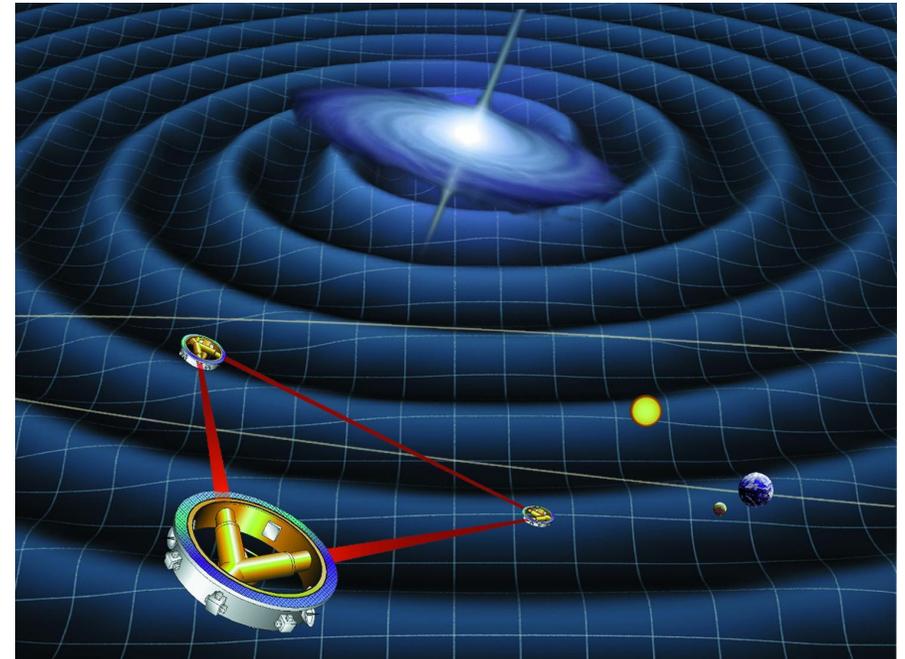


# Gravitational wave physics

- We are interested in gravitational waves from black hole/neutron star mergers
- Precision of waveform template needs to be improved by orders of magnitude for future GW detectors (LISA, Einstein Telescope, Taiji, TianQin, ...)

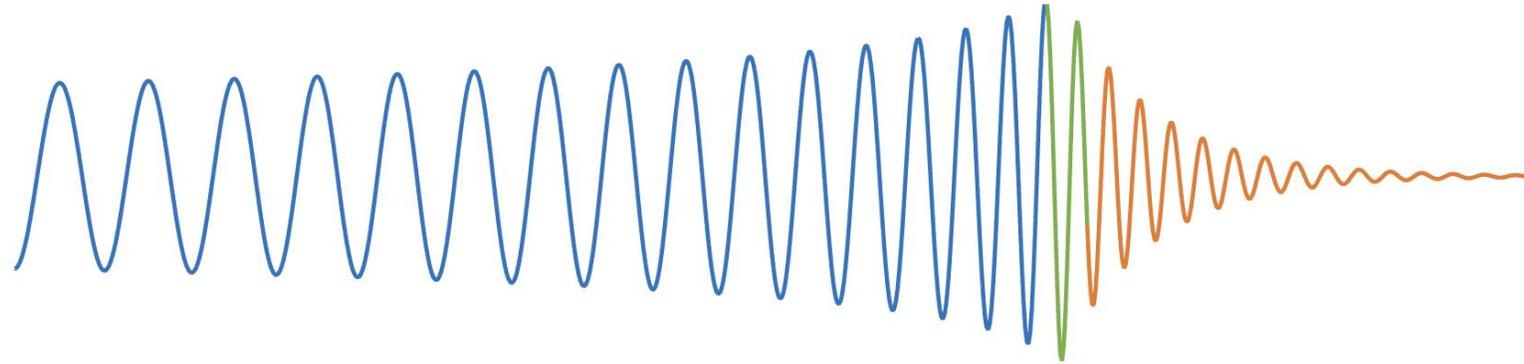


Credit: MARK GARLICK / SCIENCE PHOTO / Getty images



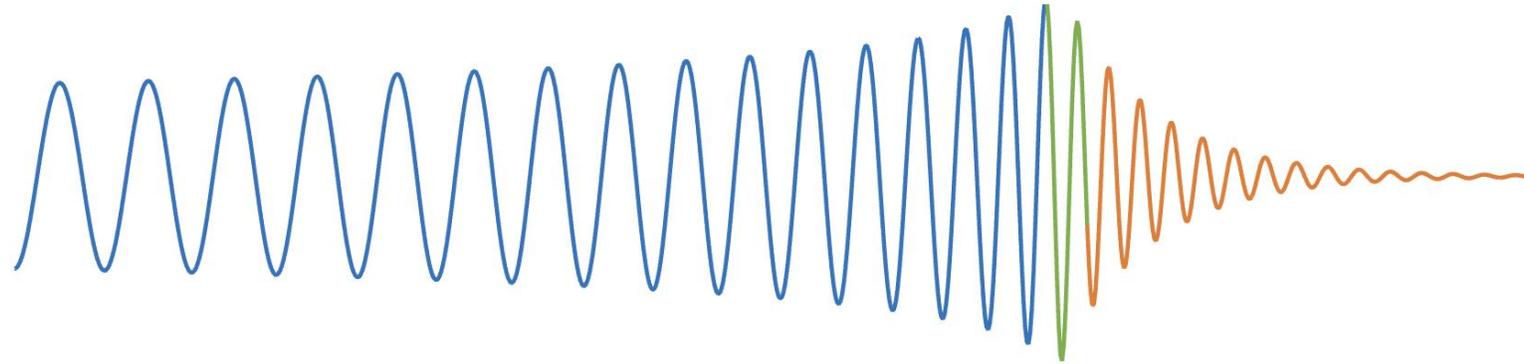
# Perturbation regime of 2-body systems

- 3 phases of 2-body mergers: **inspiral**, **merger**, **ringdown**



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- We are interested in the **inspiral phase/scattering** of a binary black hole where perturbation theory is applicable  $r \gg r_{\text{Schw}} (\gg \lambda_{\text{Comp}})$
- Post-Minkowskian expansion:  $n\text{PM} \sim \mathcal{O}(G^n)$



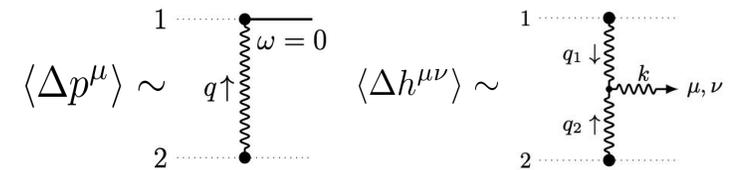
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  - How to compute various observables efficiently?
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- Worldline-based methods [Plefka, Mogull, Jakobsen, Porto, Kälin, Liu, Goldberger, Rothstein, ...]

$$S_{\text{pm}} = -\frac{m}{2} \int_{-\infty}^{\infty} d\tau (g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + 1)$$



- Scattering amplitude-based methods

- EFT matching [Solon, Cheung, Rothstein, et al]

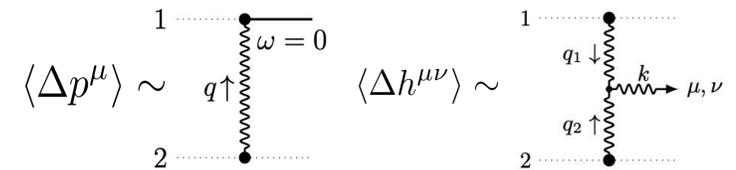
- KMOC  $\langle \Delta p_1^\mu \rangle = \langle \psi | S^\dagger \mathbb{P}_1^\mu S - \mathbb{P}_1^\mu | \psi \rangle$  [Kosower, Maybee, O'Connell]

- eikonal exponentiation  $\tilde{\mathcal{S}}(s, b) = (1 + \dots) e^{2i\delta(s, b)}$  [Di Vecchia, Heissenberg, Russo, Veneziano, ...]

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*Shortcut to all observables:  
a minimal set of classical **generating functions**?*

# Radial action as a generating function

- For **spinless** binary, in the COM frame the **conservative** radial action is

$$I_r := \frac{1}{2\pi} \int_{\mathcal{C}} p_r(r) dr \quad p_r(r) : \text{radial momentum}$$

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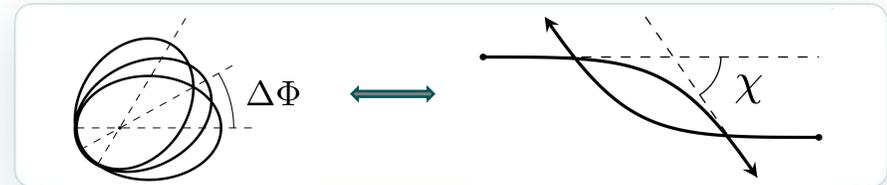
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- Bonus: periastron advance via analytic continuation [Kälin, Porto, Liu, Cho, ...]

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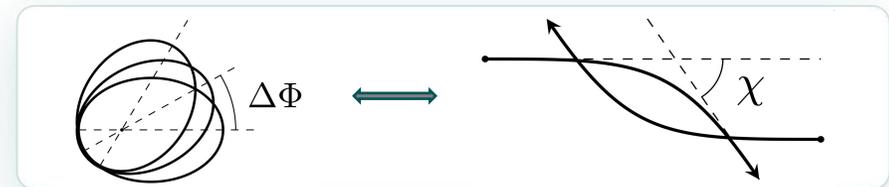
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- Amplitude-action relation (up to 3PM) [Bern, Parra-Martinez et al 21']

$$e^{iI_r} \sim \lim_{\hbar \rightarrow 0} \int dq \mathcal{A}(p_1, p_2, q) e^{\frac{i}{\hbar} b \cdot q} \quad b: \text{impact parameter}$$

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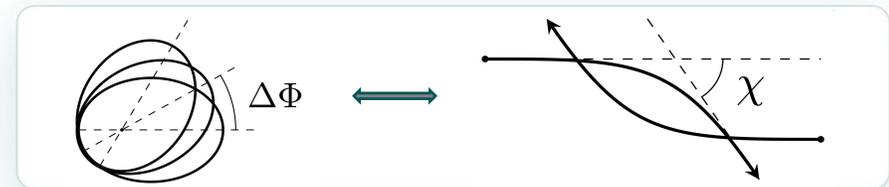
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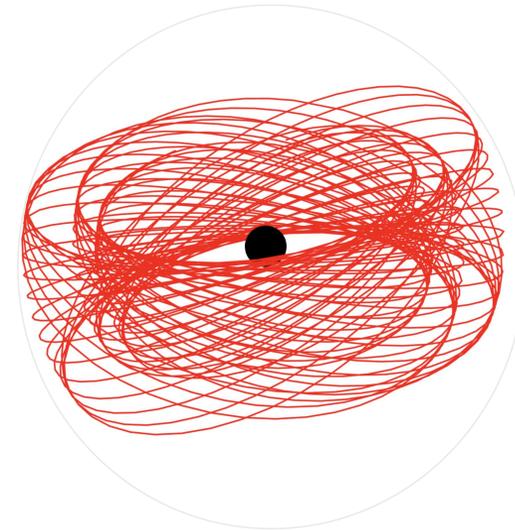
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- Generalize to **spinning** and **radiative** effects?

# Spinless probe in Kerr

- Radial action with spin  $I_r(\mathcal{E}, l, a, l_Q)$  [Gonzo, CS 2304.06066]

Even geodesic is complicated

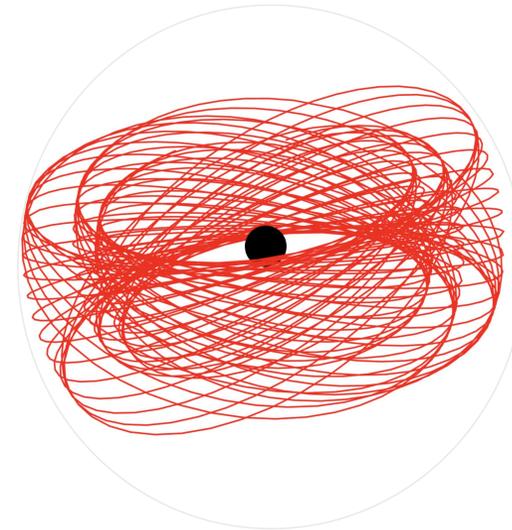


# Spinless probe in Kerr

- Radial action with spin  $I_r(\mathcal{E}, l, a, l_Q)$  [Gonzo, CS 2304.06066]
- Analytic continuation to bound orbit

$$I_r^{\text{bound}}(\mathcal{E}, l, a, l_Q) \stackrel{\mathcal{E} \leq 0}{=} I_r(\mathcal{E}, l, a, l_Q) - I_r(\mathcal{E}, -l, -a, -l_Q)$$

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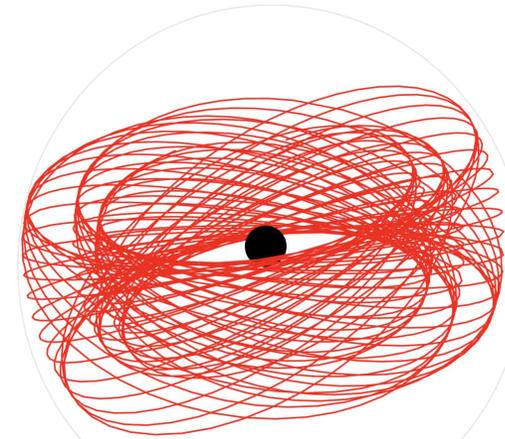
- From Hamilton's principal function, we also need the **polar** action to obtain the azimuthal angle

$$I_\theta = \frac{1}{2\pi} \int_{\mathcal{C}_\theta} p_\theta d\theta$$

$$\frac{\chi_{\text{azimuthal}} + \pi}{2\pi} = -\frac{\partial I_r}{\partial L} - \frac{\partial I_\theta}{\partial L}$$

- Puzzle: the radial action should contain **all information** about conservative scattering

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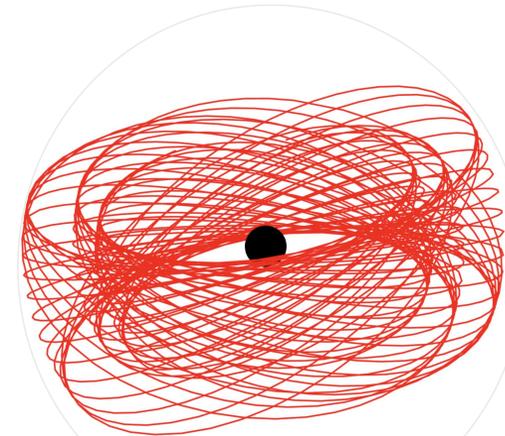
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*New way to connect the radial action and observables*

# Master formula for scattering observables

- Global scattering observables

$$\Delta O := O(+\infty) - O(-\infty) = \Delta O|_{\text{con}} + \Delta O|_{\text{rad}}$$

e.g.: momentum impulse  $\Delta p^\mu$ , spin kick  $\Delta s^\mu$ , orbital angular impulse  $\Delta L^{\mu\nu}$ , ...

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- **Conservative**

[Gonzo, CS 2405.09687]

$$\tilde{\mathcal{K}}^{\text{cl}} = I_r(b, v_1, v_2, s_1, s_2)$$

- **Radiative**

[Gonzo, Alessio, CS 2506.03249]

$\tilde{\mathcal{K}}_{n,\mathcal{R}}^{\text{cl}}(b_i, v_i, s_i; k_1, \dots, k_n)$  : (n-4)-graviton kernel

$$\Delta O|_{\text{con}} = \sum_{j=1} \frac{1}{j!} \underbrace{\{\tilde{\mathcal{K}}^{\text{cl}}, \{\tilde{\mathcal{K}}^{\text{cl}}, \dots, \{\tilde{\mathcal{K}}^{\text{cl}}, O\} \dots\}}}_{j \text{ times}}$$

$$\begin{aligned} \Delta O|_{\text{rad}} = & \frac{i}{2!} \int_k \left( \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(k) \{\tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}}(k), O\} - \text{c.c.} \right) \\ & + \frac{i}{3!} \int_k \left( \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(k) \{\tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}}(k), \{\tilde{\mathcal{K}}^{\text{cl}}, O\}\} - \text{c.c.} \right) \\ & + \frac{2i}{3!} \int_k \left( \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(k) \{\tilde{\mathcal{K}}^{\text{cl}}, \{\tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}}(k), O\}\} - \text{c.c.} \right) \\ & + \frac{i}{3!} \int_k \left( \{\tilde{\mathcal{K}}^{\text{cl}}, \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(k)\} \{\tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}}(k), O\} - \text{c.c.} \right) \\ & + \frac{1}{3!} \int_{k_1, k_2} \left( \{\tilde{\mathcal{K}}_{6,\mathcal{R}}^{B\text{cl}}(k_1, k_2), O\} \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(k_1) \tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}}(k_2) + \text{c.c.} \right) \\ & + \dots \end{aligned}$$

# Conservative case

- We start from the **exponential** representation of S-matrix [Damgaard, Plante et al, 21']

$$\hat{S} = \exp\left(\frac{i}{\hbar}\hat{N}\right) \Rightarrow \Delta\hat{O} := \hat{S}^\dagger\hat{O}\hat{S} - \hat{O} = \sum_{j \geq 1} \frac{(-i)^j}{\hbar^j j!} \underbrace{[\hat{N}, [\hat{N}, \dots, [\hat{N}, \hat{O}]]]}_{j \text{ times}}$$

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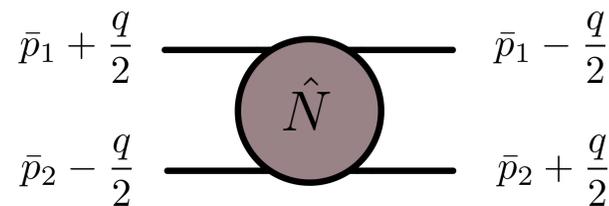
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- In the classical limit, the 2-to-2 N-matrix elements are finite. [Bjerrum-Bohr, Plante et al, 21']

Naively, we take  $[\hat{N}, \hat{O}] \rightarrow i\hbar\{\tilde{\mathcal{K}}^{cl}, O\}$

$$\Delta O|_{\text{con}} = \sum_{j=1} \frac{1}{j!} \underbrace{\{\tilde{\mathcal{K}}^{cl}, \{\tilde{\mathcal{K}}^{cl}, \dots, \{\tilde{\mathcal{K}}^{cl}, O\} \dots\}}}_{j \text{ times}}$$

$$\tilde{\mathcal{K}}^{cl}(p_1, p_2; b) = \lim_{\hbar \rightarrow 0} \int d^4q \delta(p_1 \cdot q) \delta(p_2 \cdot q) e^{\frac{i}{\hbar} b \cdot q} \left\langle \bar{p}_1 + \frac{q}{2}, \bar{p}_2 - \frac{q}{2} \left| \hat{N} \right| \bar{p}_1 - \frac{q}{2}, \bar{p}_2 + \frac{q}{2} \right\rangle$$



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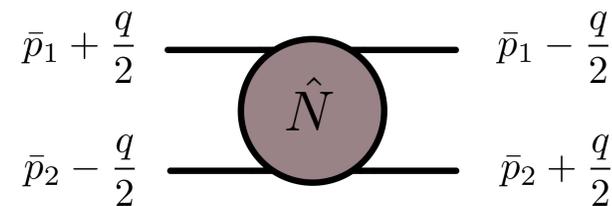
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$\tilde{\mathcal{K}}^{cl}(p_1, p_2; b)$  is **real** by definition (unlike amplitude)

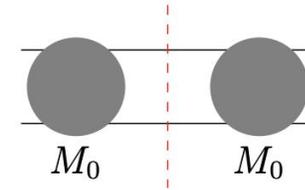


# N-matrix elements

- N-matrix elements from amplitudes [Damgaard, Hansen, Planté, Vanhovec 23']

$$\hat{S} = \mathbf{1} + i\hat{T} \Rightarrow \frac{\hat{N}}{\hbar} = \hat{T} - \frac{1}{2}i\hat{T}^2 - \frac{\hat{T}^3}{3} + \dots$$

$$N_1 = M_1 - \frac{i}{2}$$



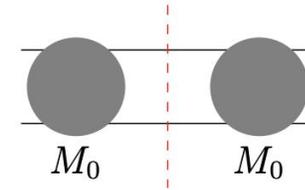
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- N-matrix elements from Magnus series [Kim, Kim, 24'; Brandhuber, Brown et al, 25']

$$iN^{(2)} = \frac{(-i)^2}{2} \int d^4x_1 d^4x_2 \theta_{12} [\mathcal{H}_1, \mathcal{H}_2]$$

$$iN_4^{(2)} = \sum_{\text{perms}} (-i)^2 \left[ \omega \left( \begin{array}{c} \diagup \quad \diagdown \\ \leftarrow \end{array} \right) \begin{array}{c} \diagdown \quad \diagup \\ \leftarrow \end{array} + \omega \left( \begin{array}{c} \diagup \quad \diagdown \\ \rightarrow \end{array} \right) \begin{array}{c} \diagdown \quad \diagup \\ \rightarrow \end{array} \right]$$

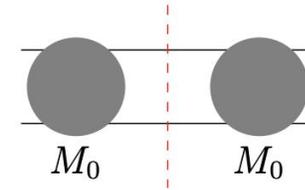
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retarded advanced

- For **spinless** case, one recovers the formula for scattering angle

$$\chi = -2\pi \frac{\partial \tilde{\mathcal{K}}^{\text{cl}}}{\partial L} - \pi$$

- The Fourier-transformed 4-pt element is the radial action [Bjerrum-Bohr, Plante et al, 21'; Kim, Patil, 25']

$$\tilde{\mathcal{K}}^{\text{cl}} = \text{radial action}$$

# Solving constraints

- Naively, the commutators are replaced with Poisson brackets  $[\hat{N}, \hat{O}] \rightarrow i\hbar\{\tilde{\mathcal{K}}^{cl}, O\}_{\text{P.B.}}$ .

$$\{b_i^\mu, v_i^\nu\}_{\text{P.B.}} = \frac{\eta^{\mu\nu}}{m_i},$$

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- However, Lorentz covariance introduces **non-physical** degree of freedom. To account for that, we choose to impose the constraints

$$\text{covariant spin supplementary conditions: } v_{i,\mu} S_i^{\mu\nu} = 0$$

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*The brackets are interpreted as **Dirac** brackets!*

# 2-body Dirac brackets

The Poisson brackets are promoted to **Dirac** brackets

$$\{f, g\}_{\text{D.B.}} = \{f, g\}_{\text{P.B.}} - \sum_{j,k=1} \{f, \phi_j\}_{\text{P.B.}} (M^{-1})_{j,k} \{\phi_k, g\}_{\text{P.B.}}, \quad M_{j,k} = \{\phi_j, \phi_k\}_{\text{P.B.}}$$

$$\{b_i^\mu, v_j^\nu\}_{\text{D.B.}} = \delta_{ij} \frac{\eta^{\mu\nu}}{m_j} + (-1)^{\delta_{ij}} \frac{\mathcal{P}_i^{\mu\nu}}{m_j},$$

$$\{b_i^\mu, s_j^\nu\}_{\text{D.B.}} = \delta_{ij} \frac{s_i^\mu v_i^\nu}{m_j} + (-1)^{\delta_{ij}} \frac{\mathcal{P}_i^{\mu\rho} s_{j\rho}^\nu}{m_j},$$

$$\begin{aligned} \{b_i^\mu, b_j^\nu\}_{\text{D.B.}} &= \frac{v_i \cdot v_j}{\sigma^2 - 1} \left( \text{sgn}_i \frac{b^\mu v_j^\nu}{m_i} - \text{sgn}_j \frac{b^\nu v_i^\mu}{m_j} \right) \\ &\quad + \delta_{ij} \tilde{S}_i^{\mu\nu} + (-1)^{\delta_{ij}} \left( \mathcal{P}_i^{\mu\rho} \tilde{S}_{j\rho}^\nu - \mathcal{P}_j^{\nu\rho} \tilde{S}_{i\rho}^\mu \right), \end{aligned}$$

$$\{s_i^\mu, s_j^\nu\}_{\text{D.B.}} = \delta_{ij} \tilde{S}_i^{\mu\nu},$$

$$\mathcal{P}_1^{\mu\nu} := \frac{v_1^\mu (v_1^\nu - \sigma v_2^\nu)}{\sigma^2 - 1}, \quad \mathcal{P}_2^{\mu\nu} := \frac{v_2^\mu (v_2^\nu - \sigma v_1^\nu)}{\sigma^2 - 1}, \quad \text{sgn}_1 := -1, \quad \text{sgn}_2 := +1.$$

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The brackets only act on **incoming kinematics**.

# Spinning probe in Kerr

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$$\text{Energy: } E = -\pi_\mu \xi_t^\mu + \frac{1}{2} S^{\alpha\beta} \nabla_\beta (\xi_t)_\alpha$$

$$\text{Angular momentum: } L = \pi_\mu \xi_\phi^\mu - \frac{1}{2} S^{\alpha\beta} \nabla_\beta (\xi_\phi)_\alpha$$

$$\text{Rüdiger constant: } K = K_{\mu\nu} \pi^\mu \pi^\nu - 2 \pi^\mu S^{\rho\sigma} (Y^\nu{}_\sigma \nabla_\nu Y_{\mu\rho} - Y^\nu{}_\mu \nabla_\nu Y_{\sigma\rho})$$

$$\text{Rüdiger linear invariant: } s_{\parallel} = -\frac{1}{2} \frac{\sqrt{-\det g_{\mu\nu}} \epsilon_{\mu\nu\lambda\rho} Y^{\mu\sigma} \pi_\sigma S^{\nu\lambda} \pi^\rho}{m^3}$$

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- Integrability ensures the **separability** of variables, simple integration gives  $I_r(b^\mu, v_i^\mu, s_i^\mu)$
- Results: radial action to **all orders** in  $G$ , in terms of hypergeometric function.
- Explicit expressions are obtained up to  $\mathcal{O}(G^6 s_1^1 s_2^4)$

Novel results:

$$\Delta v_1^\mu(b^\mu, s_i^\mu, v_i^\mu) \Big|_{\mathcal{O}(G^6 s_1 s_2^4)}, \Delta s_1^\mu(b^\mu, s_i^\mu, v_i^\mu) \Big|_{\mathcal{O}(G^6 s_1 s_2^4)}$$

# Spinning probe in Kerr – Bound state

- Using action-angle variables, we define the fundamental frequencies

$$\omega_i = \frac{\partial H}{\partial J_i}, \quad i = r, \theta, \phi, \phi_S.$$

$$J_t = -E, \quad J_\phi = L, \quad J_{\phi_S} = S_{\parallel},$$

$$J_r = \frac{1}{2\pi} \oint p_r dr, \quad J_\theta = \frac{1}{2\pi} \oint p_\theta d\theta$$

Analytic continuation of the radial action

$$I_r(\mathcal{E}, l, a, l_Q, s_{\parallel}) = \frac{I_r^>(\mathcal{E}, l, a, l_Q, s_{\parallel}) - I_r^>(\mathcal{E}, -l, -a, -l_Q, -s_{\parallel})}{2\pi}$$

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- Frequency ratios

$$K^{\theta r} = \frac{\omega_\theta}{\omega_r}, \quad K^{\phi r} = \frac{\omega_\phi}{\omega_r}, \quad K^{\phi_S r} = \frac{\omega_{\phi_S}}{\omega_r}$$

- In the equatorial limit  $\vec{s}_1 \parallel \vec{s}_2 \parallel \vec{L}^G$

$$K^{\phi r} \Big|_{\mathcal{O}(G^6 s_1 s_2^4)}$$

# Radiative contributions

[Gonzo, Alessio, CS 2506.03249]

- Again, we start from the exponential representation of S-matrix  $\hat{S} = \exp\left(\frac{i}{\hbar}\hat{N}\right)$   
[Damgaard, et al, JHEP 2021]

$$\Delta\hat{O} := \hat{S}^\dagger \hat{O} \hat{S} - \hat{O} = \sum_{j \geq 1} \frac{(-i)^j}{\hbar^j j!} \underbrace{[\hat{N}, [\hat{N}, \dots, [\hat{N}, \hat{O}]]]}_{j \text{ times}}$$

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- 4-pt element is not enough.
- Consider a  $2 \rightarrow 2$  scalar scattering, the relevant matrix elements up to  $O(G^5)$

$$\mathcal{K}(p_1, p_2; q_1) = \begin{array}{c} p_1 \text{ --- } \text{---} p_1 + q_1 \\ \text{---} \hat{N} \text{---} \\ p_2 \text{ --- } \text{---} p_2 + q_2 \end{array}$$

$$\mathcal{K}_{5,\mathcal{R}}(p_1, p_2, k_1; q_1, q_2) = \begin{array}{c} p_1 \text{ --- } \text{---} p_1 + q_1 \\ \text{---} \hat{N} \text{---} \text{---} k_1 \\ p_2 \text{ --- } \text{---} p_2 + q_2 \end{array}$$

$$\mathcal{K}_{6,\mathcal{R}}^A(p_1, p_2, k_1, k_2; q_1, q_2) = \begin{array}{c} p_1 \text{ --- } \text{---} p_1 + q_1 \\ \text{---} \hat{N} \text{---} \text{---} k_1 \\ \text{---} \text{---} k_2 \\ p_2 \text{ --- } \text{---} p_2 + q_2 \end{array}$$

$$\mathcal{K}_{6,\mathcal{R}}^B(p_1, p_2, k_1, k_2; q_1, q_2) = \begin{array}{c} p_1 \text{ --- } \text{---} p_1 + q_1 \\ \text{---} k_2 \text{---} \hat{N} \text{---} \text{---} k_1 \\ p_2 \text{ --- } \text{---} p_2 + q_2 \end{array}$$

# Coherent state expansion of semi-classical S-matrix

- In the heavy-mass limit, the super-classical terms ( $1/h^n$ ) exponentiate!  
[Brandhuber et al, 23'; Damgaard et al 19']

$$\exp\left(\frac{i}{\hbar}\hat{N}\right)|p_1p_2\rangle \stackrel{\hbar\rightarrow 0}{\sim} \hat{S}^{\text{cl}}|p_1p_2\rangle$$

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- We expand the semi-classical S-matrix in coherent states.

$$\hat{S}^{\text{cl}} = \exp\left\{\frac{i}{\hbar}\left[\tilde{\mathcal{K}}^{\text{cl}}(b) + \sum_{\sigma_1} \frac{1}{\sqrt{\hbar}} \int_{k'_1} \hat{\alpha}_{5,\mathcal{R}}^{\text{cl}}(k'_1) + \sum_{N=2}^{+\infty} \frac{1}{N!\hbar^{N/2}} \sum_{\sigma_1,\dots,\sigma_N} \int_{k'_1,\dots,k'_N} \hat{\alpha}_{4+N,\mathcal{R}}^{\text{cl}}(k'_1,\dots,k'_N)\right]\right\},$$

$$\hat{\alpha}_{5,\mathcal{R}}^{\text{cl}}(k'_1) = \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(b_1, b_2; k'_1) a_{\sigma_1}^\dagger(k'_1) + \text{h.c.}$$

$$\begin{aligned} \hat{\alpha}_{6,\mathcal{R}}^{\text{cl}}(k'_1, k'_2) &= \tilde{\mathcal{K}}_{6,\mathcal{R}}^{\text{Acl}}(b_1, b_2; k'_1, k'_2) a_{\sigma_1}^\dagger(k'_1) a_{\sigma_2}^\dagger(k'_2) \\ &\quad + \tilde{\mathcal{K}}_{6,\mathcal{R}}^{\text{Bcl}}(b_1, b_2; k'_1, k'_2) a_{\sigma_2}^\dagger(k'_2) a_{\sigma_1}(k'_1) \\ &\quad + \text{h.c.} \end{aligned}$$

$$\mathcal{K}(q_1, q_2) \stackrel{\int e^{iq_i \cdot b_i / \hbar} \hat{\delta}(2p_i \cdot q_i)}{\implies} \tilde{\mathcal{K}}(b_1, b_2)$$

All  $\tilde{\mathcal{K}}_{\mathcal{R}}(b_1, b_2)$  are finite in the classical limit.

# Observables from semi-classical S-matrix

- An arbitrary observable in terms of semi-classical S-matrix

$$\Delta\hat{O} = (\hat{S}^{\text{cl}})^\dagger \hat{O} \hat{S}^{\text{cl}} - \hat{O} = \sum_{n=1} \frac{1}{n!} \left( \frac{-i}{\hbar} \right)^n \underbrace{\left[ \left( \tilde{\mathcal{K}}^{\text{cl}} + \sum_{j=1}^N \hat{\alpha}_{4+j, \mathcal{R}} \right), \left[ \left( \tilde{\mathcal{K}}^{\text{cl}} + \sum_{j=1}^N \hat{\alpha}_{4+j, \mathcal{R}} \right), \dots, \left[ \left( \tilde{\mathcal{K}}^{\text{cl}} + \sum_{j=1}^N \hat{\alpha}_{4+j, \mathcal{R}} \right), \hat{O} \right] \right] \right]}_{n \text{ times}}$$

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- Taking expectation value, the commutators are taken in a hybrid way,

$$\text{graviton modes: } [a_\sigma(k), a_{\sigma'}^\dagger(k')] = 2E_k \delta_{\sigma\sigma'} \hat{\delta}^{(3)}(k - k')$$

$$\text{massive dynamic: } [f(\cdot), g(\cdot)] \rightarrow i\hbar \{f(\cdot), g(\cdot)\}_{\text{D.B.}}$$

For example,

$$[\tilde{\mathcal{K}}^{\text{cl}}, f(b_i, p_i) a_\sigma] \rightarrow i\hbar a_\sigma \{ \tilde{\mathcal{K}}^{\text{cl}}, f(b_i, p_i) \}_{\text{D.B.}}$$

$$[a_{\sigma_1}^\dagger, f(b_i, p_i) a_\sigma] \rightarrow f(b_i, p_i) [a_{\sigma_1}^\dagger, a_\sigma]$$

# Radiative contribution to observables

We have the master formula for global observables

$$\begin{aligned}
 \langle \Delta \lambda^\mu \rangle \Big|_{\mathcal{O}(G^5)} &= \left\{ \tilde{\mathcal{K}}^{\text{cl}}, \lambda^\mu \right\} + \frac{1}{2!} \left\{ (\tilde{\mathcal{K}}^{\text{cl}})^2, \lambda^\mu \right\} + \frac{1}{3!} \left\{ (\tilde{\mathcal{K}}^{\text{cl}})^3, \lambda^\mu \right\} \quad \leftarrow \text{conservative} \\
 &+ \frac{i}{2!} \int_k \left( \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(k) \left\{ \tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}}(k), \lambda^\mu \right\} - \text{c.c.} \right) \quad \leftarrow \text{LO radiative} \\
 &+ \frac{i}{3!} \int_k \left( \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(k) \left\{ \tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}}(k), \left\{ \tilde{\mathcal{K}}^{\text{cl}}, \lambda^\mu \right\} \right\} - \text{c.c.} \right) \\
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 &+ \frac{1}{3!} \int_{k_1, k_2} \left( \left\{ \tilde{\mathcal{K}}_{6,\mathcal{R}}^{B\text{cl}}(k_1, k_2), \lambda^\mu \right\} \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(k_1) \tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}}(k_2) + \text{c.c.} \right) \\
 &+ \dots
 \end{aligned}$$

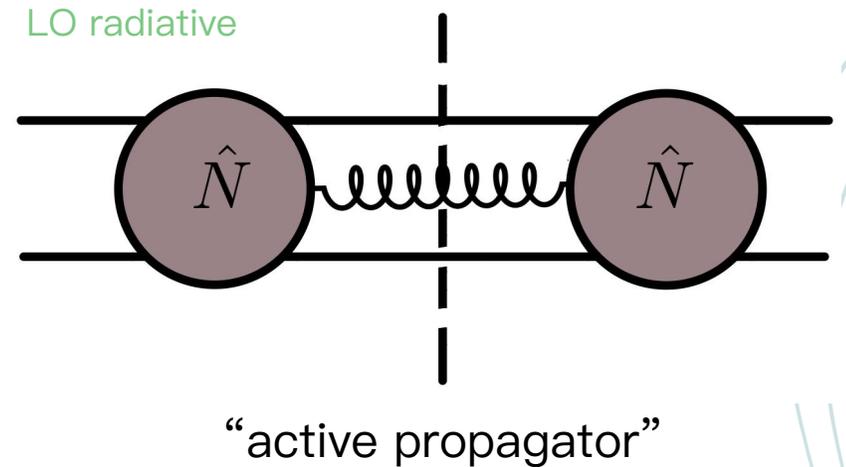
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# Angular momentum @ 2PM ( $\mathcal{O}(G^2)$ )

- We can study the angular momentum change of a **single** BH systematically

$$J_i^{\mu\nu} = 2m_i b_i^{[\mu} v_i^{\nu]} + S_i^{\mu\nu}, \quad i = 1, 2.$$

$$\langle \Delta J_i^{\mu\nu} \rangle = 2m_i (b_i^{[\mu} + \langle \Delta b_i^{[\mu} \rangle) (v_i^{\nu]} + \langle \Delta v_i^{\nu]} \rangle) + \langle \Delta S_i^{\mu\nu} \rangle.$$

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- In dual vector form in the COM frame, it is

$$J_i^\mu := \frac{1}{2E} \epsilon^\mu{}_{\nu\rho\sigma} J_i^{\nu\rho} (m_1 v_1 + m_2 v_2)^\sigma \quad \langle \Delta J_i^\mu \rangle = \langle \Delta J_i^\mu \rangle (\langle \Delta b_i^\mu \rangle, \langle \Delta v_i^\mu \rangle, \langle \Delta s_i^\mu \rangle)$$

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- The change of impact parameter depends on the **IR divergence** of the radial action, so we subtract the divergencies and define

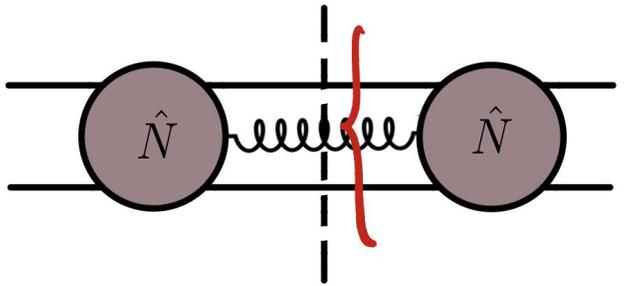
$$\tilde{\mathcal{K}}^{\text{cl}}|_{\mathcal{O}(Gs_1^0 s_2^0)} \equiv -\frac{2Gm_1 m_2 (2\sigma^2 - 1) \log |b|}{\sqrt{\sigma^2 - 1}}$$

# Angular momentum @ 2PM ( $\mathcal{O}(G^2)$ )

- $\langle \Delta b_i^\mu \rangle$  conservative: up to  $\mathcal{O}(G^2 s_1^{j_1} s_2^{j_2})$  with  $j_1 + j_2 \leq 11$  [Bohnenblust, Cangemi, Johansson, Pichini 24']

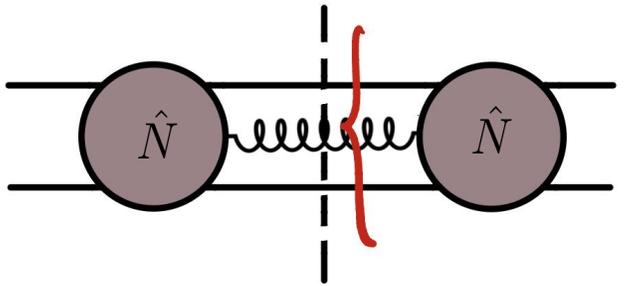
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- $\langle \Delta b_i^\mu \rangle$  radiative (from zero-frequency graviton):  $\tilde{\mathcal{K}}_{5,\mathcal{R}} = i \frac{\kappa}{2} \sum_{j=1,2,1',2'} \frac{\varepsilon_{\mu\nu}(k) p_j^\mu p_j^\nu}{\eta_j p_j \cdot k - i0}$  Weinberg's soft factor

$$\langle \Delta b_i^\mu \rangle|_{\text{rad}} = \text{Diagram} , \left. \begin{matrix} b_i^\mu \\ \text{D.B.} \end{matrix} \right\} = G \left( \frac{8 - 5\sigma^2}{3(\sigma^2 - 1)} + \frac{\sigma(2\sigma^2 - 3)}{(\sigma^2 - 1)^{\frac{3}{2}}} \text{arccosh } \sigma \right) \Delta p_1^\mu$$


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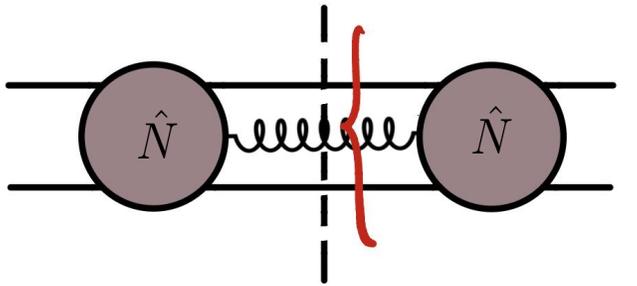
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$$\langle \Delta b_i^\mu \rangle|_{\text{rad}} = \text{Diagram} , \left. \begin{matrix} b_i^\mu \\ \text{D.B.} \end{matrix} \right\} = G \left( \frac{8 - 5\sigma^2}{3(\sigma^2 - 1)} + \frac{\sigma(2\sigma^2 - 3)}{(\sigma^2 - 1)^{\frac{3}{2}}} \text{arccosh } \sigma \right) \Delta p_1^\mu$$


- We obtain  $\langle \Delta b_i^\mu \rangle, \langle \Delta v_i^\mu \rangle, \langle \Delta s_i^\mu \rangle$  up to  $\mathcal{O}(G^2 s_1^{j_1} s_2^{j_2})$  with  $j_1 + j_2 \leq 11$ .

# Angular momentum @ 2PM ( $\mathcal{O}(G^2)$ )

- $\langle \Delta b_i^\mu \rangle$  conservative: up to  $\mathcal{O}(G^2 s_1^{j_1} s_2^{j_2})$  with  $j_1 + j_2 \leq 11$  [Bohnenblust, Cangemi, Johansson, Pichini 24']
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- At 2PM,  $\langle \Delta v_i^\mu \rangle, \langle \Delta s_i^\mu \rangle$  are conserved, radiated angular momentum is only from  $\langle \Delta b_i^\mu \rangle|_{\text{rad}}$
- Interestingly, the fact that the radiative effects come exclusively from  $\langle \Delta b_i^\mu \rangle$  implies its sensitivity to the **BMS** frame choice.

# Waveform

- Similar to the global observables, we can also compute the **waveform**

$$\langle \kappa h_{\mu\nu}(x) \rangle = \frac{\kappa \hbar^{3/2}}{4\pi |\vec{x}|} \sum_{\sigma} \int \frac{d\omega}{2\pi} \left( e^{i\omega u} \varepsilon_{\mu\nu}^{\sigma}(\hat{n}) \langle a_{\sigma}(\omega \hat{n}) \rangle + e^{-i\omega u} \varepsilon_{\mu\nu}^{*\sigma}(\hat{n}) \langle a_{\sigma}^{\dagger}(\omega \hat{n}) \rangle \right)$$

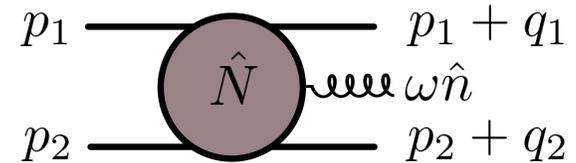
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- Explicitly working out the waveform in terms of the kernels

$$\begin{aligned} \langle \kappa h_{\mu\nu}(x) \rangle &\sim \frac{\kappa \hbar^{3/2}}{4\pi |\vec{x}|} \sum_{\sigma} \int \frac{d\omega}{2\pi} e^{i\omega u} \left( i \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(\omega \hat{n}) \leftarrow \text{LO: 5-pt amplitude} \right) \\ &+ \frac{1}{2!} i \{ \tilde{\mathcal{K}}^{\text{cl}}, \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(\omega \hat{n}) \} + \frac{1}{3!} i \{ (\tilde{\mathcal{K}}^{\text{cl}})^2, \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(\omega \hat{n}) \} \\ &+ \frac{1}{2} \int_k \tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}}(k) \tilde{\mathcal{K}}_{6,\mathcal{R}}^{A*\text{cl}}(\omega \hat{n}, k) \\ &- \frac{1}{4} \int_k (\tilde{\mathcal{K}}_{6,\mathcal{R}}^{B\text{cl}}(\omega \hat{n}, k) + \tilde{\mathcal{K}}_{6,\mathcal{R}}^{B*\text{cl}}(\omega \hat{n}, k)) \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(k) + \text{c.c.} \end{aligned}$$



# Spinning fluxes

- Total radiated momentum

$$\mathbb{K}^\mu = \sum_\sigma \int_k k^\mu a_\sigma^\dagger(k) a_\sigma(k)$$

explicitly

$$\begin{aligned} \langle \mathbb{K}^\mu \rangle \Big|_{\mathcal{O}(G^5)} &= \int_k k^\mu \tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}} \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}} \\ &+ \frac{1}{2} \int_k k^\mu \left[ \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}} \{ \tilde{\mathcal{K}}^{\text{cl}}, \tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}} \}_{DB} + \tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}} \{ \tilde{\mathcal{K}}^{\text{cl}}, \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}} \}_{DB} \right] \\ &+ \frac{1}{2} \int_{k_1, k_2} k_1^\mu \left[ \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(k_1) \tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}}(k_2) \tilde{\mathcal{K}}_{6,\mathcal{R}}^{B*\text{cl}}(k_1, k_2) \right. \\ &\quad \left. + \tilde{\mathcal{K}}_{5,\mathcal{R}}^{*\text{cl}}(k_1) \tilde{\mathcal{K}}_{5,\mathcal{R}}^{\text{cl}}(k_2) \tilde{\mathcal{K}}_{6,\mathcal{R}}^{B\text{cl}}(k_1, k_2) \right]. \end{aligned}$$

momentum conservation

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momentum conservation

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- radiated angular momentum

$$\mathbb{J}^{\mu\nu} = \sum_{\sigma=\pm} \int_k \varepsilon_\sigma^{\alpha\alpha'}(k) a_\sigma^\dagger(k) \left[ (\mathcal{J})_{\alpha\alpha'\beta\beta'}^{\mu\nu} \right] \varepsilon_\sigma^{*\beta\beta'}(k) a_\sigma(k),$$

$$\mathcal{J}_{\alpha\alpha'\beta\beta'}^{\mu\nu} = -i\eta_{\alpha\beta}\eta_{\alpha'\beta'} k^{[\mu} \frac{\overleftrightarrow{\partial}}{\partial k_{\nu]}} - 2i\eta_{\alpha'\beta'} \delta_\alpha^{[\mu} \delta_\beta^{\nu]},$$

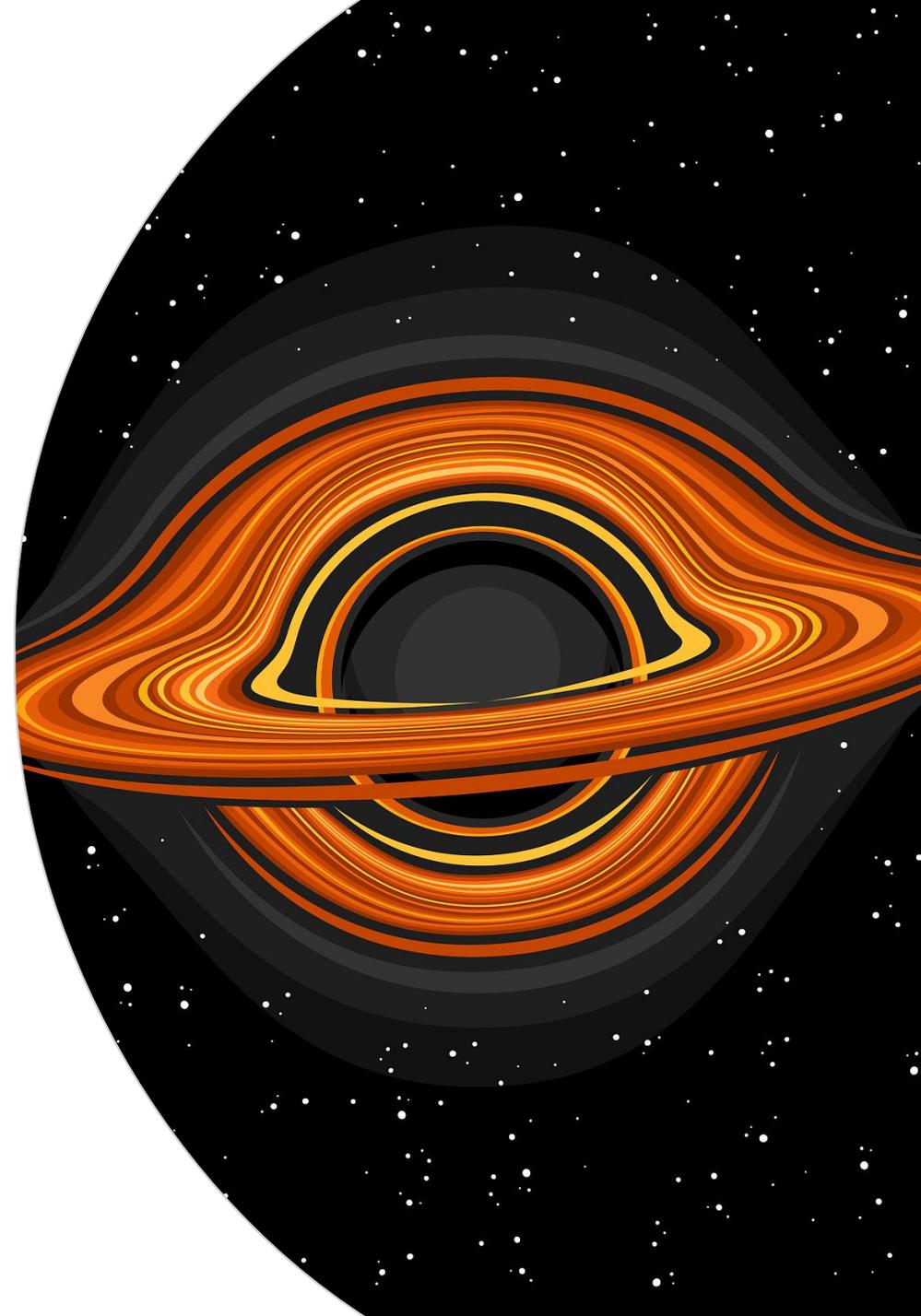
- We verify the conservation in the COM frame

$$\mathbb{J}^\mu := \frac{1}{2E} \epsilon^\mu{}_{\nu\rho\sigma} \mathbb{J}^{\nu\rho} (m_1 v_1 + m_2 v_2)^\sigma,$$

$$\langle \mathbb{J}^\mu \rangle + \Delta J_1^\mu + \Delta J_2^\mu = 0$$

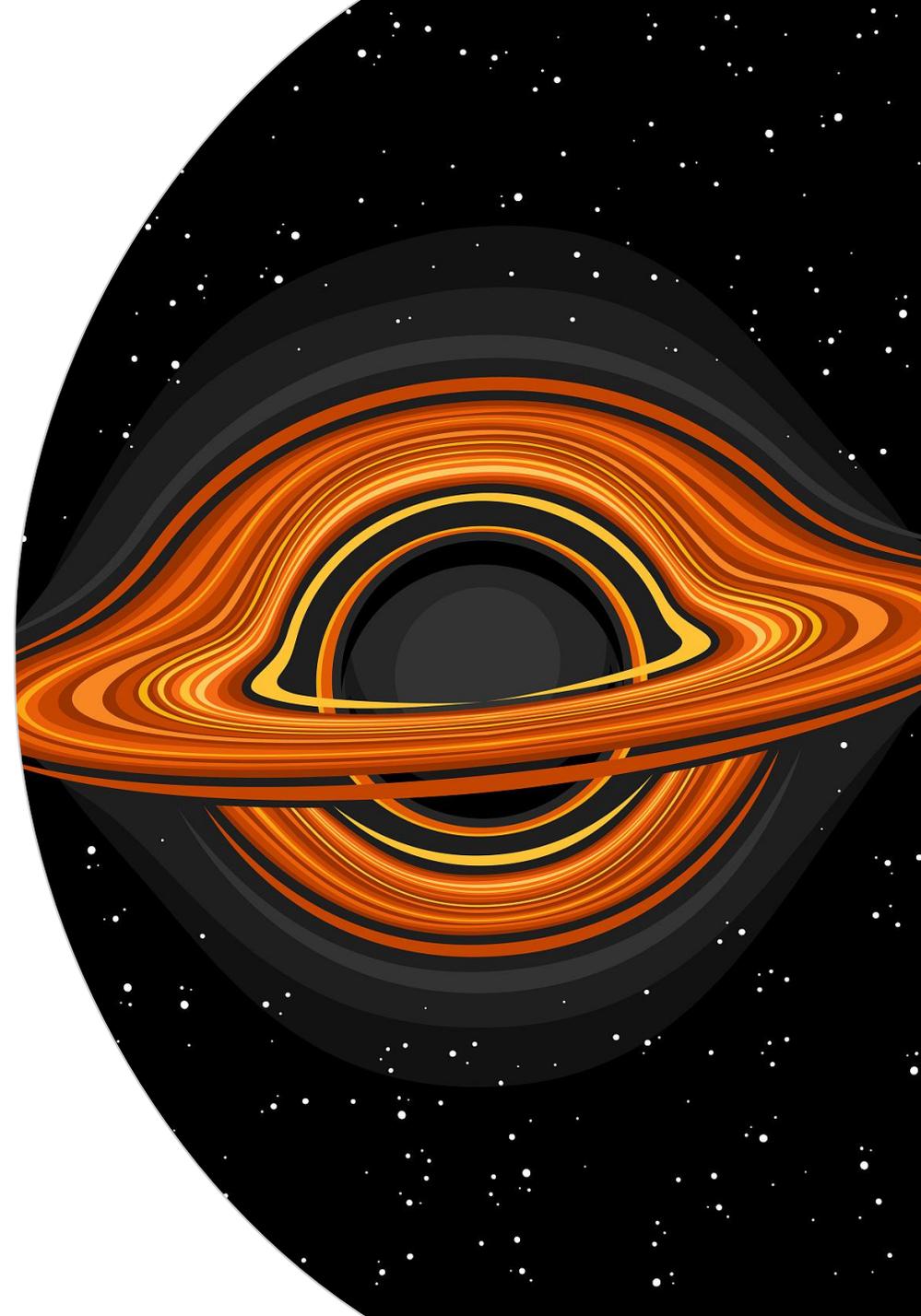
# Conclusion & Outlook

- Conclusion:
  - classical  $\text{Log}(S)$  matrix elements  $\sim$  generating functions
  - conservative & radiative observables in Dirac brackets
  - compatible with PM, PN & SF



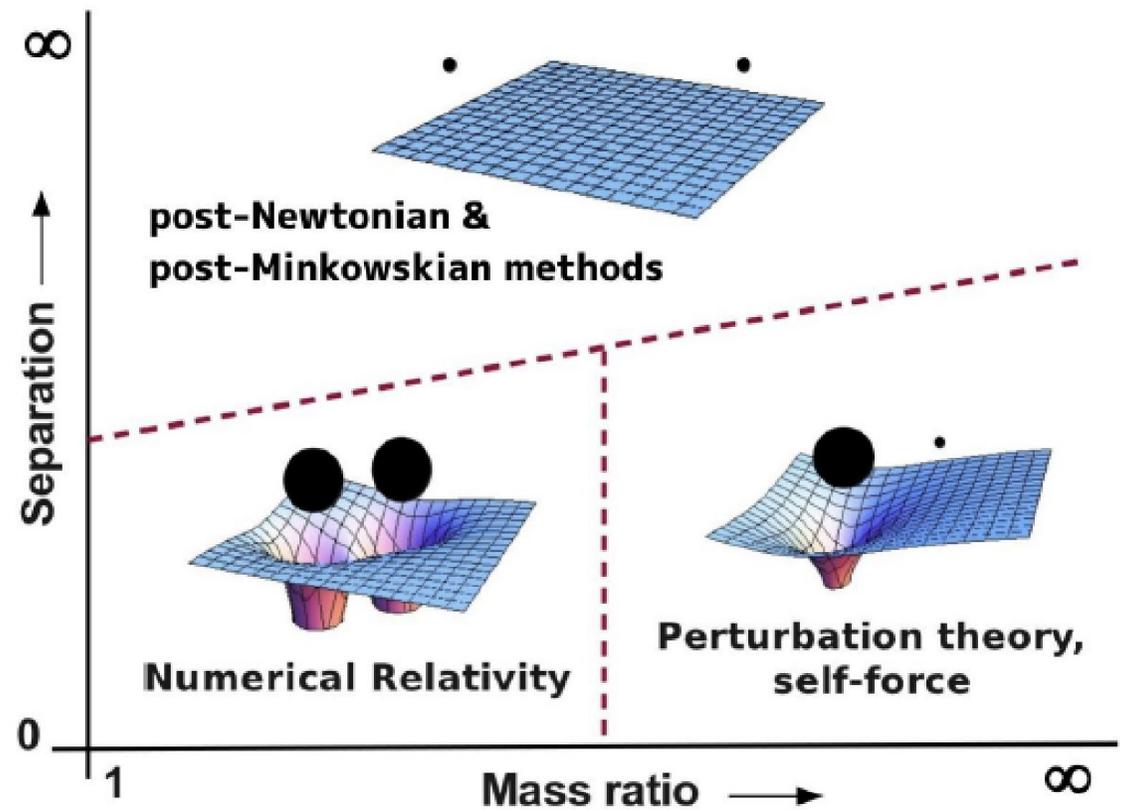
# Conclusion & Outlook

- Conclusion:
  - classical  $\text{Log}(S)$  matrix elements  $\sim$  generating functions
  - conservative & radiative observables in Dirac brackets
  - compatible with PM, PN & SF
- Outlook
  - Ongoing: proof for the master formula (from Wigner-Weyl transform)
  - re-summed form of the master formula?
  - BMS frame and  $\langle \Delta b_i^\mu \rangle$
  - bound state?



# Approaches to the two-body problem

- Post-Newtonian (PN): slow velocity,  $v \ll c$
- Post-Minkowskian (PM): in Newton's constant  $G_N$
- Self-force (SF):  $m_1/m_2$
- Numerical relativity
- ...



Credit: L. Barack & A. Pound

# Observables from scattering amplitudes

Quantum amplitudes should have all scattering information

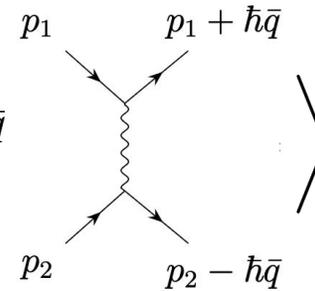
- KMOC:

$$\begin{aligned} \langle \Delta p_1^\mu \rangle &= \langle \psi | S^\dagger \mathbb{P}_1^\mu S | \psi \rangle - \langle \psi | \mathbb{P}_1^\mu | \psi \rangle \\ &= \langle \psi | i [\mathbb{P}_1^\mu, T] | \psi \rangle + \langle \psi | T^\dagger [\mathbb{P}_1^\mu, T] | \psi \rangle \end{aligned}$$

[Kosower, Maybee, O'Connell; Bern, Zeng, Shen, Teng, Roiban, Cheung, Solon, Scheopner, Luna, Kosmopoulos, Parra-Martinez, ... ]

- Take classical limit: **sharply peaked** wavefunction, and  $\hbar \rightarrow 0$

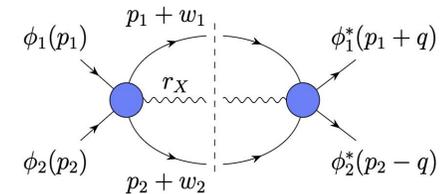
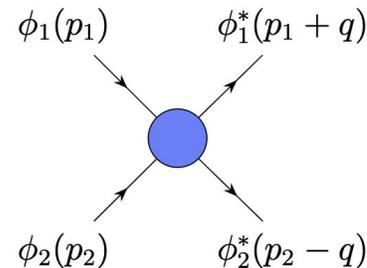
$$\Delta p_1^{\mu, (0)} = i\kappa^2 m_1 m_2 \left\langle \int \hat{d}^4 \bar{q} \hat{\delta}(\bar{q} \cdot p_1) \hat{\delta}(\bar{q} \cdot p_2) e^{-i \frac{1}{\hbar} b \cdot \bar{q}} \right\rangle$$



- We need to combine two terms to have the full result:

$$I_{(1)}^\mu \equiv \langle \psi | i [\mathbb{P}_1^\mu, T] | \psi \rangle \sim 1/\hbar^\#$$

$$I_{(2)}^\mu \equiv \langle \psi | T^\dagger [\mathbb{P}_1^\mu, T] | \psi \rangle \sim 1/\hbar^\#$$



# Other attempts

- 3PM eikonal [Bern, Luna, Roidan, Shen, Zeng 20'; Kosmopoulos, Luna 21'; Gatica 23']

$$\Delta \mathbf{p}_\perp = -\{\mathbf{p}_\perp, \chi\} - \frac{1}{2} \{\chi, \{\mathbf{p}_\perp, \chi\}\} - \mathcal{D}_{SL}(\chi, \{\mathbf{p}_\perp, \chi\}) + \frac{1}{2} \{\mathbf{p}_\perp, \mathcal{D}_{SL}(\chi, \chi)\},$$

$$\Delta \mathbf{S}_1 = -\{\mathbf{S}_1, \chi\} - \frac{1}{2} \{\chi, \{\mathbf{S}_1, \chi\}\} - \mathcal{D}_{SL}(\chi, \{\mathbf{S}_1, \chi\}) + \frac{1}{2} \{\mathbf{S}_1, \mathcal{D}_{SL}(\chi, \chi)\}.$$

$$\mathcal{D}_{SL}(f, g) \equiv - \sum_{a=1,2} \epsilon^{ijk} S_a^k \frac{\partial f}{\partial S_a^i} \frac{\partial g}{\partial L^j}$$

- From KMOC and stationary phase [Luna, Moynihan, O'Connell, Ross 23']

$$\Delta \mathbb{O}_1 = \mathbb{O}_1(Q) - \{\mathbb{O}_1(p_1), \chi\} - \frac{1}{2} \{\chi, \{\mathbb{O}_1(p_1), \chi\}\} + \dots$$

- [Kim, Kim, Lee 24'; Kim, Kim, Kim, Lee 24'; Kim 25'; Kim, Lee, Lee 25'; Kim 25']

$$\Delta_{(1)} O = \{\chi_{(1)}, O\},$$

$$\Delta_{(2)} O = \{\chi_{(2)}, O\} + \frac{1}{2} \{\chi_{(1)}, \{\chi_{(1)}, O\}\},$$

$$\Delta_{(3)} O = \{\chi_{(3)}, O\} + \frac{1}{2} \{\chi_{(2)}, \{\chi_{(1)}, O\}\} + \frac{1}{2} \{\chi_{(1)}, \{\chi_{(2)}, O\}\}$$

$$+ \frac{1}{6} \{\chi_{(1)}, \{\chi_{(1)}, \{\chi_{(1)}, O\}\}\}.$$

radiative correction

$$P_{H,\text{out}}^\mu = \frac{1}{2} \{\chi_{(1.5)}^{H^1}, \{\chi_{(1.5)}^{H^1}, P_H^\mu\}\}$$

$$\{H_{\mu\nu}(q), H_{\alpha\beta}(k)\} = iP_{\mu\nu;\alpha\beta} \text{sgn}(k^0) \delta(k^2) \delta^D(q+k)$$

# Two-body Dirac brackets

If the functions  $f(b_i, v_i, s_i), g(b_i, v_i, s_i)$  are invariant under

$$b_1 \rightarrow b_1 + v_1, \quad b_2 \rightarrow b_2 + v_2,$$

the Dirac brackets  $\{f, g\}_{\text{D.B.}}$  are equivalently

$$\begin{aligned} \{b_i^\mu, v_j^\nu\}' &= \delta_{ij} \frac{\eta^{\mu\nu}}{m_j}, & \{b_i^\mu, b_j^\nu\}' &= \delta_{ij} \frac{S_i^{\mu\nu}}{m_i^2}, \\ \{b_i^\mu, s_j^\nu\}' &= \delta_{ij} \frac{2s_i^{[\mu} v_i^{\nu]}}{m_j}, & \{s_i^\mu, s_j^\nu\}' &= \delta_{ij} \frac{S_i^{\mu\nu}}{m_i^2}. \end{aligned}$$

This drastically simplifies the calculation.

# Example: spinning probe in Kerr

- $\mathcal{N} = 2$  worldline as a spinning point particle coupled to Kerr metric

[Gibbons, Rietdijk, Holton; Jacobson, et al]

$$S[x^\mu(\tau), p_\mu(\tau), \psi_a(\tau)] = \int d\tau \left[ p_\mu \dot{x}^\mu + i\bar{\psi}_a \dot{\psi}^a - eH - i\bar{\chi}\mathcal{Q} - i\chi\bar{\mathcal{Q}} - a\mathcal{J} \right]$$

$$\begin{aligned} H &= \frac{1}{2} [g^{\mu\nu} \pi_\mu \pi_\nu + m^2 + \mathcal{O}(S^2)] \\ \mathcal{Q} &= \psi^a e_a^\mu \pi_\mu \\ \bar{\mathcal{Q}} &= \bar{\psi}^a e_a^\mu \pi_\mu \\ \mathcal{J} &= \bar{\psi}^a \psi_a - q \end{aligned}$$

- The spin tensor is described by the auxiliary Grassmann field

$$S^{\mu\nu} = -2ie_a^\mu e_b^\nu \bar{\psi}^{[a} \psi^{b]}$$

- The EOM is the same with the MPD equation.
- Truncated at **the linear order in spin**, the system is **integrable**. Carefully choosing the tetrad allows us to integrate out the radial momentum, written in terms of the conserved quantities,

$$I_r^>(\mathcal{E}, l, a, l_Q, s_{||})$$