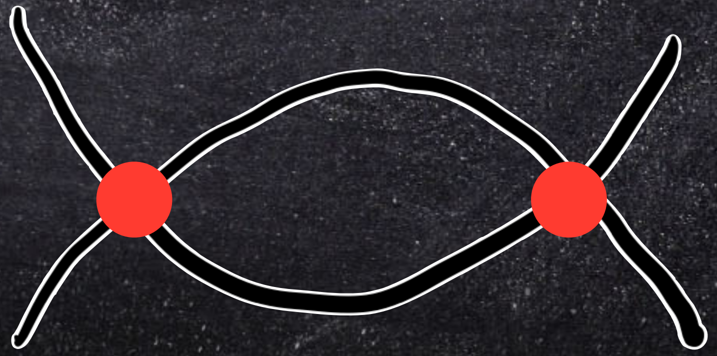
Long is the Path, Close is the Goal

Finite-coupling effects on positivity bounds



Francesco Riva
(Geneva University)

Bellazzini,Elias-Miro,Rattazzi,Riembau,FR'20

Bellazzini,Riembau,FR '22

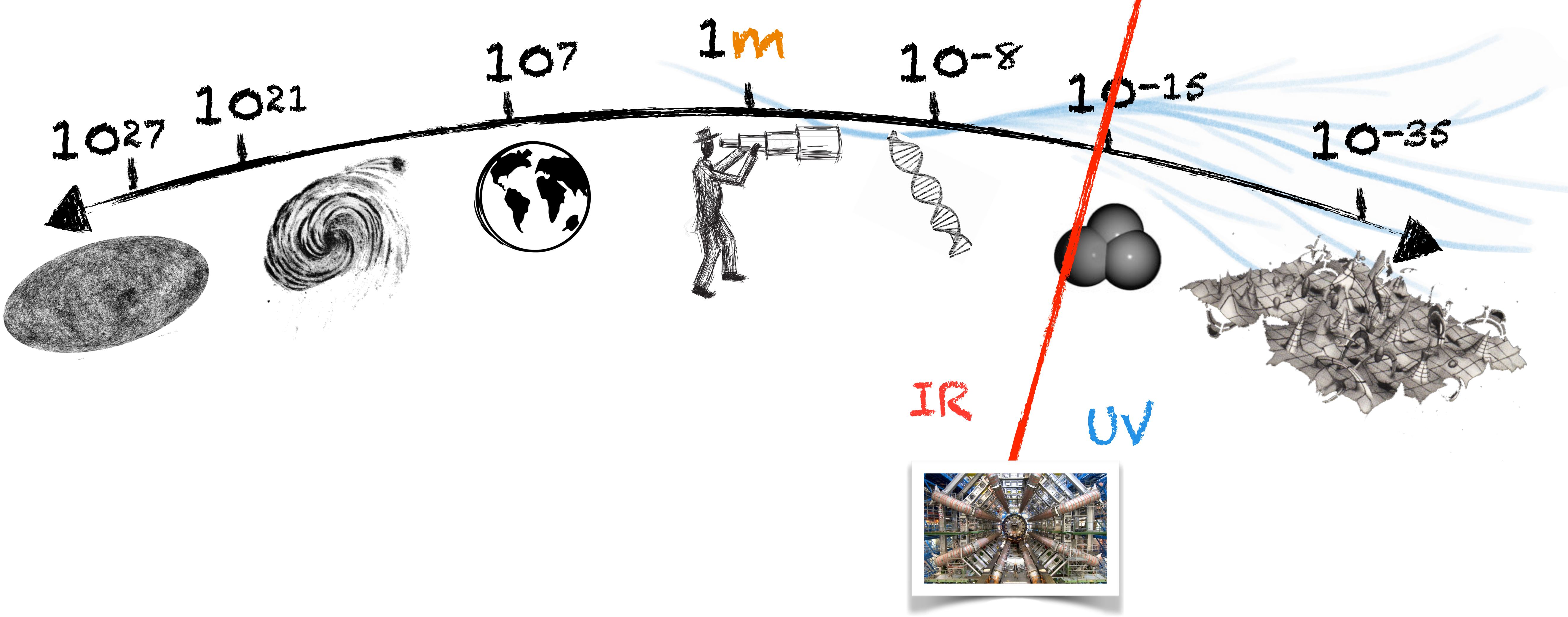
Beadle,Isabella,Perrone,Ricossa,FR,Serra'24-'25

Bellazzini,Berman,Isabella,FR,Romano,Sciotti'25

Effective Field Theories

Capture low-energy universal limit of infinitely microscopic theories

They are why we can do physics, without knowing Quantum Gravity



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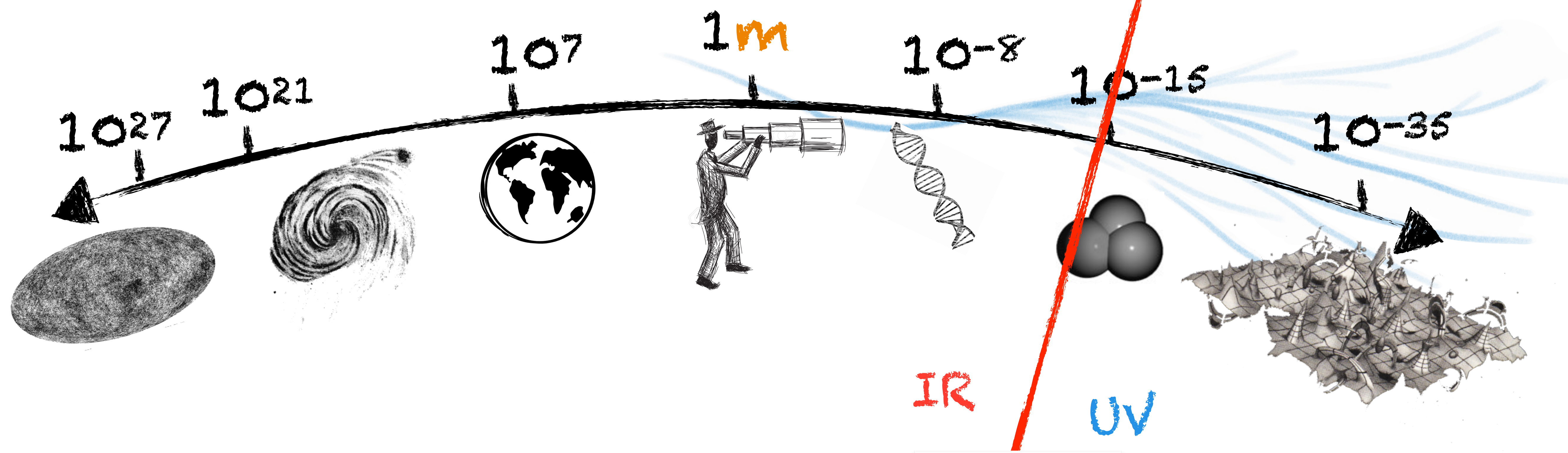
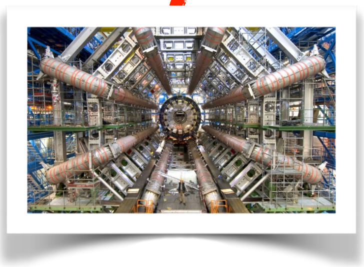
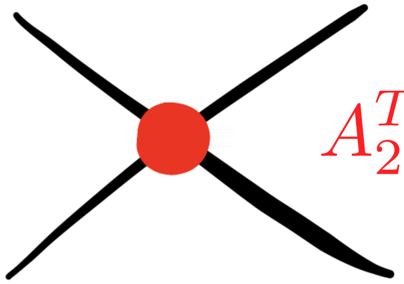




- Lore: EFT = Anything goes



$$A^{Tree}_{2\to2} = g_0 + g_2(s^2+t^2+u^2) - g_3 stu + g_4(s^2+t^2+u^2)^2 + \cdots$$

any value

UV/IR Relations

(Positivity/Bootstrap)

$$A_{2\to 2}^{Tree} = g_0 \quad + \quad g_2(s^2+t^2+u^2) - g_3 stu + g_4\,(s^2+t^2+u^2)^2 + \cdots$$

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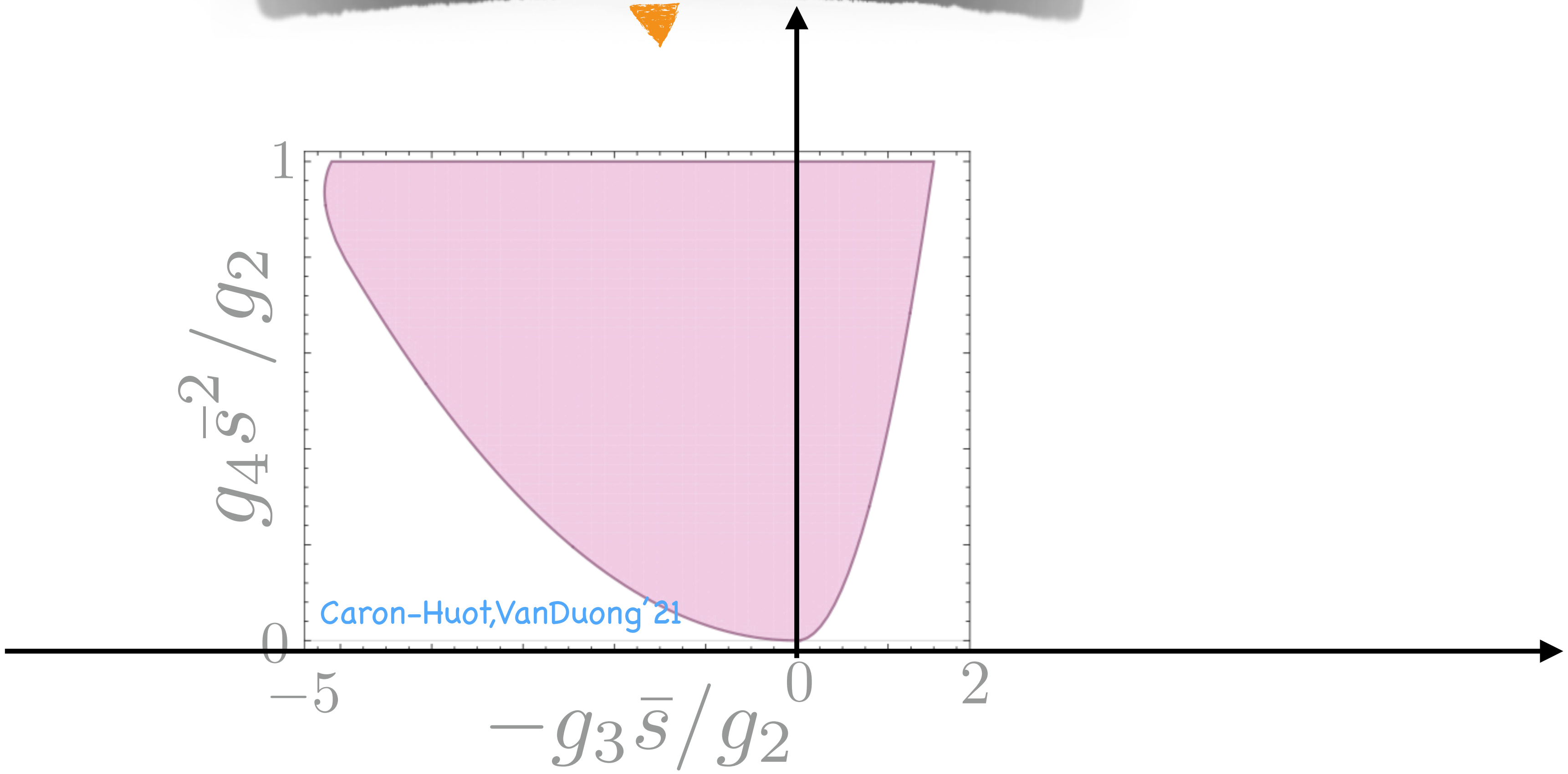
UV theory is Unitary and Causal



Caron-Huot,VanDuong'21

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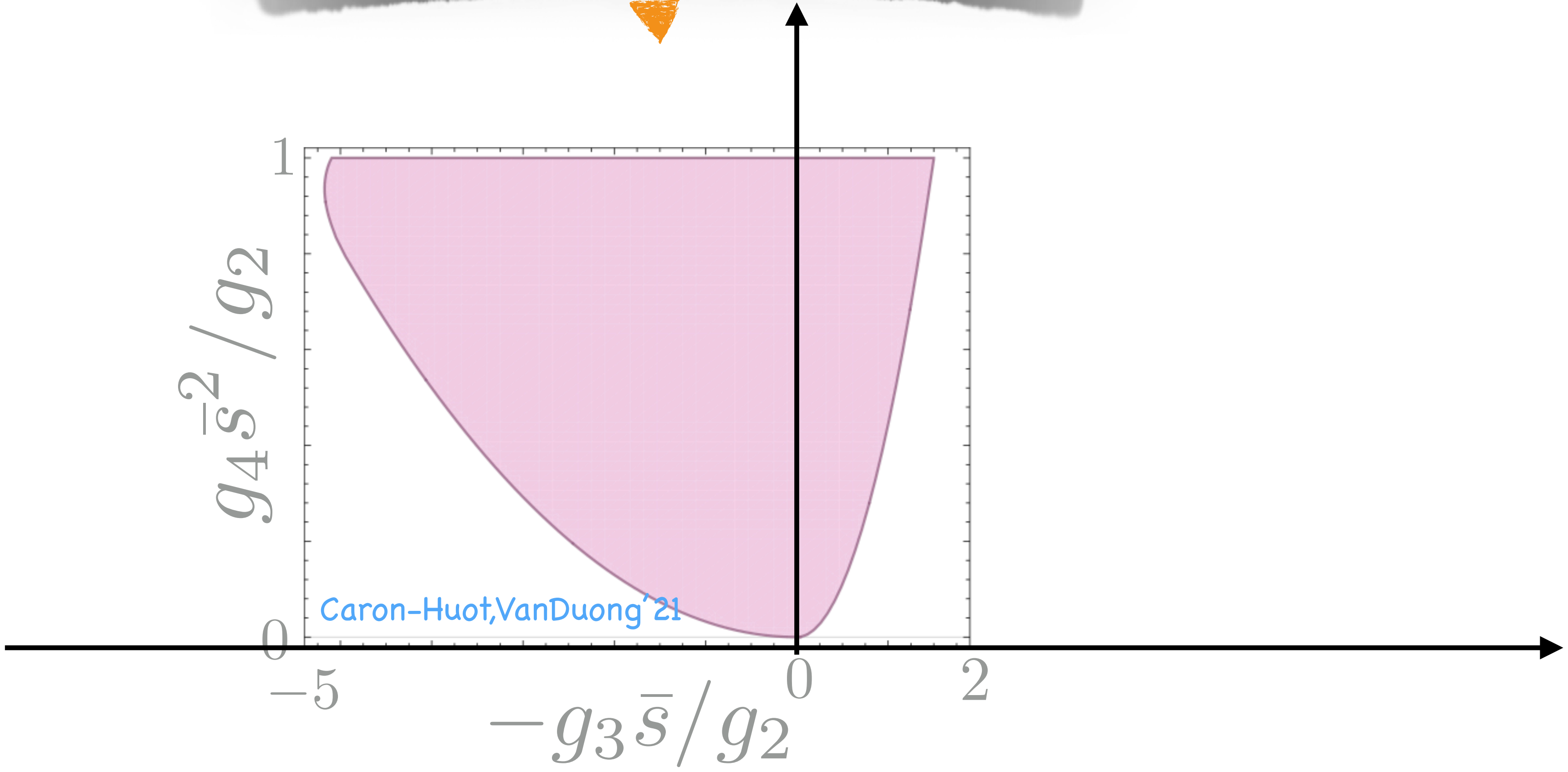

- Not all EFTs are UV completable!

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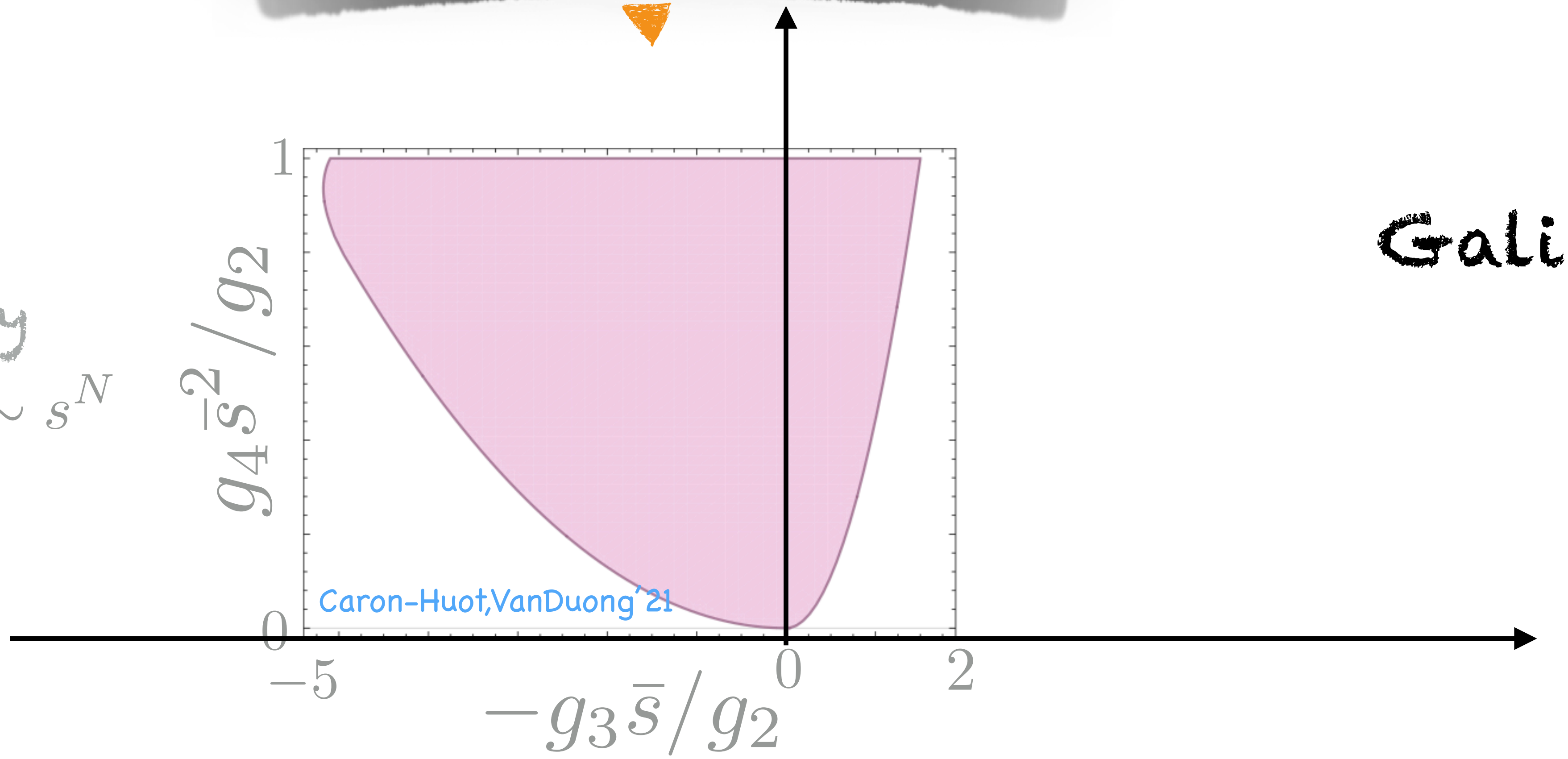
UV theory is Unitary and Causal

Galileons

$\mathcal{L}(\pi)$ with symmetry

$\pi \to \pi + \alpha x^n \rightarrow A \sim s^N$



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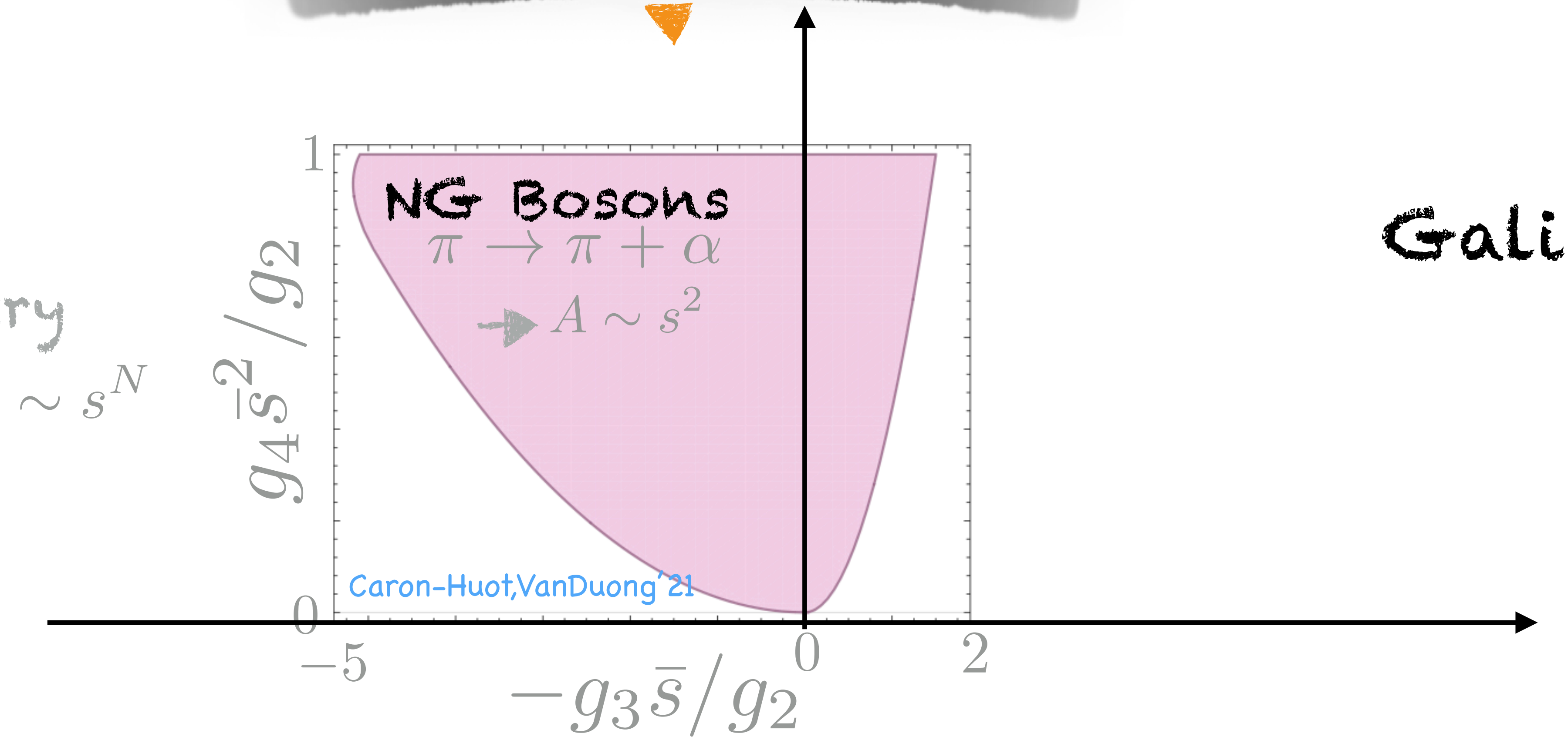

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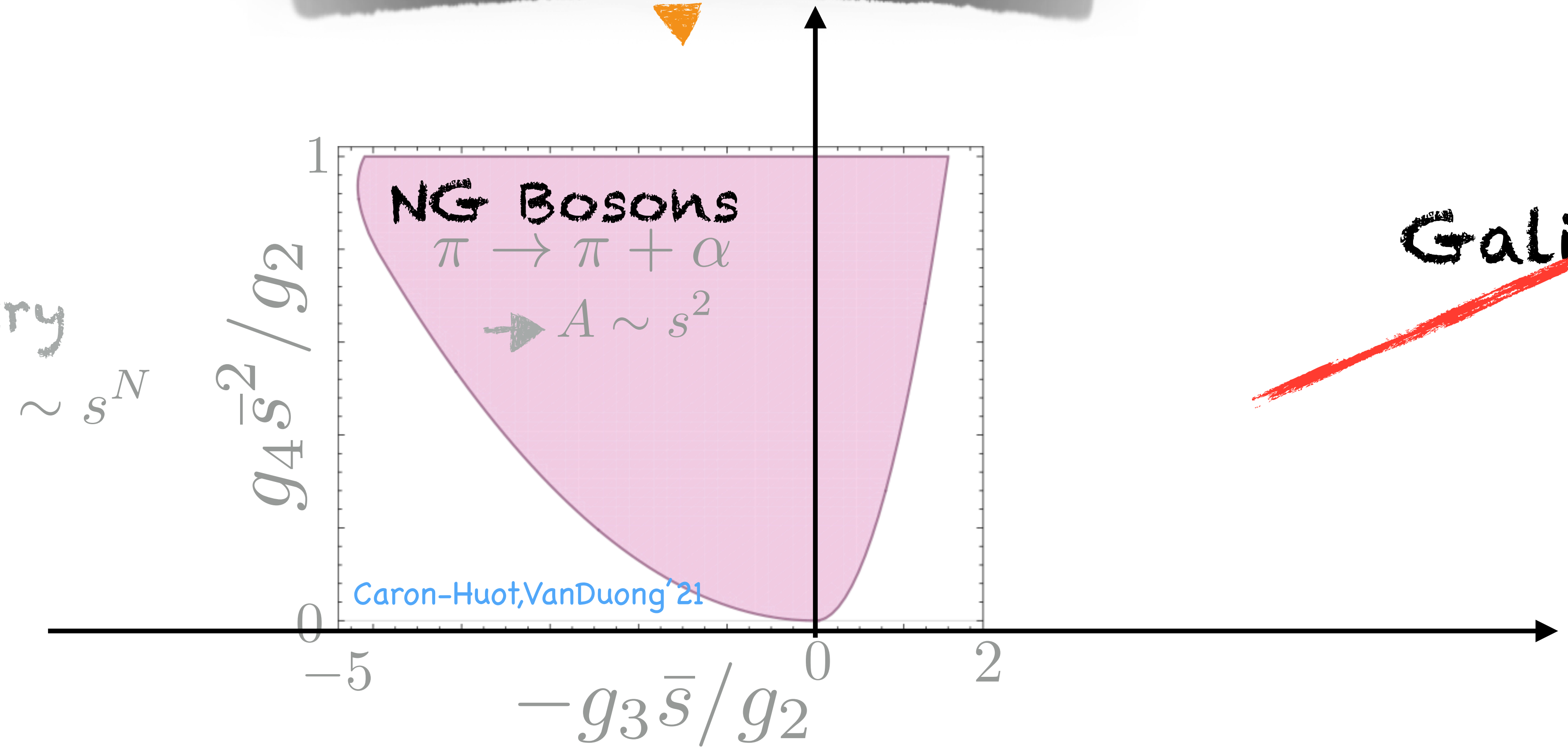

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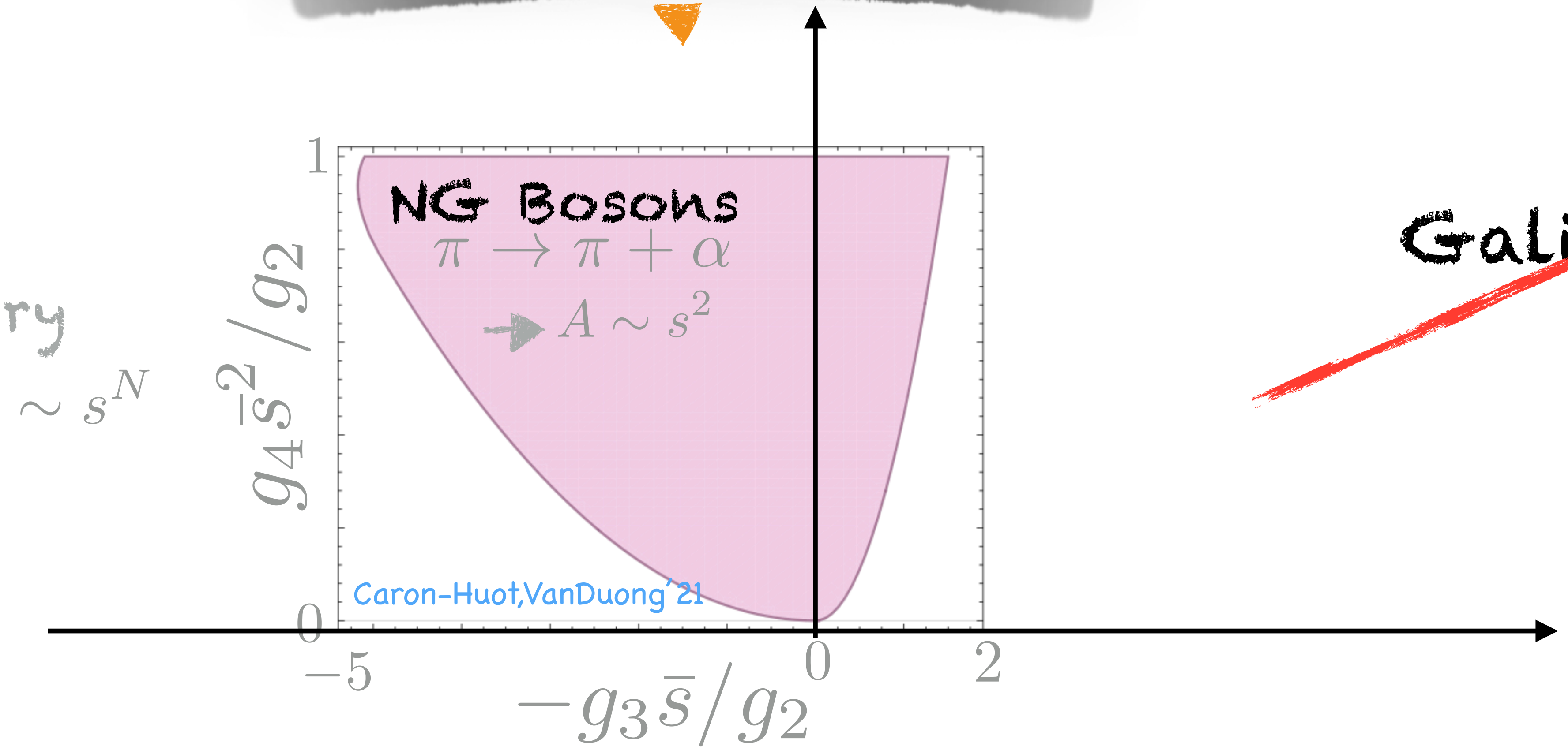

~~Galileons~~

- Not all EFTs are UV completable!

Massive Gravity

Bellazzini,Isabella,Ricossa,FR'24

Isolated massive Higher-Spin j>2

Bellazzini,FR,Serra,Sgarlata'19

Theory with large high-derivative interactions

Caron-Huot,VanDuong'21, Elias-Miro,Bellazzini,Rattazzi,Riembau,FR'21,Arkani-Hamed,Huang,Huang'21,...

UV/IR Relations

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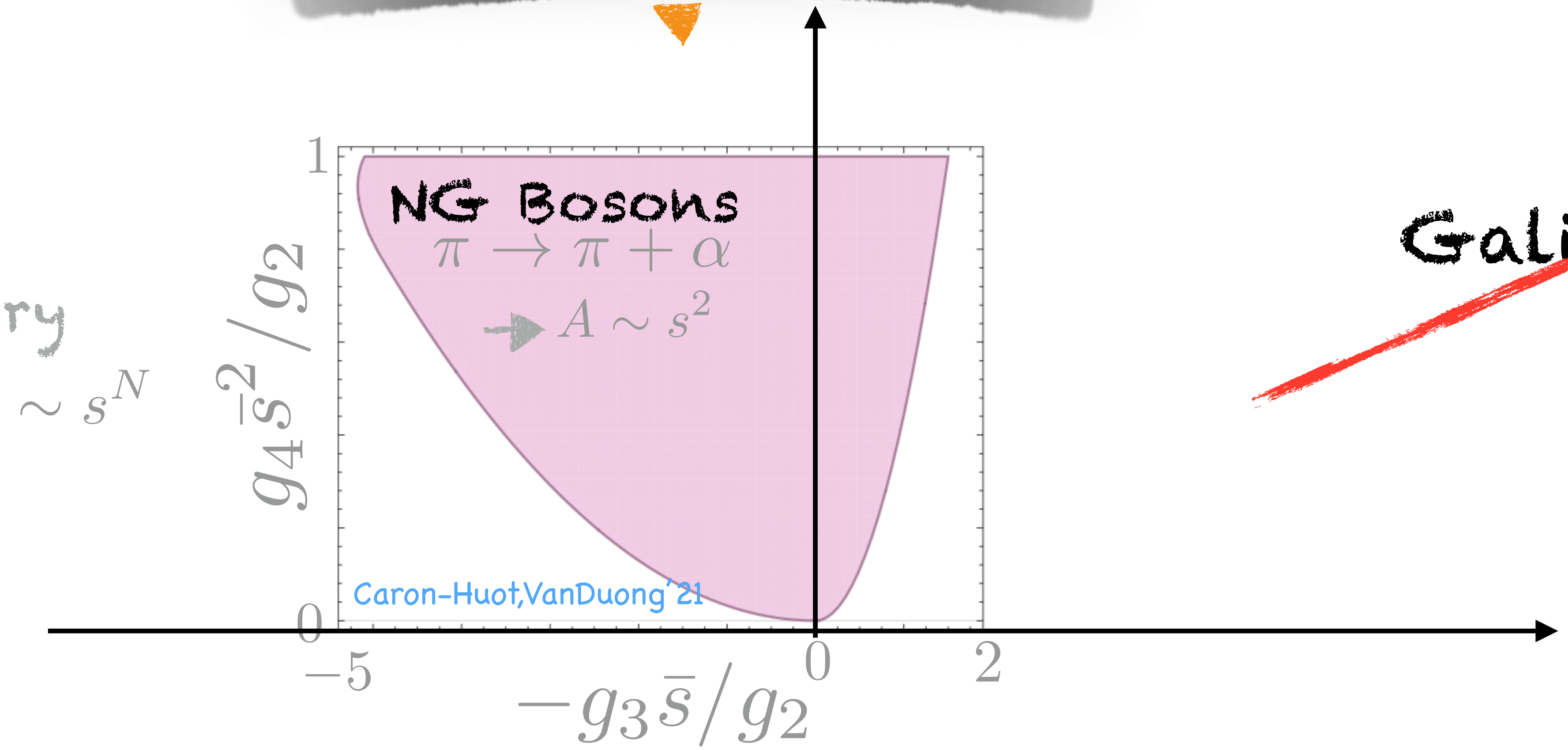

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Bellazzini,Isabella,Ricossa,FR'24

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Caron-Huot,VanDuong'21, Elias-Miro,Bellazzini,Rattazzi,Riembau,FR'21,Arkani-Hamed,Huang,Huang'21,...

UV/IR Relations

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Most results in the literature based on tree-level EFT

$$g_n \to 0 \qquad\qquad \frac{g_n}{g_m} = finite$$

UV/IR Relations

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This talk: finite couplings and loop effects in EFT

$$g_n \neq 0 \qquad \alpha \neq 0 \qquad G_N \neq 0$$

$$\forall D$$

UV/IR Relations

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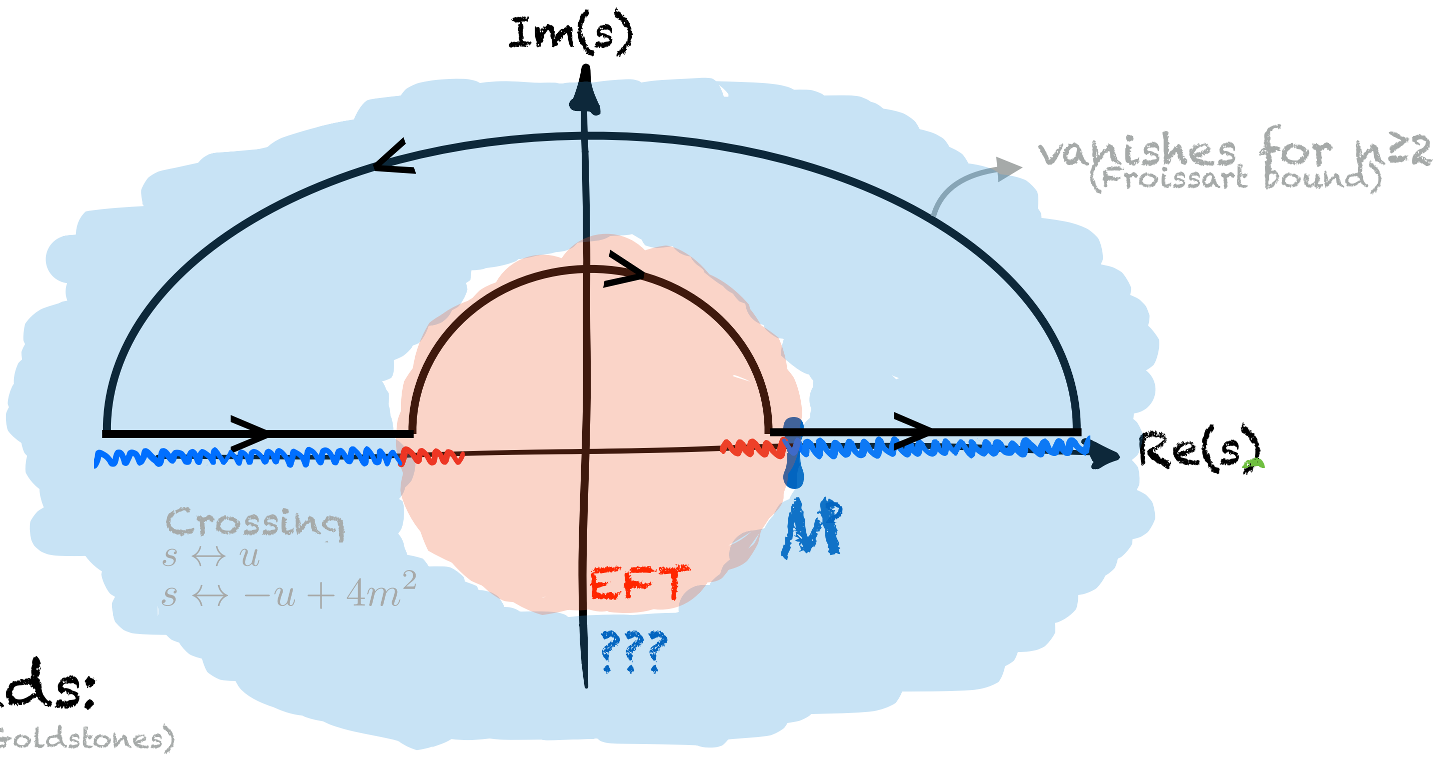
"In weakly coupled theories, arent loop effects small?"

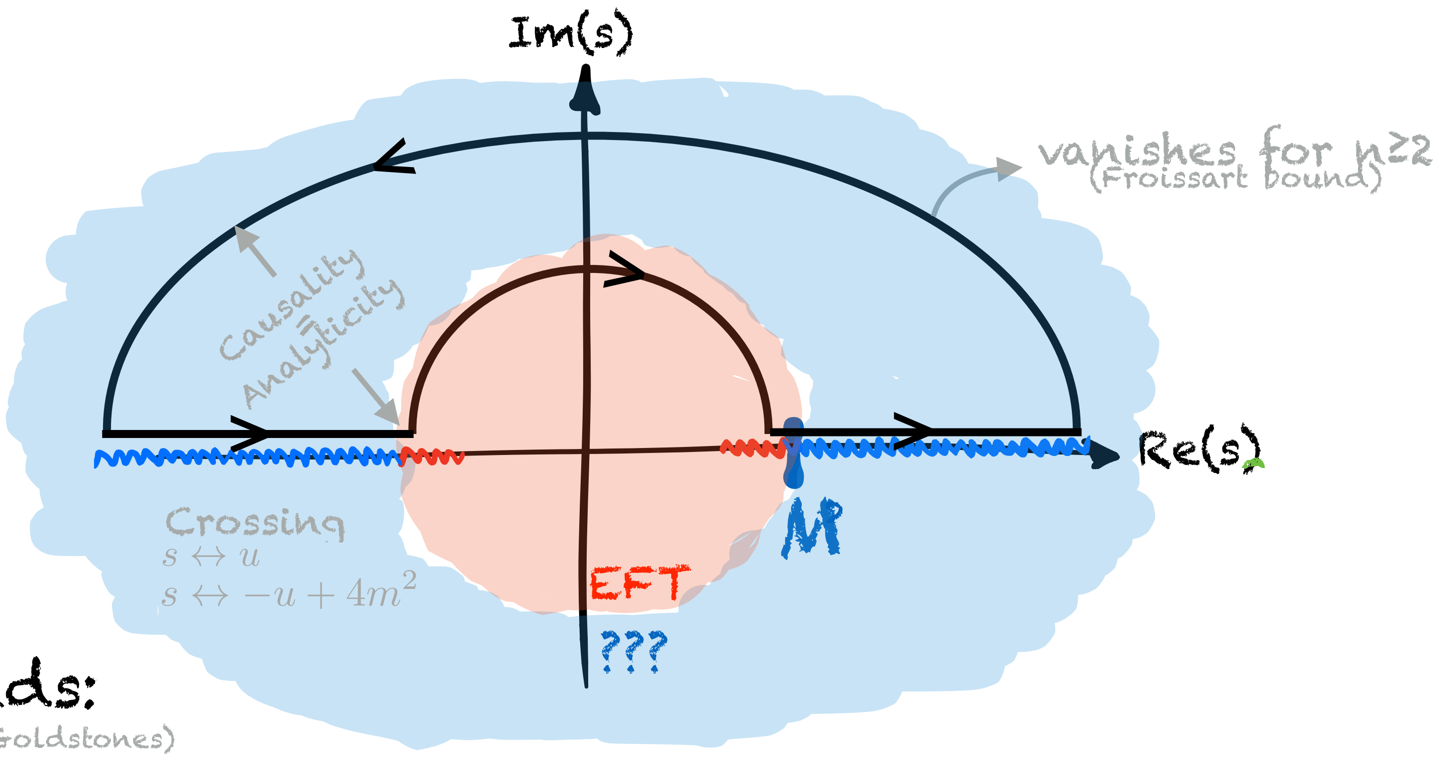
IR-UV Connection



Forward (t=0) bounds:

(for theories that have it, like Goldstones)

IR-UV Connection



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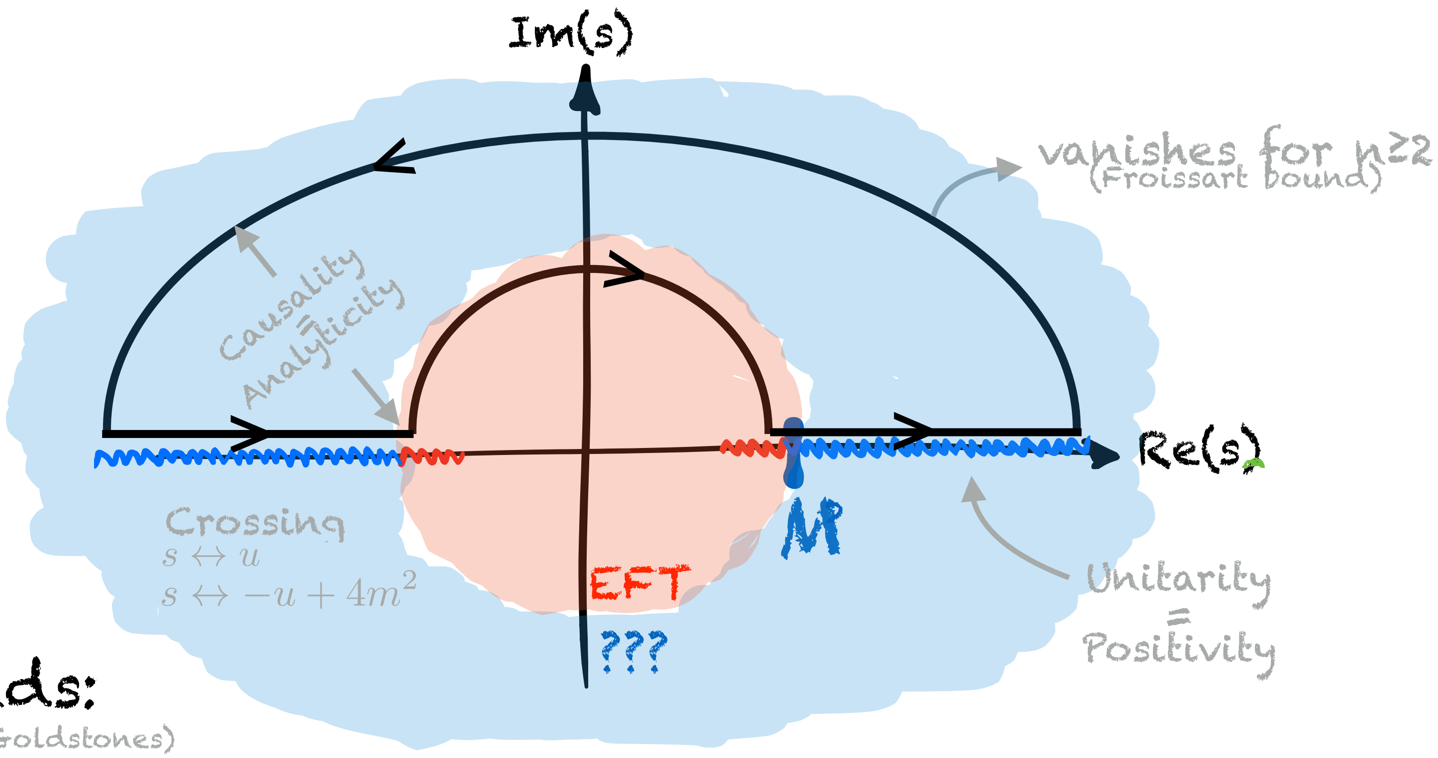

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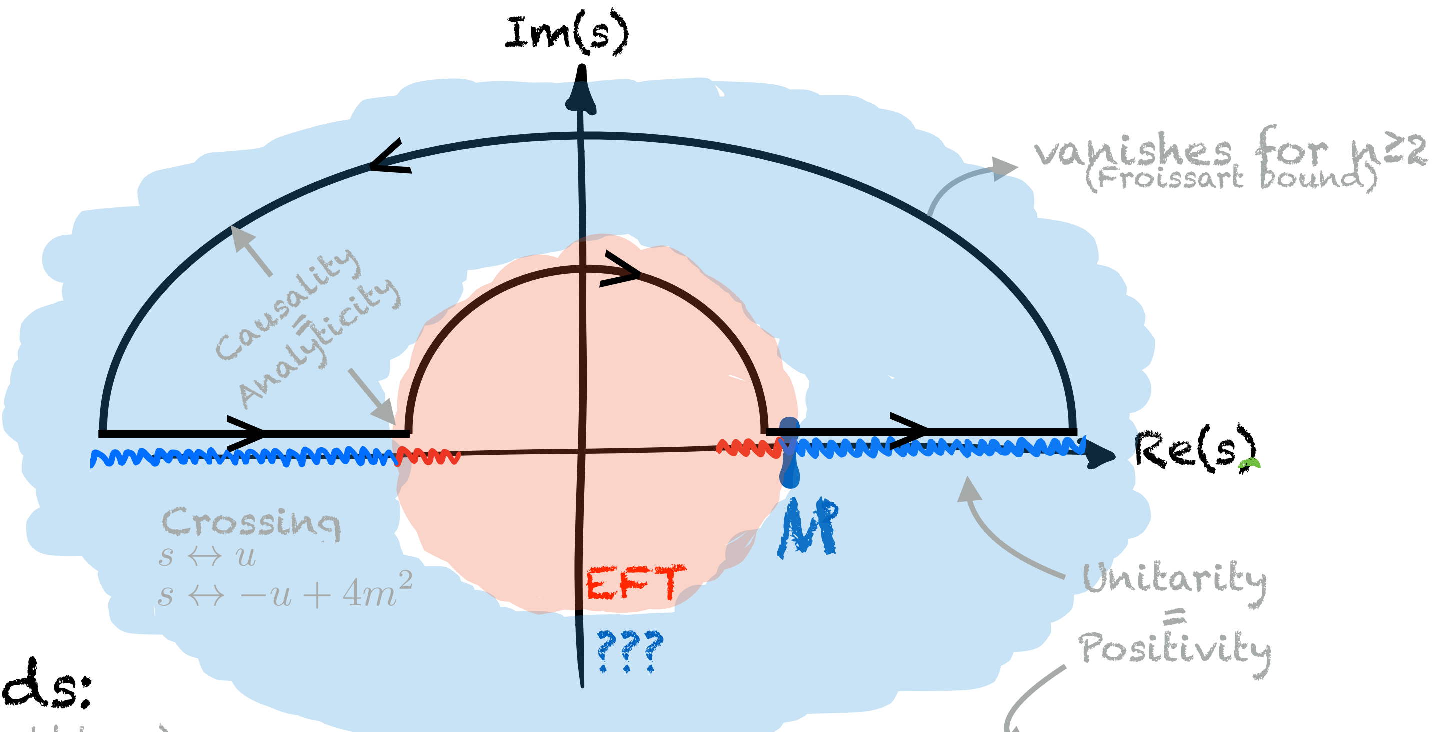

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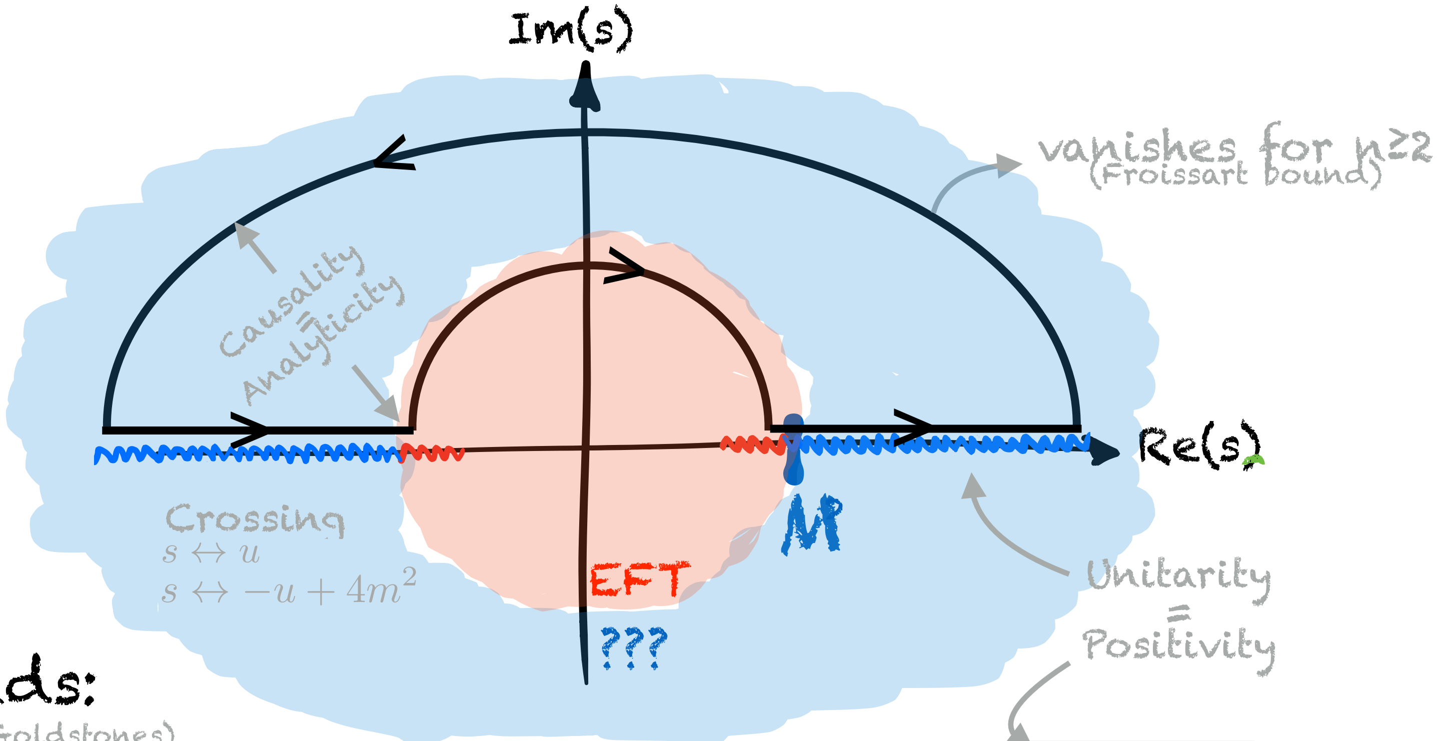
Arcs:

$$\mathcal{A}_n(\bar{s}) \equiv \int_{\cap_{\bar{s}}} \frac{ds}{\pi i} \frac{A(s)}{s^{n+1}} = \frac{2}{\pi} \int_{\bar{s}}^{\infty} ds \frac{\mathrm{Im} A(s)}{s^{n+1}}$$

Cutoff M²

IR-UV Connection



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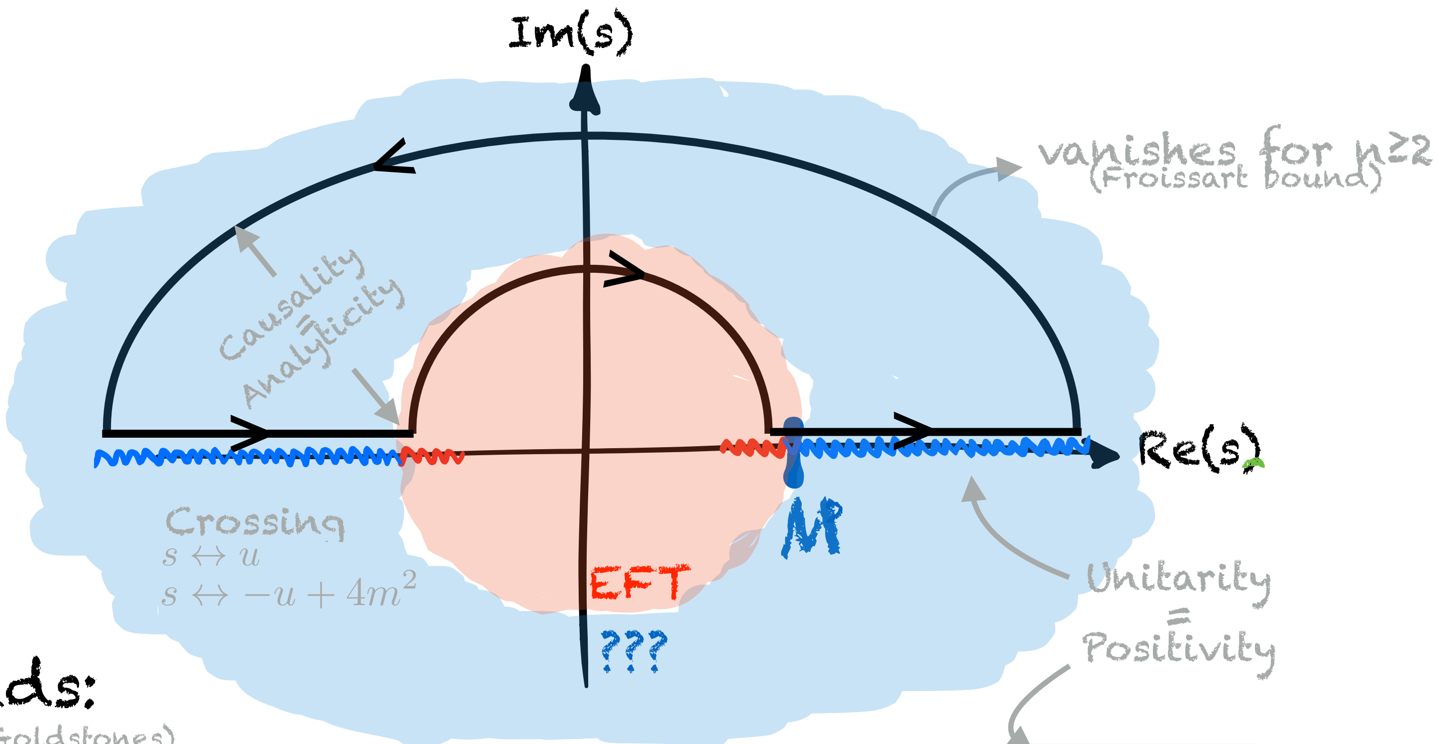
IR-UV Connection



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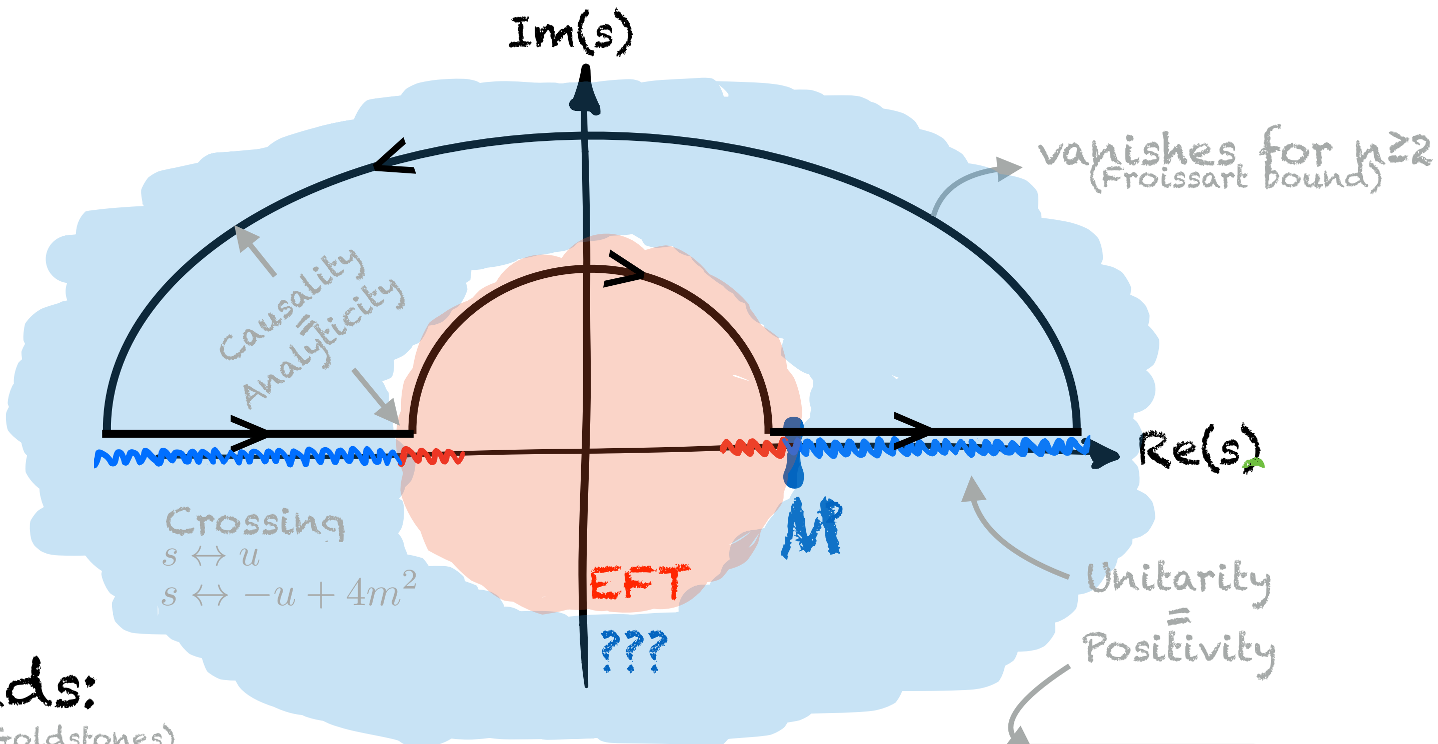

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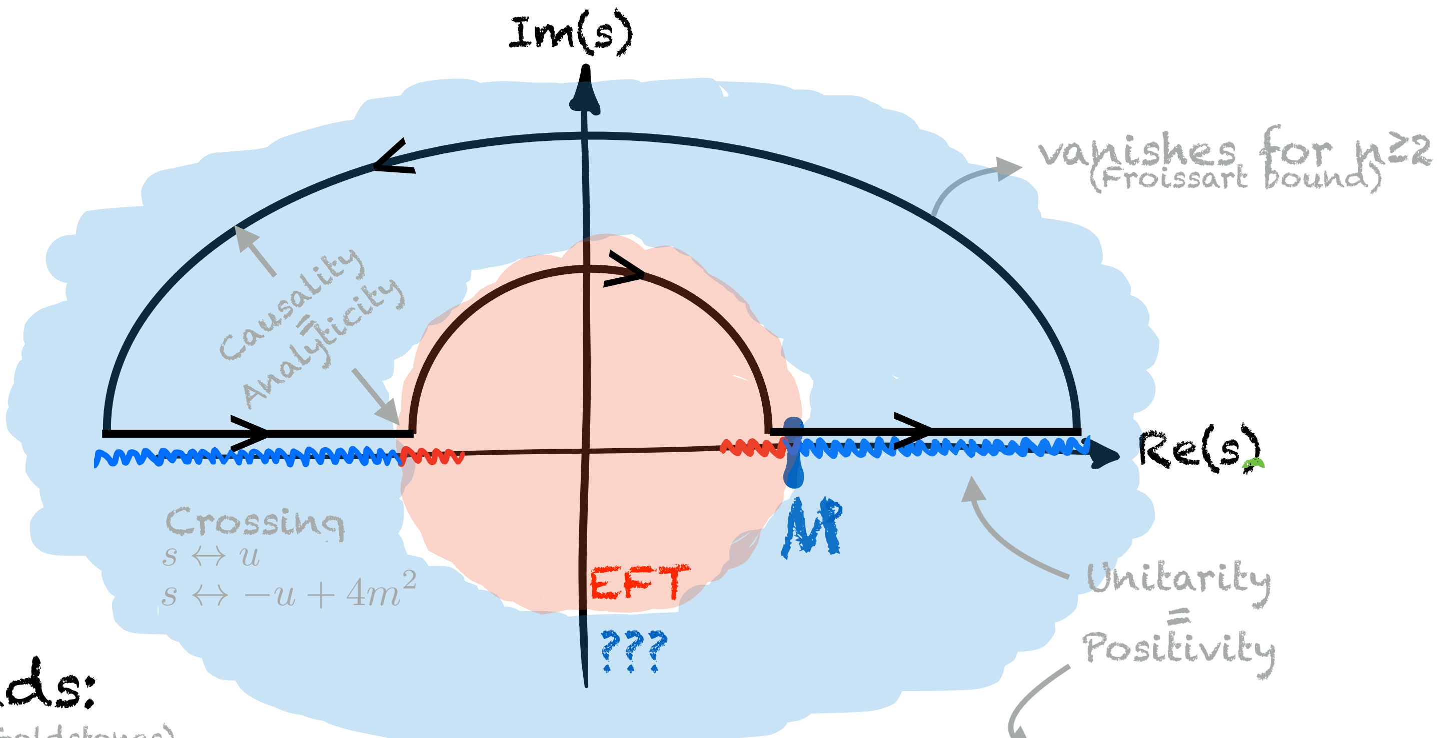

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Cutoff M²

$$\Downarrow$$

Calculable in EFT, e.g. $\mathcal{A}_2 = g_2$

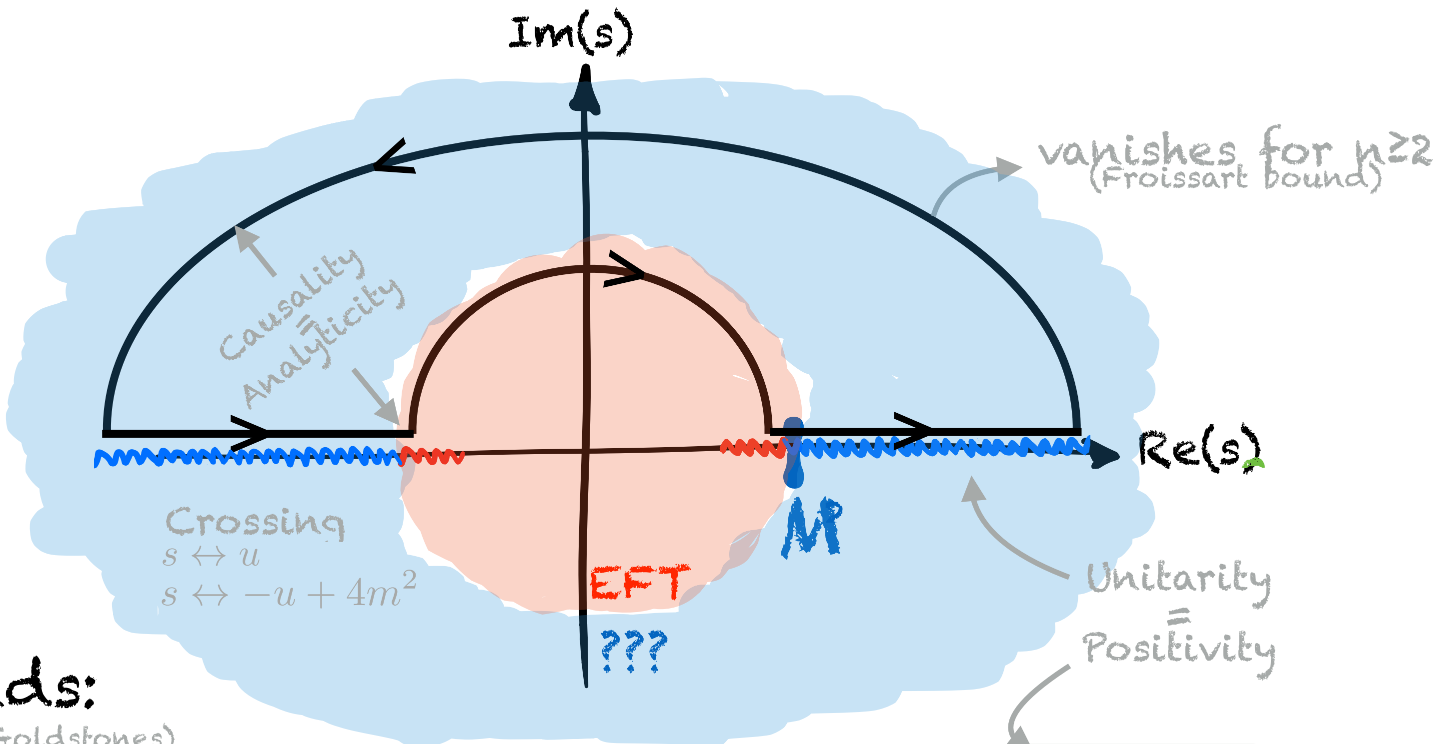
$$\Uparrow$$

$$\mathcal{A}_n > 0 \qquad (n \geq 2)$$

Adams,Arkani-Hamed,Dubovsky,Nicolis,Rattazzi’06,

- Consistency condition for EFTs

IR-UV Connection



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$$\mathcal{A}_2 \geq \bar{s}^2 \mathcal{A}_4 \geq \bar{s}^4 \mathcal{A}_6 \geq \cdots \geq 0$$

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi'06,

Bellazzini, Elias-Miro, Rattazzi, Riembau, FR'20

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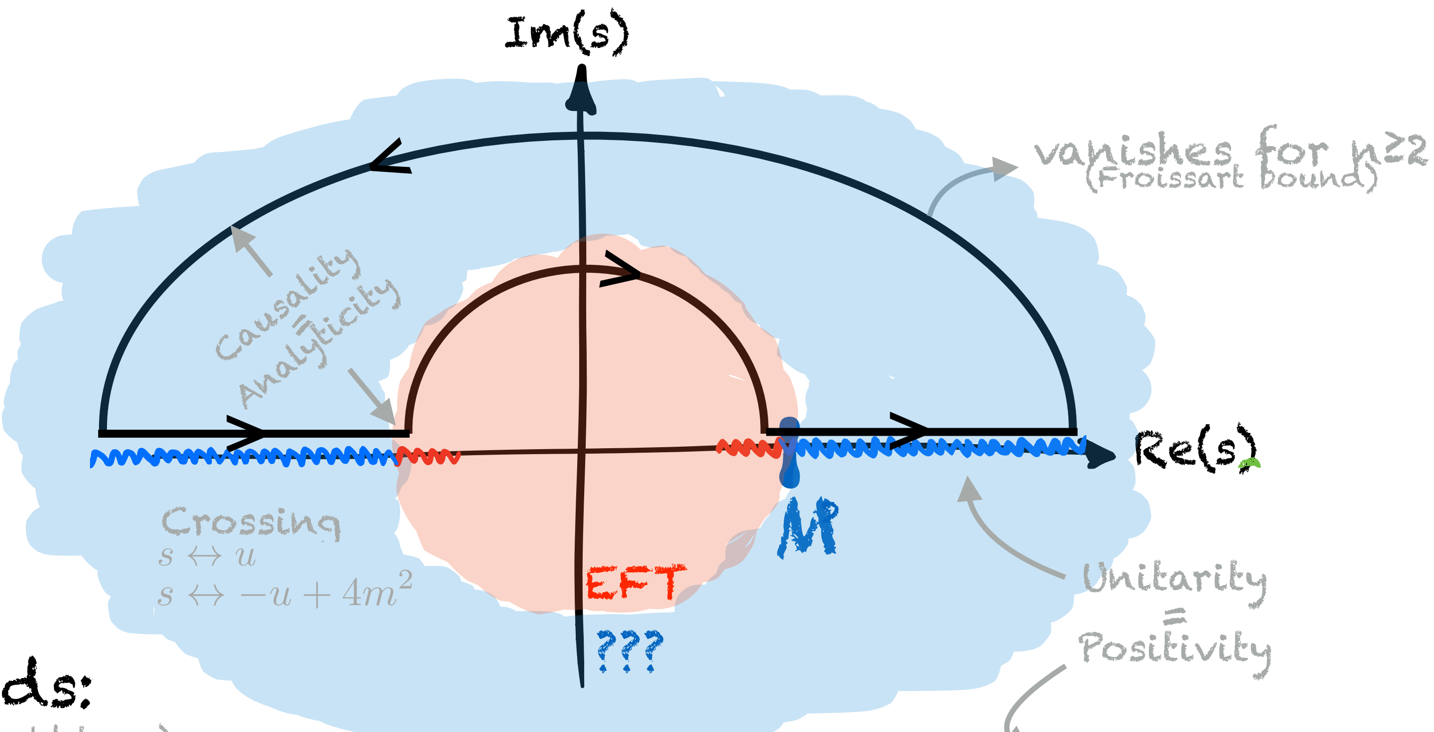

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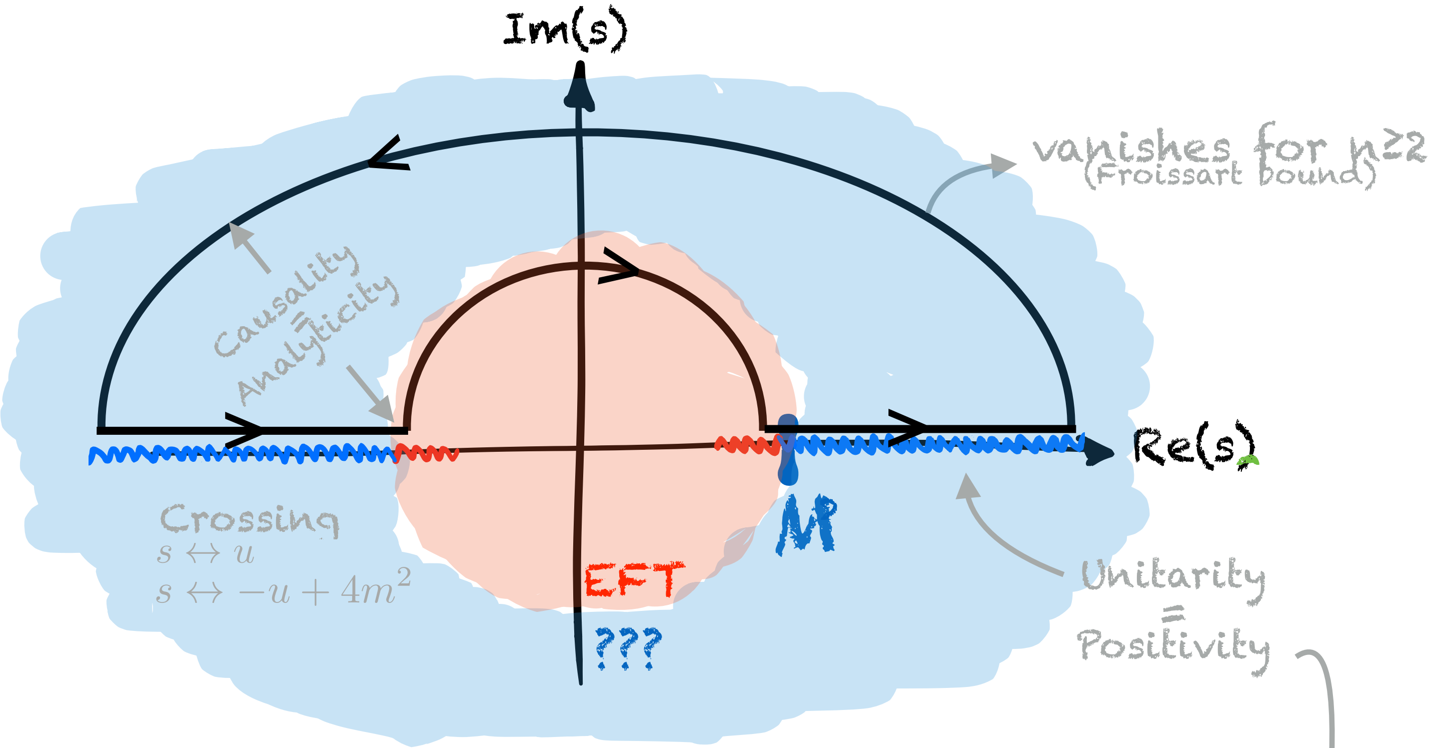

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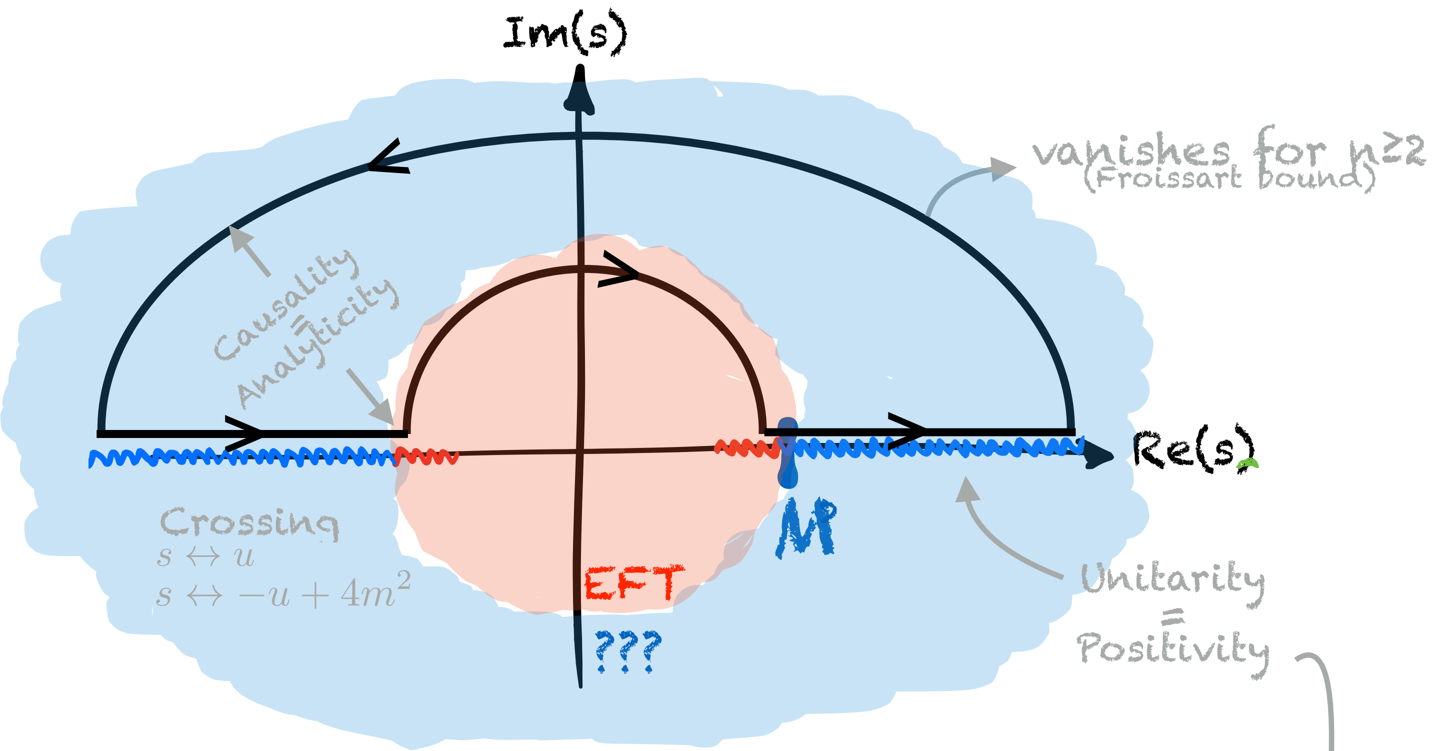
IR-UV Connection



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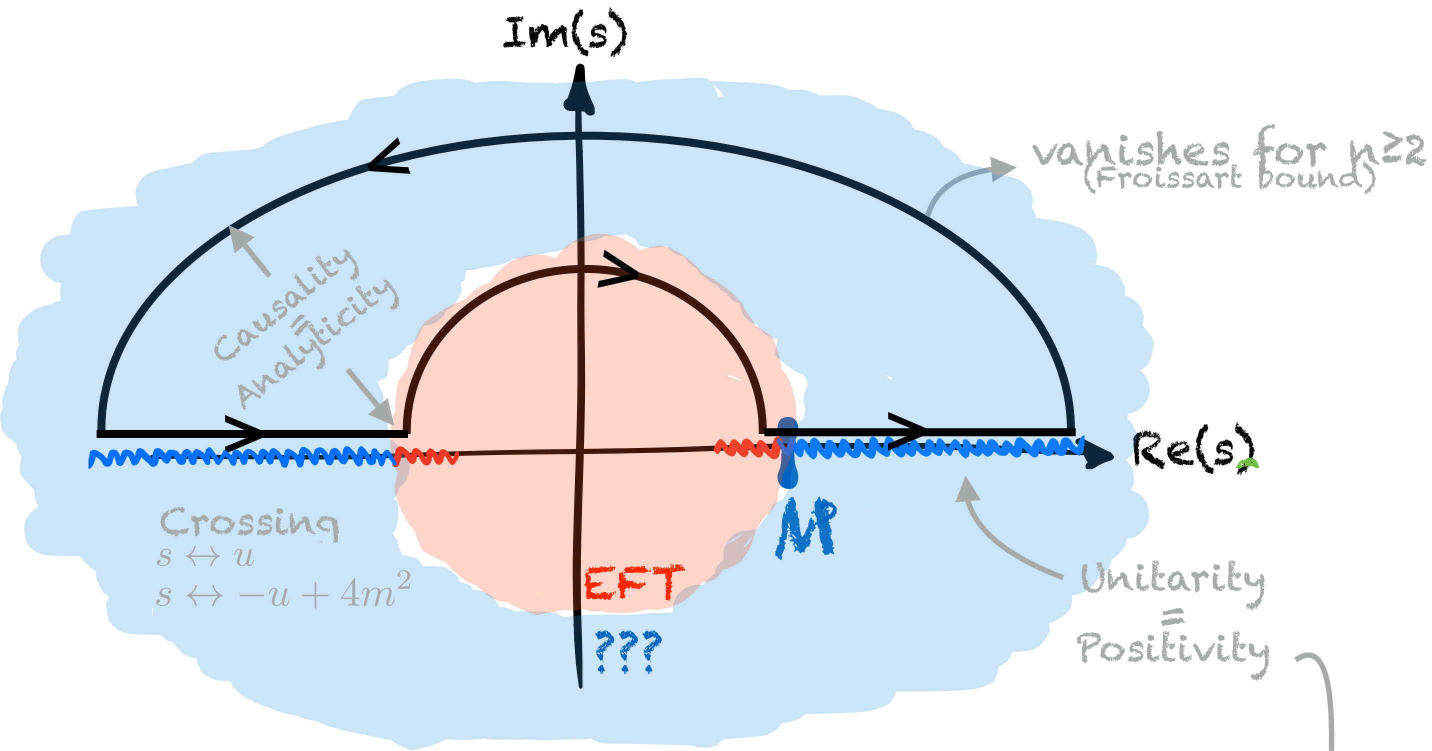
$$\left(\partial_t \mathcal{A}_n + \frac{n+1}{2}\mathcal{A}_n\right)\Big|_{t=0} \geq 0$$

deRham,Melville,Tolley,Zhou'17
Bellazzini,Elias-Miro,Rattazzi,Riembau,FR'20

IR-UV Connection



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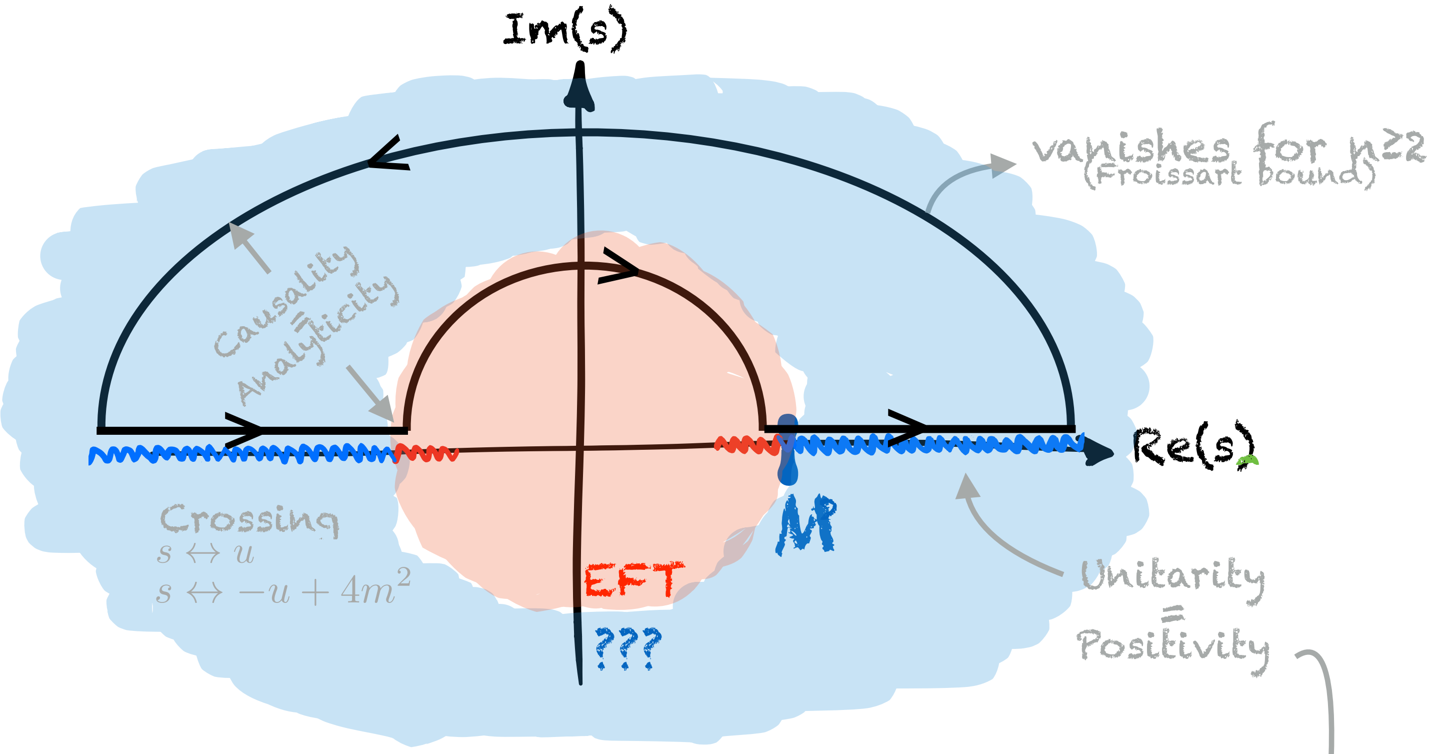

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deRham,Melville,Tolley,Zhou'17
Bellazzini,Elias-Miro,Rattazzi,Riembau,FR'20

$$A(s,t) = \cdots g_4 s^4 + g_4 s^2 t^2 + \cdots$$

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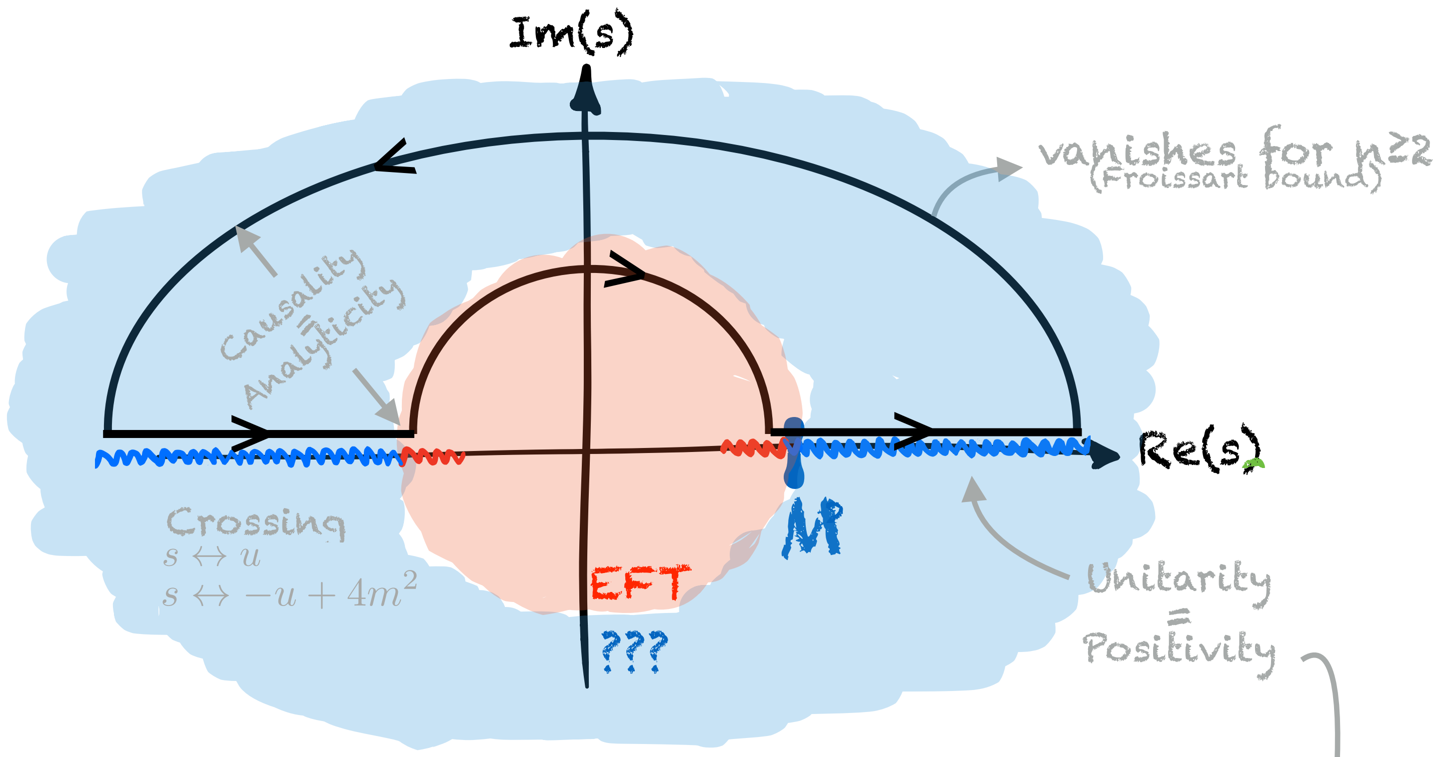

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deRham,Melville,Tolley,Zhou'17
Bellazzini,Elias-Miro,Rattazzi,Riembau,FR'20

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Caron-Huot,VanDuong'20
Tolley,Wang,Zhou'20

"Null constraints"

UV/IR Relations

$$A_{2\to 2}^{Tree} = g_2(s^2+t^2+u^2) - g_3 stu + g_4\,(s^2+t^2+u^2)^2 + \cdots +$$

- Unitary, analytic 2→2 scattering amplitude

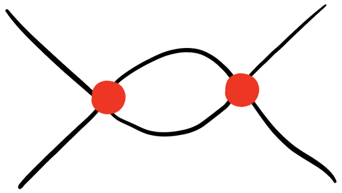
$$A(s,t)$$

- Arc ↔ Couplings

$$\partial_t^k \Big|_{t=0} \mathcal{A}_n = g_{n+k,k}$$

- Null Constraints $\partial_t^2 \mathcal{A}_2|_{t=0} = \mathcal{A}_4|_{t=0}$

UV/IR Relations - Beyond L.O.



$$A_{2\to 2} = g_2(s^2+t^2+u^2) - g_3\, stu + g_4\,(s^2+t^2+u^2)^2 + \cdots + \frac{g_2^2}{16\pi^2} s^2 t^2 \log\frac{t}{s} + \cdots$$

- Unitary, analytic 2→2 scattering amplitude

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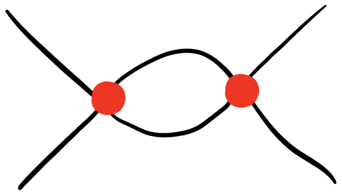
UV/IR Relations - Beyond L.O.



$$A_{2\to 2} = g_2(s^2+t^2+u^2) - g_3 stu + g_4\,(s^2+t^2+u^2)^2 + \cdots + \frac{g_2^2}{16\pi^2} s^2 t^2 \log\frac{t}{s} + \cdots$$

- Unitary, analytic 2→2 scattering amplitude

$$A(s,t)$$

- Arc ↔ Couplings

$$\partial_t^k\Big|_{t=0} \mathcal{A}_n = g_{n+k,k} + \sum_{n,m}^{\infty} \#_{n,m}\, g_n g_m$$

- Null Constraints $\partial_t^2 \mathcal{A}_2|_{t=0} = \mathcal{A}_4|_{t=0}$

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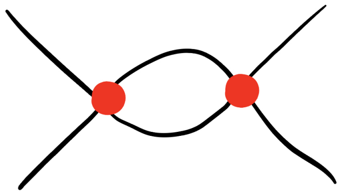

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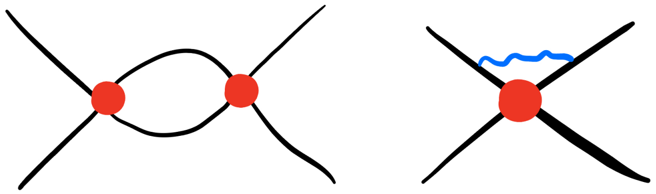

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$$A(s,t) \underset{\uparrow}{=} \frac{1}{\infty}$$

long-range forces, D=4

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Is it possible that for massless particles most bounds disappear as soon as

$$g_n, \alpha, G_N \neq 0 \qquad ?$$

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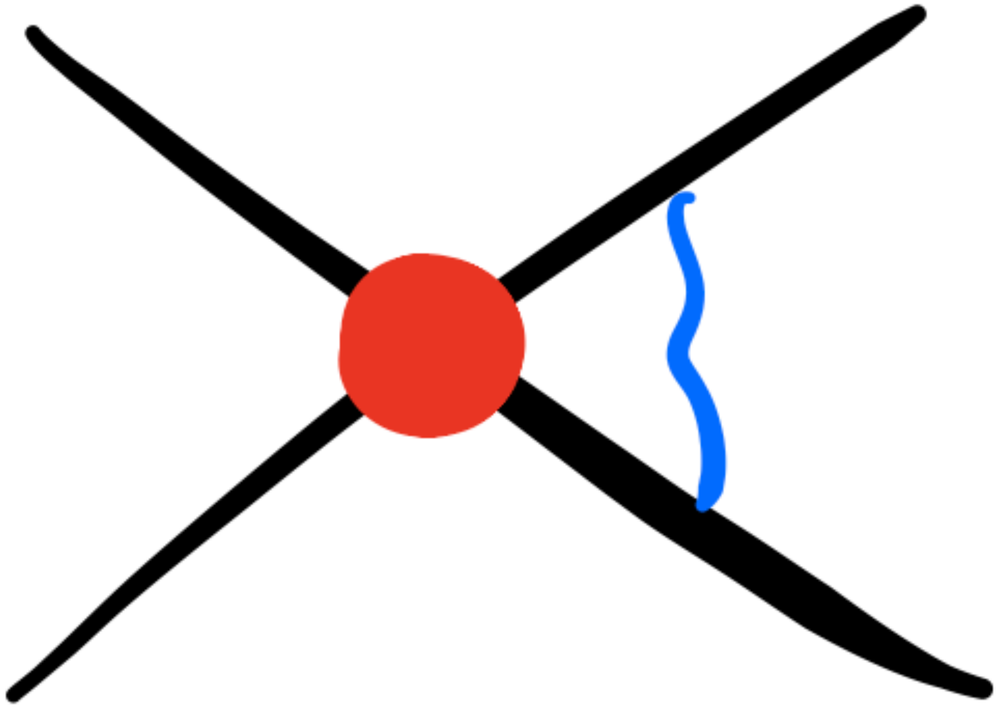
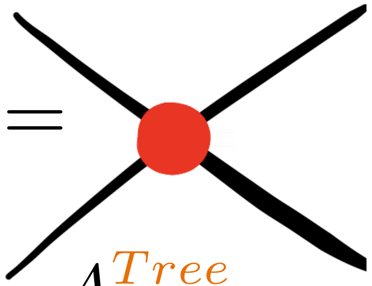
Goal of this talk:

Discover dispersion relations for which NLO effects are truly NLO, and compute them



Beyond L.O.
long-range Interactions

Esclusive Amplitudes: Unitary and Analytic but...

$A_{2\to 2} =$ 

$A^{Tree}_{2\to 2}$

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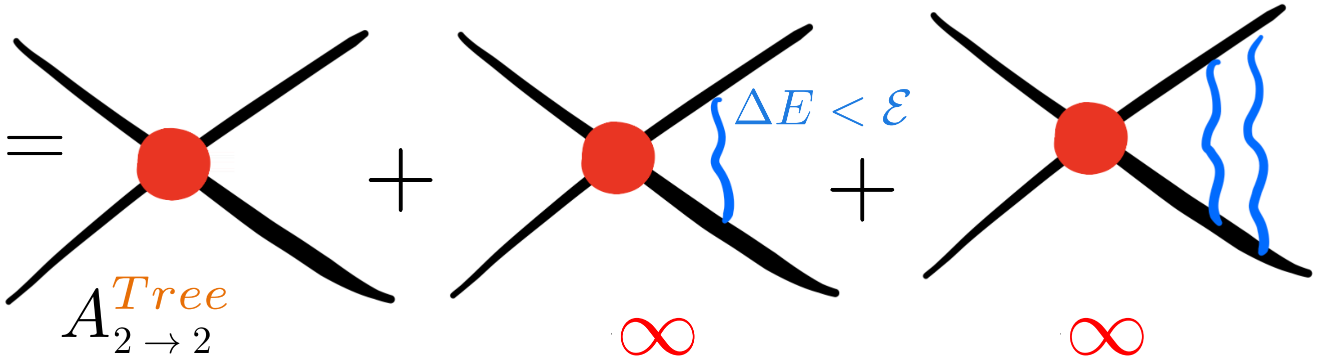

$+ \dots = \mathcal{W}_{\mathcal{E}}\left(A^{Tree}_{2\to2} + sub\right) = 0$

∞

$\uparrow$

$e^{-\infty}$

Beyond L.O.
long-range Interactions

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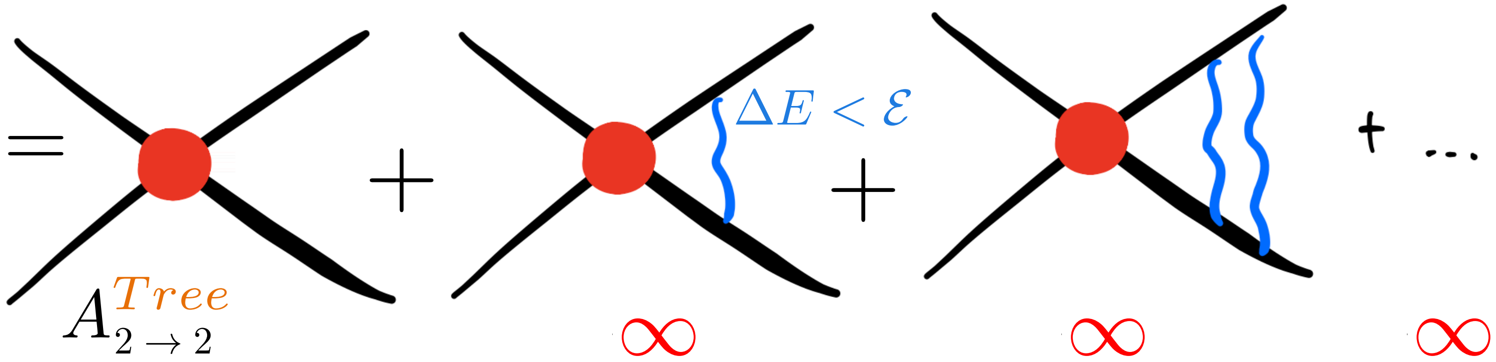
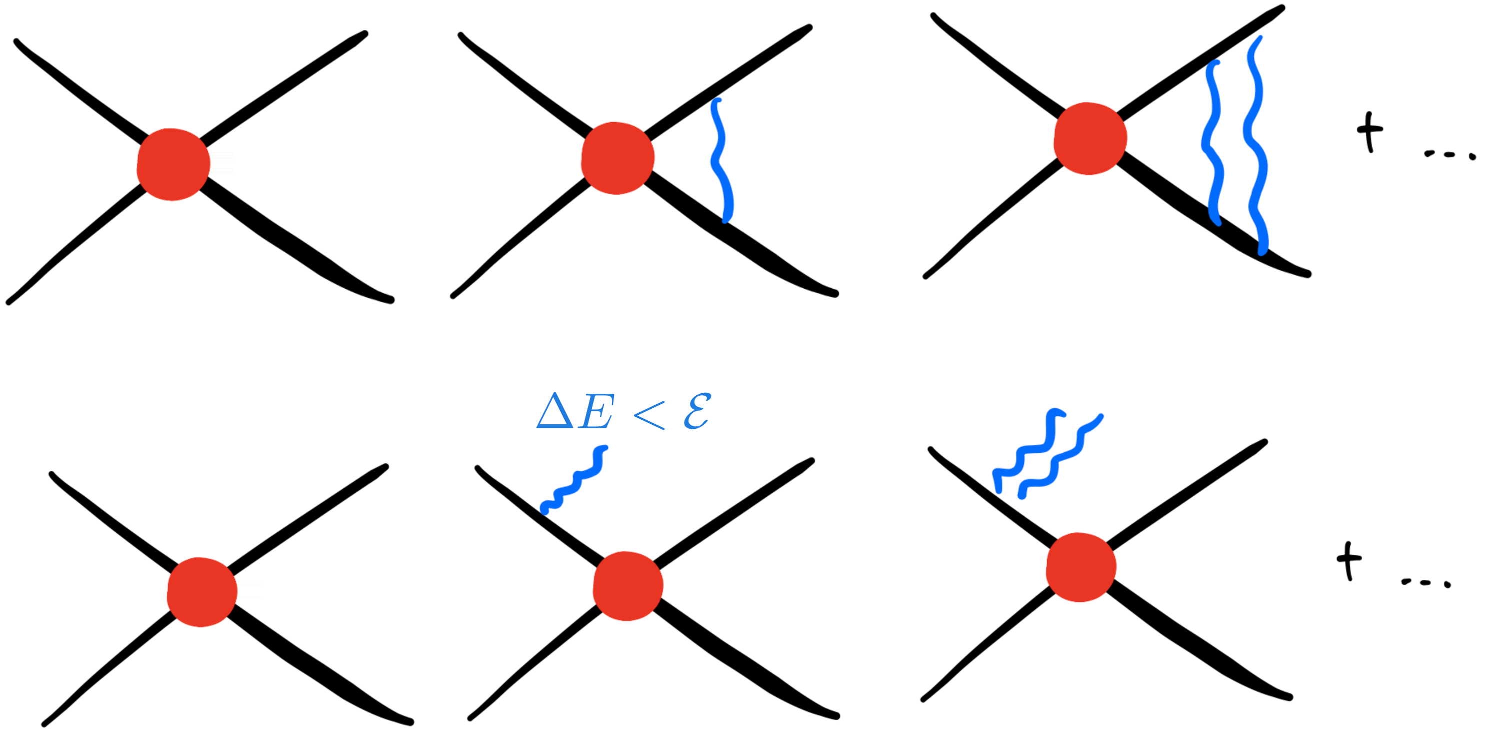


$= \mathcal{W}_{\mathcal{E}}\,(A^{Tree}_{2\to2} + sub) \;= 0$

$e^{-\infty}$

Inclusive Cross-sections: IR finite, but no analyticity

$$\sigma(\Delta E < \mathcal{E}) = \sum^{\infty} |A|^2$$



Stripped Amplitudes: IR Finite

Inclusive Cross-sections	Esclusive Amplitudes
$\sigma(\Delta E < \mathcal{E})$	A
IR Safe ✗	✗ Unitary/Analytic

Stripped Amplitudes: IR Finite

Inclusive Cross-sections

$$\sigma(\Delta E < \mathcal{E})$$

IR Safe

✗

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Unitary/Analytic

In this work

Stripped Amplitudes

$$A_{\mathcal{E}} \equiv \lim_{\epsilon \to 0} \frac{A^{\epsilon}}{\mathcal{W}^{\epsilon}_{\mathcal{E}}}$$

IR Safe

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Stripped Amplitudes: IR Finite

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$\epsilon = D - 4$

IR Safe

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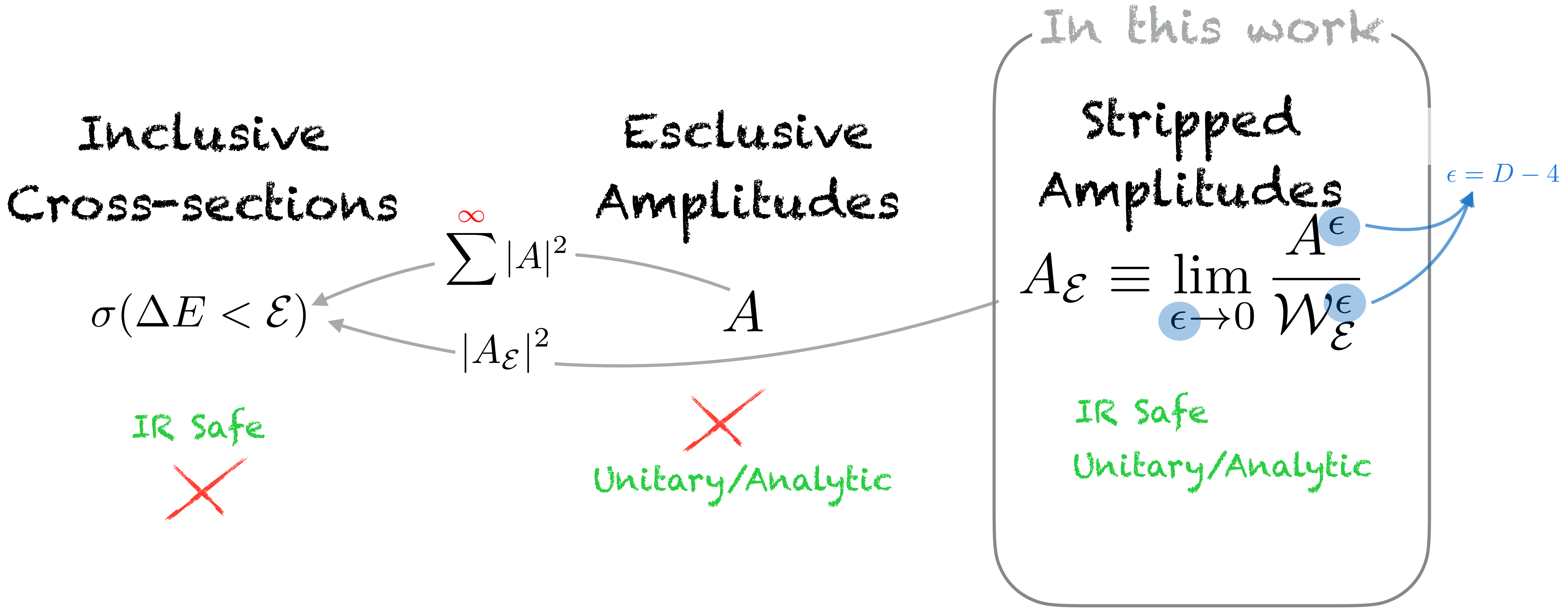

Inclusive Cross-sections

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IR Safe ✗

$\sum^{\infty} |A|^2$

$|A_{\mathcal{E}}|^2$

Esclusive Amplitudes

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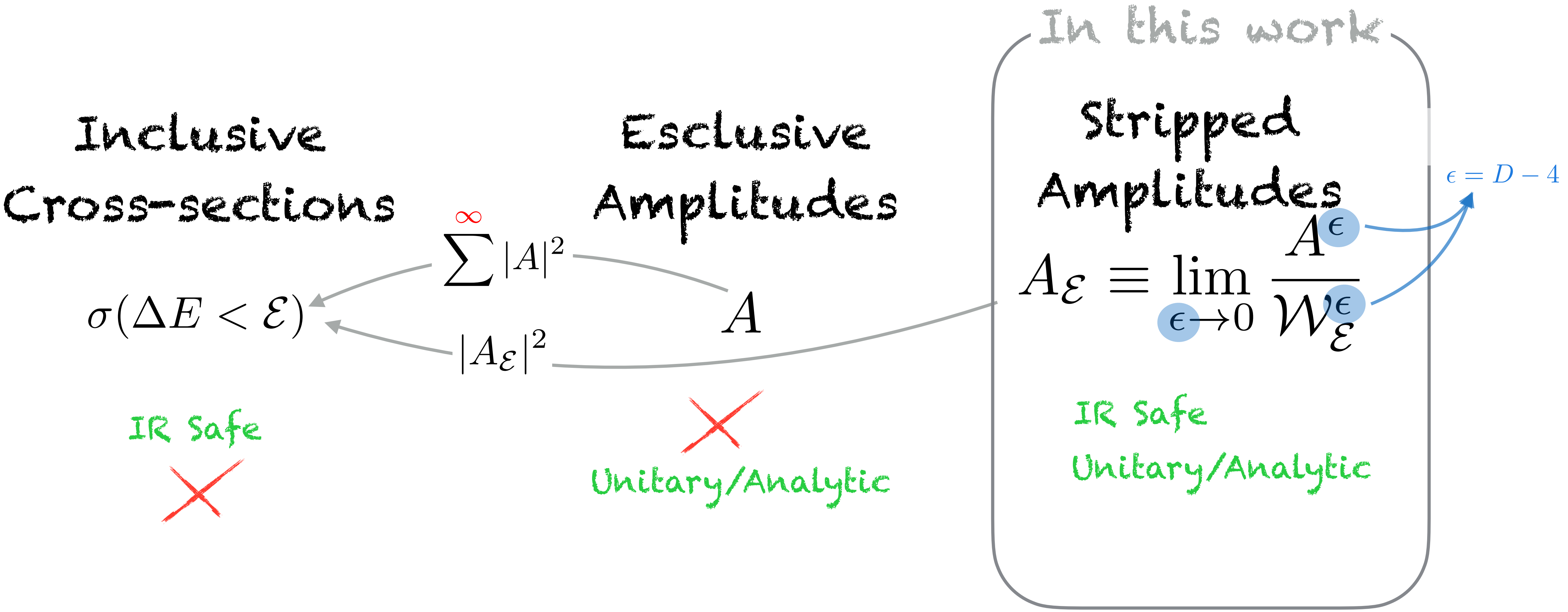

In this work

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Unitary/Analytic

- $A_\mathcal{E}$ are formal amplitudes of finite size $\mathcal{E}^{-1}$ experiments

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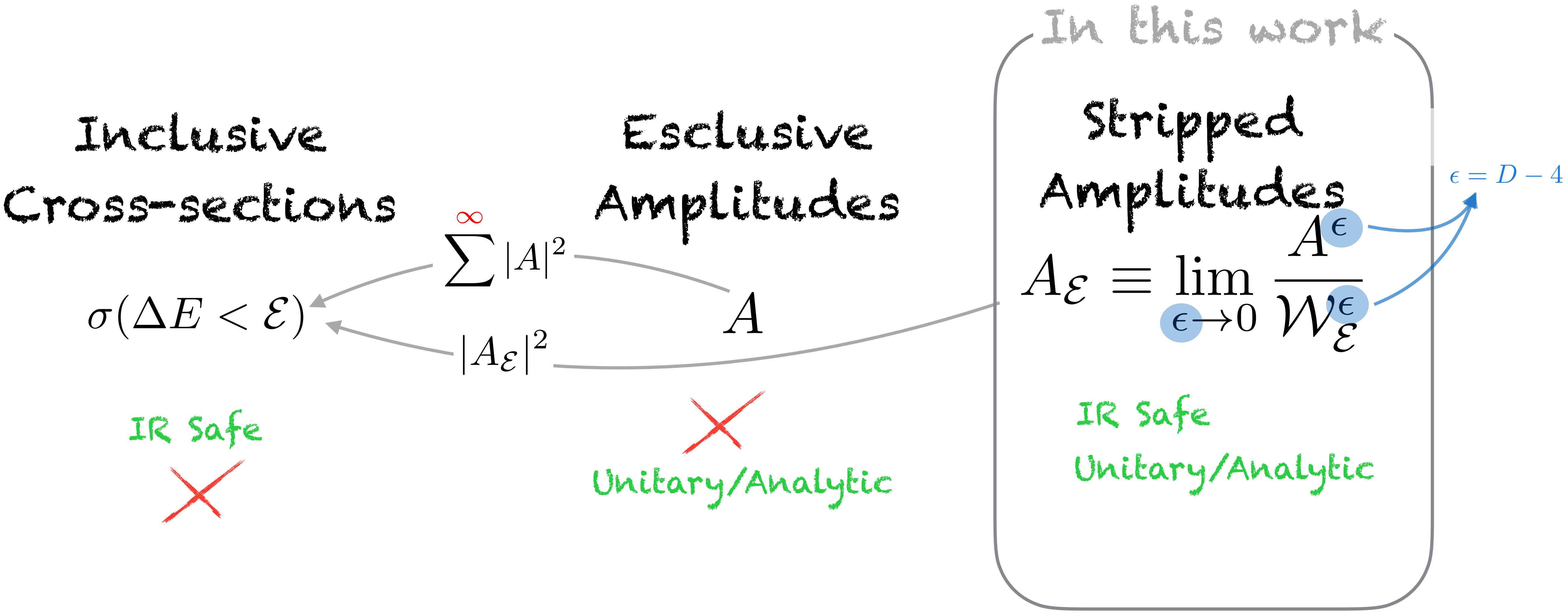

In this work

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IR Safe

Unitary/Analytic

- $A_{\mathcal{E}}$ are formal amplitudes of finite size $\mathcal{E}^{-1}$ experiments
- Regulator ϵ disappears, physical $\mathcal{E}$ remains: it specify initial/final states!

Stripped Amplitudes: Analytic

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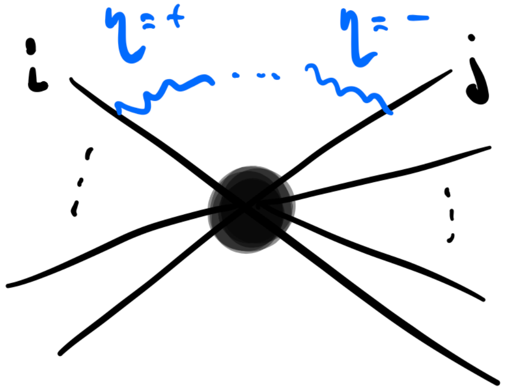
QED:

$$\mathcal{W} = \exp\left\{ \frac{\alpha}{8\pi} \frac{(\mathcal{E}/\mu)^{2\epsilon}}{\epsilon} \left(\sum_{i,j} \eta_i \eta_j q_i q_j \frac{1}{\beta_{ij}} \log \frac{1+\beta_{ij}}{1-\beta_{ij}} - i \sum_{\substack{i\neq j \\ \eta_i\eta_j=1}} q_i q_j \frac{2\pi}{\beta_{ij}} + O(\epsilon) \right) \right\}$$

Weinberg'65



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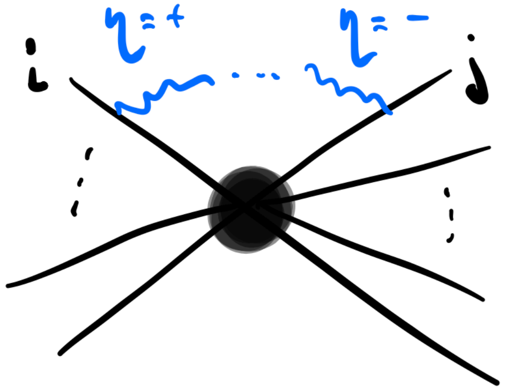

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Bellazzini,Berman,Isabella,FR,Romano,Sciotti'25

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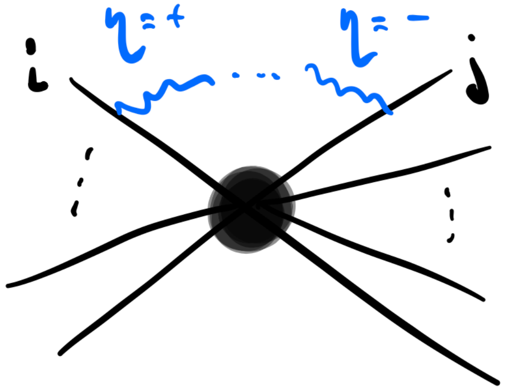

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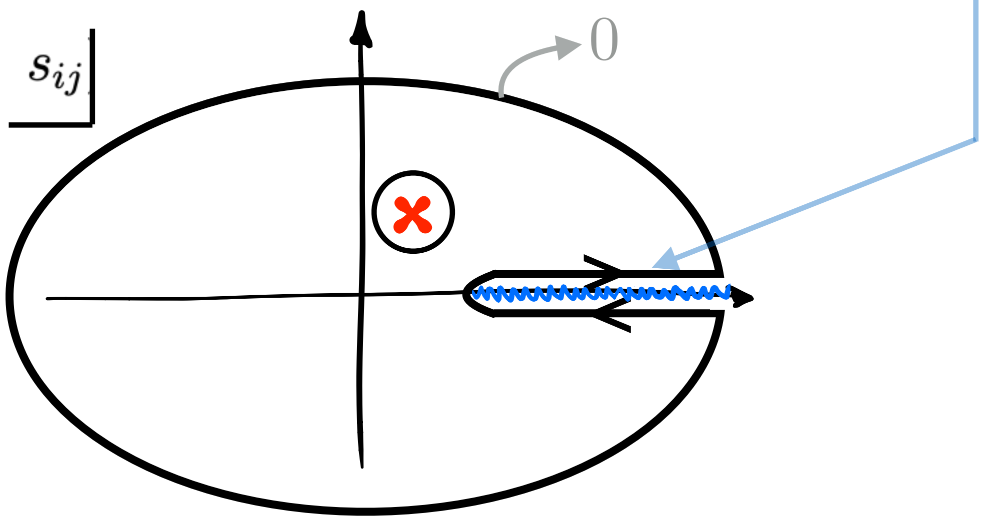

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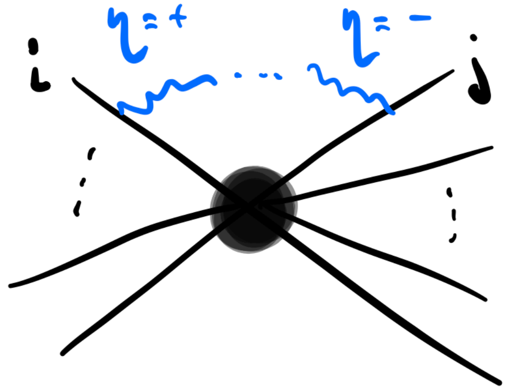

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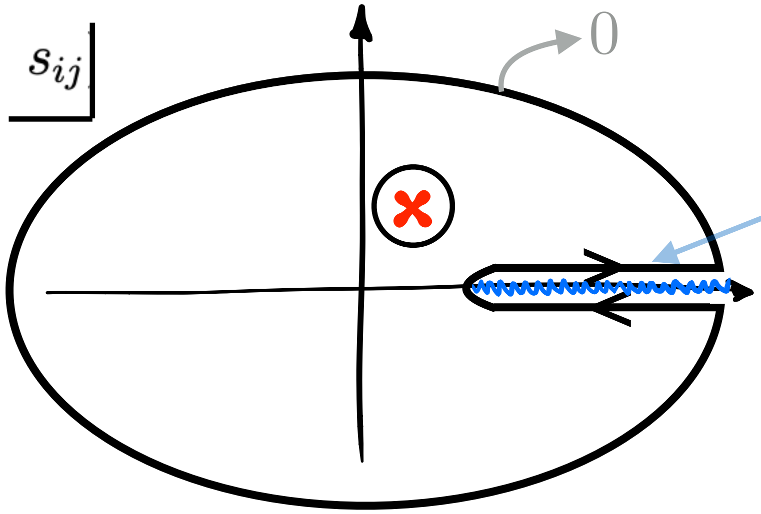

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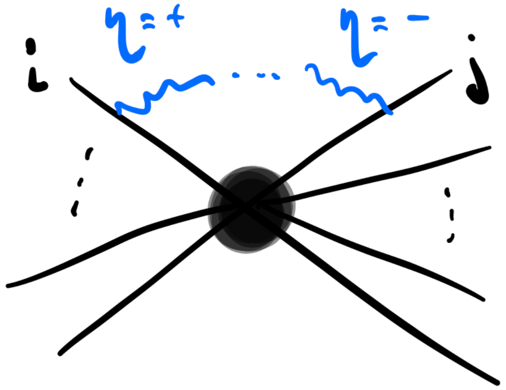

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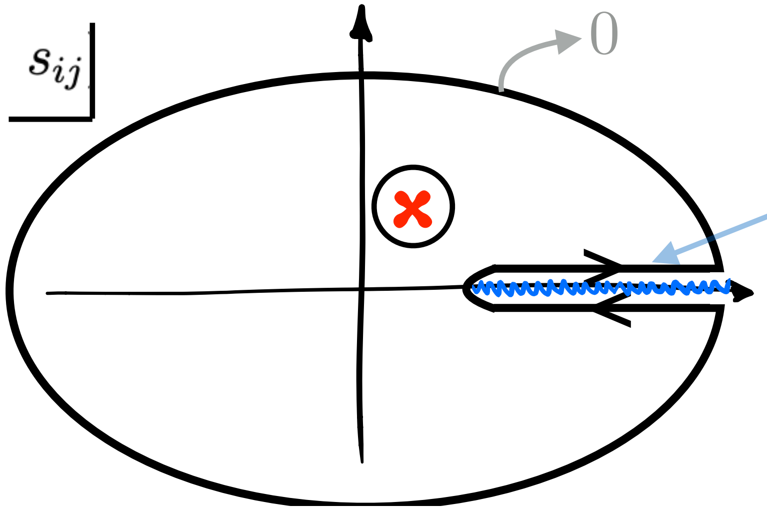

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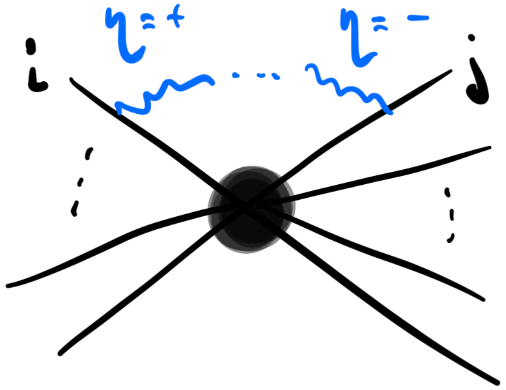

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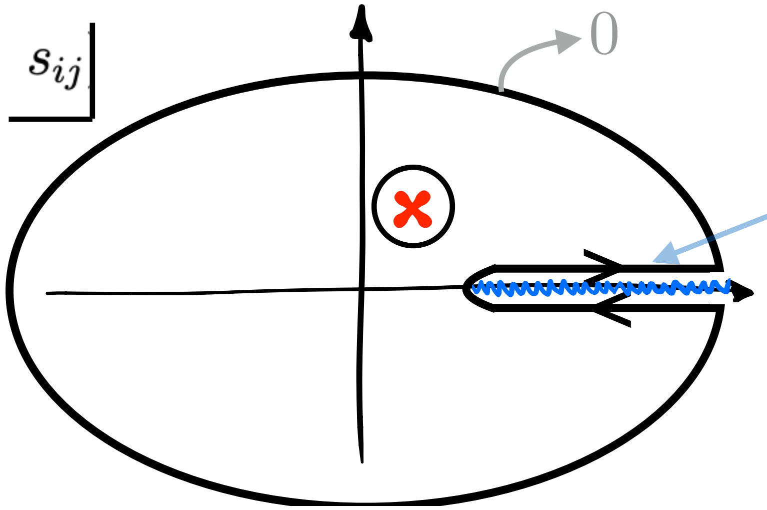

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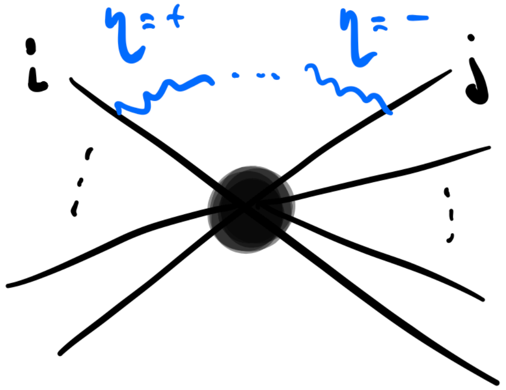

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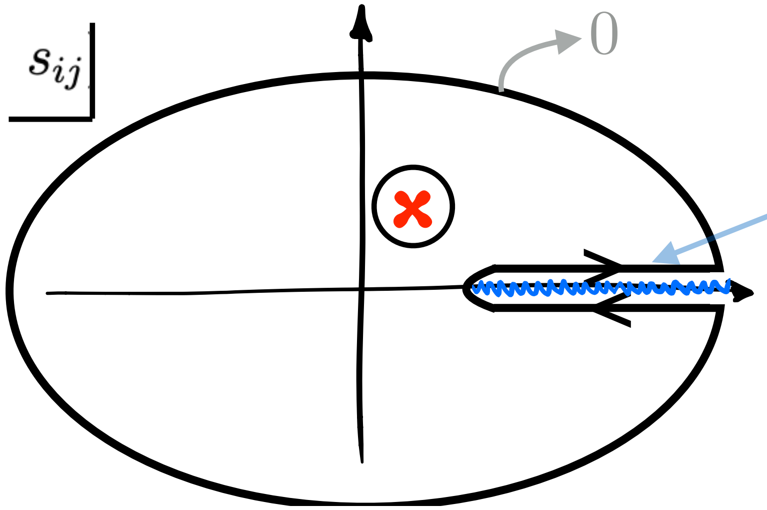

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▶ $A_{\mathcal{E}}$ has same analyticity as A and respects Froissart $A/s^2 \xrightarrow[s\to\infty]{} 0$!

... and similarly in Gravity!

Stripped Amplitudes: Unitary and Positive

Effectively 2 QED couplings:

α

$\alpha_{\mathcal{E}} \equiv \alpha \log M/\mathcal{E}$

Captures finite perturbative effects

Captures leading IR effects

Stripped Amplitudes: Unitary and Positive

Effectively 2 QED couplings:

α $\qquad\qquad$ $\alpha_{\mathcal{E}} \equiv \alpha \log M/\mathcal{E}$ $\qquad\qquad$ ⩔

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Stripped Amplitudes: Unitary and Positive

Effectively 2 QED couplings:

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In gravity: $G \to 0\,, \;\; \mathcal{E} \to 0\,, \quad G_{\mathcal{E}} \equiv GM^2 \log M/\mathcal{E} = \text{fixed}$

Stripped Amplitudes: Unitary and Positive

Effectively 2 QED couplings:

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⩔

Captures finite perturbative effects

Captures leading IR effects

In gravity: $G \to 0\,,\ \ \mathcal{E} \to 0\,,\quad G_{\mathcal{E}} \equiv GM^2 \log M/\mathcal{E} = \text{fixed}$

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$$A_{\mathcal{E}} = A_{\mathcal{E}}^{\text{hard}} + \delta_{\text{sub. soft}}$$

$E_\gamma > \mathcal{E}$

Stripped Amplitudes: Unitary and Positive

Effectively 2 QED couplings:

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$E_\gamma > \mathcal{E}$

Is unitary (equivalent to $m_\gamma = \mathcal{E}$ theory)

- Leads to positivity at all orders in $\alpha_{\mathcal{E}}$ and L.O. in α

$$\mathcal{A}_n(s,t) = \int_{M^2}^{+\infty} \frac{ds}{\pi}\left[\frac{\overline{\text{Im}}\, A^s_{\mathcal{E}}(s,t)}{s^{3+n}} + \frac{(-1)^n \overline{\text{Im}}\, A^u_{\mathcal{E}}(s,t)}{(s+t-4m^2)^{3+n}}\right]$$

e.g.

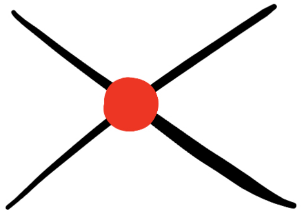
$$\mathcal{A}_n \geq 0 + O(\alpha)$$

$$\partial_t|_{t=0}\mathcal{A}_n \geq 0 + O(\alpha)$$

Bounds for charged/neutral pions

$\pi^+\pi^0 \to \pi^+\pi^0$

Tree level:



don't enter twice-subtracted relations

$$A^{\text{tree}}_{+0} = \boxed{c_{0,0} + c_{1,1}t} + c_{2,0}(s - m_\pm^2 + t/2)^2 + \boxed{c_{2,2}t^2} + c_{3,1}t(s - m_\pm^2 + t/2)^2 + \ldots$$

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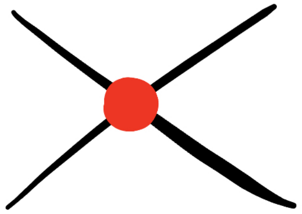

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Positivity Bounds:

$$\mathcal{A}_2 \geq 0 \qquad \bar{s}\partial_{t=0}\mathcal{A}_2 + \frac{3}{2}\mathcal{A}_2 \geq 0$$

Cutoff M²

Bounds for charged/neutral pions

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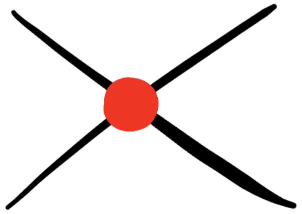

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▶

$$c_{20} > 0$$

$$(M^2 - m_\pm^2)\frac{c_{31}}{c_{20}} \geq -\frac{3}{2}$$

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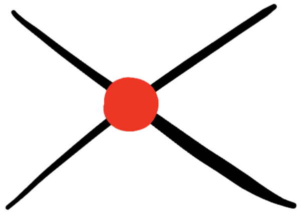

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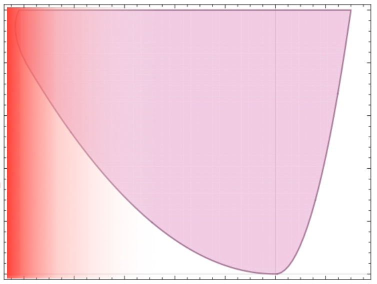

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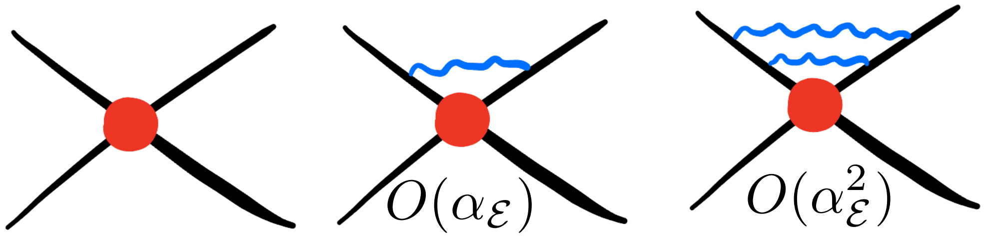

+ .. $= 0$ exclusively...

$$A_{\mathcal{E}} = \left[c_{0,0} + c_{1,1}t + c_{2,0}(s - m_\pm^2 + t/2)^2 + c_{2,2}t^2 + c_{3,1}t(s - m_\pm^2 + t/2)^2 + \ldots\right] \times$$

$$\left[1 + \frac{\alpha}{2\pi}\log\left(\frac{m_\pm^2(4m_\pm^2 - t)}{\mathcal{E}^4}\right)\frac{t - 2m_\pm^2}{\sqrt{t(4m_\pm^2 - t)}}\arctan\left(\frac{\sqrt{t}}{\sqrt{4m_\pm^2 - t}}\right) + \frac{\alpha}{2\pi}\log\frac{4\pi\mu^2}{\mathcal{E}^2}\right] + O(\alpha_{\mathcal{E}}^2)$$

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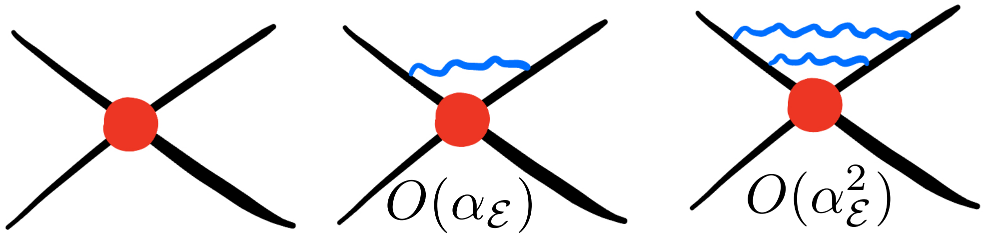

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▶

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$$c_{2,0}\left[\frac{3}{2} - \frac{(M^2-m_\pm^2)}{m_\pm^2}\frac{\alpha}{3\pi}\log\left(\frac{\mathcal{E}}{m_\pm}\right)\right] + (M^2-m_\pm^2)c_{3,1} + O(\alpha_{\mathcal{E}}^2,\alpha) \geq 0$$

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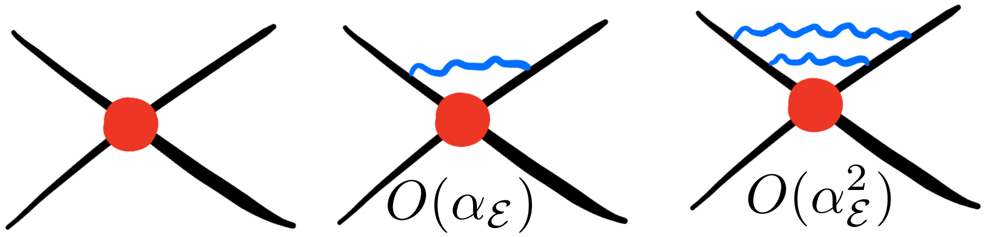

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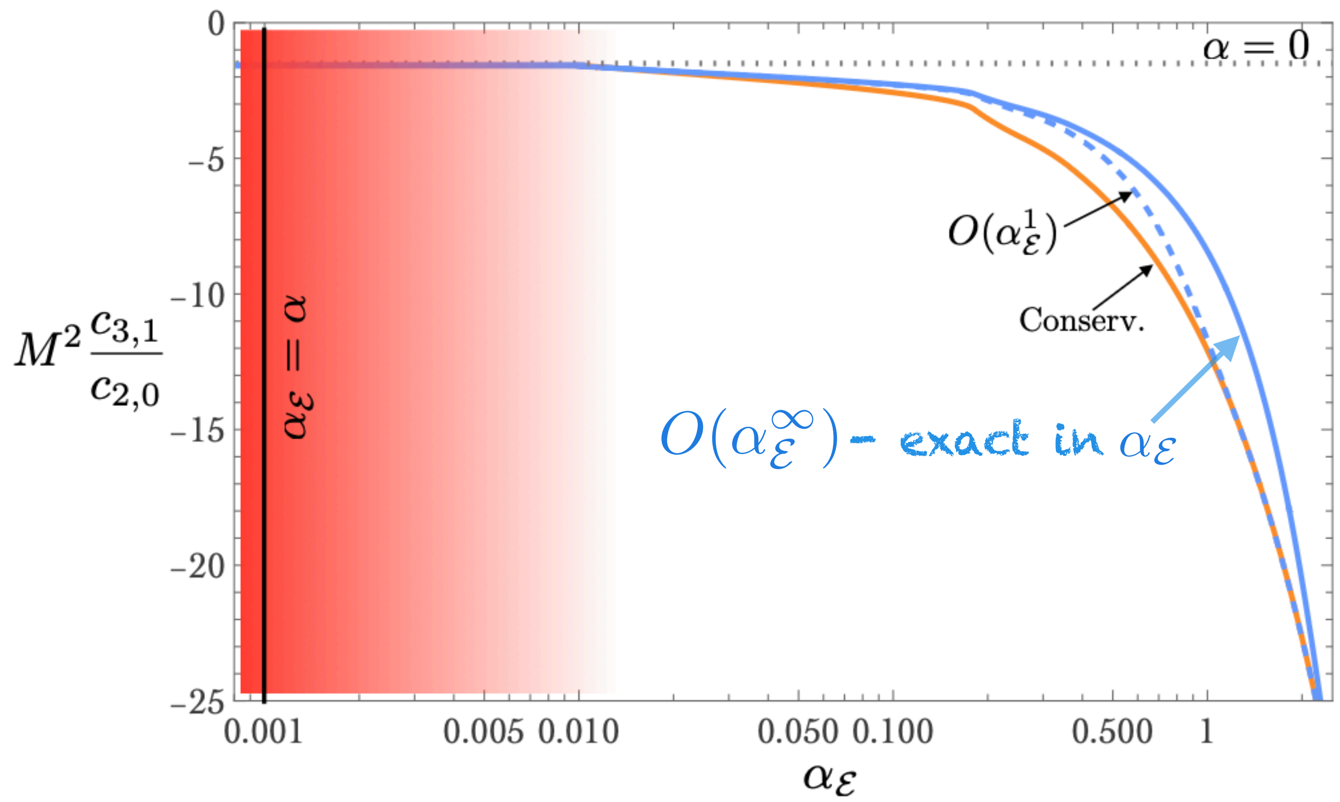

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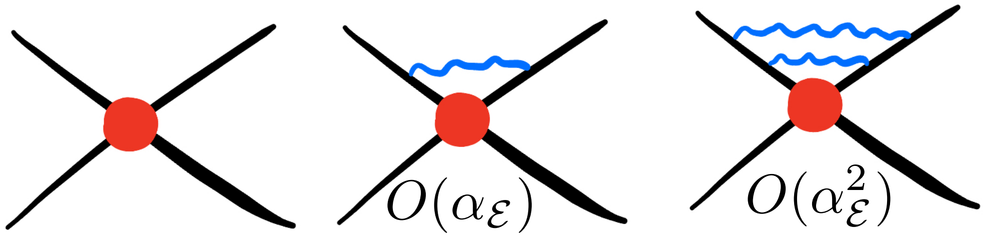

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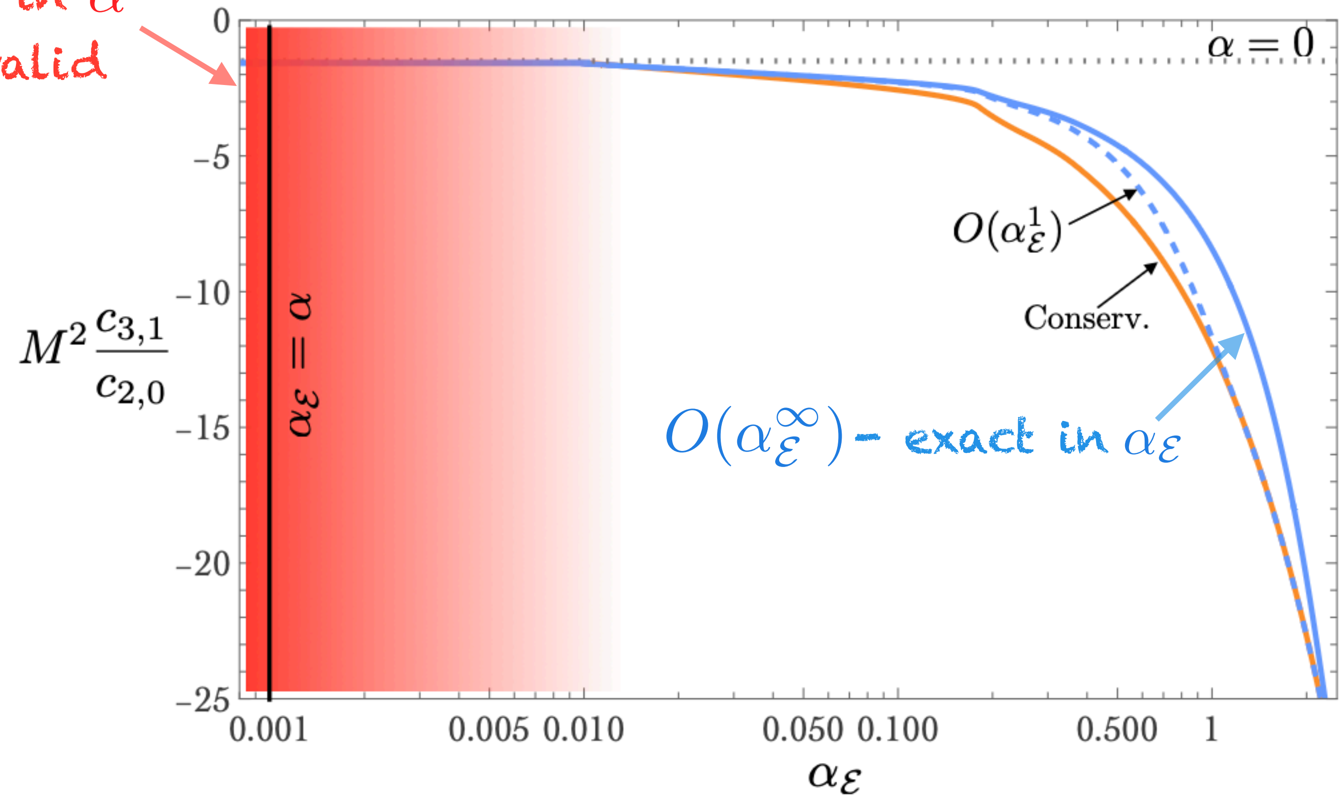
approx in α not valid



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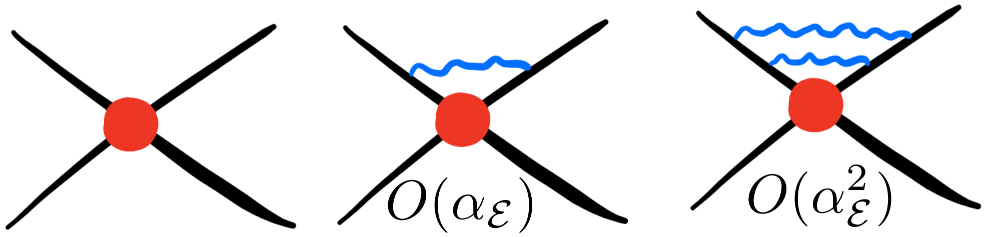

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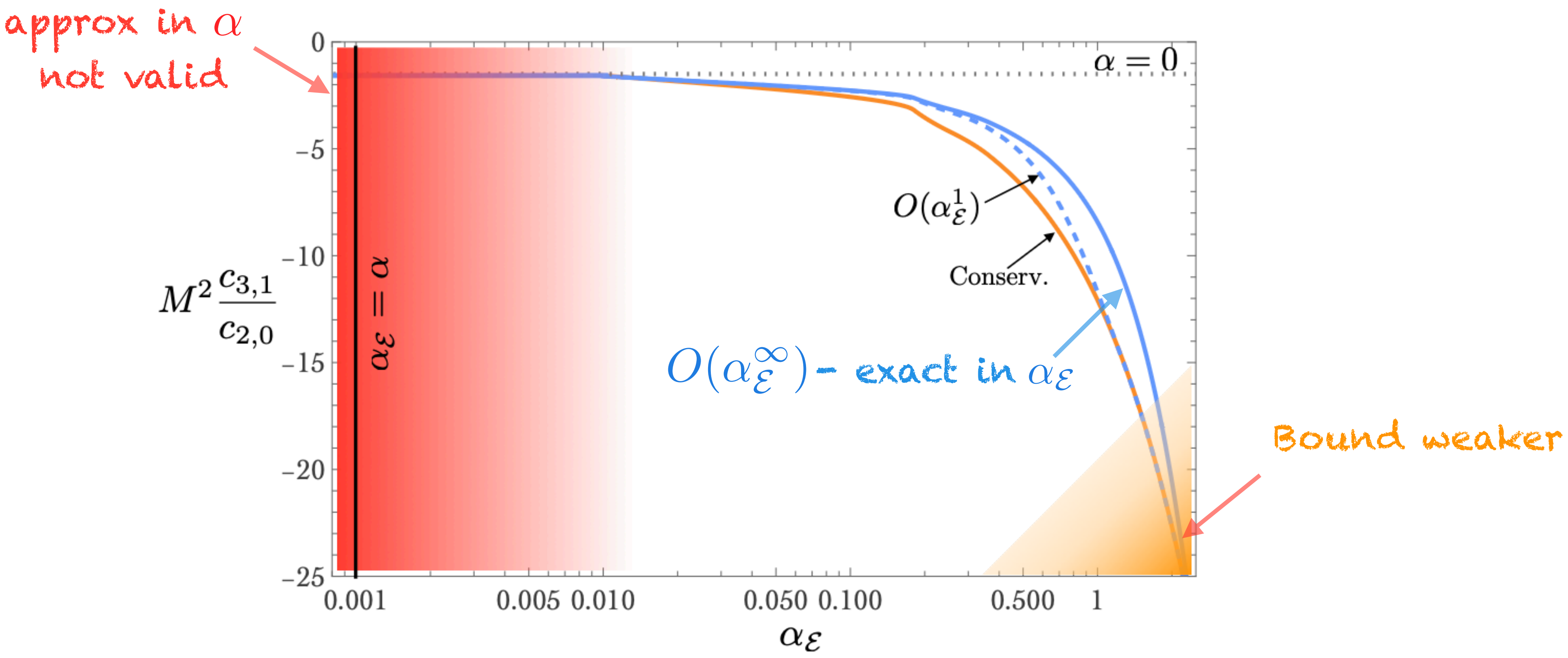

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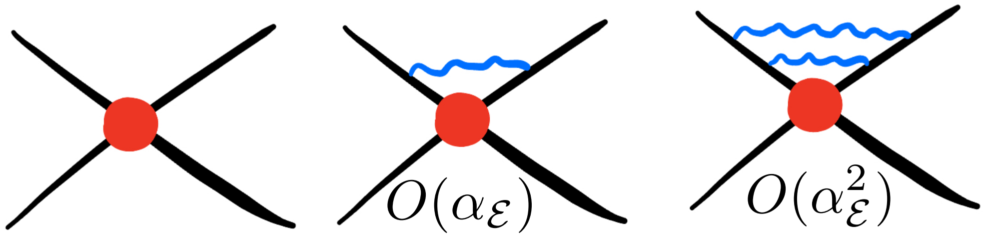

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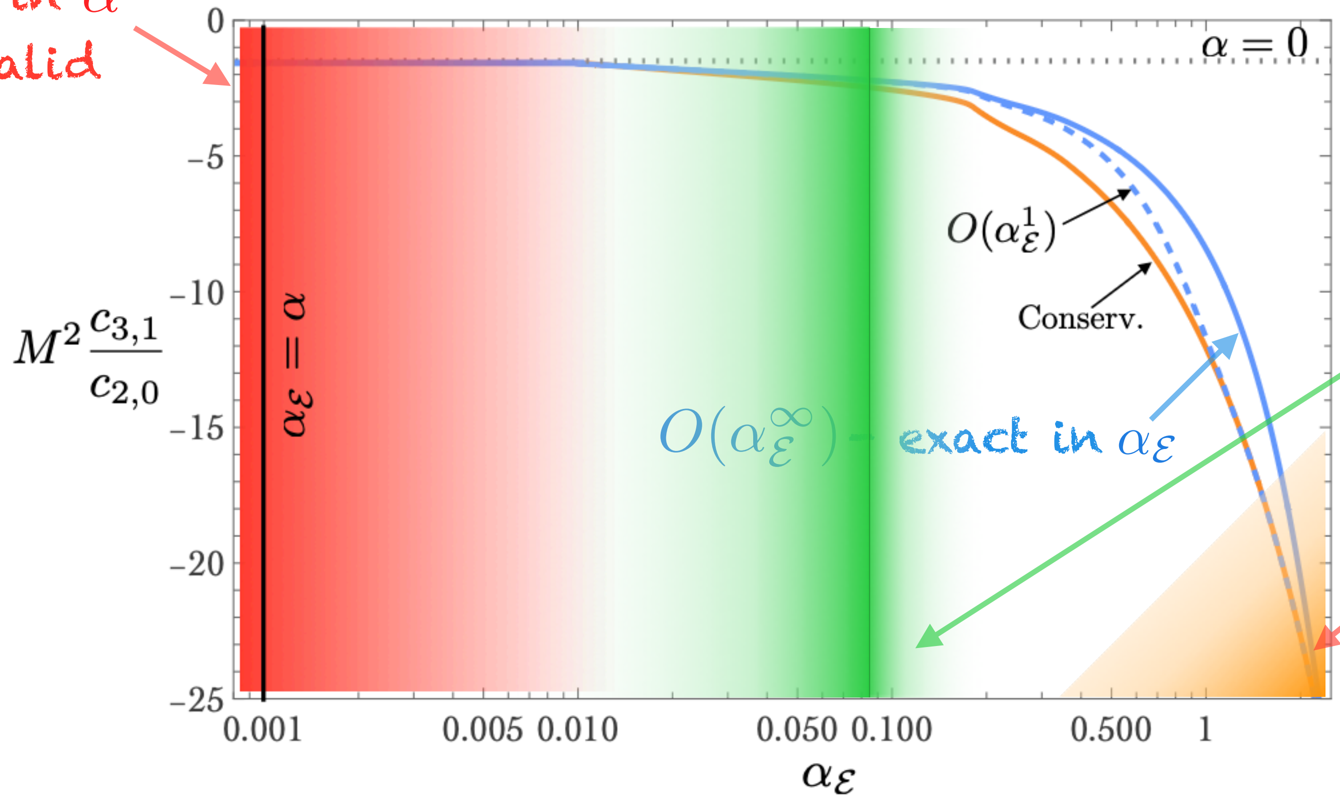


Optimal experiment
has size $1/\mathcal{E}$ of this order

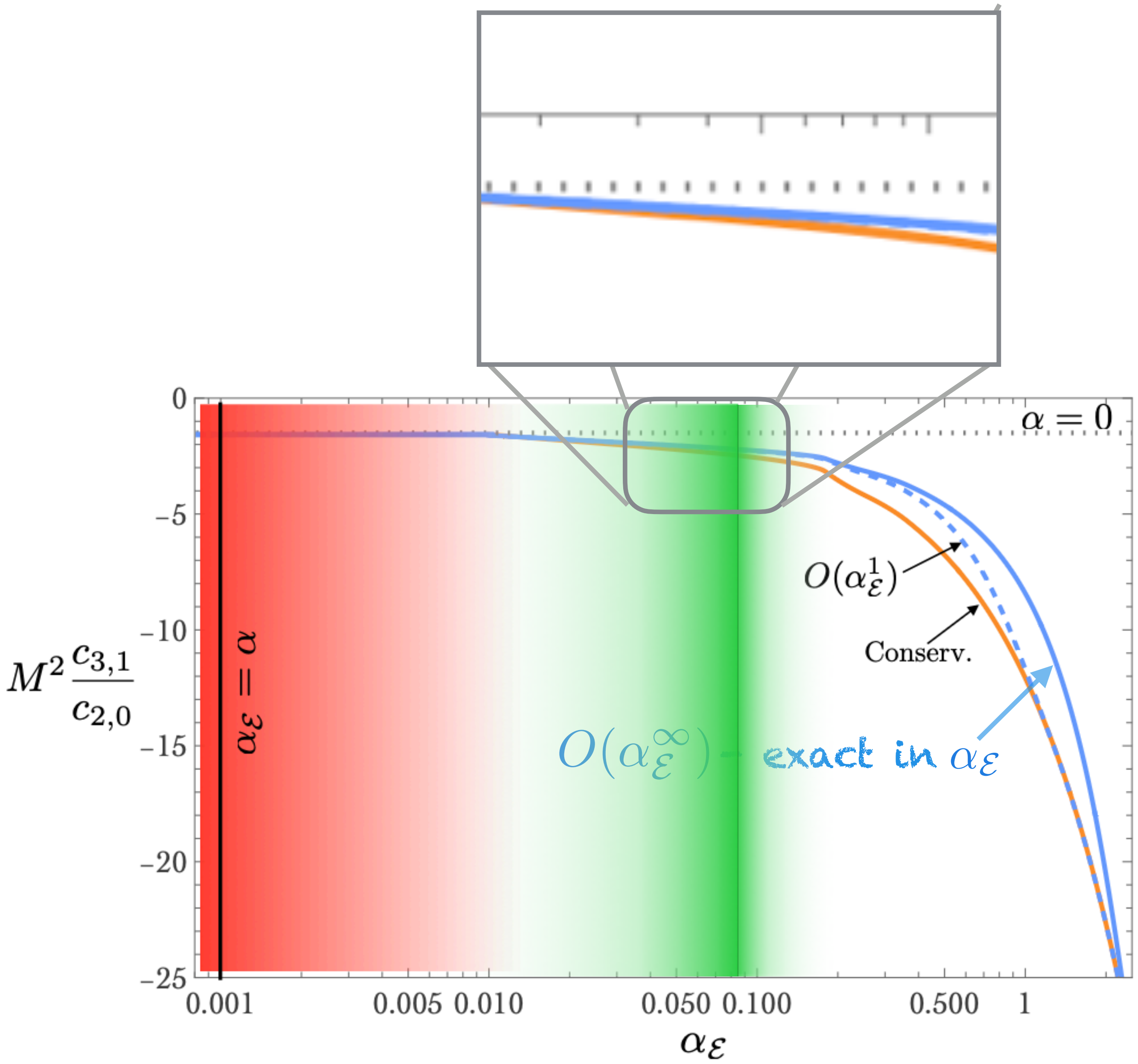
Bound weaker

Bounds for charged/neutral pions

$\pi^+\pi^0 \to \pi^+\pi^0$

Positivity bounds survive to $\alpha \neq 0$ also in d=4...

QED corrections are perturbative.



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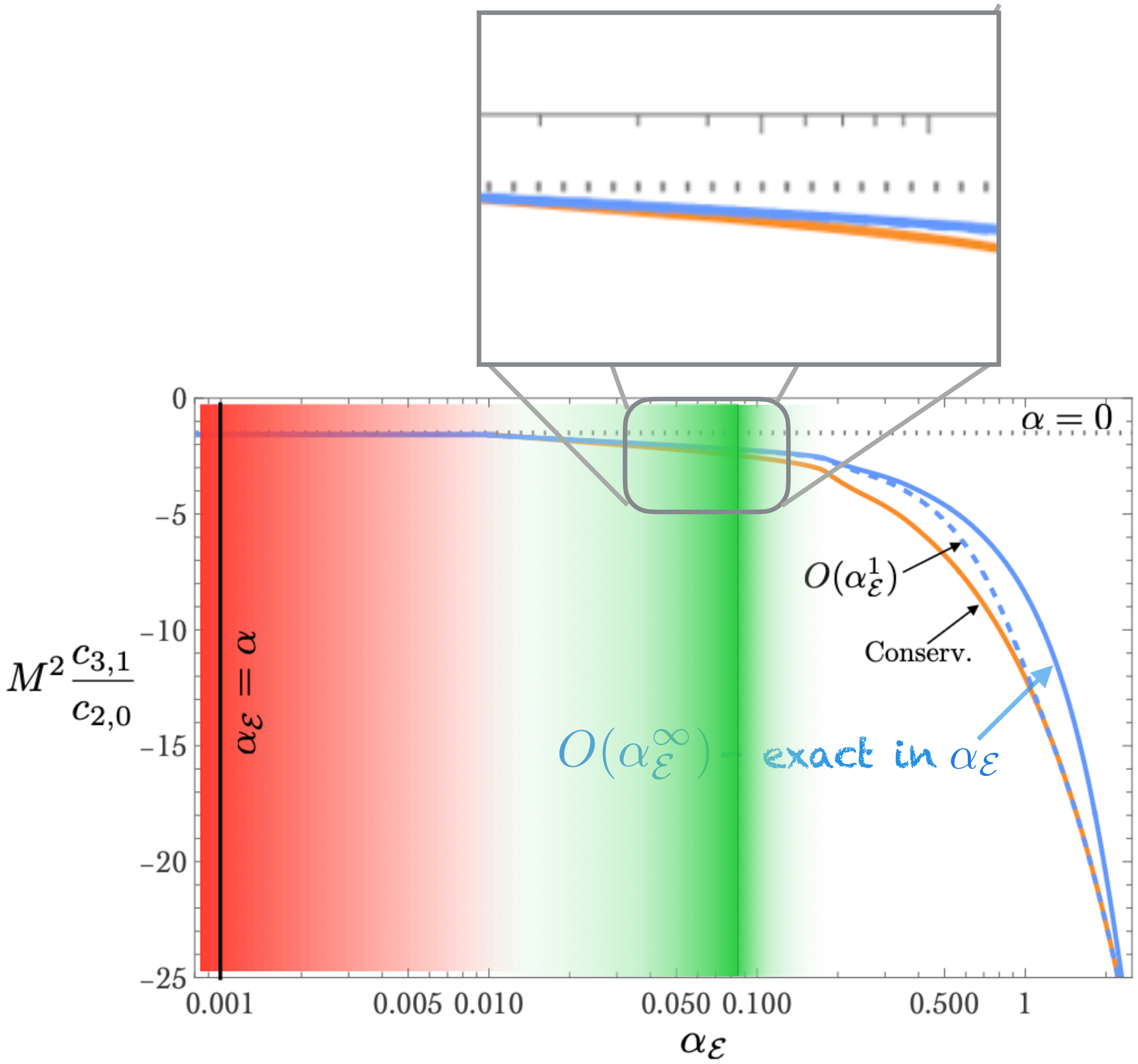
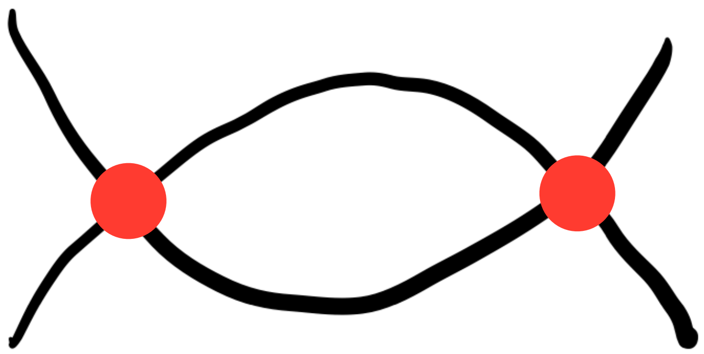
"Long is the path, close is the goal"





Negativity

Not all tree-level bounds are affected in the same way...

Bellazzini,Elias-Miro,Riembau,Rattazzi,FR'20
Bellazzini,Riembau,FR '22
Beadle,Isabella,Perrone,Ricossa,FR,Serra'24-'25

$\pi^0\pi^0 \to \pi^0\pi^0$

$$A_{2\to 2}^{Tree} = g_0 + g_2(s^2+t^2+u^2) - g_3 stu + g_4(s^2+t^2+u^2)^2 + \cdots$$

$$0 \leq \mathcal{A}_2 = g_2$$

$$0 \leq \mathcal{A}_4 = g_4$$

$$0 \leq \mathcal{A}_6 = g_6$$

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- Each coupling individually bounded, independently of others!

Negativity

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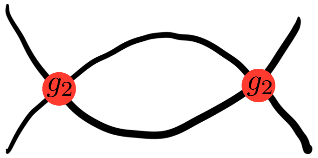

$\beta_4 = \frac{7}{10}\frac{g_2^2}{16\pi^2}$

$$-\frac{\beta_4}{2}s^2\left(s^2-\frac{tu}{21}\right)\log(-s) + (s\leftrightarrow t,u) + \cdots$$

$$0 \le \mathcal{A}_2 = g_2$$

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Negativity

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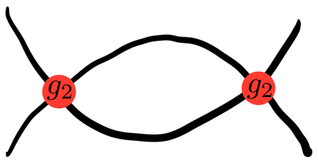

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- In amplitude: like running $g_4(s)$. In dispersion relation: more...

$$0 \leq \mathcal{A}_2 = g_2 + \frac{\beta_4}{2}s^2$$

$$0 \leq \mathcal{A}_4 = g_4(s)$$

$$0 \leq \mathcal{A}_6 = \frac{\beta_4}{2s^2} + g_6$$

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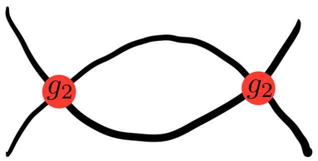

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- Negativity for irrelevant interactions
  (running vs irrelevant are indistinguishable)

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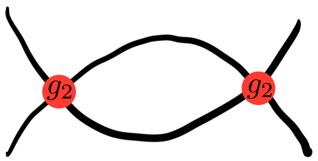
$\pi^0\pi^0 \to \pi^0\pi^0$

$$A_{2\to 2} = g_0 + g_2(s^2+t^2+u^2) - g_3 stu + g_4(s^2+t^2+u^2)^2 + \cdots$$



$\beta_4 = \frac{7}{10}\frac{g_2^2}{16\pi^2}$

$$-\frac{\beta_4}{2}s^2\left(s^2 - \frac{tu}{21}\right)\log(-s) + (s \leftrightarrow t, u) + \cdots$$

- In amplitude: like running $g_4(s)$. In dispersion relation: more...

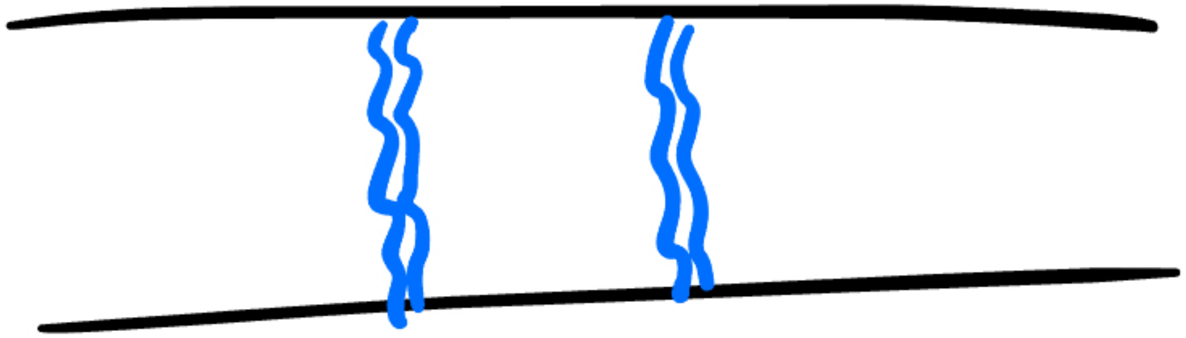
$$0 \le \mathcal{A}_2 = g_2 + \frac{\beta_4}{2}s^2 + \#\frac{g_2g_3}{16\pi^2}s^3 + \#\frac{g_2g_4}{16\pi^2}s^4 + \#\frac{g_2g_5}{16\pi^2}s^5 + \cdots$$

$$0 \le \mathcal{A}_4 = g_4(s) + \#\frac{g_2g_3}{16\pi^2}s + \#\frac{g_2g_4}{16\pi^2}s^2 + \#\frac{g_2g_5}{16\pi^2}s^3 + \cdots$$

$$0 \le \mathcal{A}_6 = \boxed{\frac{\beta_4}{2s^2}} + g_6 + \#\frac{g_2g_3}{16\pi^2 s} + \#\frac{g_2g_4}{16\pi^2} + \#\frac{g_2g_5}{16\pi^2}s + \cdots$$

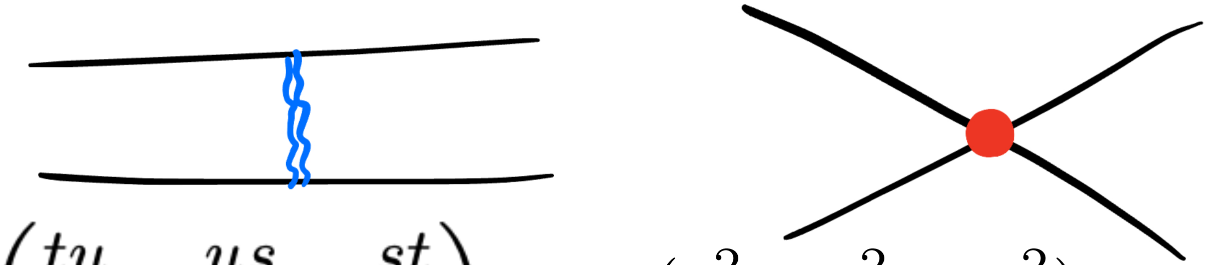
- Negativity for irrelevant interactions
(running vs irrelevant are indistinguishable)

- All coefficients enter! Must assume that effects not considered are small



Bounds in Gravity

Tree-level: $G_N \to 0$ $g_2, g_3, \cdots \to 0$ $\frac{g}{G_N} = finite$



$$A(s,t) = G_N\left(\frac{tu}{s} + \frac{us}{t} + \frac{st}{u}\right) + g_2(s^2 + t^2 + u^2) + g_3 stu + \cdots$$

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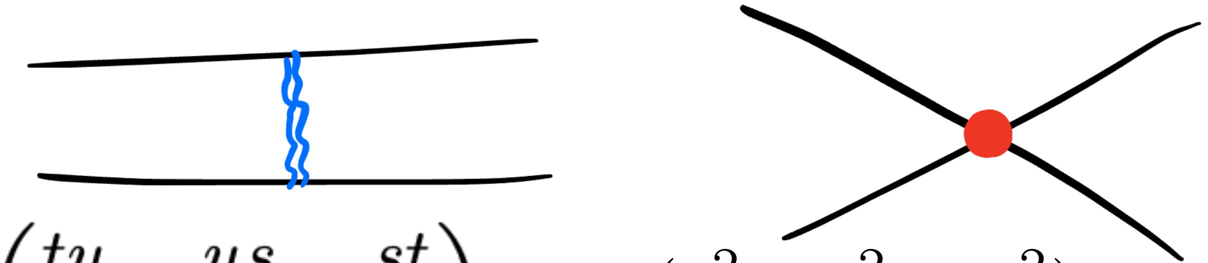

$$A(s,t) = G_N\left(\frac{tu}{s}+\frac{us}{t}+\frac{st}{u}\right) + g_2(s^2+t^2+u^2) + g_3 stu + \cdots$$

$$\int_{\cap_{\bar{s}_t}} \frac{ds}{\pi i}\frac{A(s,t)}{s^3} \qquad = \qquad \frac{2}{\pi}\int_{\bar{s}}^{\infty} ds \sum_{\ell} \mathrm{Im} f_\ell(s) \frac{P_\ell(1+\frac{2t}{s})}{s^3}$$

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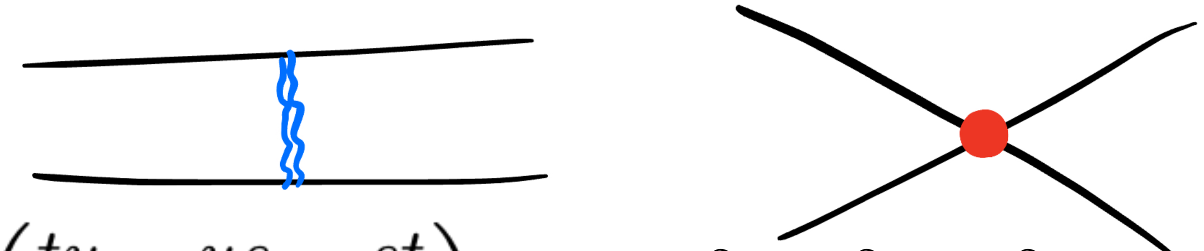

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When A not analytic, Partial wave expansion holds only distributionally

$$\int d\mu(t) \int_{\cap_{\bar{s}_t}} \frac{ds}{\pi i}\frac{A(s,t)}{s^3} = \int d\mu(t)\, \frac{2}{\pi}\int_{\bar{s}}^{\infty} ds \sum_{\ell} \mathrm{Im} f_\ell(s) \frac{P_\ell(1+\frac{2t}{s})}{s^3}$$

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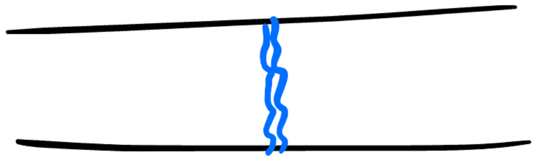

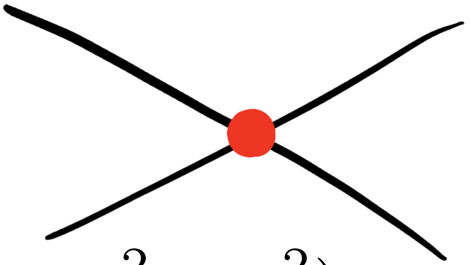

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Find $d\mu(t) = f(t)dt$ **so that** $> 0 \quad \forall \ell, s$

Caron-Huot,Mazac,Rastelli,Simons-Duffin'22

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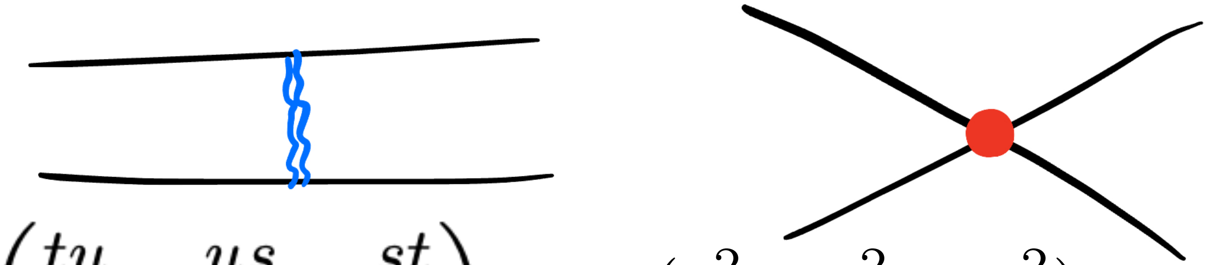

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Caron-Huot,Mazac,Rastelli,Simons-Duffin'22

$$\int d\mu(t)\left(-\frac{\kappa^2}{t} + g_2 + g_3 t + g_4 t^2 + \cdots\right) > 0$$

▶Positivity condition in IR

Bounds in Gravity

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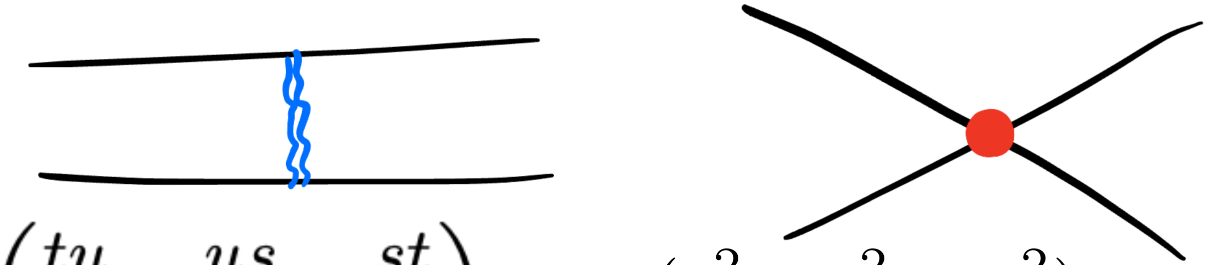

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Caron-Huot,Mazac,Rastelli,Simons-Duffin'22

Also works in gravity!

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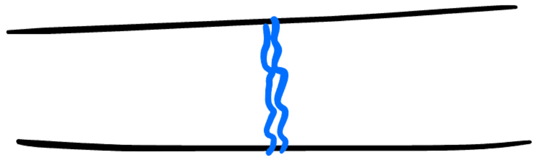

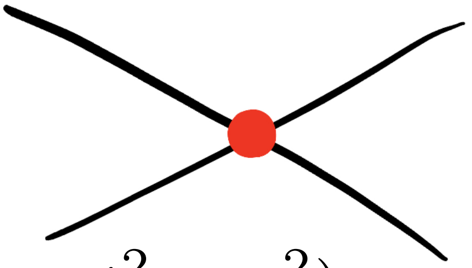

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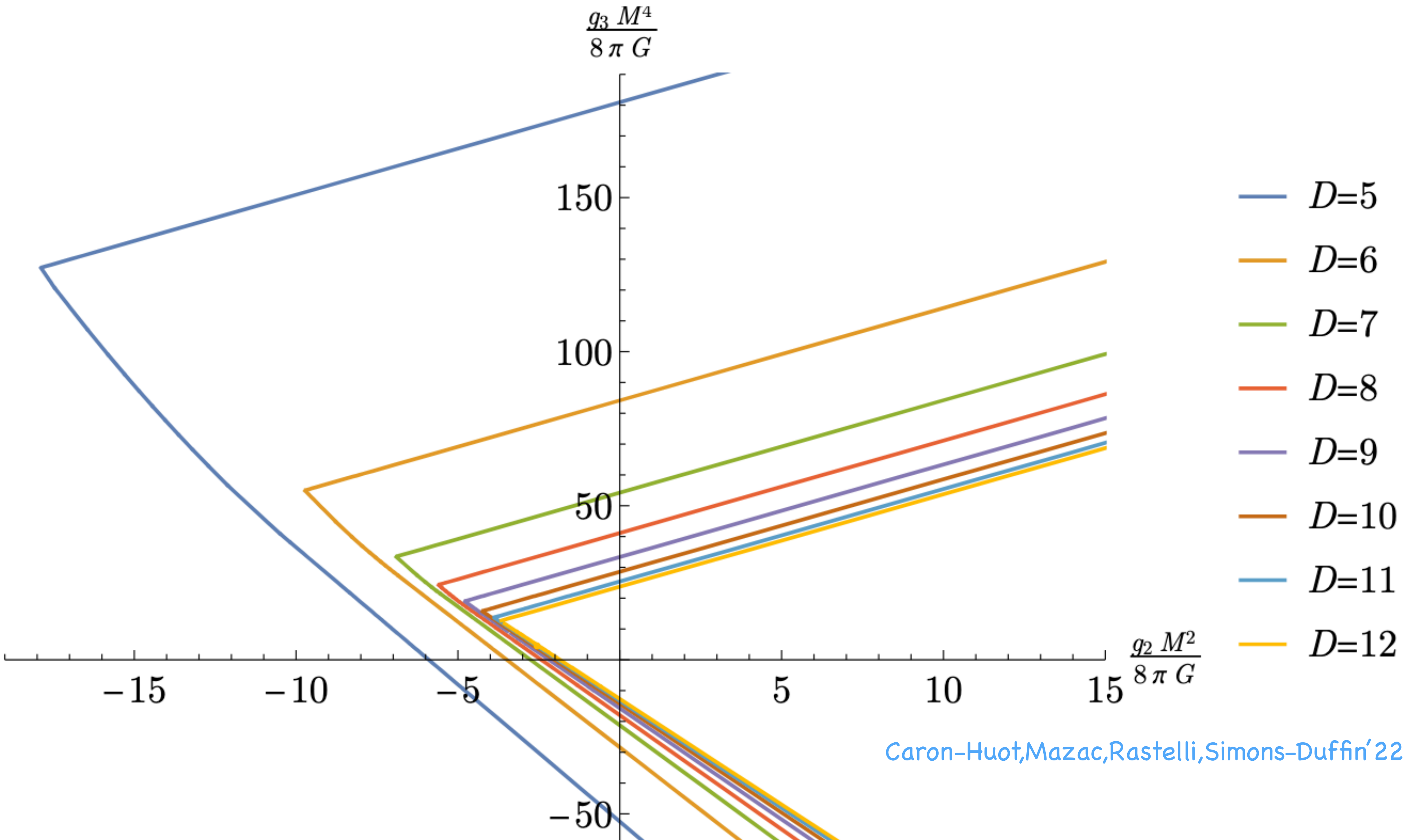

Caron-Huot,Mazac,Rastelli,Simons-Duffin'22

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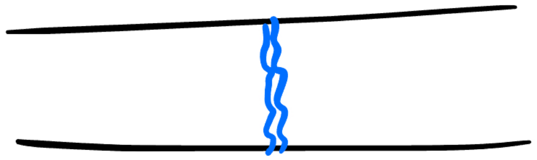

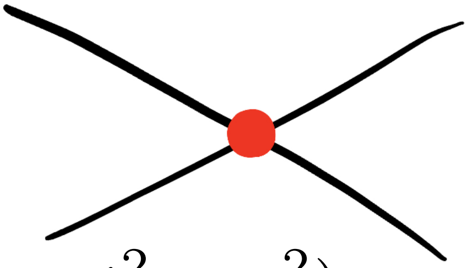

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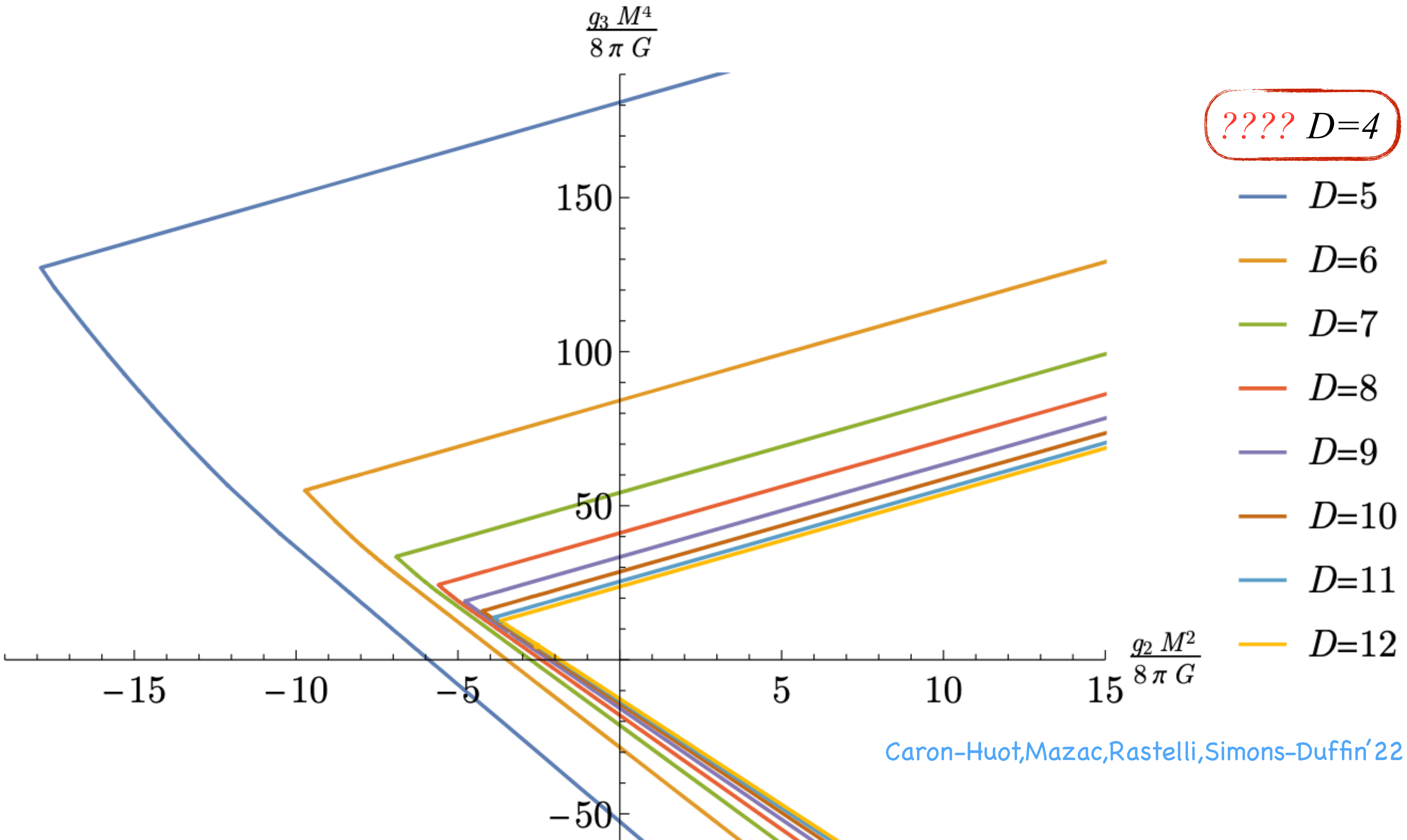
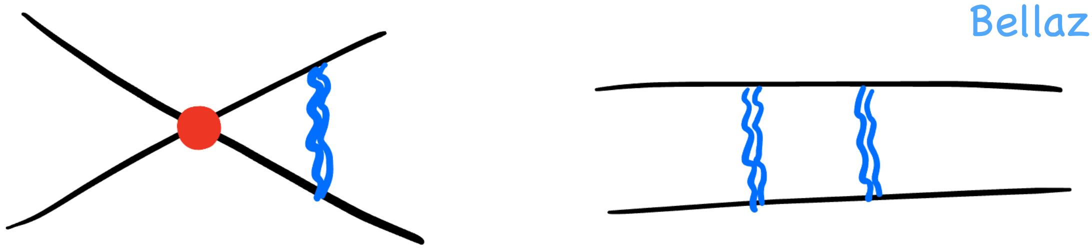


Caron-Huot,Mazac,Rastelli,Simons-Duffin'22

Bounds in Gravity

Real World d=4: $G \to 0\,,\ \ \mathcal{E} \to 0\,,\quad G_{\mathcal{E}} \equiv G M^2 \log M/\mathcal{E} = \text{fixed}$

Bellazzini,Berman,Isabella,FR,Romano,Sciotti'25



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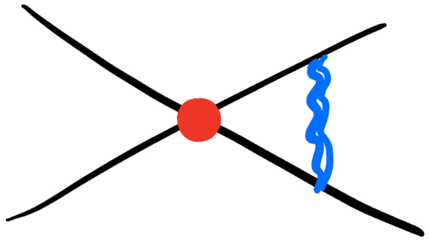

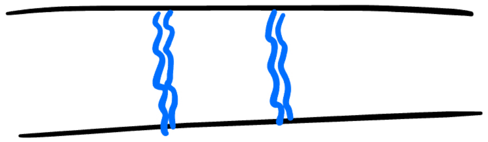

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Stripped Amplitude: $A_{\mathcal{E}} = \left.\frac{A}{\mathcal{W}_{GR}}\right|_{\epsilon\to 0}$

$$\mathcal{W}_{\rm GR}(s,t) = \exp\left[-G\frac{(\mathcal{E}/\mu)^{2\epsilon}}{\pi\epsilon}\left(s\log\left(-s/\mu^2\right) + t\log\left(-t/\mu^2\right) + u\log\left(-u/\mu^2\right)\right)\right]$$

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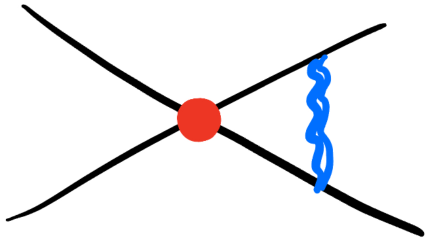

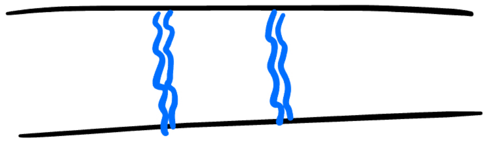

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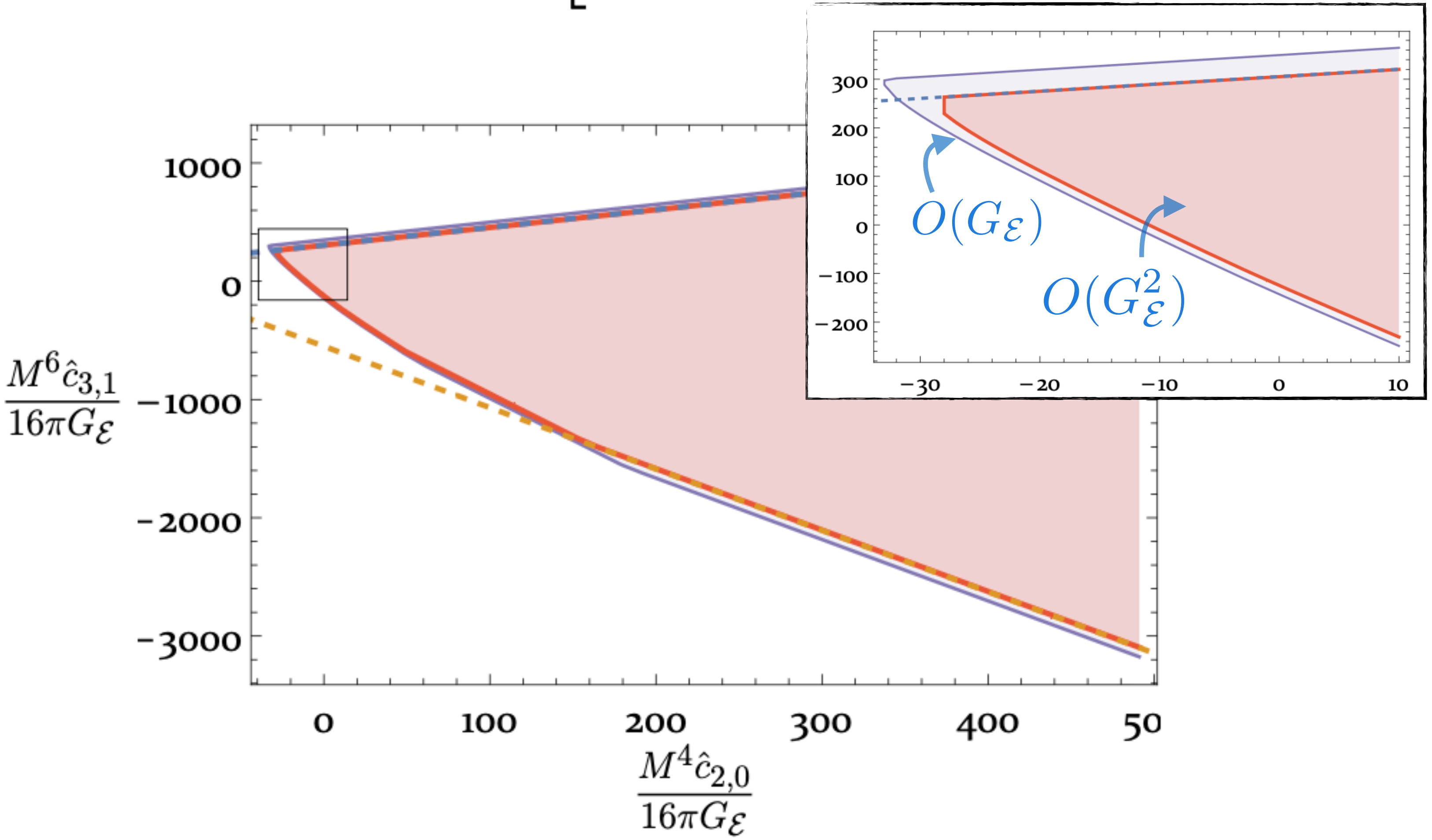
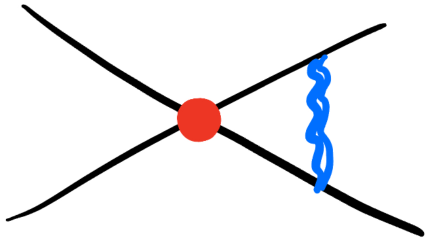
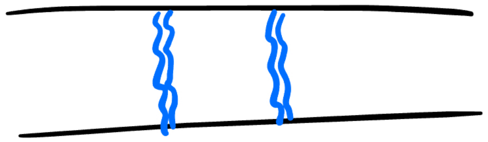


Bounds in Gravity

Unreal World d>4: $G_N = \kappa^2 = finite$ $\quad g_n = finite$

Beadle,Isabella,Perrone,Ricossa,FR,Serra'24-'25
Chang,Parra-Martinez'25





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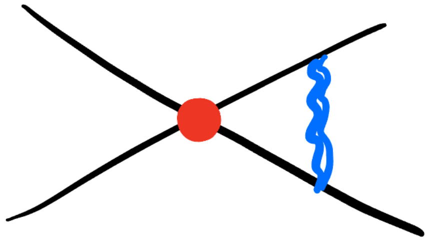

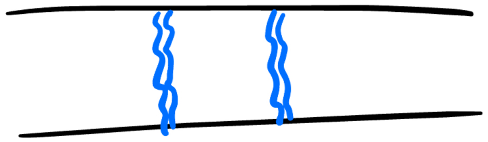

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$$loop \sim G_N^2\,(-t)^{\frac{d-6}{2}}\, s^3 \begin{cases}\log(-t) & d \text{ is even,}\\ 1 & d \text{ is odd}\end{cases}$$

Appears in all arcs

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Unreal World d>4: $G_N = \kappa^2 = finite$ $\quad g_n = finite$

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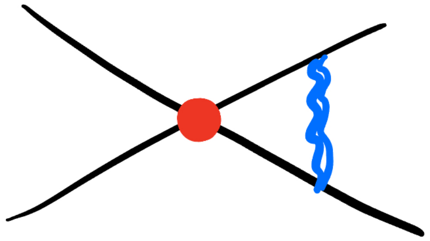

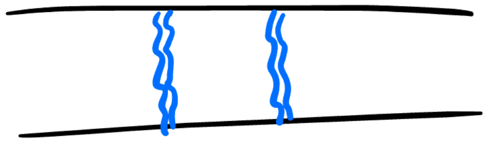

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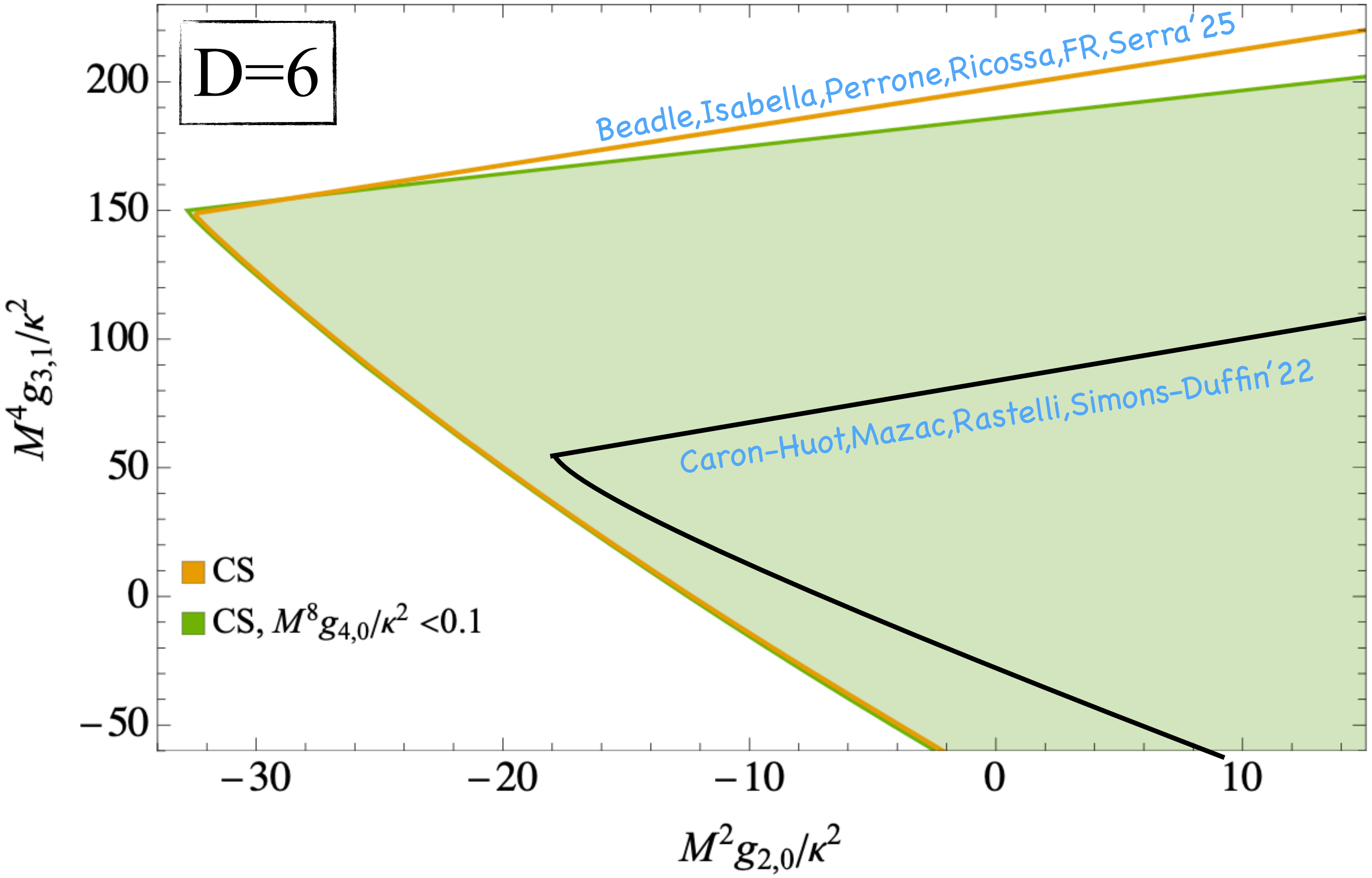

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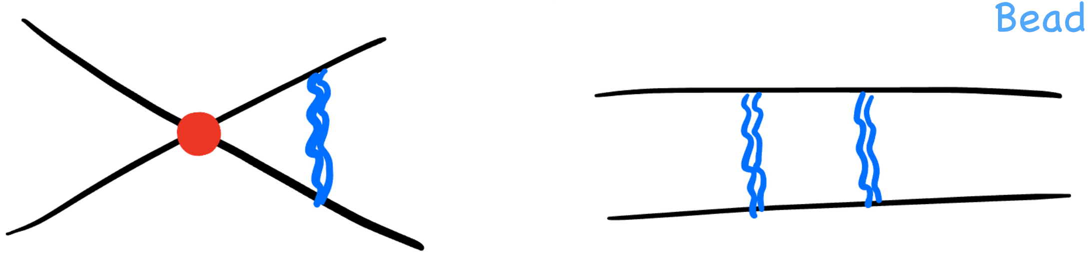

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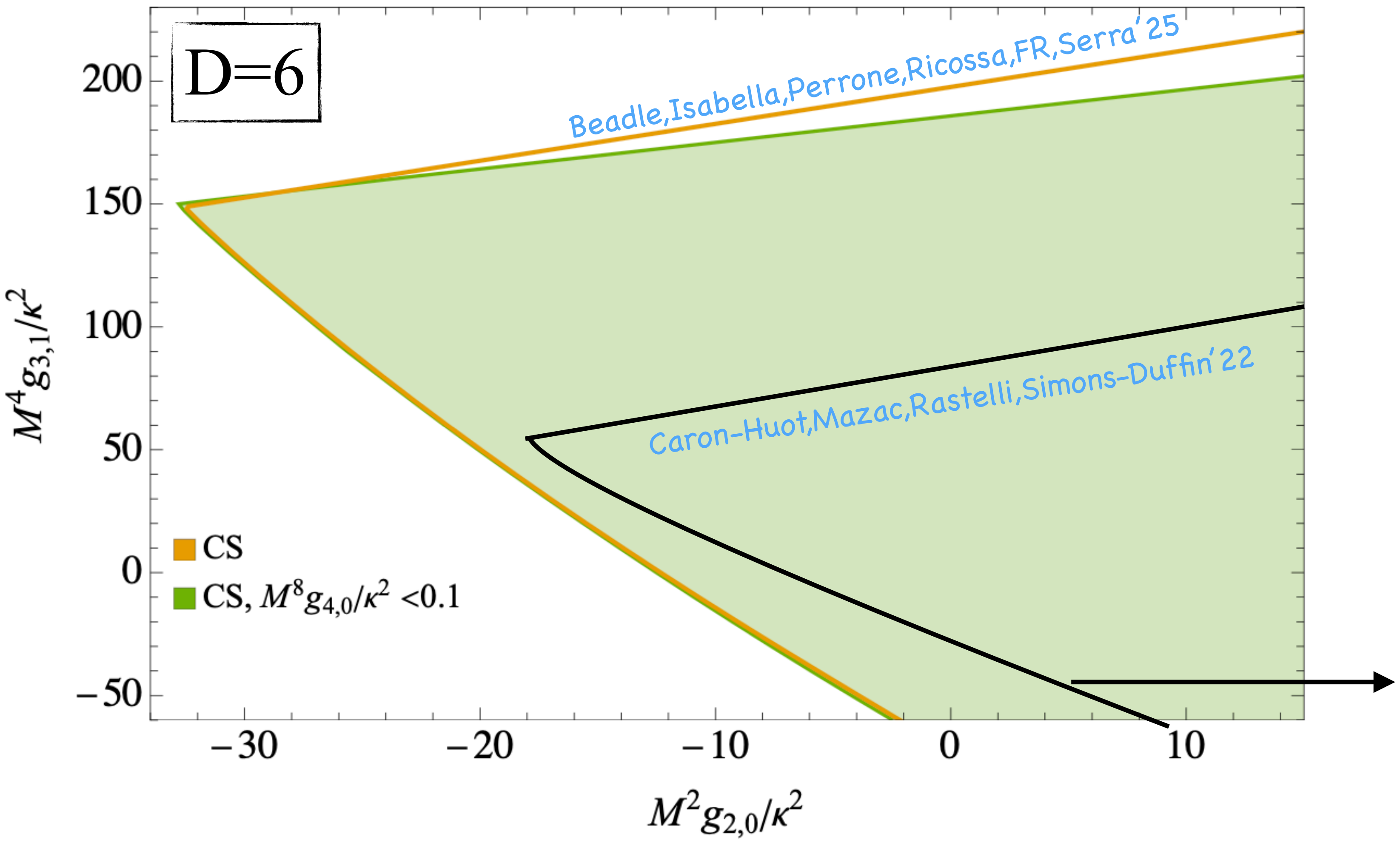


Beadle,Isabella,Perrone,Ricossa,FR,Serra'25

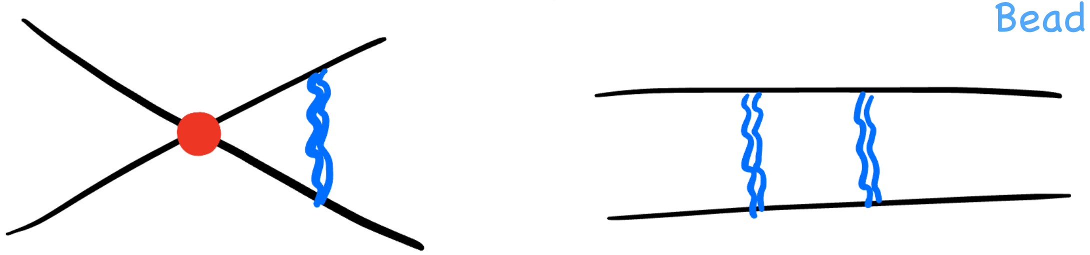
Caron-Huot,Mazac,Rastelli,Simons-Duffin'22

Size of loops along edge: ∞

Bounds in Gravity

Unreal World d>4: $G_N = \kappa^2 = finite \quad g_n = finite$

Beadle,Isabella,Perrone,Ricossa,FR,Serra'24-'25
Chang,Parra-Martinez'25



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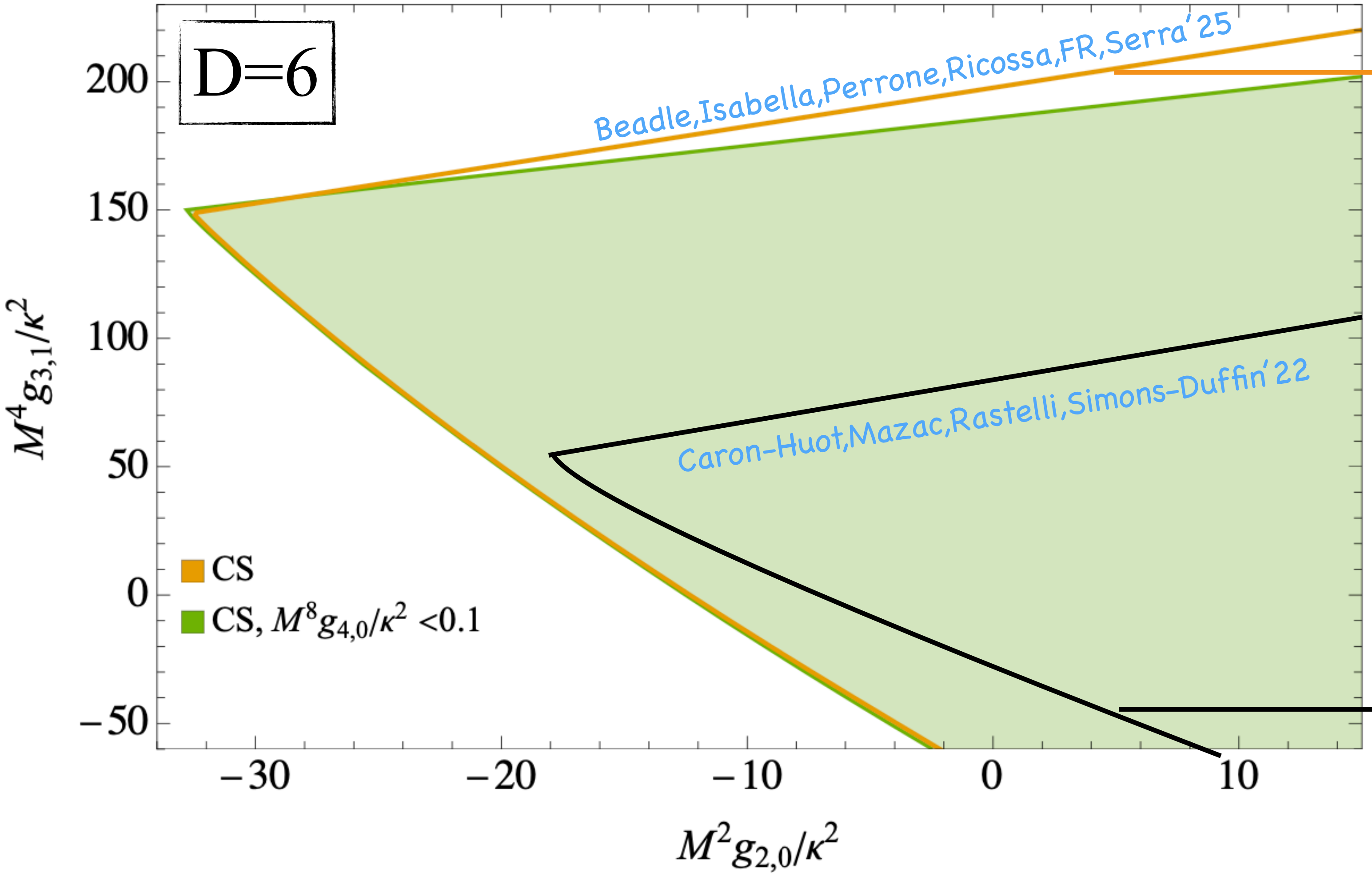
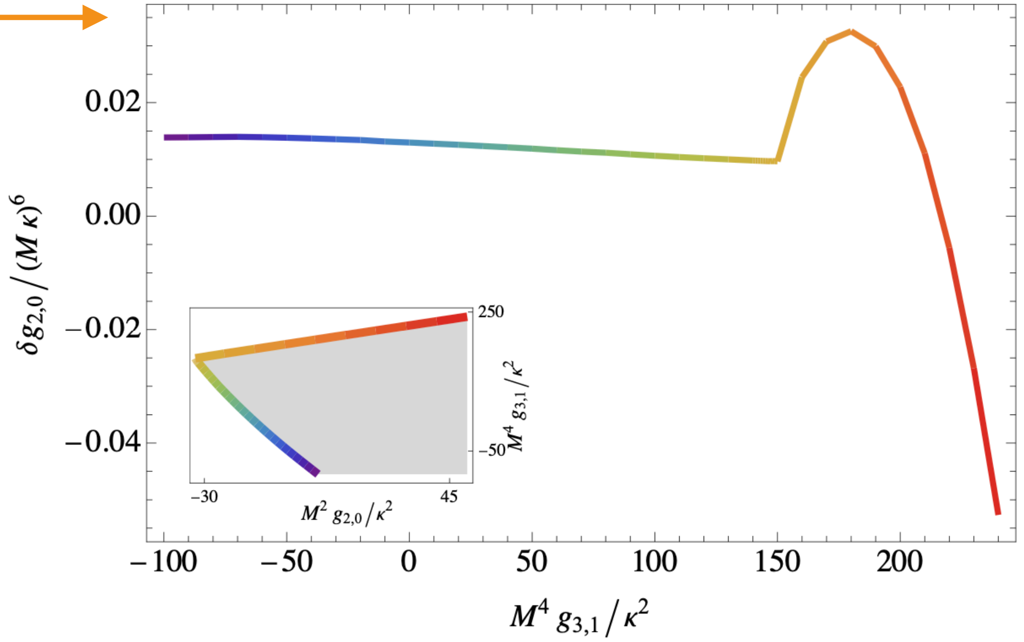


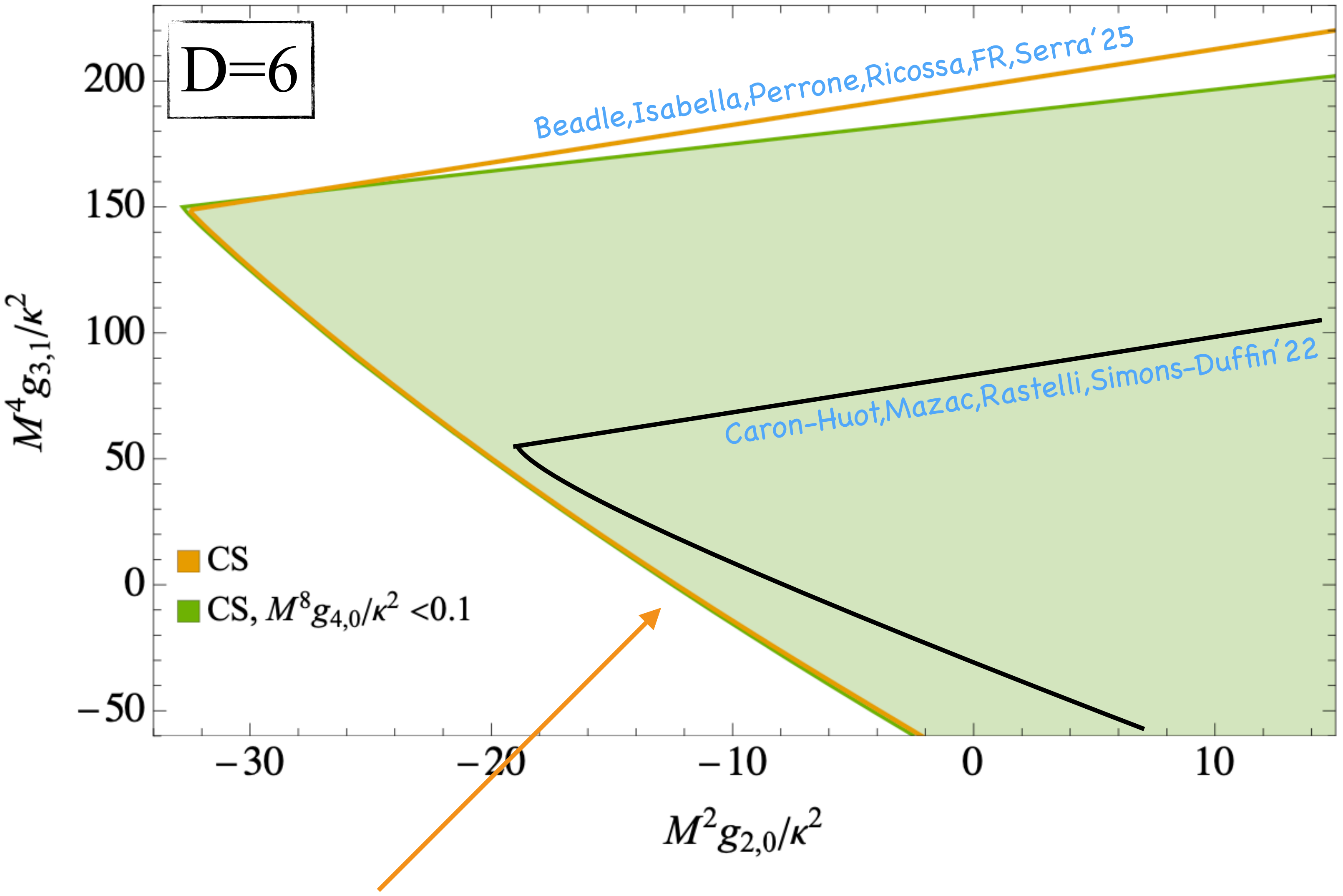
Size of loops along edge:



Size of loops along edge: ∞

"The Path is the Goal"

- Positivity bounds for which loop effects are finite, have a different shape even at tree-level!



Tree-level bounds, consistent in loop perturbation theory

"The Path is the Goal"

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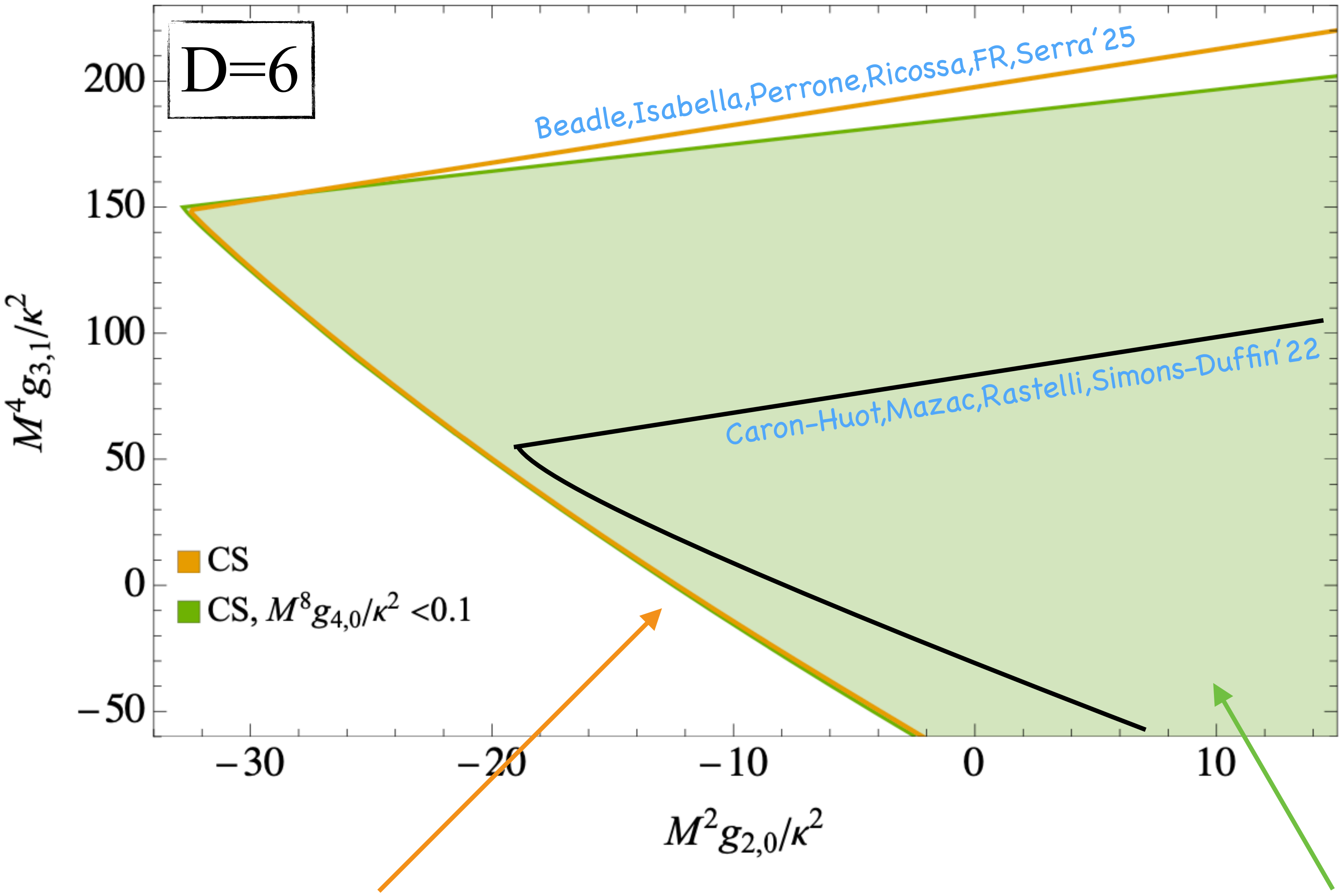
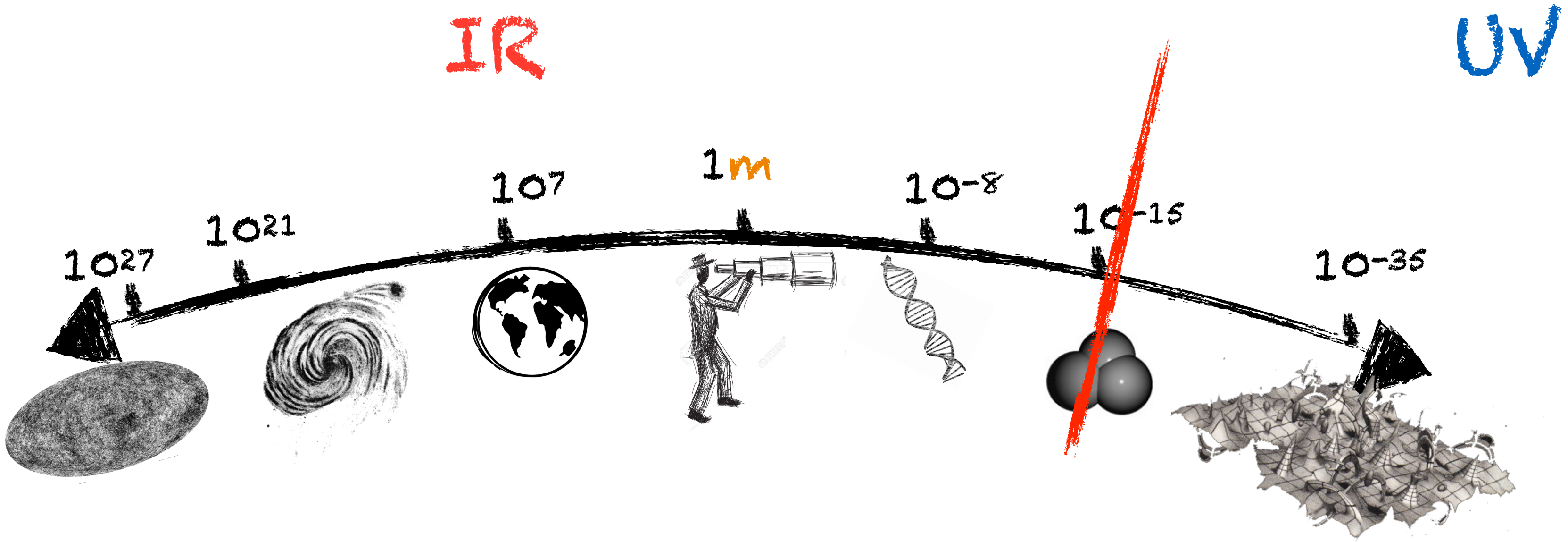


Tree-level bounds, consistent in loop perturbation theory

Tree-level bounds, consistent in loop and EFT perturbation theory

Summary



Causality
Unitarity

Constrained EFTs ←

- With appropriate Dispersion Relations, Positivity bounds are modified only perturbatively by loop effects – as it should be
- With long-Range interactions, physical experiment size enters results
- Bounds on very irrelevant operators washed away. Assumptions needed.
- IR safe bounds might differ from naive tree-level

Improvement

- Combination of arcs can have finite sums and circumvent Münts-Szàsz:

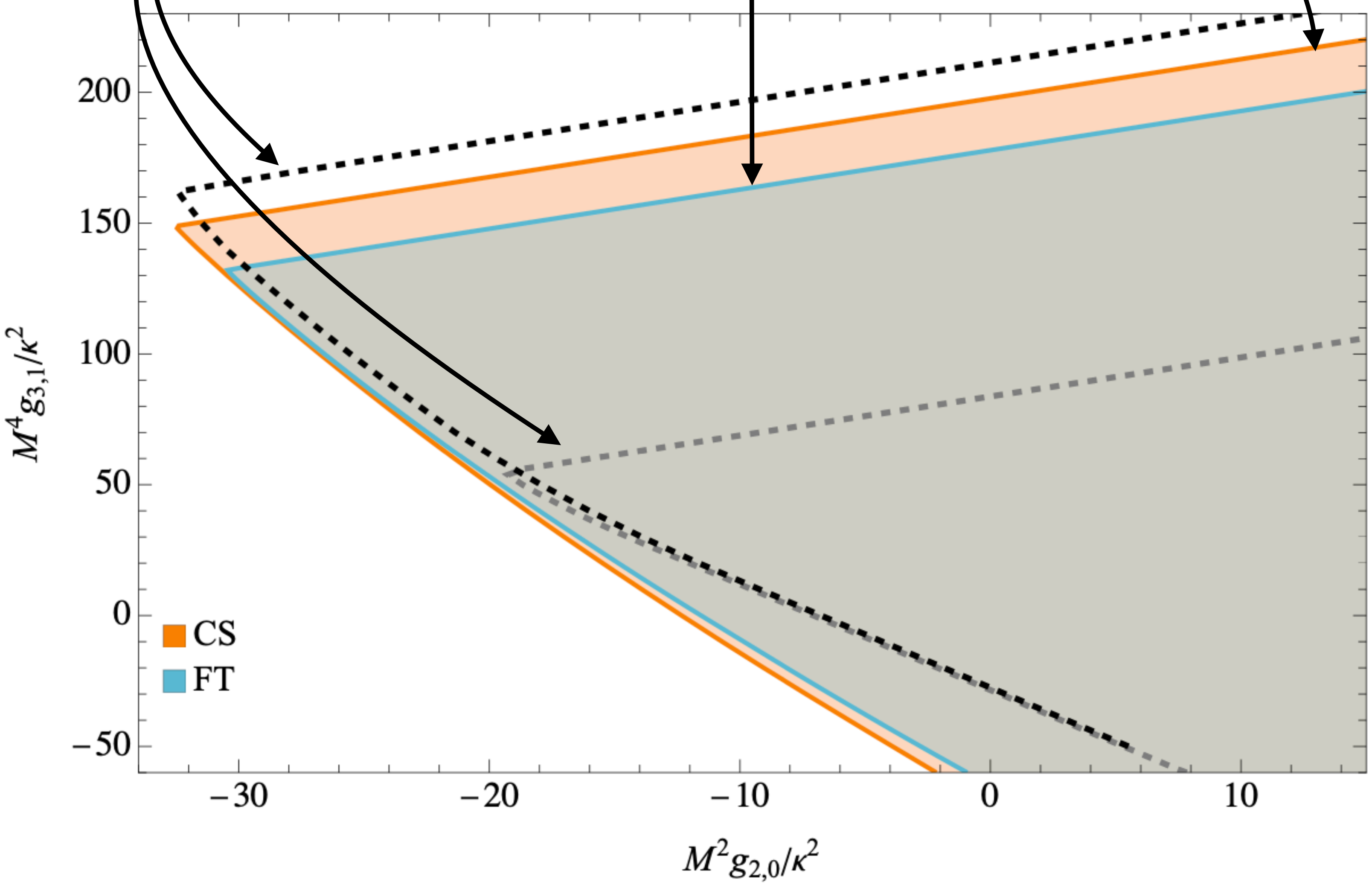
$$\int d\mu(t)\left(-\frac{\kappa^2}{t}+g_2+g_3 t\right) > 0$$

Caron-Huot,Mazac,Rastelli,Simons-Duffin'22

Beadle,Isabella,Perrone,Ricossa,FR,Serra'24

Beadle,Isabella,Perrone,Ricossa,FR,Serra'25
Chang,Parra-Martinez'25

$|t| \leq t_{\max} \equiv M^2\frac{(\sqrt{17}-1)}{8}$



Improvement

- Combination of arcs can have finite sums and circumvent Müntz-Szàsz:

$$\int d\mu(t)\left(-\frac{\kappa^2}{t}+g_2+g_3t\right)>0$$

Caron-Huot,Mazac,Rastelli,Simons-Duffin'22

Beadle,Isabella,Perrone,Ricossa,FR,Serra'24

Beadle,Isabella,Perrone,Ricossa,FR,Serra'25
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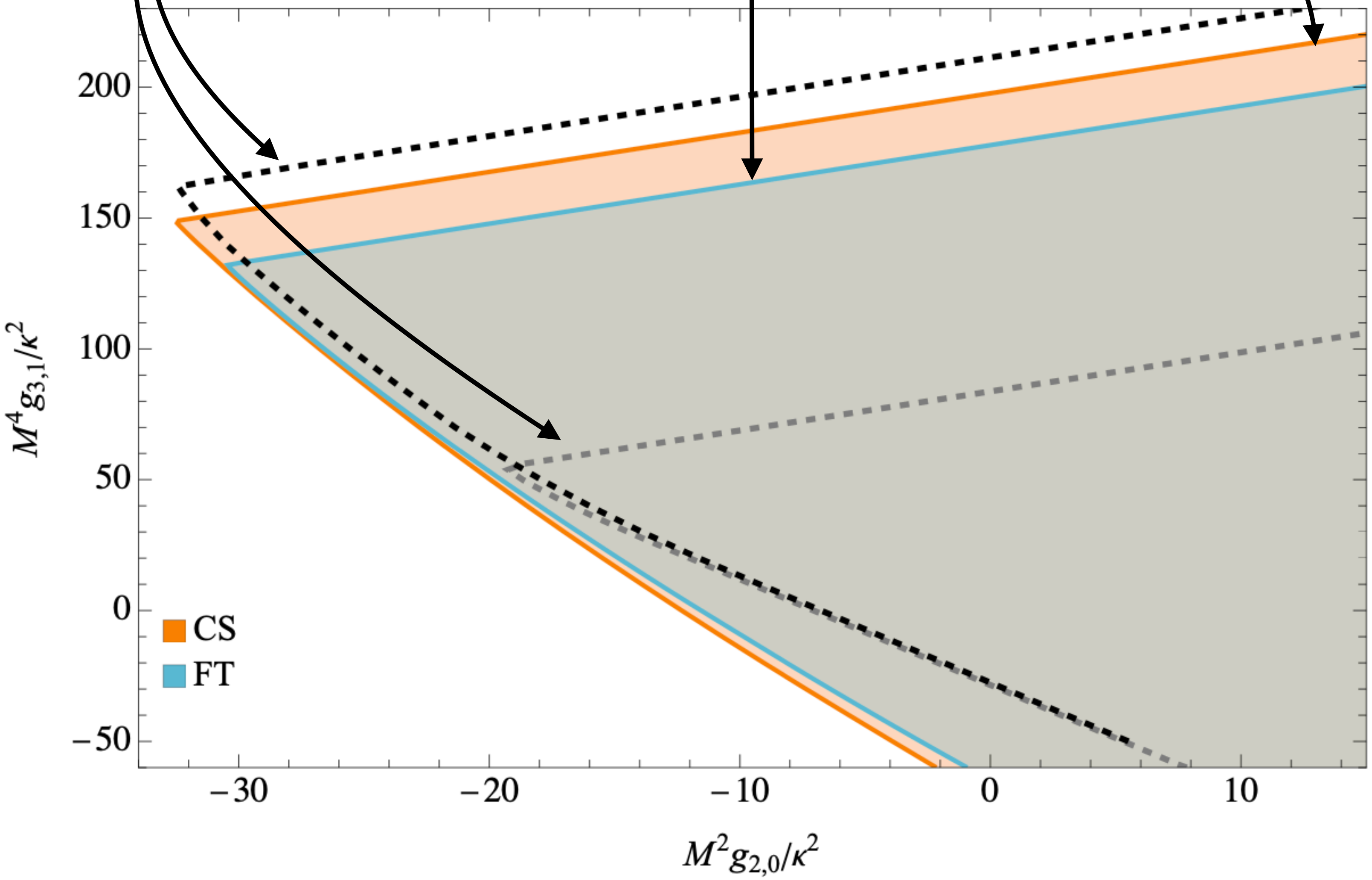
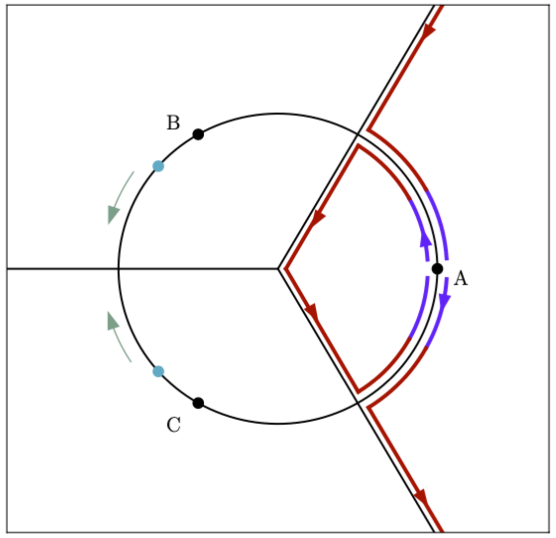
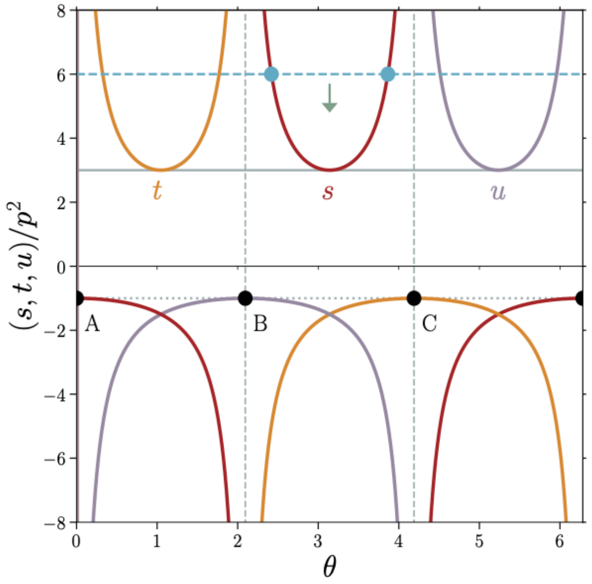


CSDR

$$\frac{2stu}{(s^2+t^2+u^2)} = p^2 \qquad p^2 \leq \frac{M^2}{3}$$

$$s(z,p) = -\frac{3p^2 z}{1+z+z^2}\,, \quad t(z,p) = s(z\,\xi,p)\,, \quad u(z,p) = s(z\,\xi^2,p)\,,$$





$$a_n^{CS}(p) = \int_{M^2}^{\infty} \frac{ds}{2\pi} s^{-3n-4}\left(3p^2+2s\right)\left(p^2+s\right)^n \mathrm{Disc}\mathcal{M}(s,p^2)\,.$$

$$\frac{\kappa^2}{p^2} + g_{2,0} + g_{3,1}p^2 = \left\langle (2s+3p^2)\mathcal{P}_\ell\left(\sqrt{\frac{s-3p^2}{s+p^2}}\right)\right\rangle_{CS}\,.$$

Bounds ↔ Positive Polynomials

Bellazzini,Elias-Miro,Rattazzi,Riembau,FR'20

Bounds on EFT Coeff. ↔ Bounds on Moments ↔ Positive polynomials in [0,1]

e.g. $\int_0^1 d\mu(x)\ 1-x>0 \qquad \Rightarrow \qquad \mathcal{A}_0 - s^2\mathcal{A}_1 > 0$

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$$p = x^n(1-x)^m$$

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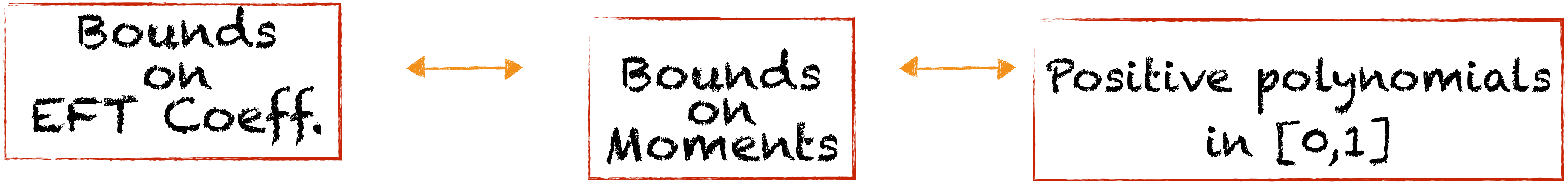

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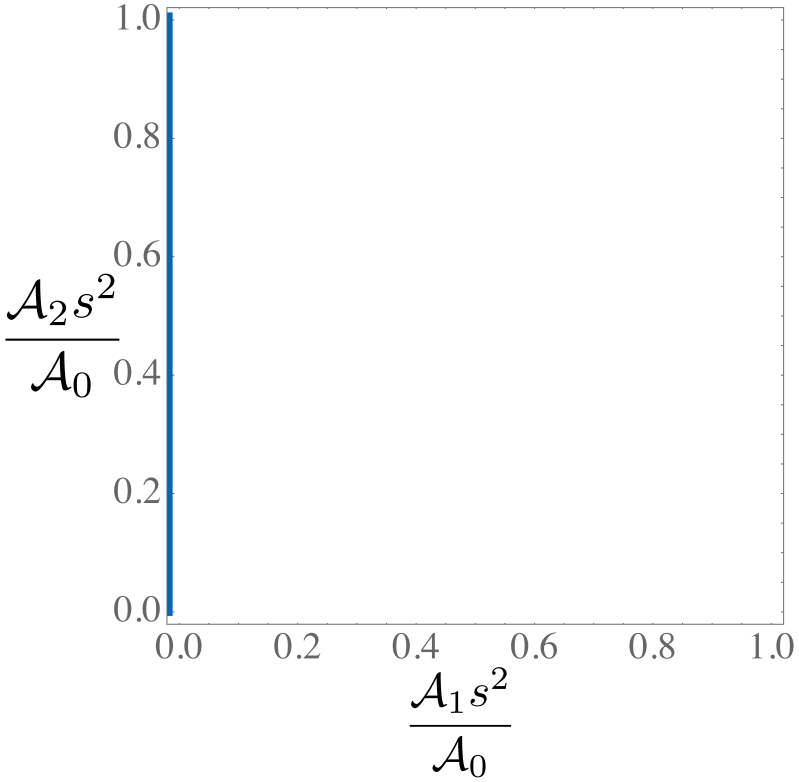

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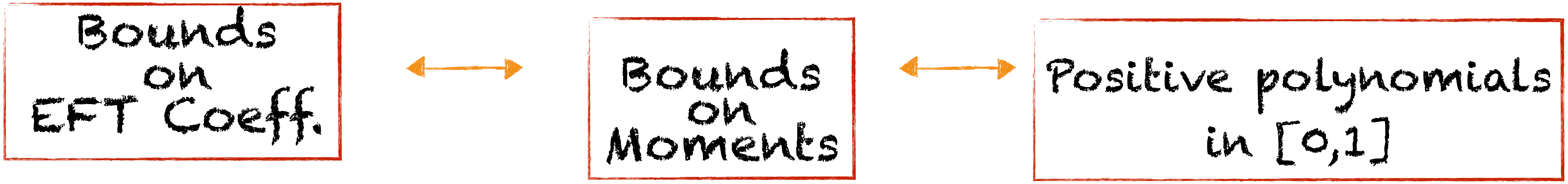

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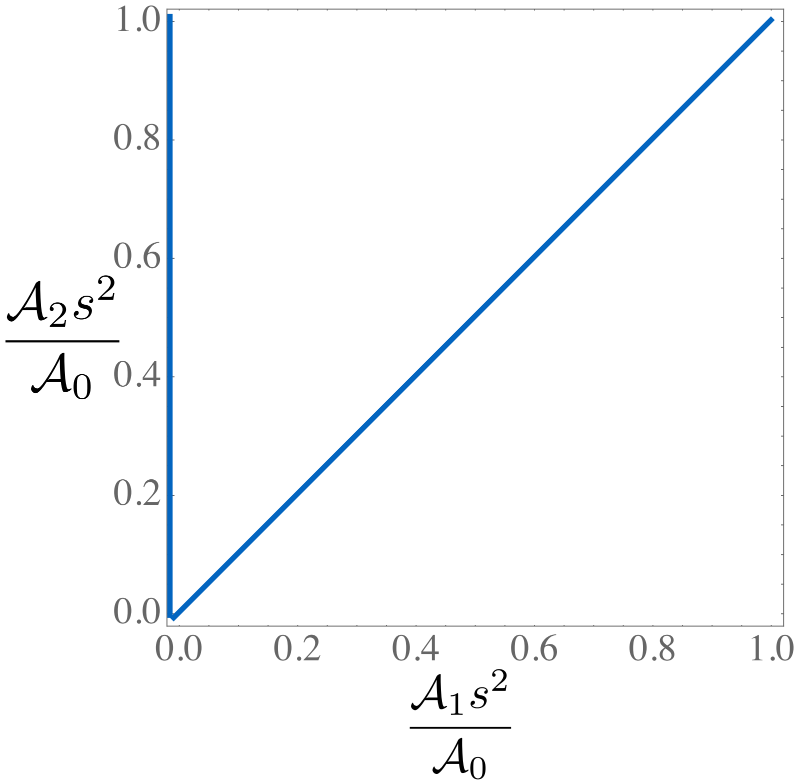

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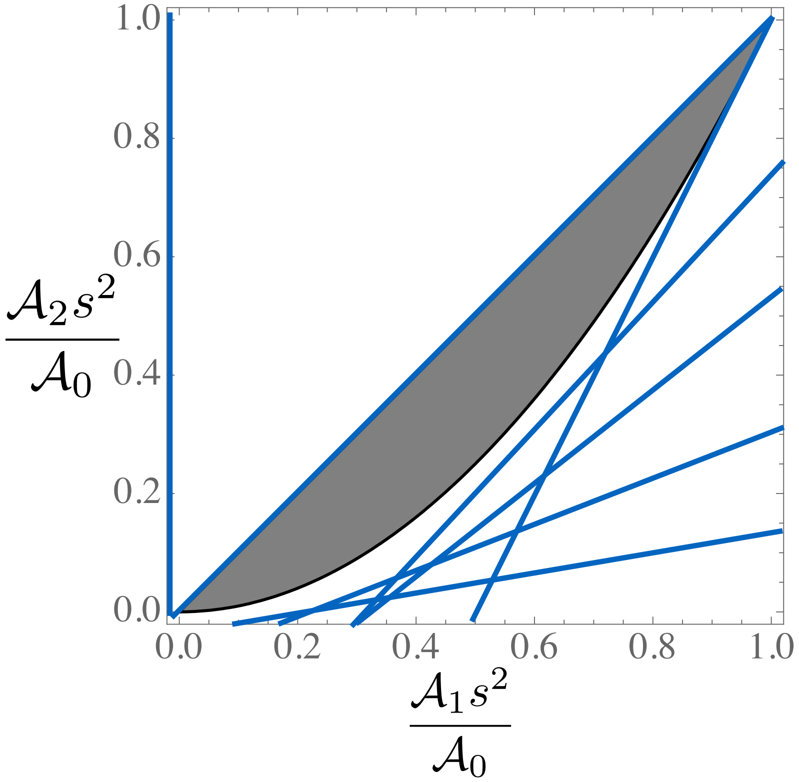

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"Sum of Squares" representation

$$p = q_1^2 + x q_2^2 + (1-x)q_3^2 + x(1-x)q_4^2$$

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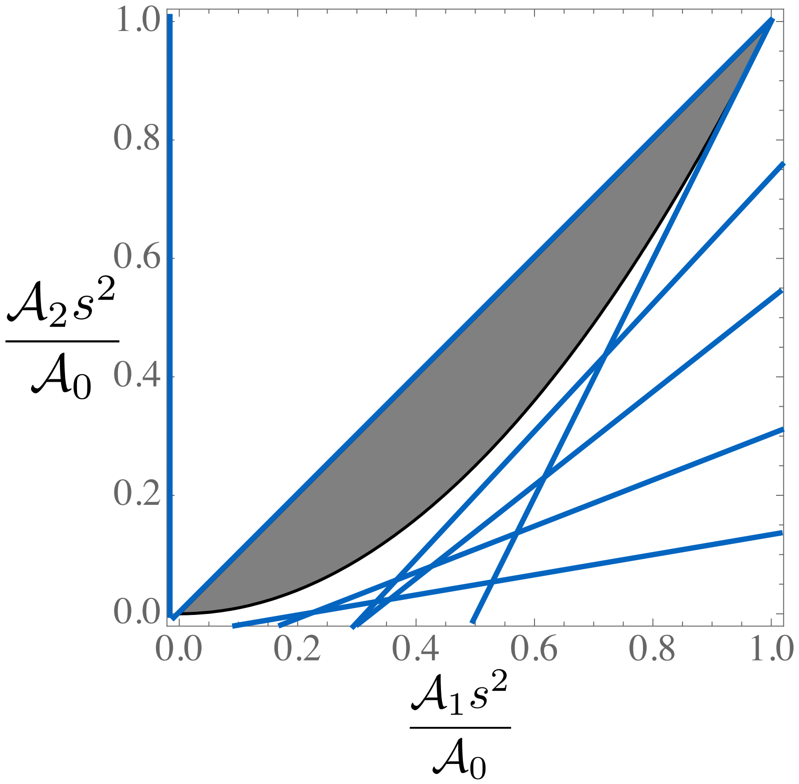

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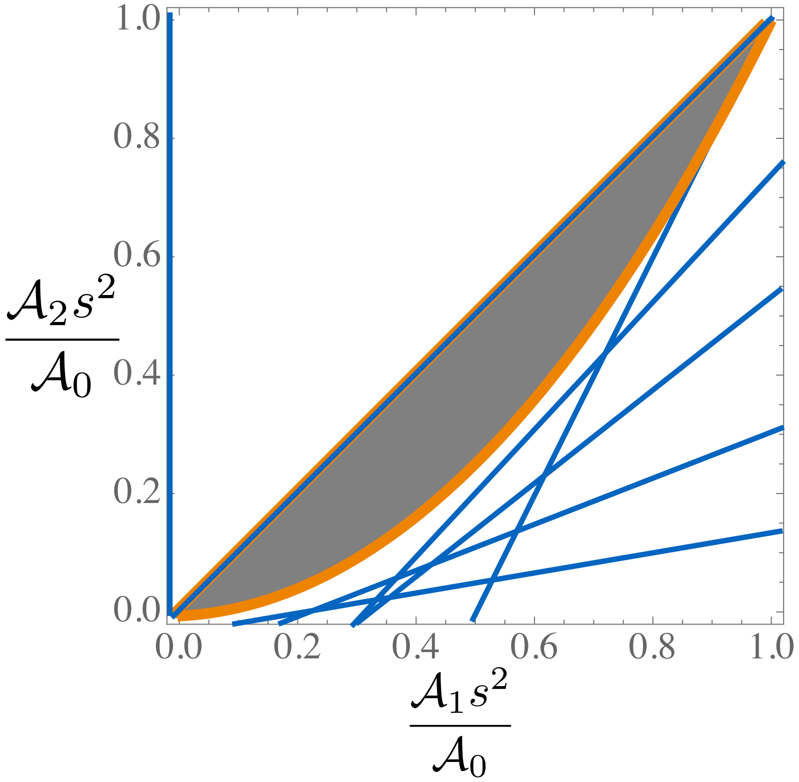

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$$0 < \int_0^1 q_1(x)^2 d\mu(x) = \sum_{i,j=0}^{N/2} \alpha_i\, \mathcal{A}_{i+j}\, \alpha_j \qquad q_1^2 = \sum \alpha_i x^i$$

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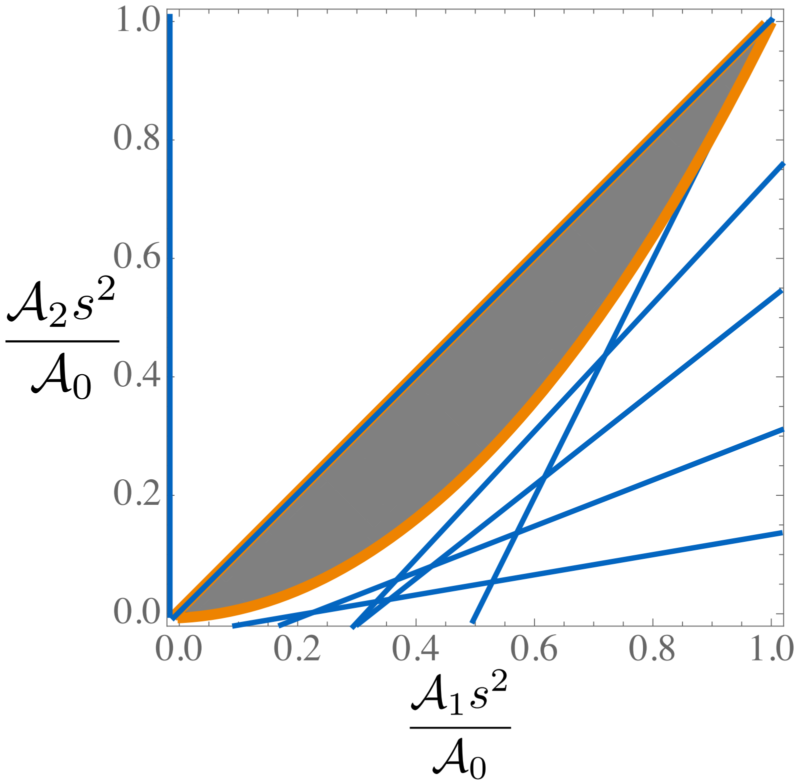

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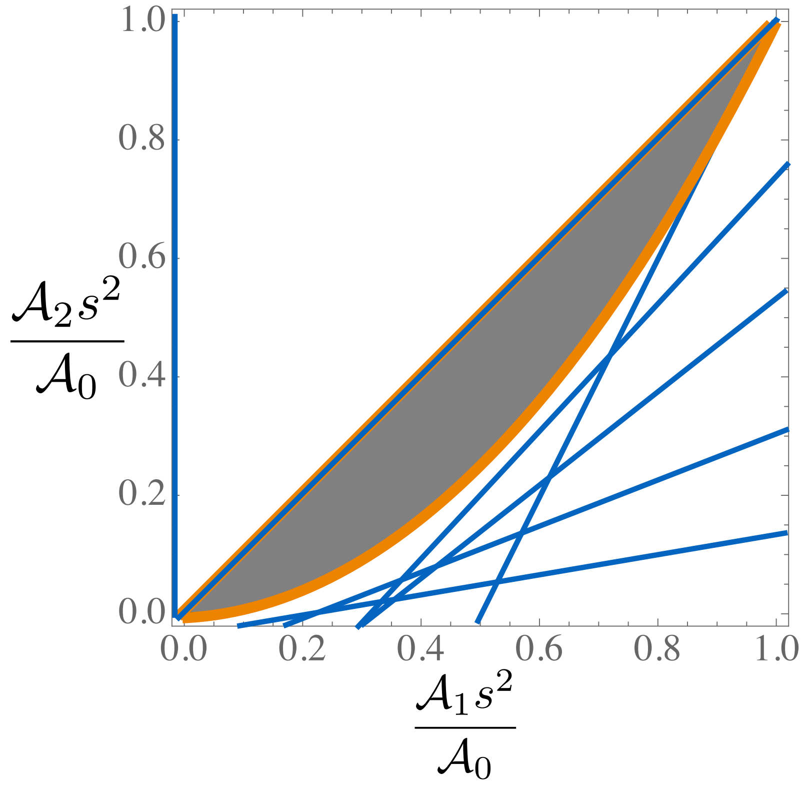

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Positive Definite Hankel Matrix

$$H_N^0 \succ 0$$
$$H_N^1 \succ 0$$
$$H_{N-1}^0 - \hat{s}^2 H_N^1 \succ 0$$
$$H_{N-1}^1 - \hat{s}^2 H_N^2 \succ 0$$

e.g. $H_4^0 \equiv \begin{pmatrix} \mathcal{A}_0 & \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{A}_3 \\ \mathcal{A}_2 & \mathcal{A}_3 & \mathcal{A}_4 \end{pmatrix}$

Bounds ↔ Positive Polynomials

Bellazzini,Elias-Miro,Rattazzi,Riembau,FR'20

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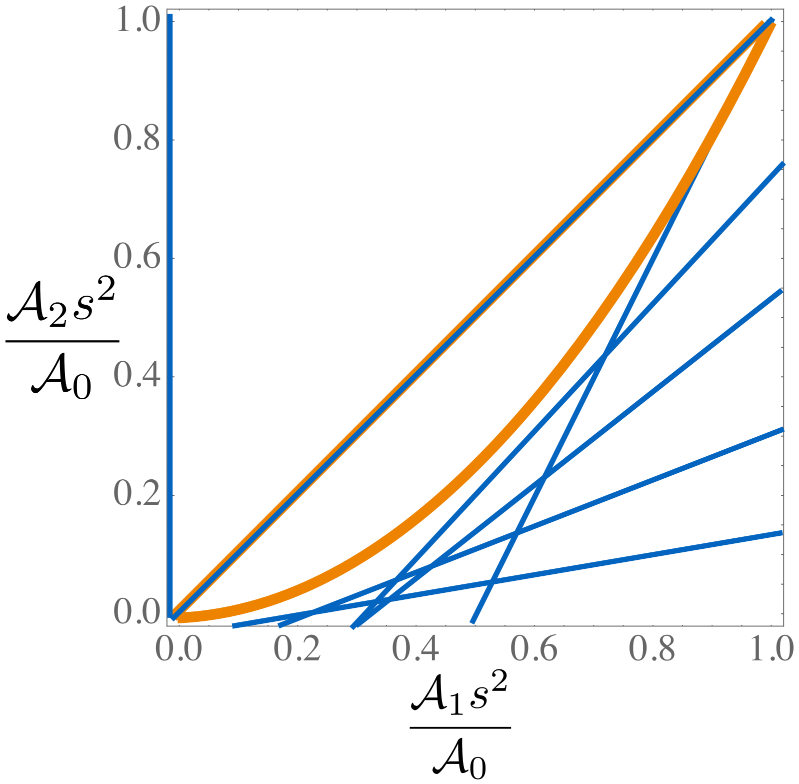
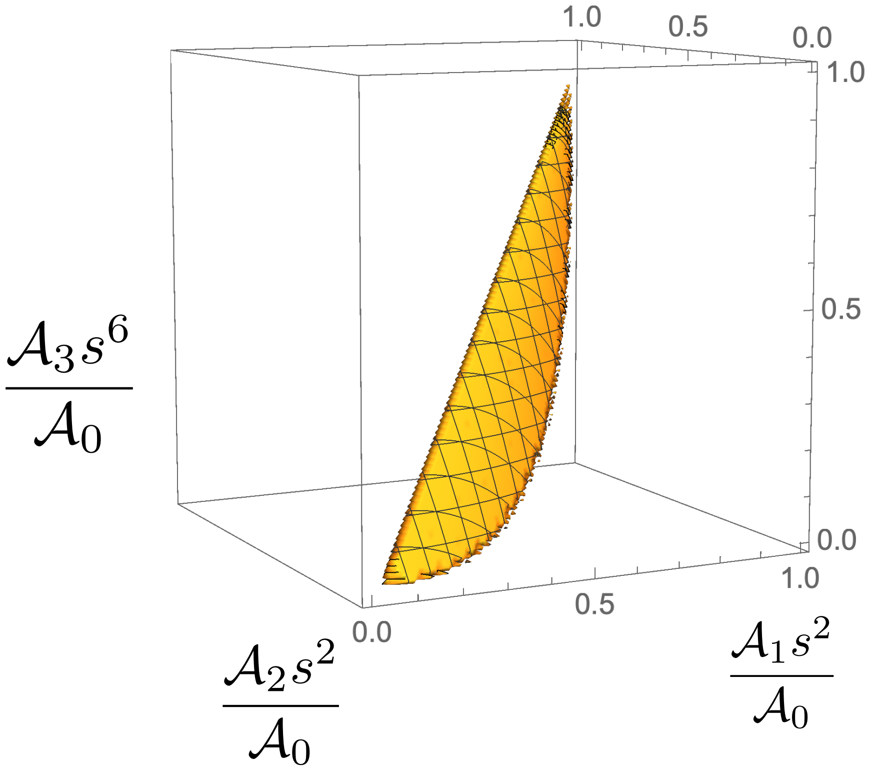


$$\begin{aligned} H_N^0 &\succ 0 \\ H_N^1 &\succ 0 \\ H_{N-1}^0 - \hat{s}^2 H_N^1 &\succ 0 \\ H_{N-1}^1 - \hat{s}^2 H_N^2 &\succ 0 \end{aligned}$$

...up to 4 arcs...



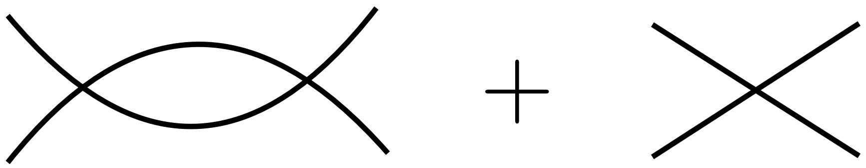
3. IR Divergences in massless theories

Bellazzini,Riembau,FR ’21

EFTs are Weakly Coupled $\neq$ Tree-Level: $\partial_t|_{t=0}\text{disp. rel.}$

always assumed in literature

Caron-Huot,van’Duong’20



$$= g_2 s^2 + g_3 s^2 t \; + \cdots \; \frac{g_2^2}{16\pi^2} s^2 t^2 \log \frac{t}{s}$$

- Dispersion relations at finite-t:

$$\mathcal{A}_0(t) \equiv \int_{\cap_{\bar{s}_t}} \frac{ds}{\pi i} \frac{A(s,t)}{(s+\frac{t}{2})^3} = \frac{2}{\pi} \int_{\bar{s}}^{\infty} ds \sum_{\ell} \mathrm{Im} f_\ell(s) \frac{P_\ell(1+\frac{2t}{s})}{(s+\frac{t}{2})^3}$$

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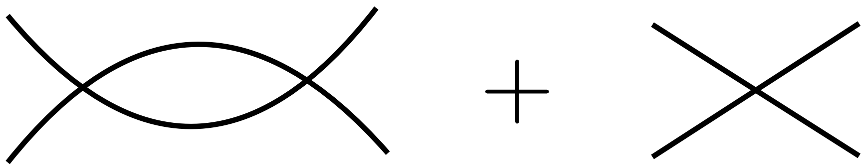

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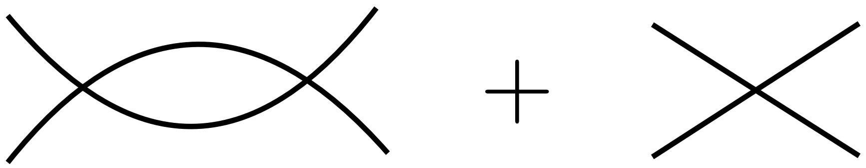

$$+ \qquad = g_2 s^2 + g_3 s^2 t + \cdots \quad \frac{g_2^2}{16\pi^2} s^2 t^2 \log\frac{t}{s}$$

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Not Positive ☹

Positive

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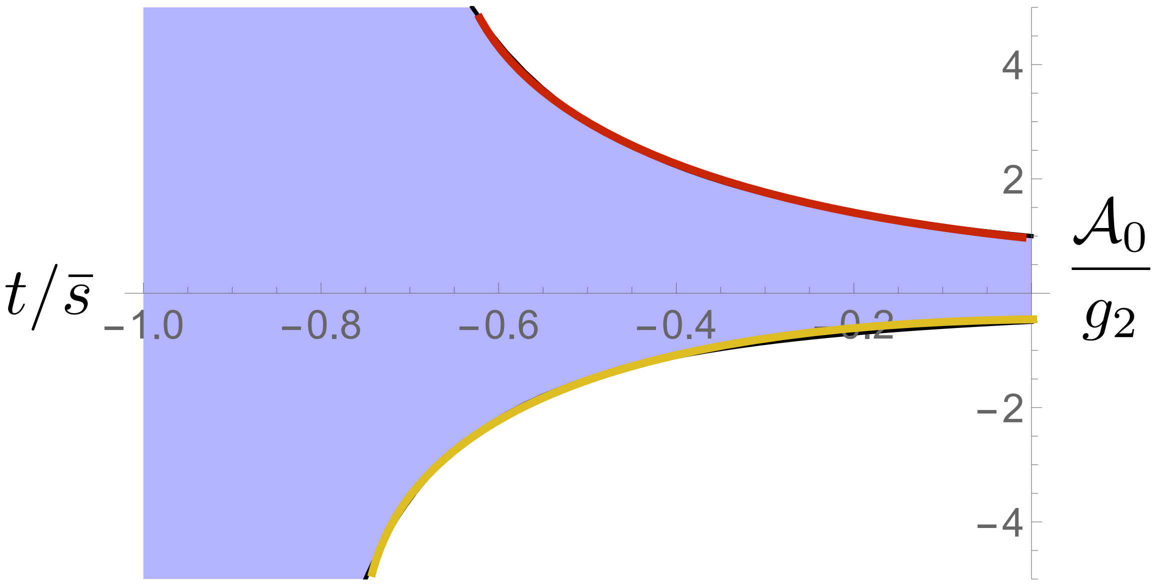
...but bounded!

$-1/2 < \bullet < 1$



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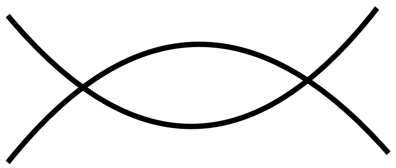

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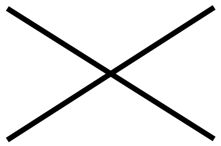

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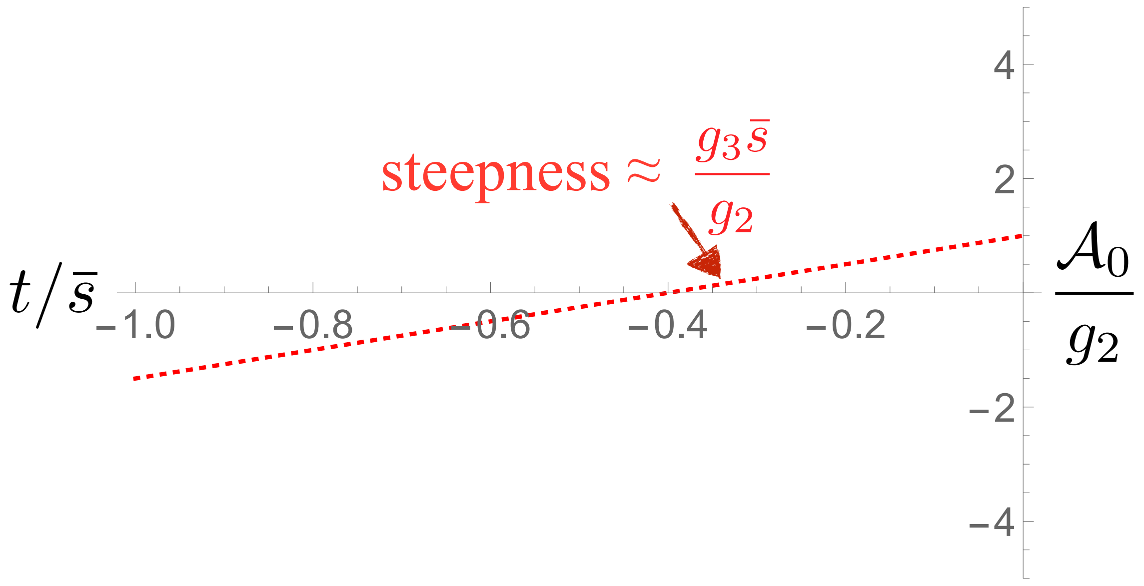

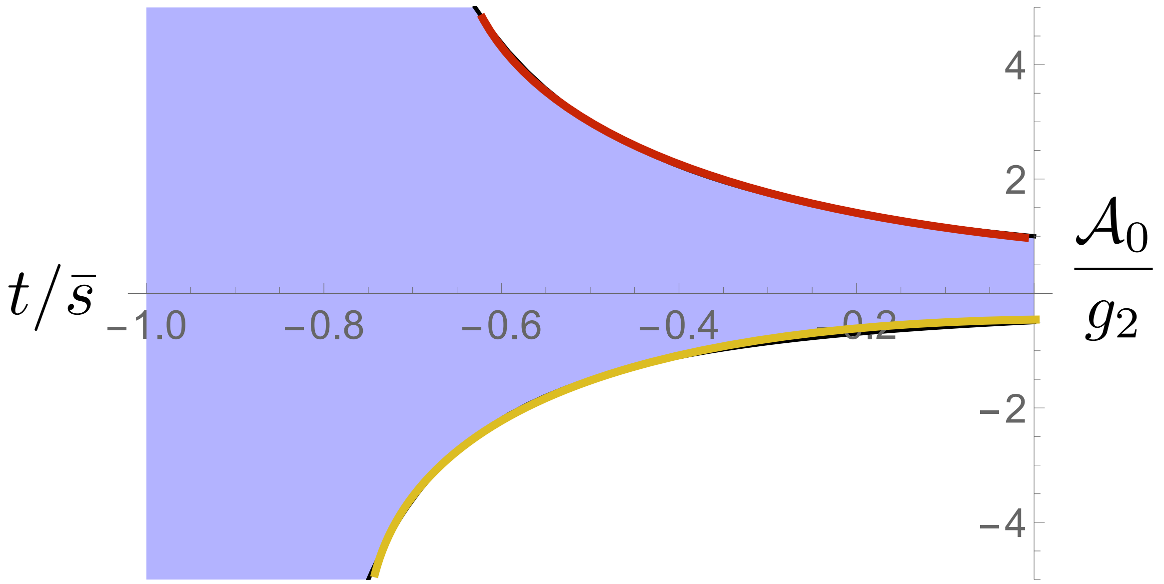

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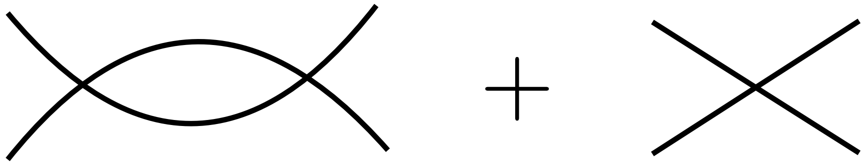

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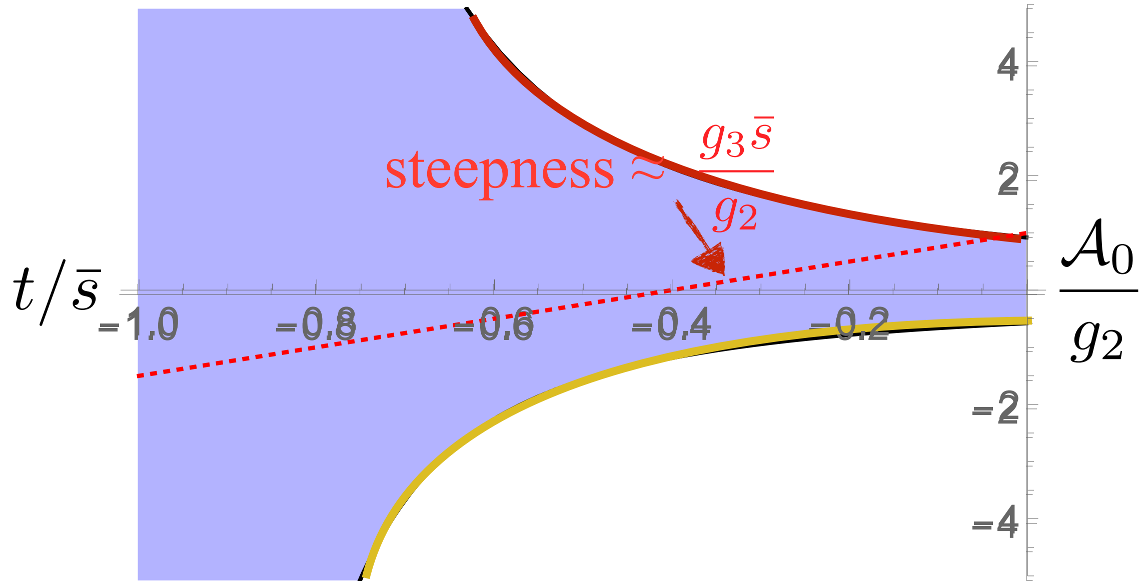

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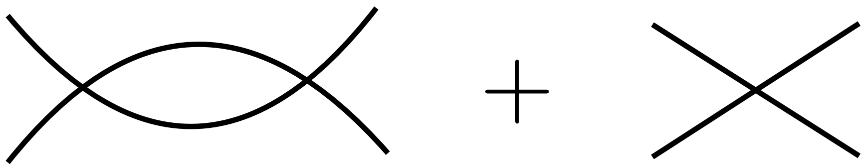

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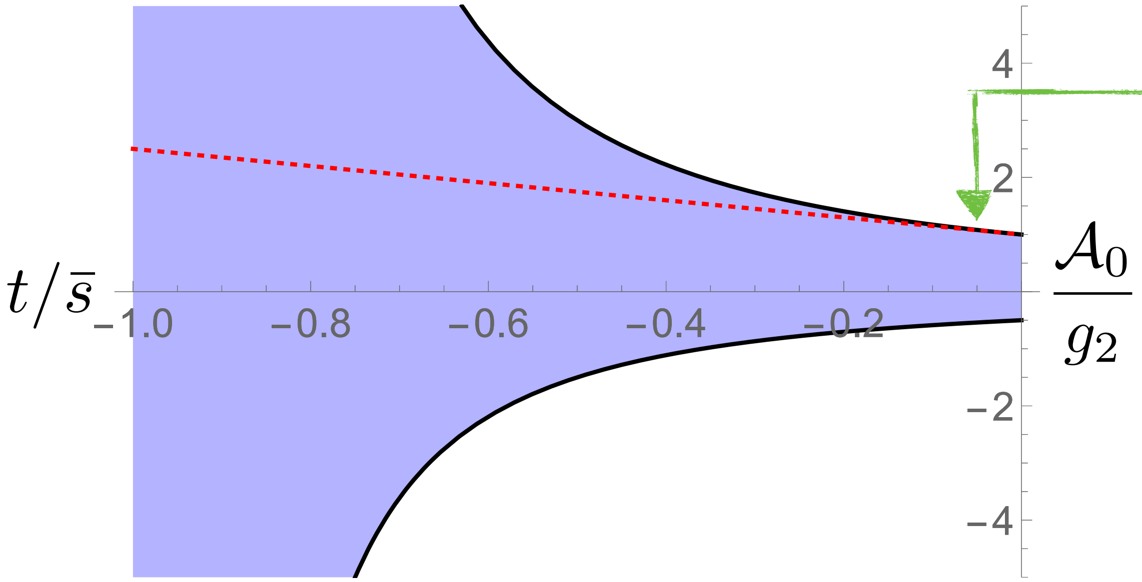
Lower bound from t=0 relies on $\partial_t \mathcal{A}_0$



$$\boxed{-\frac{3}{2}} < \frac{g_3 \bar{s}}{g_2}$$

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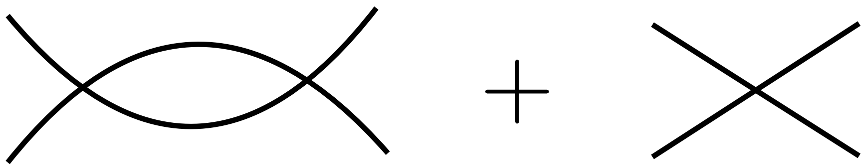

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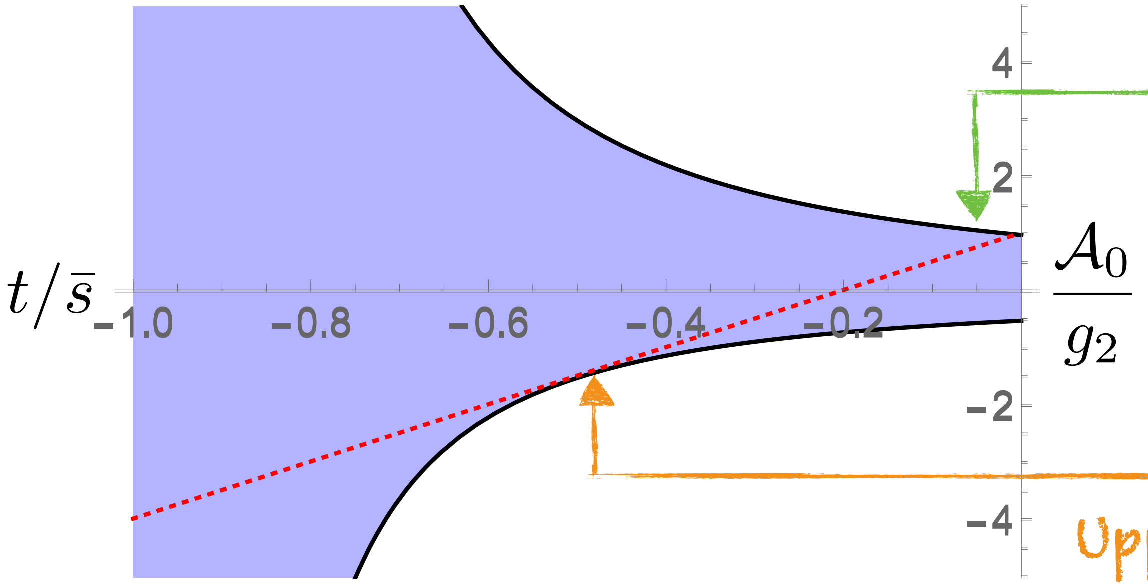

Lower bound from t=0 relies on $\partial_t \mathcal{A}_0$

$$\boxed{-\frac{3}{2}} < \frac{g_3 \bar{s}}{g_2} \lesssim \boxed{5} + \frac{61}{42}\frac{g_2^2}{16\pi^2}\bar{s}^4$$

Upper bound from t≈-0.5s

Loop effects small!