

# Fourier Integrals for Gravitational Radiation

Giacomo Brunello

- **G.B.**, Chestnov, Crisanti, Giroux, Smith [2510.26874]
- **G.B.**, De Angelis, Kosower [2511.05412]
- **G.B.**, De Angelis [2403.08009]
- **G.B.**, Crisanti, Giroux, Mastrolia, Smith [2311.14432]



New Frontiers of Quantum Field and Gravity  
Peking University, 13/01/2026



# Many integrals appearing in theoretical physics are of the same type

**Definition.** *Physics is a part of mathematics devoted to the calculation of integrals of the form  $\int g(x)e^{f(x)}dx$ . Different branches of physics are distinguished by the range of the variable  $x$  and by the names used for  $f(x)$ ,  $g(x)$  and for the integral. For example, in classical statistical physics  $x$  runs over a symplectic manifold,  $f(x)$  is called the Hamiltonian function and the integral has the meaning of a partition function or of a correlation function. In a  $d$ -dimensional quantum field theory  $x$  runs over the space of functions on a  $d$ -dimensional manifold (the space of fields) and  $f(x)$  is interpreted as an action functional.*

Schwarz, Shapiro, 2008

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# Many integrals appearing at this conference are of the same type

Integrability

Amplitude

S-matrix

Energy correlators

Gravity

Conformal field theory

# What this talk is about

Fourier integrals

$$\int e^{f(x)} g(x) dx$$

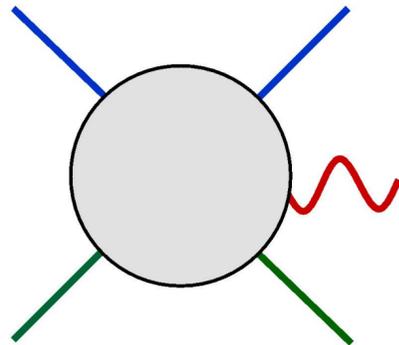
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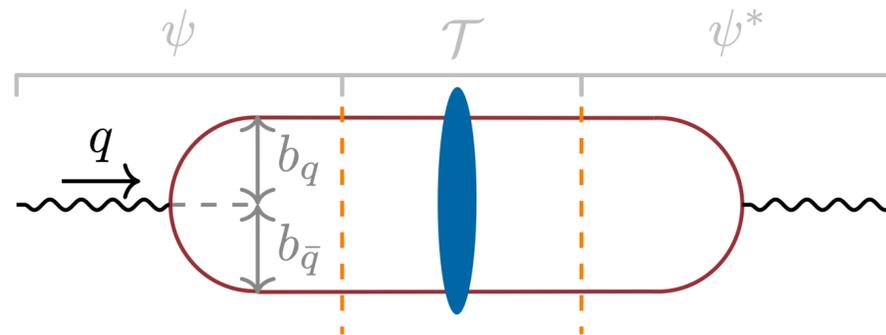
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### Gravitational Radiation

$$\mathcal{W}_h(\omega, \mathbf{n}) = \int d\hat{\mu} e^{ib \cdot q}$$

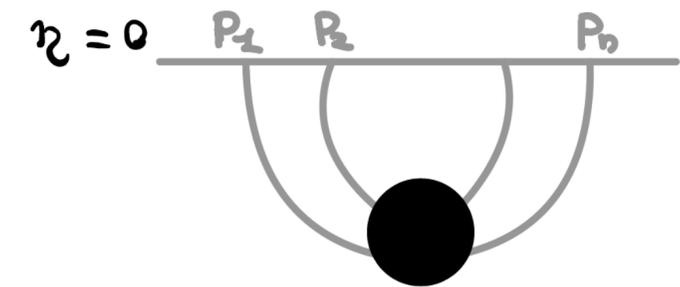


### QCD in the saturation regime



G.B., Caron-Huot, Crisanti, Giroux, Smith [2510.26874]

### Cosmological correlators



# What this talk is about (II)

We know how to deal with multi-loop **Feynman Integrals**  
Integration-by-parts identities, Vector space structure, Differential Equations, ...

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Very little is known about **Fourier Integrals**

**Today:** Export multi-loop technology to the study of Fourier integrals

# An Example from Quantum Field Theory

$$G = \int \frac{d^D q}{(2\pi)^D} \frac{e^{iq \cdot b}}{q^2 - m^2 + i\epsilon} \propto K_{D/2-1}(mb) \quad \text{Bessel integral} \quad b = \sqrt{-b^2}$$

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**Second order** differential equation

$$G(y) = y^\nu F(y) \quad (y^2 \partial_y^2 + y \partial_y - (y^2 + \nu^2))F(y) = 0 \quad \nu = \frac{D}{2} - 1 \quad y = mb$$

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System of **first order differential equations**:

$$\partial_y \begin{bmatrix} F \\ \partial_y F \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 + \frac{\nu^2}{z^2} & -\frac{1}{z} \end{bmatrix} \begin{bmatrix} F \\ \partial_y F \end{bmatrix} \quad \text{How to solve the differential equations?}$$

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How to solve the differential equations?

Fourier **integral family**

$$I_{a_1 a_2} = \int \frac{d^D q}{(2\pi)^D} \frac{e^{iq \cdot b} (iq \cdot b)^{a_1}}{(q^2 - m^2 + i\epsilon)^{a_2}}$$

How many independent integrals?

# Roadmap

I ) Gravitational waveforms from Amplitudes

II ) Fourier Integrals as Twisted Period Integrals

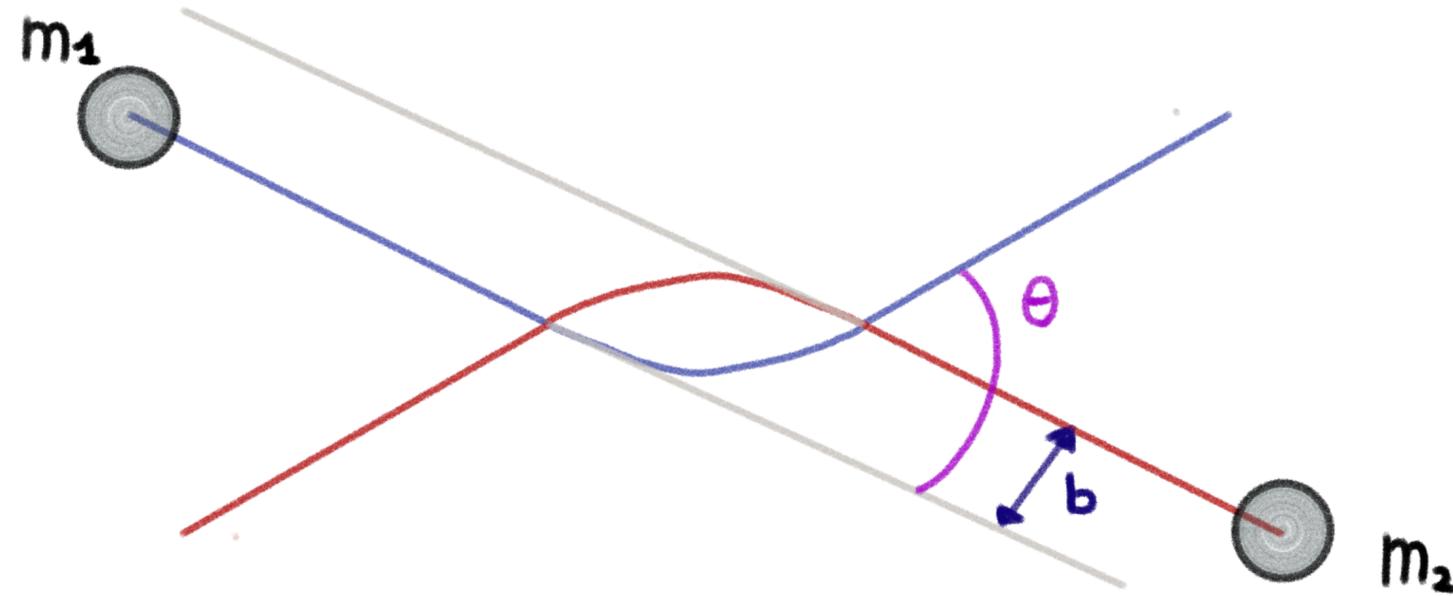
III ) Differential Equations for Fourier Integrals

IV ) Kinematic limits of Fourier Integrals

# Gravitational Waveforms from Amplitudes

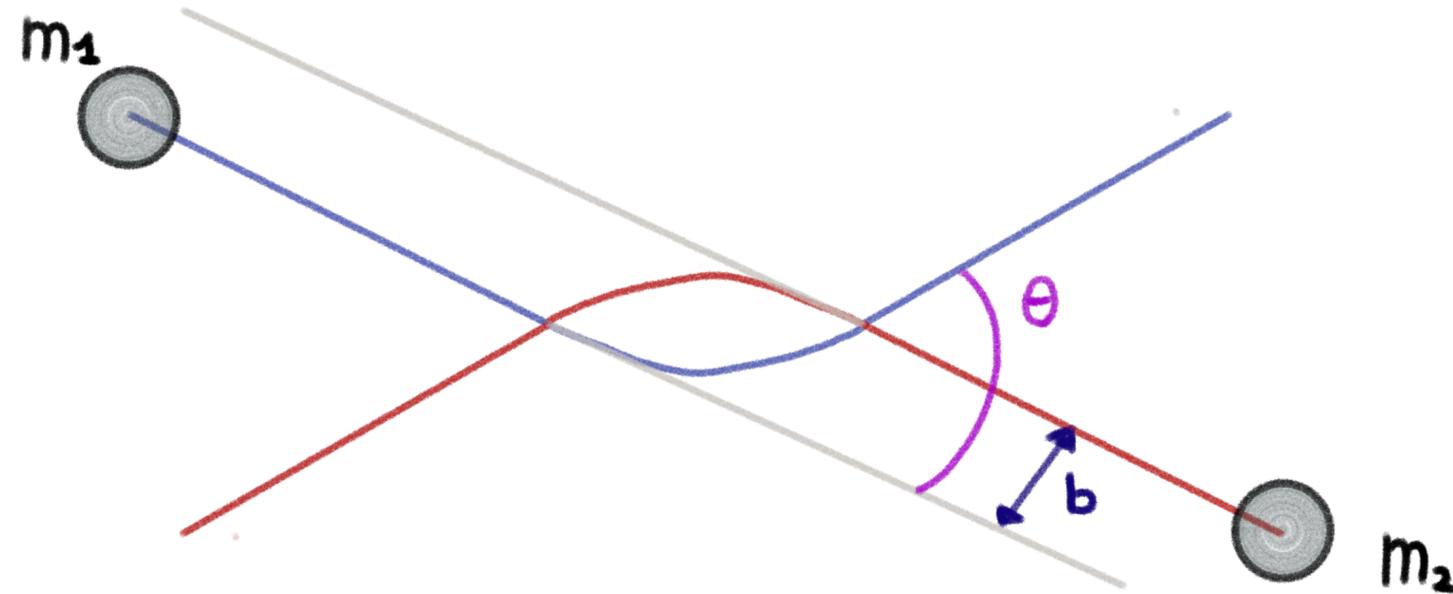
# Physical motivation: Waveform from Amplitudes

Classical Gravitational waveforms emitted during the **scattering of two compact objects**



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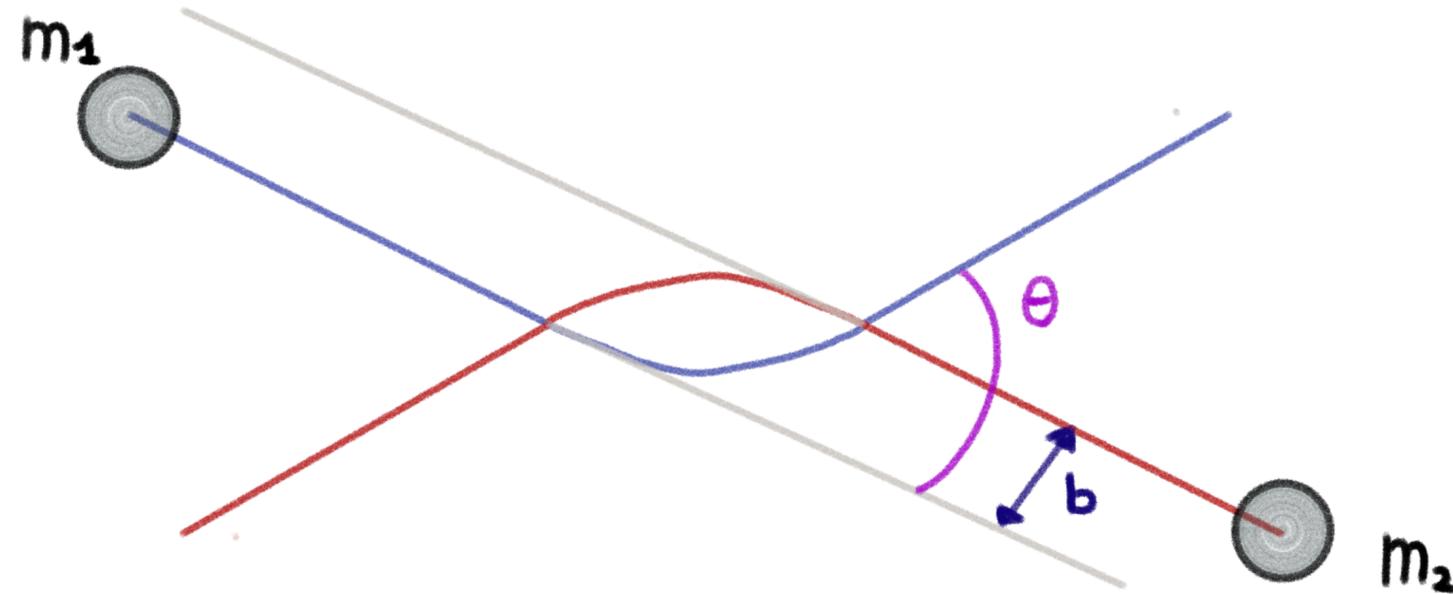
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# Physical motivation: Waveform from Amplitudes

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Hierarchy of length scales

$$\frac{1}{m} \ll G m \ll |b| \ll r \quad \frac{G m}{b}$$

Compton wavelength      Schwarzschild radius      Impact parameter      Distance      Post-Minkowskian expansion parameter

# Gravitational Waveform from Scattering Amplitudes in a Nutshell

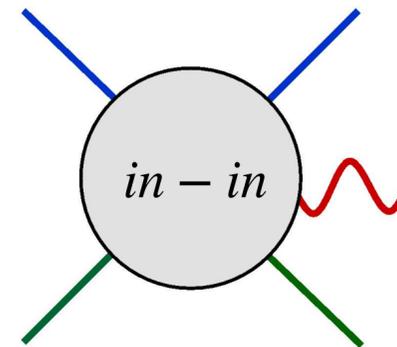
See David's talk  
See Stefano's talk

[Kosower, Maybee, O'Connell]  
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## Fourier transform of 2 → 3 scattering amplitudes

$$\mathcal{W}_h(\omega, \mathbf{n}) \propto \int \hat{d}^D q \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_2 \cdot (q - k)) e^{ib \cdot q}$$

Fourier Transform



$u_1, u_2$  classical four-velocities  
 $k$  emitted graviton momentum  
 $q$  Fourier momentum  
 $b$  impact parameter

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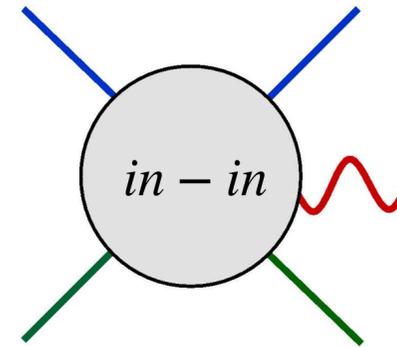
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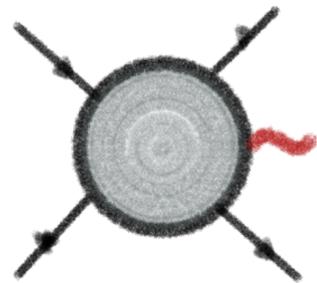
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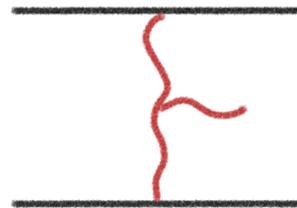


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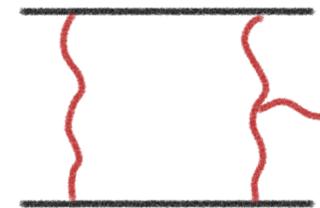
Perturbative expansion in powers of  $\frac{Gm}{b}$



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[Jakobsen, Mogull, Plefka, Steinhoff]  
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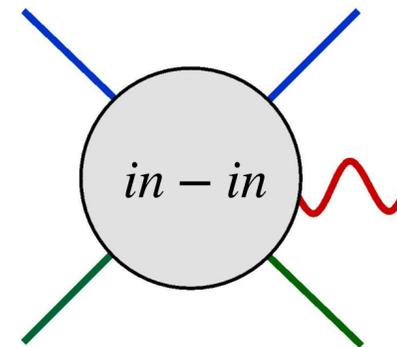
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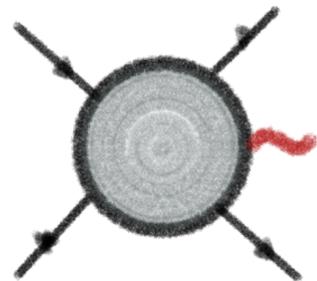
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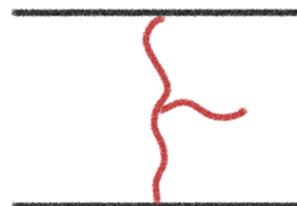


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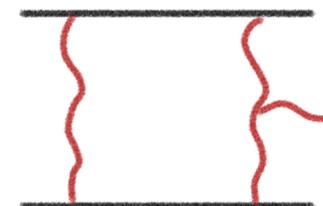
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## Kinematics

$$\text{Lorentz factor } \gamma = u_1 \cdot u_2 \quad u_i^2 = 1 \quad u_i \cdot b = 0 \quad k^2 = 0$$

$$w_1 = u_1 \cdot k \quad w_2 = u_2 \cdot k \quad b \cdot k \quad b = \sqrt{-b^2}$$

# How to compute waveforms?

See David's talk

Loop-by-loop approach is very cumbersome  
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[G.B., De Angelis]

[G.B., De Angelis, Kosower]

Fourier integrals are **twisted period integrals**:  
Finite-dimensional vector space structure, linear and quadratic relations, differential equations

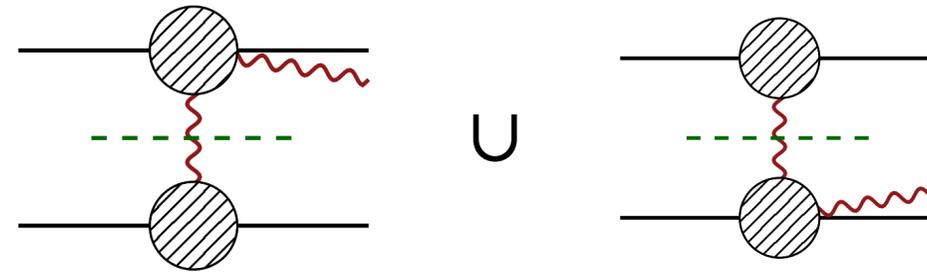
[G.B., Crisanti, Giroux, Mastrolia, Smith]

# Leading order waveform

Fourier transform of tree-level 2  $\rightarrow$  3 **scattering amplitudes**

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Fourier transform

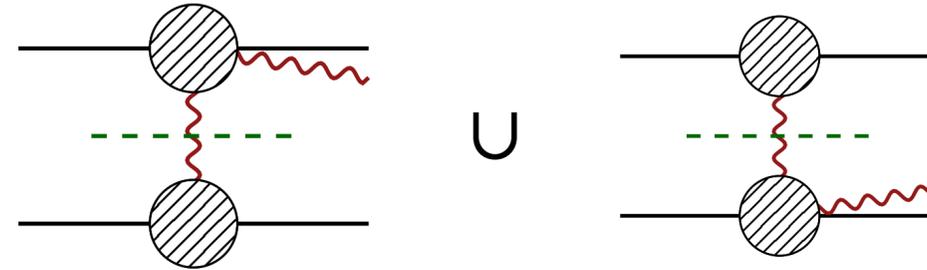


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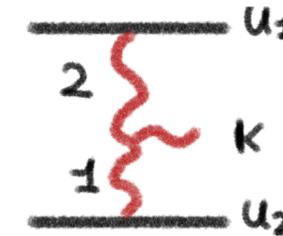
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$D$ - dimensional **Fourier integrals** need to be computed

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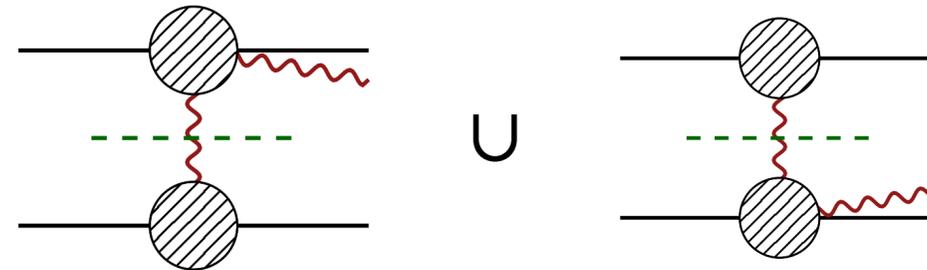


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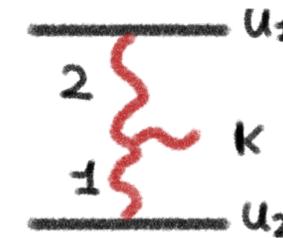
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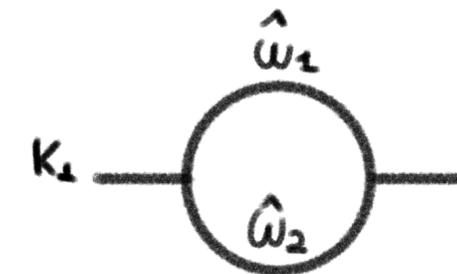
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Integrating out  $\delta$ -functions: Fourier transform of a one-loop two-point function with different masses

$$I_{a_1 a_2 a_3} = \frac{(-b^2)^{a_1 + a_2 + 1 - D/2}}{\sqrt{\gamma^2 - 1}} \int \hat{d}^{D-2} q_{\perp} e^{iq_{\perp} \cdot \hat{b}} \frac{(iq_{\perp} \cdot \hat{b})^{-a_3}}{(q_{\perp}^2 - \hat{w}_2)^{a_1} [(q_{\perp} - k_{\perp})^2 - \hat{w}_1^2]^{a_2}}$$



$$\hat{w}_i = \frac{w_i}{\sqrt{\gamma^2 - 1}} \quad k_{\perp}^2 = \hat{w}_1^2 + \hat{w}_2^2 - 2\gamma \hat{w}_1 \hat{w}_2 \quad \hat{b}^2 = -1$$

# Fourier Integrals as Twisted Period Integrals

# Integral relations for Fourier integrals

**Integration-by-parts identities:** linear relations among Fourier integrals

[Chetyrkin, Tkachov] [Laporta]

[**G.B.**, De Angelis]

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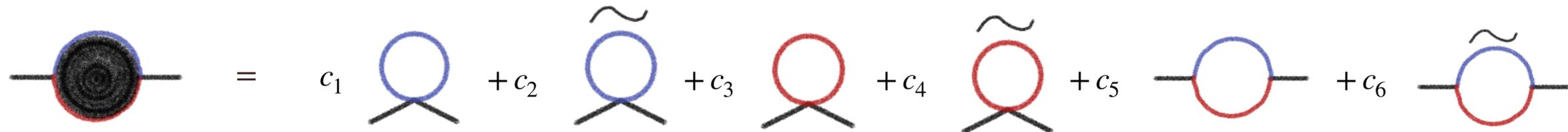
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**Decompose** any integral in terms of six irreducible Master Integrals



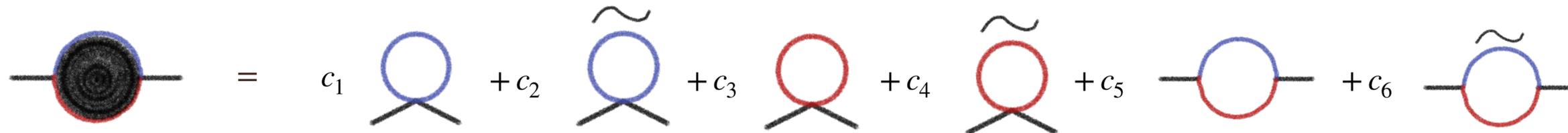
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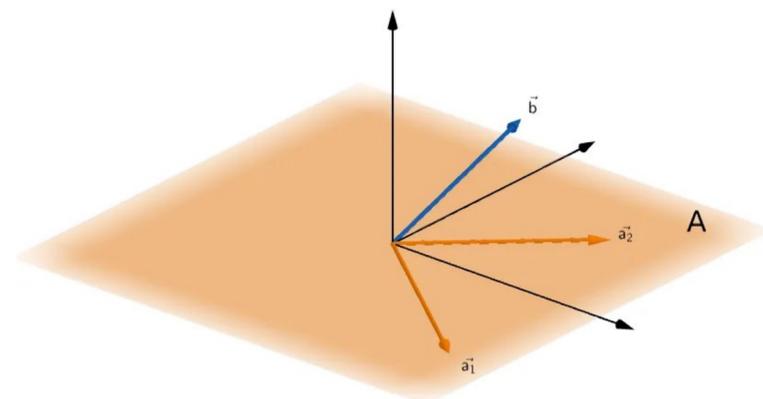
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Finite-dimensional vector space structure



$$c_5 = \text{[Complex Integral]} \cdot \text{[Blue Circle with Two External Lines]}$$

Vector space?

Dimension?

Scalar product?

# Interlude: Fourier Integrals as Twisted Period Integrals

[G.B., Crisanti, Giroux, Mastrolia, Smith]

**Baikov representation:** denominators promoted to integration variables  $D_i \rightarrow z_i$

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**Twisted period integral:** multivalued integration kernel (twist)  $u(z)$ , single-valued form  $\varphi$

$$\langle \text{form} | \text{contour} \rangle := \int_{\Gamma} u(z) \varphi$$

Twisted differential

$$\nabla_{\omega} = d + \omega \wedge \quad \omega = d \log u(z)$$

[Mastrolia, Mizera]  
[Frellesvig, Gasparotto, Laporta,  
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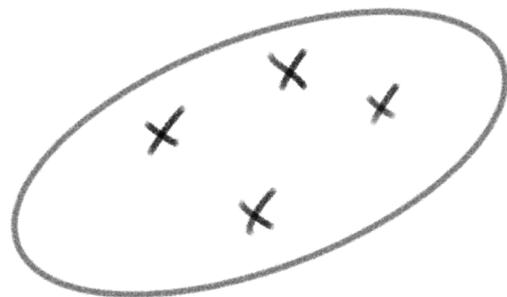
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**Ambient space** where our forms live

$$T = \mathbb{C}^3 / (\{\mathcal{B} = 0\} \cup D^{\vee})$$

Ambient Space

$$D^{\vee} = V(z_1 \cdot z_2)$$

Relative singularities

# Vector space of differential forms

[Mastrolia, Mizera]

[Frellesvig, Gasparotto, Laporta,  
Mandal, Mattiazzi, Mizera]

Differential forms satisfy **integration-by-parts identities**

$$\mathcal{B}|_{\partial\Gamma} = 0$$

$$0 = \int_{\Gamma} d(u(z) \phi) = \int_{\Gamma} u(z) \nabla_{\omega} \phi$$

$\Rightarrow$

Equivalence class

$$\varphi \sim \varphi + \nabla_{\omega} \phi$$

Closed form

$$\nabla_{\omega} \varphi = 0$$

# Vector space of differential forms

[Mastrolia, Mizera]

[Frellesvig, Gasparotto, Laporta,  
Mandal, Mattiazzi, Mizera]

Differential forms satisfy **integration-by-parts identities**

$$\mathcal{B}|_{\partial\Gamma} = 0 \quad 0 = \int_{\Gamma} d(u(z) \phi) = \int_{\Gamma} u(z) \nabla_{\omega} \phi \quad \Rightarrow \quad \begin{array}{ll} \text{Equivalence class} & \varphi \sim \varphi + \nabla_{\omega} \phi \\ \text{Closed form} & \nabla_{\omega} \varphi = 0 \end{array}$$

Differential forms belong to a (relative) **twisted de-Rham co-homology group**

$$\langle \varphi | \in H^n(T, \nabla_{\omega}) = \frac{\ker(\nabla_{\omega})}{\text{Im}(\nabla_{\omega})}$$

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**Finite-dimensional** vector-space structure

[Lee, Pomeranski]

Basis of forms  $\{\langle e_i | \}_{i=1}^{\nu}$



# Vector space of integration contours

**Twisted boundary operator:** assign a branch to every contour, the operator returns the boundary

$$\Gamma = \sum_i a_i \Gamma_i \otimes u_{\Gamma_i}(z) \quad \partial_\omega(\Gamma \otimes u_\Gamma(x)) = (\partial\Gamma) \otimes u_\Gamma|_{\partial\Gamma} \quad \text{Equivalence class} \quad \Gamma \sim \Gamma + \partial_\omega \eta$$

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**Exponential twist:** convergence depends on how  $\Gamma$  goes to infinity

[G.B., Crisanti, Giroux, Mastrolia, Smith]

$$u(z) \propto e^{z_3} \begin{matrix} \text{Re}(z_3) \rightarrow \infty \\ \rightarrow \end{matrix} \infty$$

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**Finite-dimensional** vector-space structure

$$\dim(H_n) = \nu$$

Basis of contours  $\{[\gamma_i]\}_{i=1}^\nu$

Decomposition  $[\Gamma] = \sum_{i=1}^\nu c_i [\gamma_i]$

# Bilinear pairings

## Dual forms & dual contours

$$u(z) \rightarrow u^{-1}(z)$$

$$|\varphi^\vee\rangle \in H^{n\vee}$$

$$[\Gamma^\vee] \in H_n^{rd\vee}$$

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## Intersection pairings

$$I = \langle \varphi | \Gamma \rangle$$

Integral

$$I^\vee = [\Gamma^\vee | \varphi^\vee]$$

Dual Integral

$$\langle \varphi | \varphi^\vee \rangle$$

Intersection number  
between forms

$$[\Gamma | \Gamma^\vee]$$

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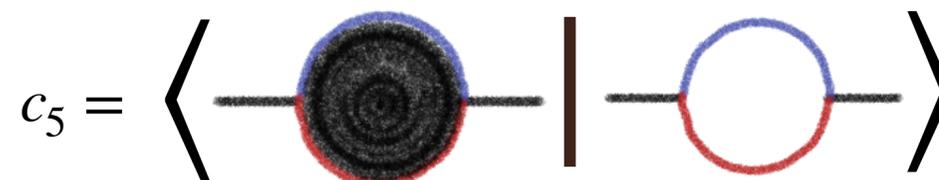
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The scalar product is  
the intersection number!

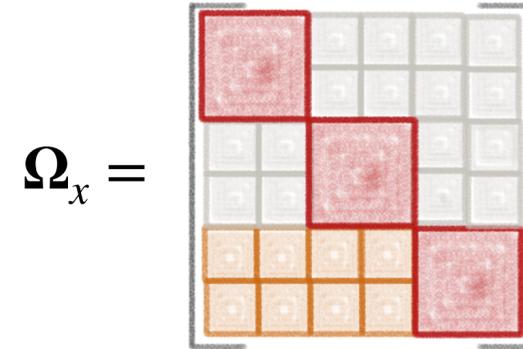
[Mastrolia, Mizera]

# Differential Equations for Fourier integrals

# Differential Equations for Fourier integrals

Master Integrals obey **systems of first order Differential Equations**

$$\partial_x \mathbf{J} = \mathbf{\Omega}_x \mathbf{J}$$
$$x \in \{\hat{w}_1, \hat{w}_2, \hat{b} \cdot k, \gamma\}$$



[Kotikov] [Bern, Dixon, Kosower]

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We can compute the integrals  
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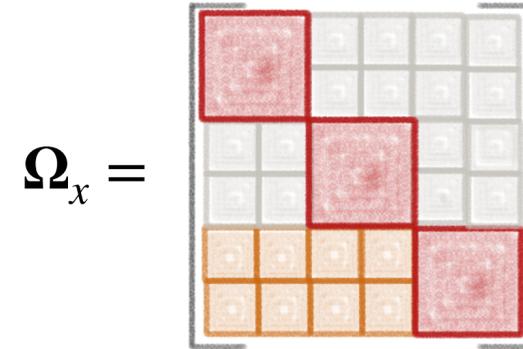
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Integrals as **Laurent expansion** around  $D = 4$

$$D = 4 - 2\epsilon \quad J_i = \frac{1}{\epsilon^n} J_i^{(-n)} + \dots + J_i^{(0)} + \epsilon J_i^{(1)} + \dots$$

# Epsilon-factorised forms

kinematic dependence factorised from dimensional one [Henn]

$$\mathbf{G} = \mathcal{U} \mathbf{J}$$

**Change of variables**

$$\partial_x \mathbf{G} = \epsilon \hat{\Omega}_x \mathbf{G}$$

$$\mathbf{G} = \left( \mathbf{1} + \epsilon \int_{\gamma} d\hat{\Omega} + \epsilon^2 \int_{\gamma} d\hat{\Omega} \int_{\gamma} d\hat{\Omega} + \dots \right) \mathbf{G} \Big|_{\partial\gamma}$$

**Iterated integrals** [Chen] [Goncharov]

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Iterated integrals: fast numerical evaluation and good analytic control

# Epsilon-factorised forms

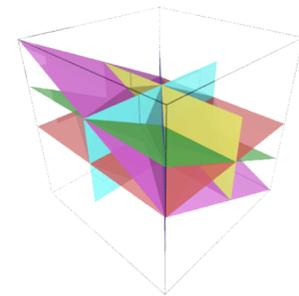
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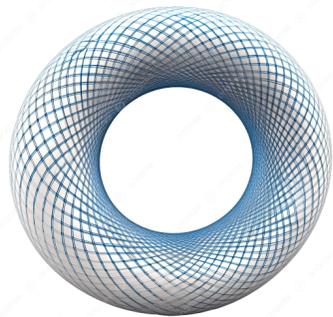
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Iterated integrals: fast numerical evaluation and good analytic control

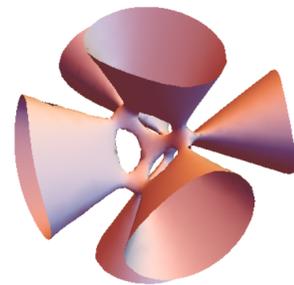
For **Feynman Integrals** this class is known:



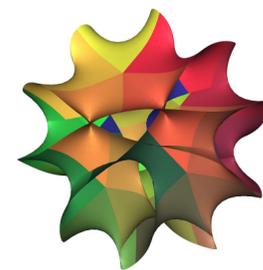
PolyLogs



Elliptic integrals



K3-surfaces

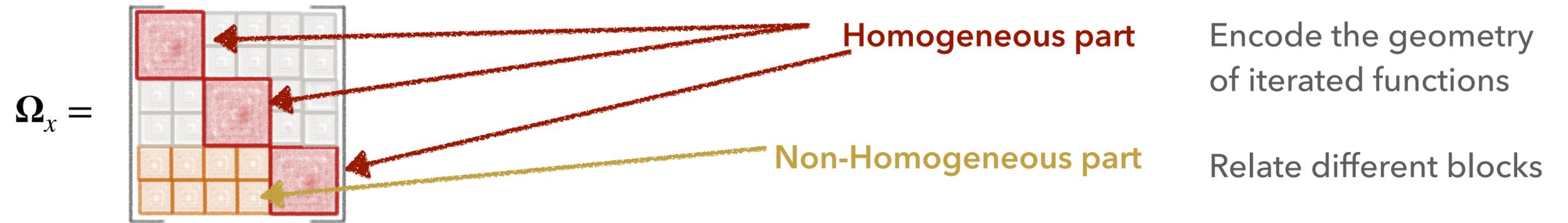


Calabi-Yau

How about Fourier integrals?

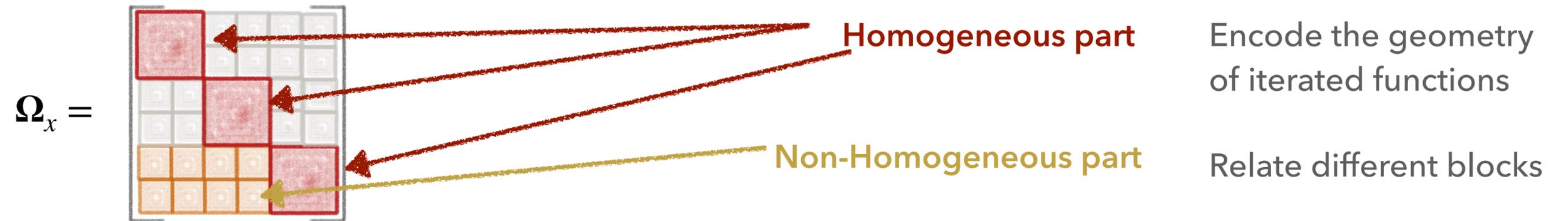
# Strategy for epsilon-factorised forms

Differential equations have a **triangular structure**



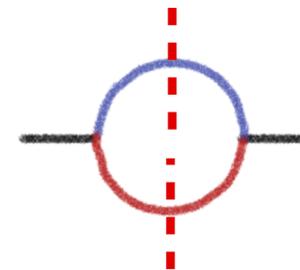
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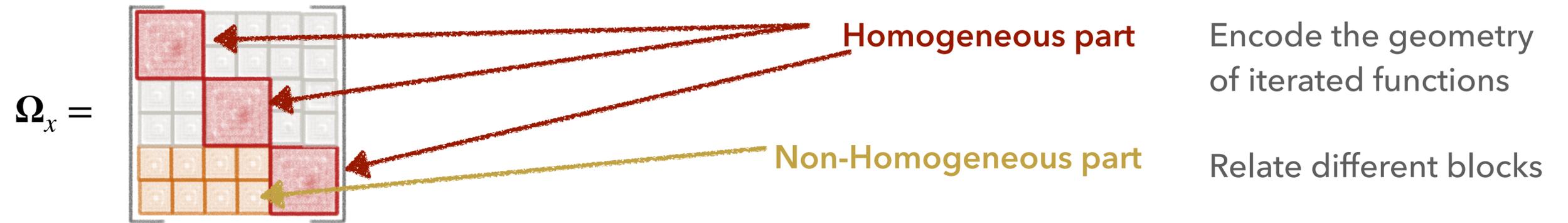
The solution to the homogeneous blocks is given by **maximal cut** integrals

$$\frac{1}{D_i} \rightarrow \delta(D_i)$$



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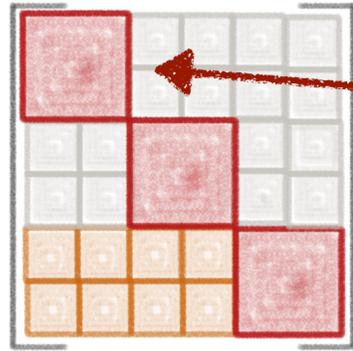
$\epsilon$ -factorised via homogenous solution in  $D=4$ : **period matrix**

[Primo, Tancredi]  
 [Frellesvig]  
 [Gorges, Nega, Tancredi, Wagner]

$$P_{ij} = \langle e_i | \eta_j \rangle := \int_{\gamma_i} u(z) \Big|_{D=4} \eta_j$$

# Epsilon-factorised form of the first block

[G.B., Chestnov, Crisanti, Giroux, Smith]



**Maximal cut**

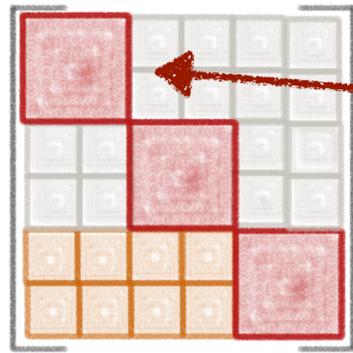
$$J_i \Big|_{D=4}^{mc} = \int u(z_3) \varphi_i^{mc}$$

$$u(z_3) = \frac{e^{z_3}}{\sqrt{z_3^2 - \hat{w}_2^2}}$$

$$\varphi_i^{mc} = \{1, z_3\} dz_3$$

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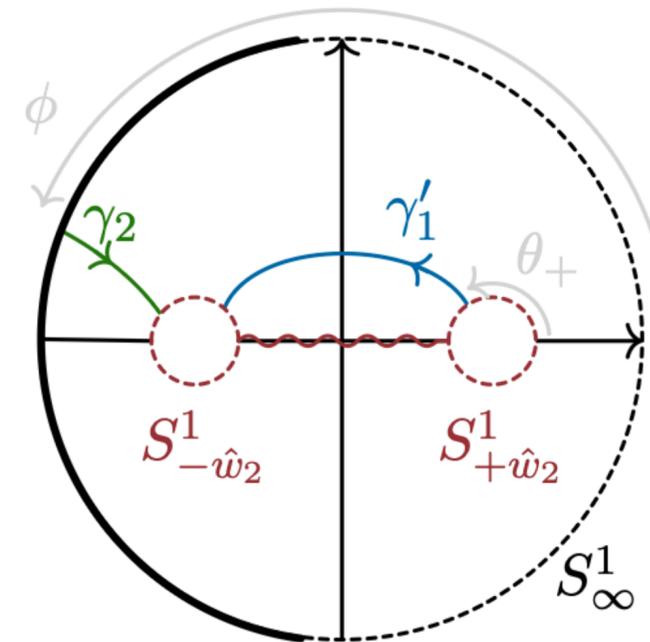
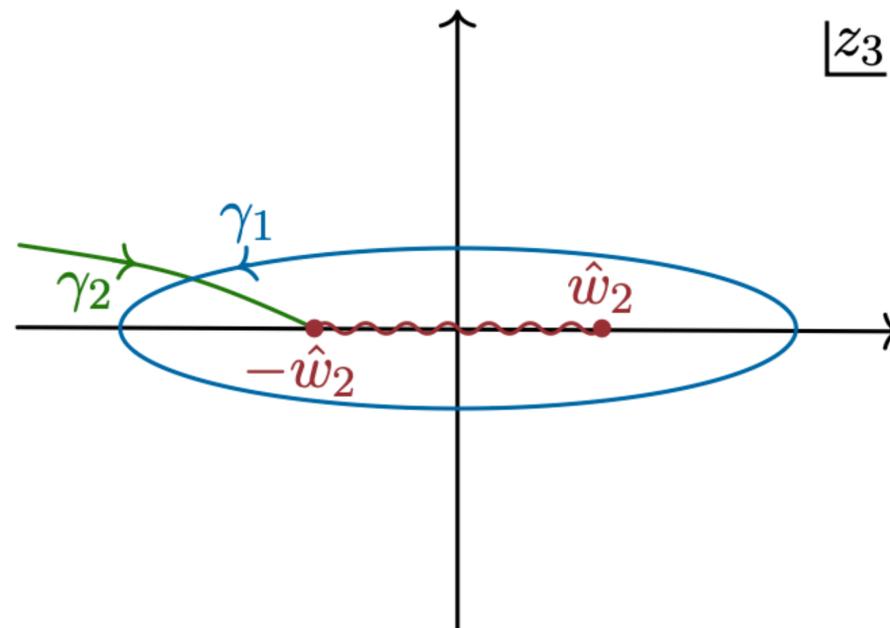


## Maximal cut

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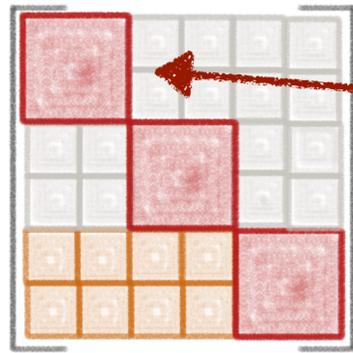
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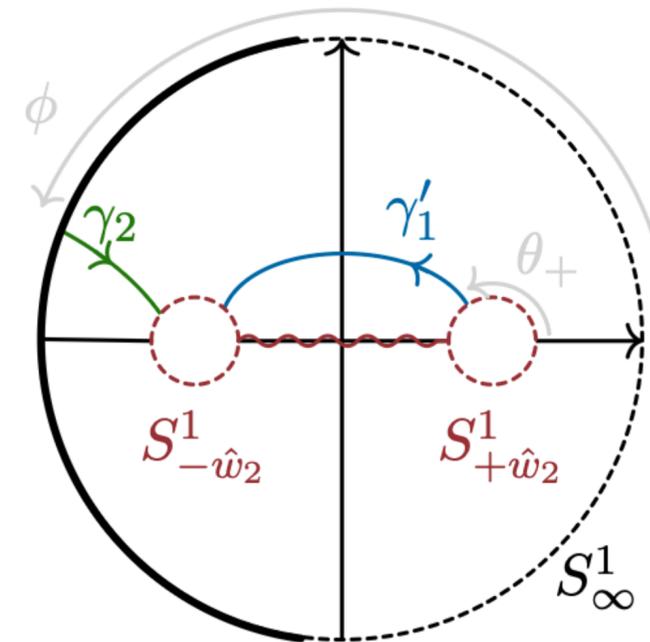
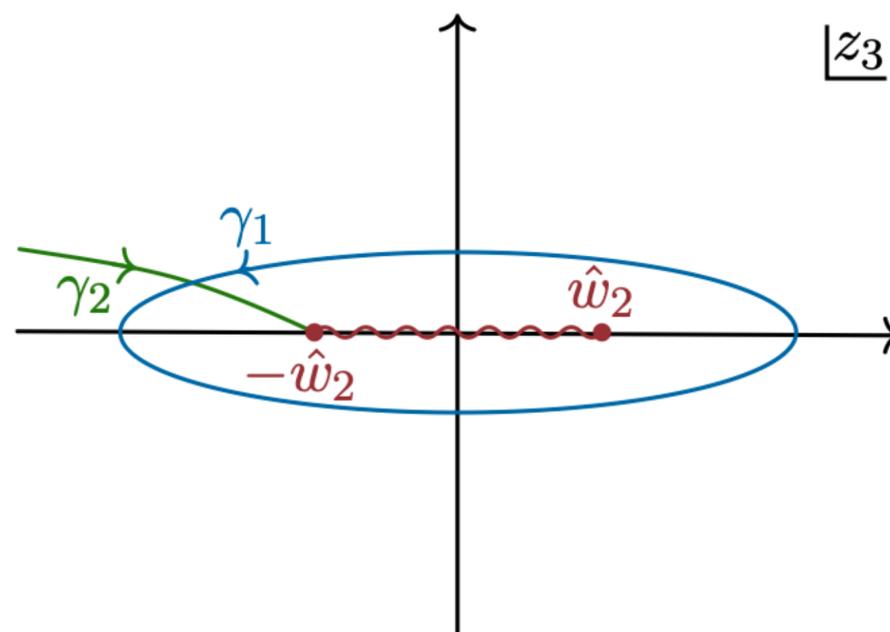


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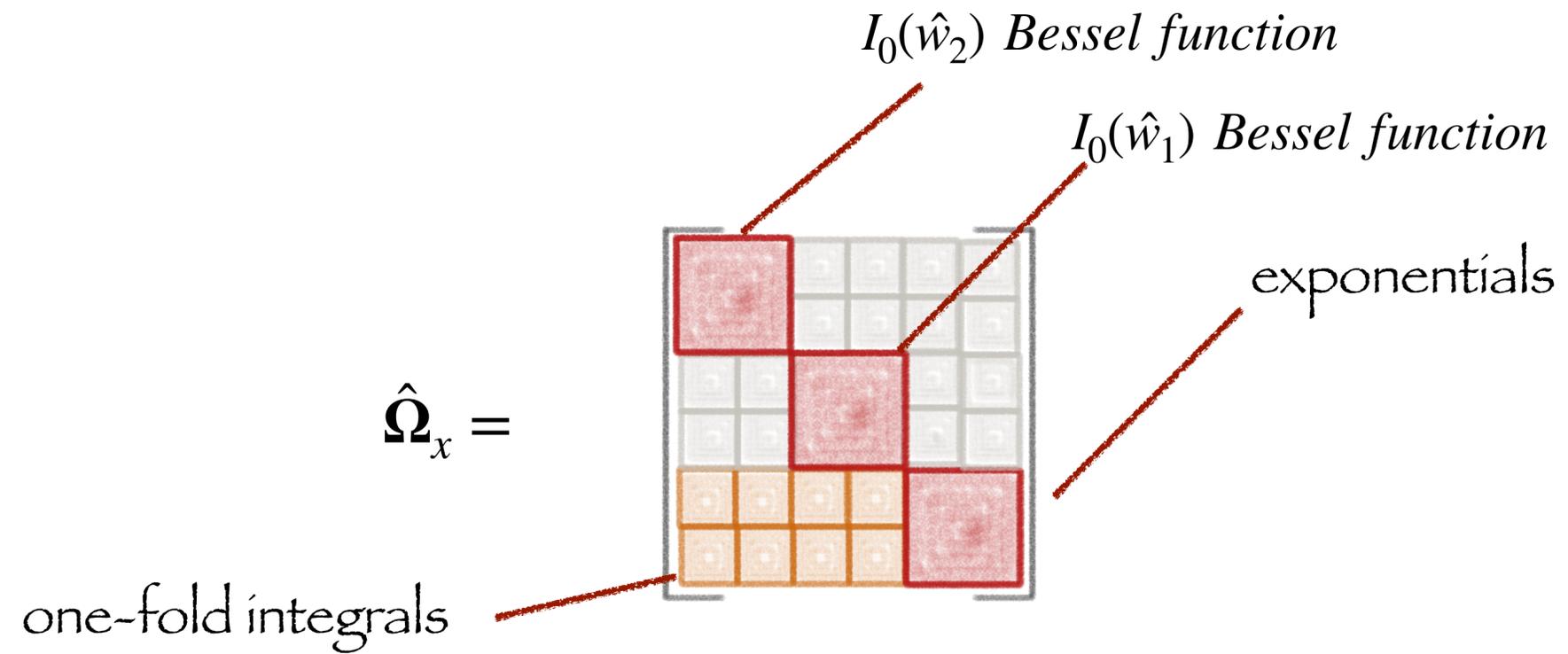
**Bessel Integration kernel**

$$d\mathbf{J}'|_{\text{block 1}} = -\varepsilon d \log(\hat{w}_2) \begin{bmatrix} 1 & \frac{1}{I_0(\hat{w}_2)^2} \\ I_0(\hat{w}_2)^2 & 1 \end{bmatrix} \cdot \mathbf{J}'|_{\text{block 1}}.$$

Same structure of elliptic integrals!

# Result and perspectives

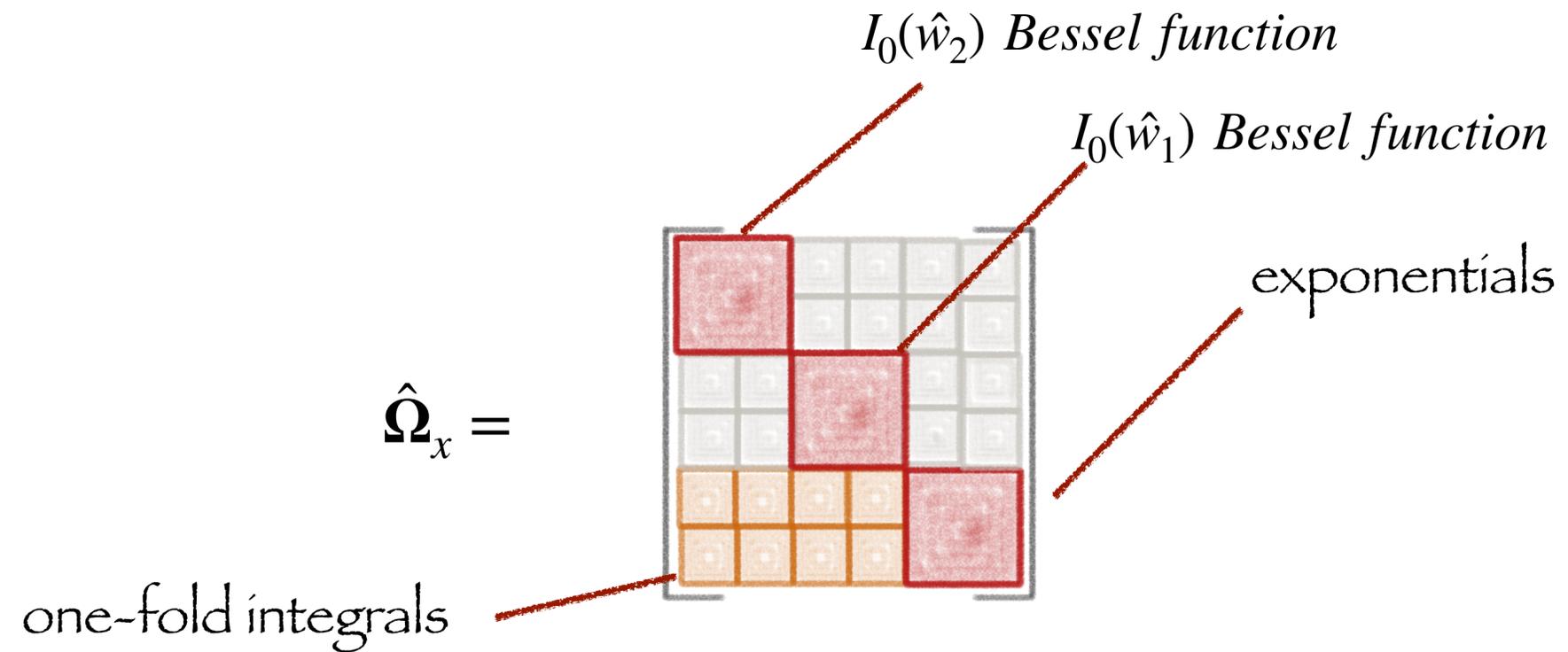
[G.B., Chestnov, Crisanti, Giroux, Smith]



Iterated integrals over Bessel-like functions, exponential integrals, one-fold integrals

# Result and perspectives

[G.B, Chestnov, Crisanti, Giroux, Smith]



Iterated integrals over Bessel-like functions, exponential integrals, one-fold integrals

**What is the algebra of these iterated integrals?**

**How can we perform fast numerical evaluations?**

# Kinematic Limits of Fourier Integrals

# Kinematic limits of integrals

From **asymptotic limits** we can get full analytic control

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Regions

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Regions

**Method of regions:** expansion under the integral sign

[Beneke, Smirnov]

# Kinematic limits from differential equations: Restriction theory

[Haraoka]

[Chestnov, Matsubara-Heo, Munch, Takayama]

Normal form of DEs

$$\Omega_1 \stackrel{x_1 \rightarrow 0}{=} \frac{1}{x_1} \Omega_1^{(-1)} + \Omega_1^{(0)} + \mathcal{O}(x_1)$$

Residue matrix

$$\Omega_{i \neq 1} \stackrel{x_1 \rightarrow 0}{=} \Omega_i^{(0)} + \mathcal{O}(x_1)$$

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$$\det(\Omega_1^{(-1)} - \lambda \mathbf{1}) \mathbf{J}^{(\lambda)} = 0 \quad \partial_a \mathbf{J}_0^{(\lambda)} = \Omega_a^{(0)} \cdot \mathbf{J}_0^{(\lambda)} \quad a \in \{x_2, \dots, x_n\}$$

Additional IBPs

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Additional IBPs

**3) Higher order recursion** for the subleading corrections  $\mathbf{J}_k^{(\lambda)}$

$$(k + \lambda + 1) \mathbf{J}_{k+1}^{(\lambda)} = \sum_{m=-1}^k \Omega_1^{(m)} \cdot \mathbf{J}_{k-m}^{(\lambda)}$$

**We can obtain as many orders as we want recursively!**

# Kinematic limits for waveform

[G.B., Chestnov, Crisanti, Giroux, Smith]

**Soft expansion:**

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**Hard region**

$$q^\mu \rightarrow q^\mu$$

**Soft region**

$$q^\mu \rightarrow \omega q^\mu$$

$$\mathbf{J} = \sum_{i=0}^{\infty} \omega^i J_i^{(0)} + x_1^{-2\epsilon} \sum_{i=0}^{\infty} \omega^i J_i^{(-2\epsilon)}$$

Polylogs

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**Soft region**

$$q^\mu \rightarrow \omega q^\mu$$

$$\mathcal{W}_{h,\text{soft}}^{(0)} = \kappa^3 m_1 m_2 \left( c_1 \frac{1}{\omega} + c_2 \log(\omega) + c_3 \omega \log(\omega) + \dots \right)$$

**Check with literature!**

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$$\mathcal{W}_{h,\text{soft}}^{(0)} = \kappa^3 m_1 m_2 \left( c_1 \frac{1}{\omega} + c_2 \log(\omega) + c_3 \omega \log(\omega) + \dots \right)$$

**Check with literature!**

**Post-Newtonian expansion:**

$$p_\infty = \sqrt{\gamma^2 - 1} \quad p_\infty \rightarrow 0$$

# Kinematic limits for waveform

[G.B., Chestnov, Crisanti, Giroux, Smith]

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$$\mathcal{W}_{h,\text{PN}}^{(0)} = \kappa^3 m_1 m_2 \left( d_1 \frac{1}{p_\infty} + d_2 + d_3 p_\infty \dots \right)$$

**Analytic result up to  $p_\infty^{30}$**

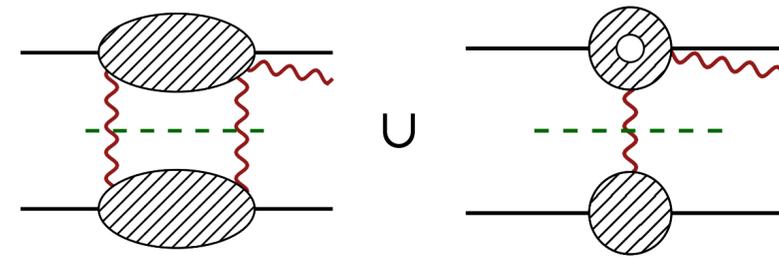
# Next-to-Leading order waveform

[G.B., De Angelis, Kosower]

Fourier transform of one-loop 2 → 3 **scattering amplitudes**

$$\mathcal{W}_h^{(0)} = \int_{\hat{q}} e^{i b \cdot q} \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_2 \cdot (q - k))$$

Fourier transform



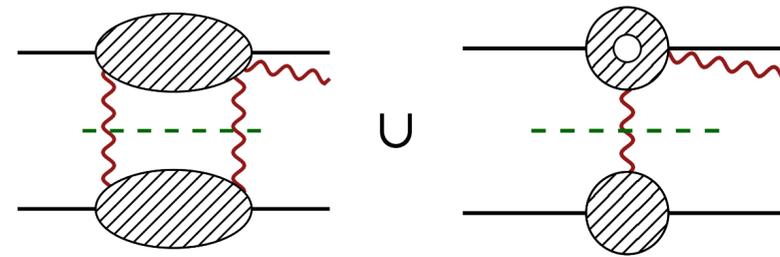
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$D$ - dimensional **Fourier-loop integrals**

$$I_{a_1 a_2 a_3 1 1 1 a_7 a_8 a_9 a_{10} a_{11}}^{(1)} = \int_{\hat{q}, \hat{\ell}} \frac{e^{D_1} D_1^{-a_1} D_{11}^{-a_{11}} \hat{\delta}(D_4) \hat{\delta}(D_5) \hat{\delta}(D_6)}{D_2^{a_2} D_3^{a_3} D_7^{a_7} D_8^{a_8} D_9^{a_9} D_{10}^{a_{10}}}$$

$$D_1 = i b \cdot q, D_2 = q^2, D_3 = (q - k)^2,$$

$$D_4 = u_1 \cdot q, D_5 = u_2 \cdot (k - q)$$

$$D_6 = u_1 \cdot \ell, D_7 = u_2 \cdot \ell, D_8 = \ell^2,$$

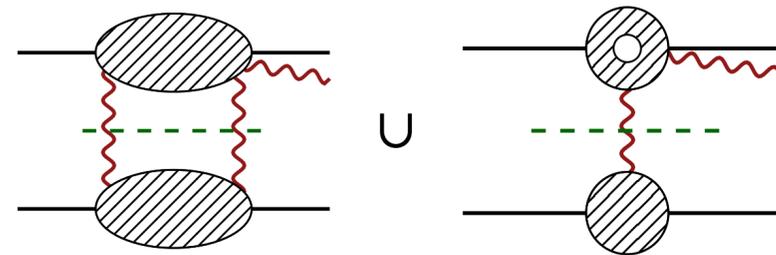
$$D_9 = (\ell - q_2)^2, D_{10} = (\ell + q_1)^2, D_{11} = i b \cdot \ell$$

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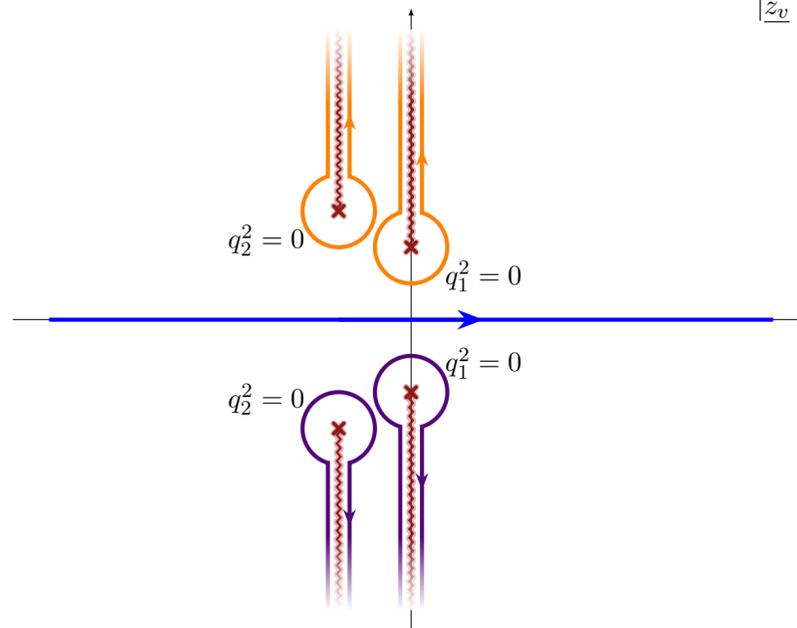
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**Fourier-loop IBPs:** decomposition in a basis of 28 integrals

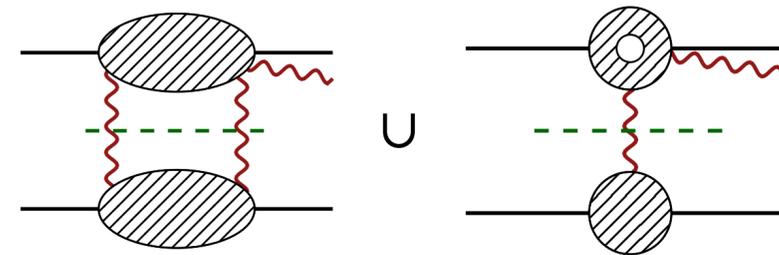


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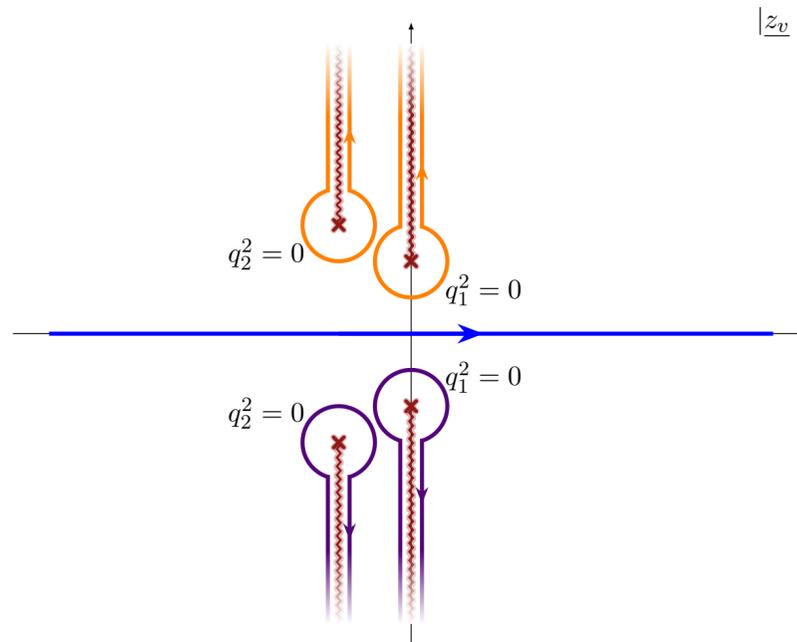
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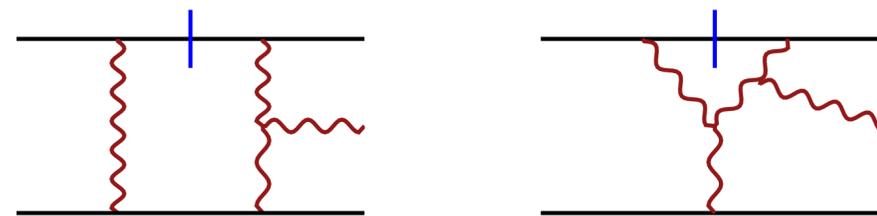
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**Fourier-loop IBPs:** decomposition in a basis of 28 integrals



**Differential Equations? Kinematic limit?**

# Six-loop Gravitational interactions at the sixth post-Newtonian order

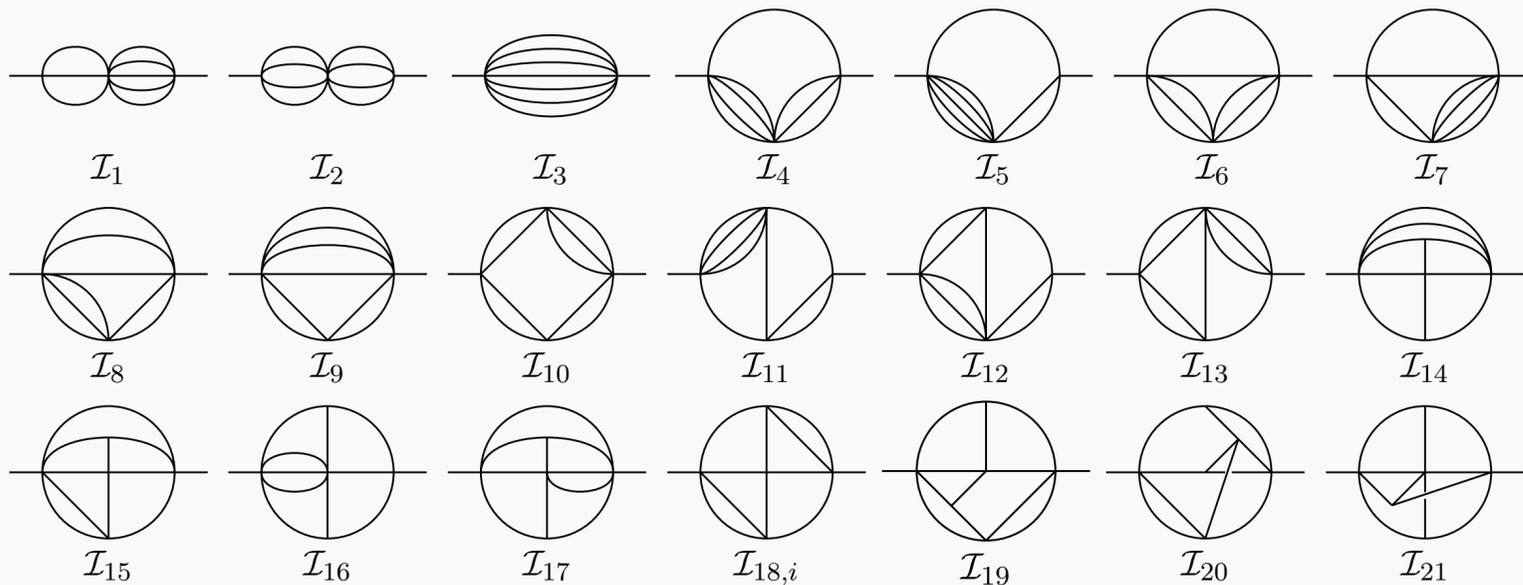
G.B., Mandal, Mastrolia, Patil, Pegorin, Ronca, Smith, Steinhoff, Torres Bobadilla [2512.19498]

See Stefano's lectures

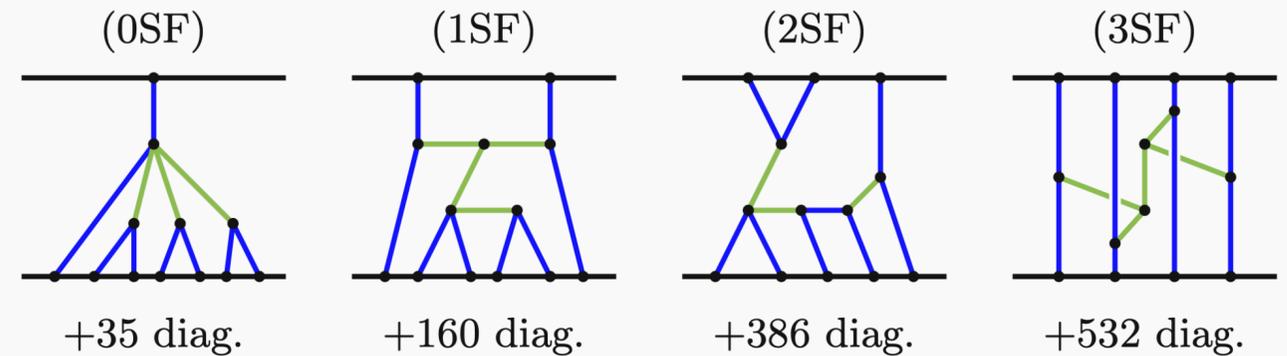
Potential from Effective Diagrams  $d = 3 + \epsilon$

$$\mathcal{V}_{\text{eff}} = \mathbf{i} \lim_{d \rightarrow 3} \int \frac{d^d \mathbf{p}}{(2\pi)^d} e^{\mathbf{i}\mathbf{p} \cdot (\mathbf{x}_{(1)} - \mathbf{x}_{(2)})} \times \text{[Diagram: Two horizontal lines with a shaded rectangular region between them.]}$$

## 21 Master Integrals



## Feynman diagrams



## Result

$$\mathcal{V}_{6\text{PN}}^{G_N^7} = -\frac{G_N^7}{r^7} \left( \frac{5}{16} m_1^7 m_2 + \frac{190}{9} m_1^6 m_2^2 + \frac{37651}{144} m_1^5 m_2^3 + \frac{5852}{9} \frac{m_1^4 m_2^4}{2} + (1 \leftrightarrow 2) \right)$$

# Outlooks

Fourier integrals are twisted period integrals

Multi-loop technology developed for Feynman integrals can be exported to the study of Fourier integrals

Epsilon-factorised form involve iterated integrals over bessel functions

Kinematic limits via restrictions

Generalization to one-loop waveform? Two-loop waveform?

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**Thanks!**

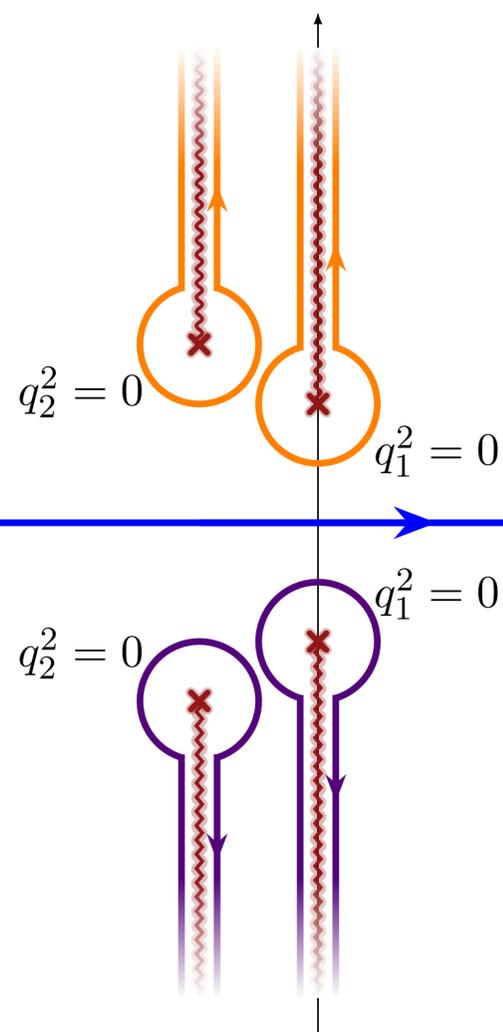
The activity was carried out within the project:  
NOTIMEFORCOSMO "No time for cosmology: Decoding  
dynamics from static cosmological correlations", Grant  
Agreement 101126304, CUP E53C23002380006

# BACKUP

# NLO Fourier integrals

[G.B., De Angelis, Kosower]

Loop-by-loop approach for Master Integrals Evaluation



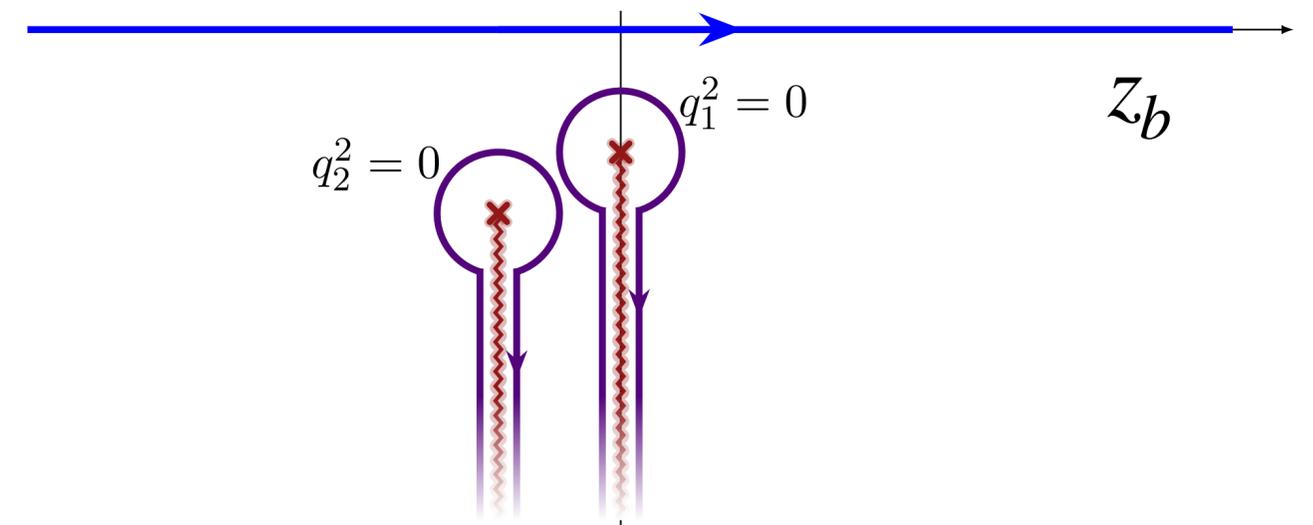
$$J_i = \int_{\hat{q}} e^{ib \cdot q} \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_2 \cdot (q - k)) \int_{\hat{\ell}} \mathcal{F}_i(q, \ell)$$

Step2: Complex analysis

Step1: Canonical Differential Equations

Fast numerically **convergent** one-fold integrals

$$J_i = \int_{-\infty}^{\infty} dz_b e^{-iz_b \sqrt{-b^2}} f(z_b)$$



# An Example from Quantum Mechanics

**Schrödinger equation** with cylindrical symmetry

$$(\nabla^2 + k^2)\psi = 0 \qquad \psi(r, \theta, z) = R_m(r)e^{im\phi}e^{ik_z z}$$

**Radial second order** differential equations

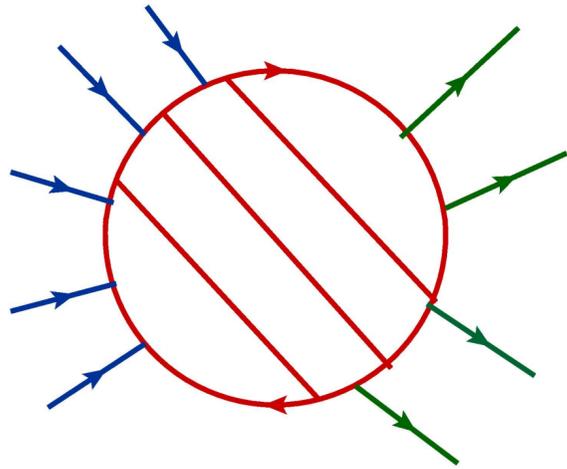
$$(r^2\partial_r^2 + r\partial_r + (k_\perp^2 r^2 - m^2))R(r) = 0 \qquad k_\perp^2 = k^2 - k_z^2$$

$$R_m = c_1 J_m(k_\perp r) + c_2 Y_m(k_\perp r) \qquad \text{Bessel Functions}$$

System of **first order differential equations**:

$$\partial_y \begin{bmatrix} R \\ \partial_r R \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{m^2}{r^2} - k_\perp^2 & -\frac{1}{r} \end{bmatrix} \begin{bmatrix} R \\ \partial_r R \end{bmatrix}$$

# Feynman Integrals



$$I_{a_1, \dots, a_n}(\{x_i\}) = \int \prod_{i=1}^L \left( \frac{d^D \ell_i}{(2\pi)^D} \right) \frac{1}{D_1^{a_1} \dots D_n^{a_n}} \quad \begin{array}{l} \{p_i\}_{i=1}^E \\ n = L \cdot E + \frac{L \cdot (L+1)}{2} \end{array}$$

Kinematic                      Measure                      Propagators

Chetyrkin, Tkachov  
Laporta

**Integration-by-parts** identities

Decomposition into a basis of **Master Integrals**

$$\int \prod_{i=1}^L \left( \frac{d^d \ell_i}{(2\pi)^d} \right) \frac{\partial}{\partial \ell_j^\mu} \left( \frac{v^\mu}{D_1^{a_1} \dots D_n^{a_n}} \right) = 0$$

$$I = \sum_{i=1}^{\nu} c_i J_i$$

Target                      Rational                      Master  
Integral                      Coefficient                      Integrals

