Analyticity of the Black Hole S-matrix

New Frontiers of Quantum Fields and Gravity

2511.11794 [M. Correia, T. Gopalka, GI, A. Wolz]

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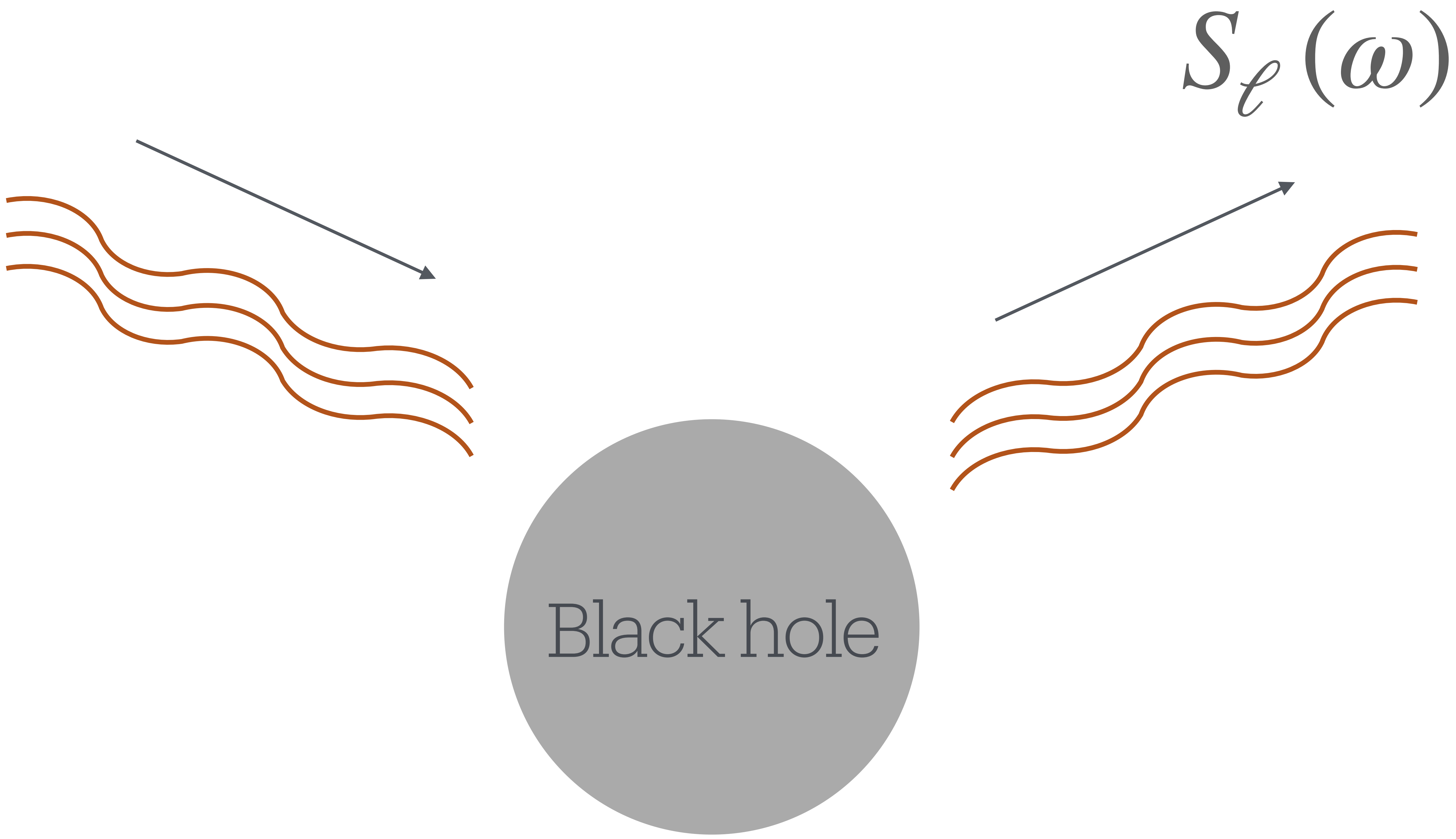
Beijing

Introduction



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Theoretical

- Poles/Branch cuts capture the response of the black hole
- S-matrix in an open system/ background breaking translations

Causality/Unitarity?

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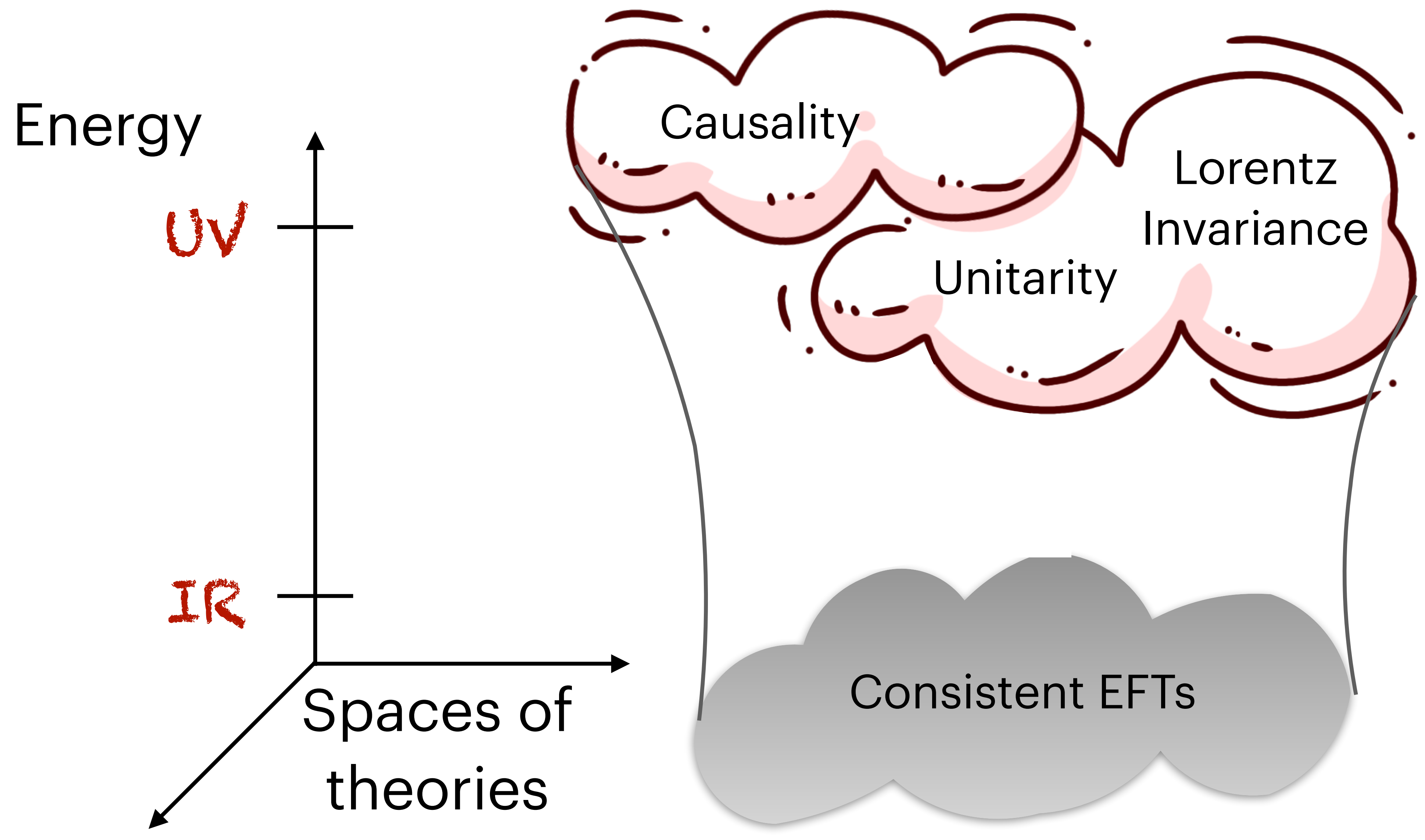
- Positivity bounds on Love numbers



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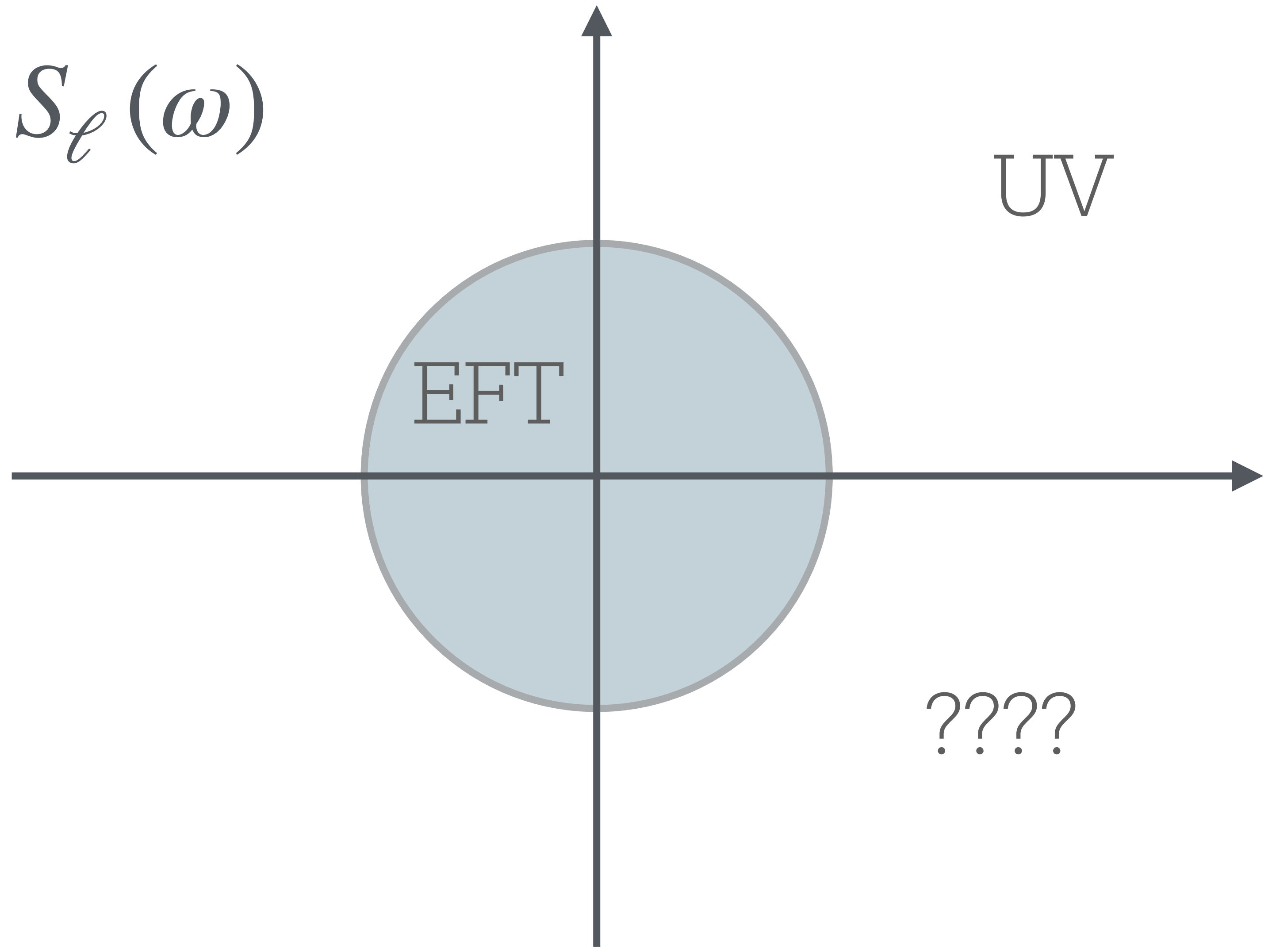

Setup

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Regge-Wheeler : $\partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) = 0$

Tortoise coordinates: $x = r + R_s \log \left(\frac{r}{R_s} - 1 \right)$

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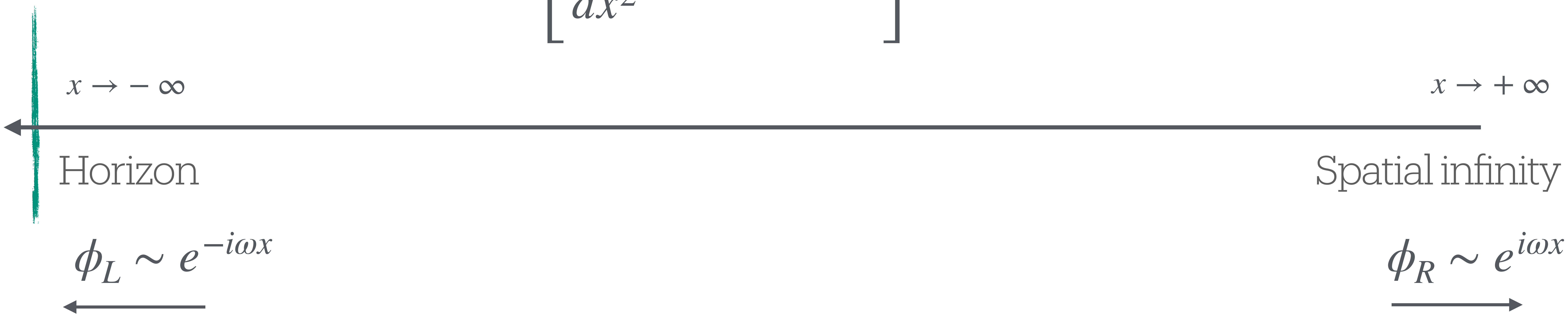

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$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x)\right]\phi(\omega, x) = 0$$

$x \to -\infty$

$x \to +\infty$

Horizon

Spatial infinity

$\phi_L \sim e^{-i\omega x}$

1+1d problem

$\phi_R \sim e^{i\omega x}$

Relation S-matrix and retarded Green's function

$\phi_L(\omega, x \to -\infty) = e^{-i\omega x}$

$\phi_R(\omega, x \to -\infty) = e^{i\omega x}$

Relation S-matrix and retarded Green's function

S-matrix

(or reflection coefficient)

$$\phi_L(\omega, x \to +\infty) = A_{\text{in}}(\omega)\, e^{-i\omega x} + A_{\text{out}}(\omega)\, e^{i\omega x}$$

$$S_\ell(\omega) = -\frac{A_{\text{out}}(\omega)}{A_{\text{in}}(\omega)}$$

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Retarded Green's function

$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x)\right] G_R(\omega, x) = \delta(x - x')$$

$$G_R(\omega, x, x') = \frac{1}{2i\omega A_{\text{in}}}\left[\phi_L(x)\, \phi_R(x')\, \theta(x' - x) + \phi_L(x')\, \phi_R(x)\, \theta(x - x')\right]$$

$$\phi_L(\omega, x \to -\infty) = e^{-i\omega x}$$

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$$G_R(\omega, x, x') \xrightarrow{x, x' \to \infty} -\frac{1}{2i\omega}\left[e^{i\omega(x - x')} - S_\ell(\omega)\, e^{i\omega(x + x')}\right]$$

Approach

1. Determine regions of analyticity for scattering on generic potentials

2. Study the solutions in known limits

Analyticity for generic potentials

In Quantum Mechanics: J.R. Taylor (1983) → We generalize to
R.G. Newton (2013)

We generalize to

1. Long range interactions
2. Dissipation/Open systems

Warmup: Volterra equations for the wavefunction

Integral solutions

$$\phi_L(\omega, x) = e^{-i\omega x} + \frac{1}{2i\omega}\int_{-\infty}^{x} \left[e^{i\omega(x-x')} - e^{i\omega(x'-x)}\right] V(x')\,\phi_L(\omega, x')\,dx'$$

$$\phi_R(\omega, x) = e^{i\omega x} + \frac{1}{2i\omega}\int_{x}^{\infty} \left[e^{i\omega(x-x')} - e^{i\omega(x'-x)}\right] V(x')\,\phi_R(\omega, x')\,dx'$$

- Right BCs
- Solutions of wave equation

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$$\phi_L(\omega, x) = e^{-i\omega x}\chi_L(\omega, x)$$

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Warmup: Volterra equations for the wavefunction

Integral solutions

$$\chi_L(\omega,x) = 1 + \int_{-\infty}^{x} G_0(\omega, x-x')\, V(x')\, \chi_L(\omega,x')\, dx'$$

$$\chi_R(\omega,x) = 1 + \int_{x}^{\infty} G_0(\omega, x-x')\, V(x')\, \chi_R(\omega,x')\, dx'$$

$$G_0(\omega, x-x') = \frac{e^{2i\omega(x-x')} - 1}{2i\omega}$$

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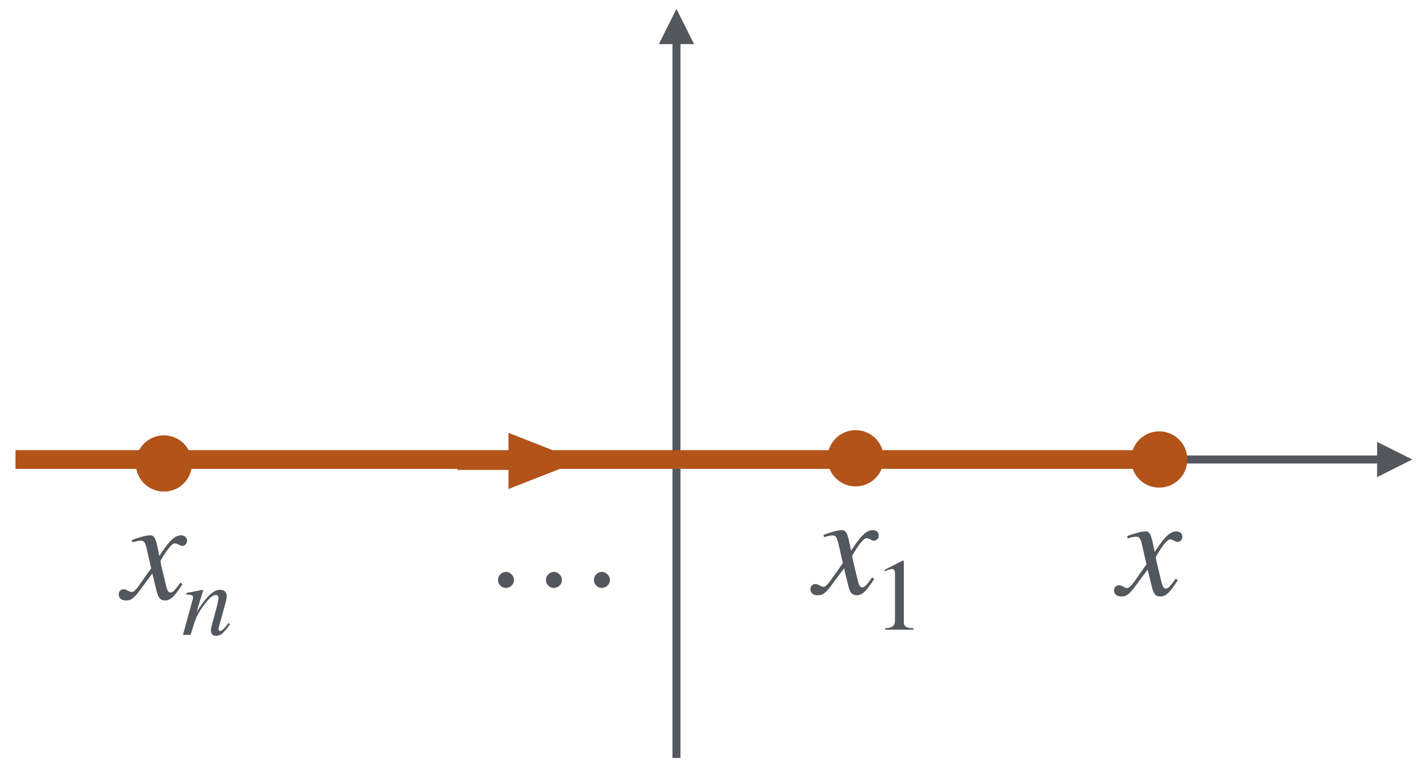
Solve the Volterra equations iteratively

$$\chi_L(\omega,x) = \sum_n \chi_L^{(n)}(\omega,x)$$

Warmup: Volterra equations for the wavefunction



$$\chi_L^{(n)}(\omega, x) = \int_{x > x_1 > \dots > x_n > -\infty} G_0(\omega, x - x_1) \cdots G_0(\omega, x_{n-1} - x_n)\, V(x_1) \cdots V(x_n)\, dx_1 \cdots dx_n$$

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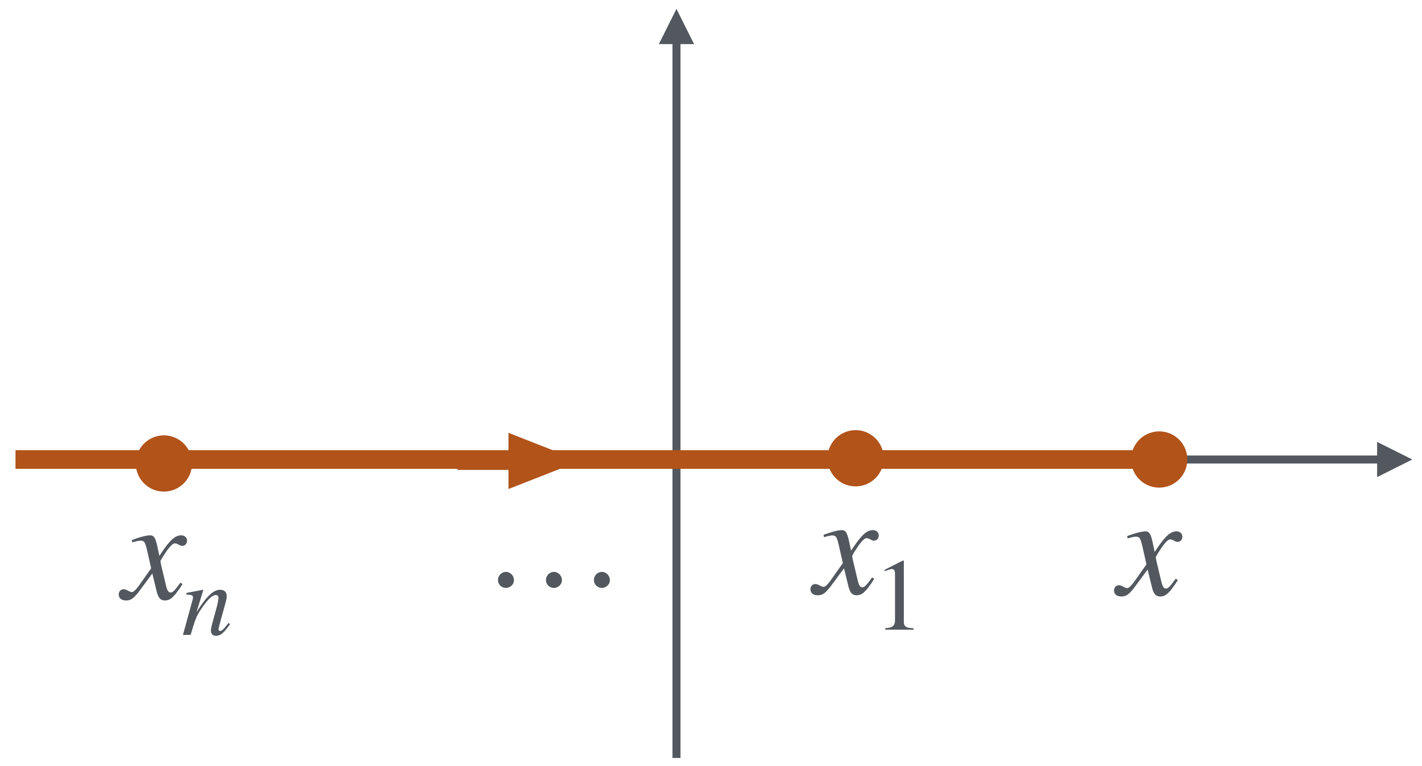

$$|\chi_L^{(n)}(\omega, x)| \leq \int_{x > x_1 > \ldots > x_n > -\infty} |G_0(\omega, x - x_1)| \cdots |G_0(\omega, x_{n-1} - x_n)| \, |V(x_1)| \cdots |V(x_n)| \, dx_1 \cdots dx_n$$

$$|G_0(\omega, x - x')| = \left| \frac{e^{2i\omega\overbrace{(x-x')}^{<0}} - 1}{2i\omega} \right|$$

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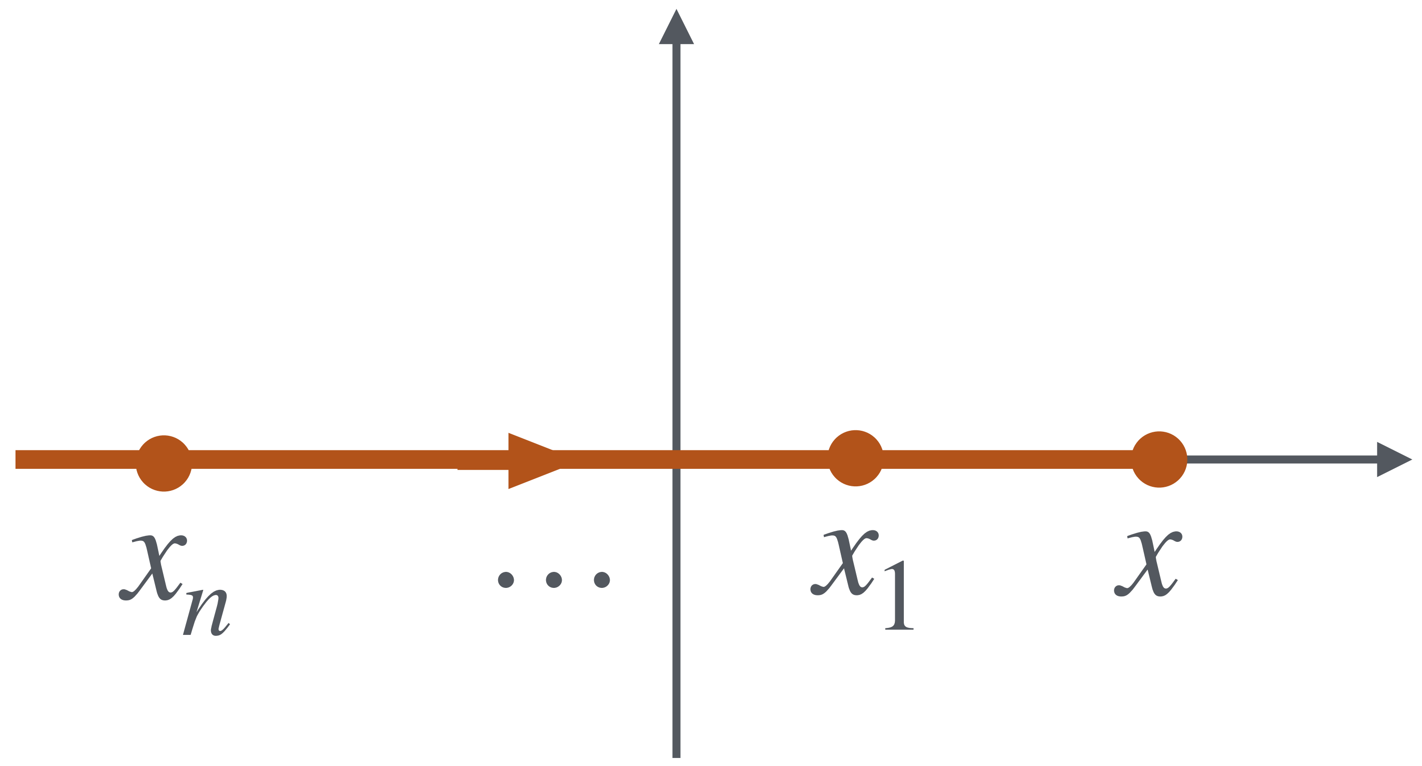

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$$|G_0(\omega, x - x')| = \left| \frac{e^{2i\omega \overbrace{(x - x')}^{<0}} - 1}{2i\omega} \right| < \frac{1}{|\omega|}$$

$\mathrm{Im}\,\omega > 0$

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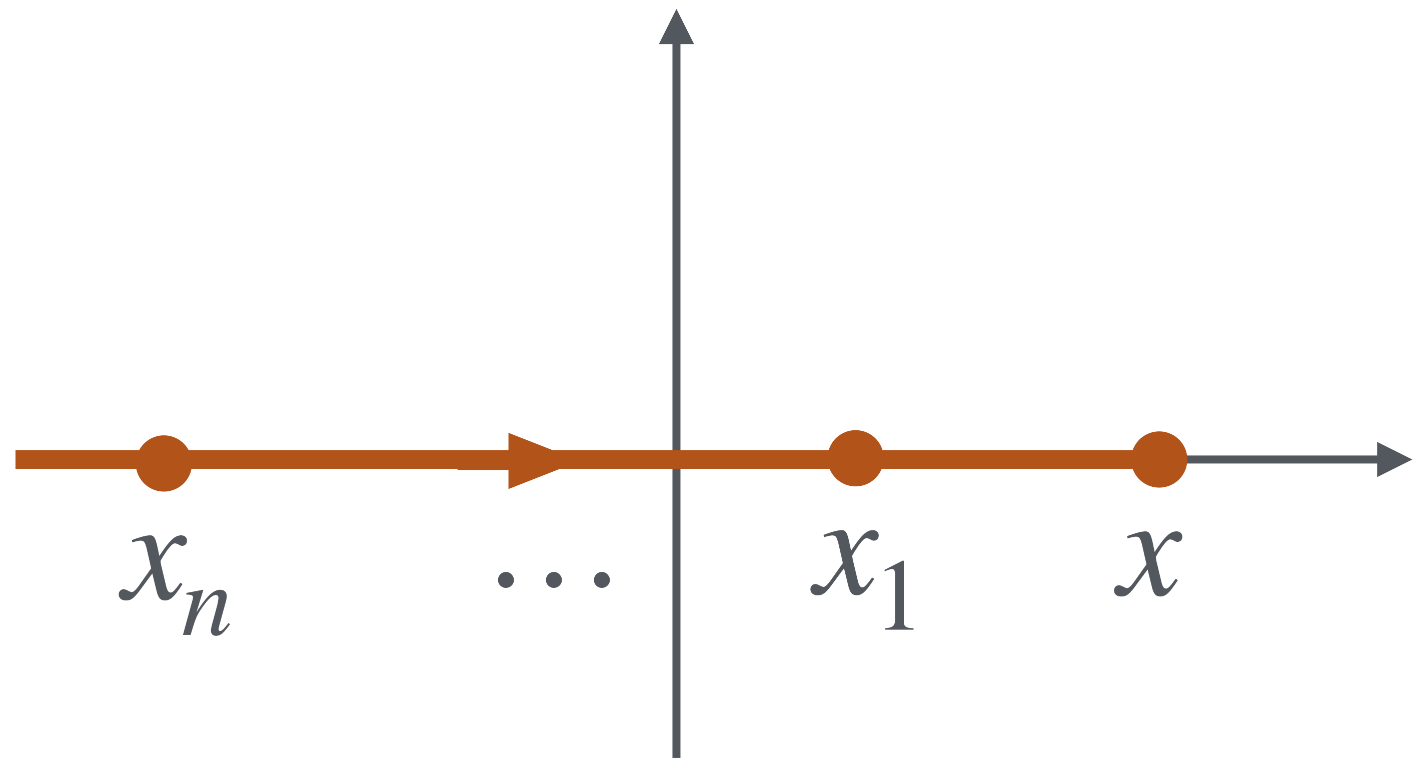

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$$\leq \frac{1}{|\omega|^n} \int_{x > x_1 > \ldots > x_n > -\infty} |V(x_1)| \cdots |V(x_n)| \, dx_1 \cdots dx_n$$

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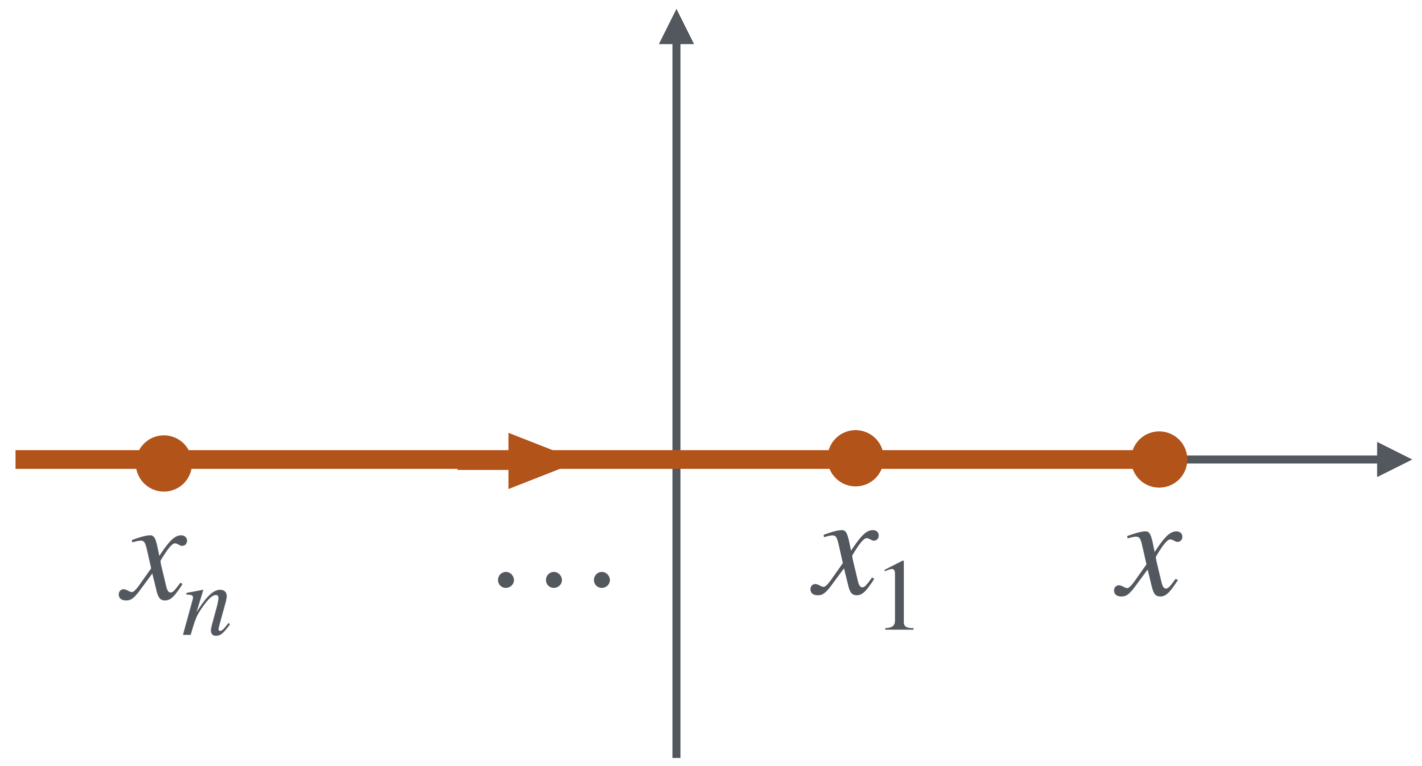

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$$\leq \frac{1}{|\omega|^n} \int_{x > x_1 > \ldots > x_n > -\infty} |V(x_1)| \cdots |V(x_n)| \, dx_1 \cdots dx_n = \frac{1}{n!} \left[\frac{1}{|\omega|} \int_{-\infty}^{x} |V(x_1)| \, dx_1 \right]^n$$

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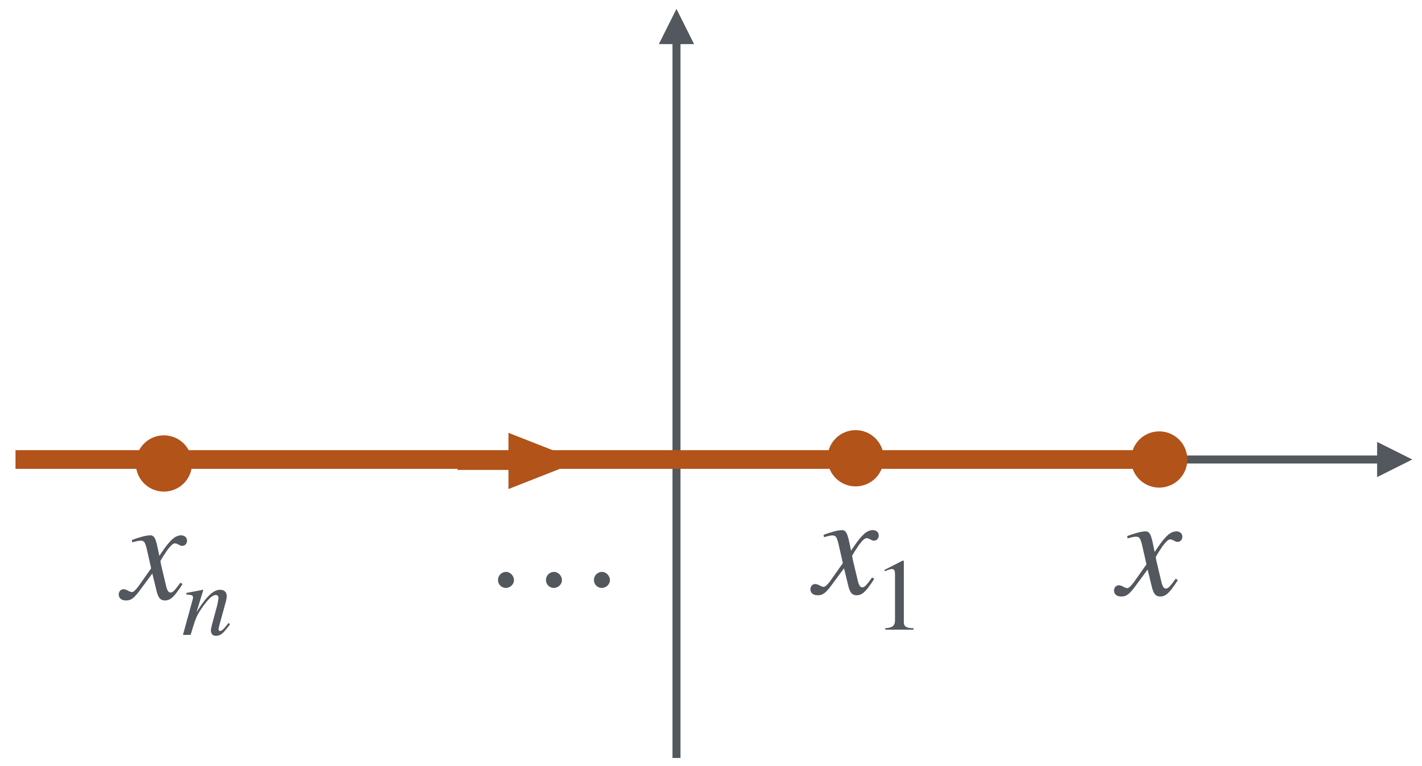
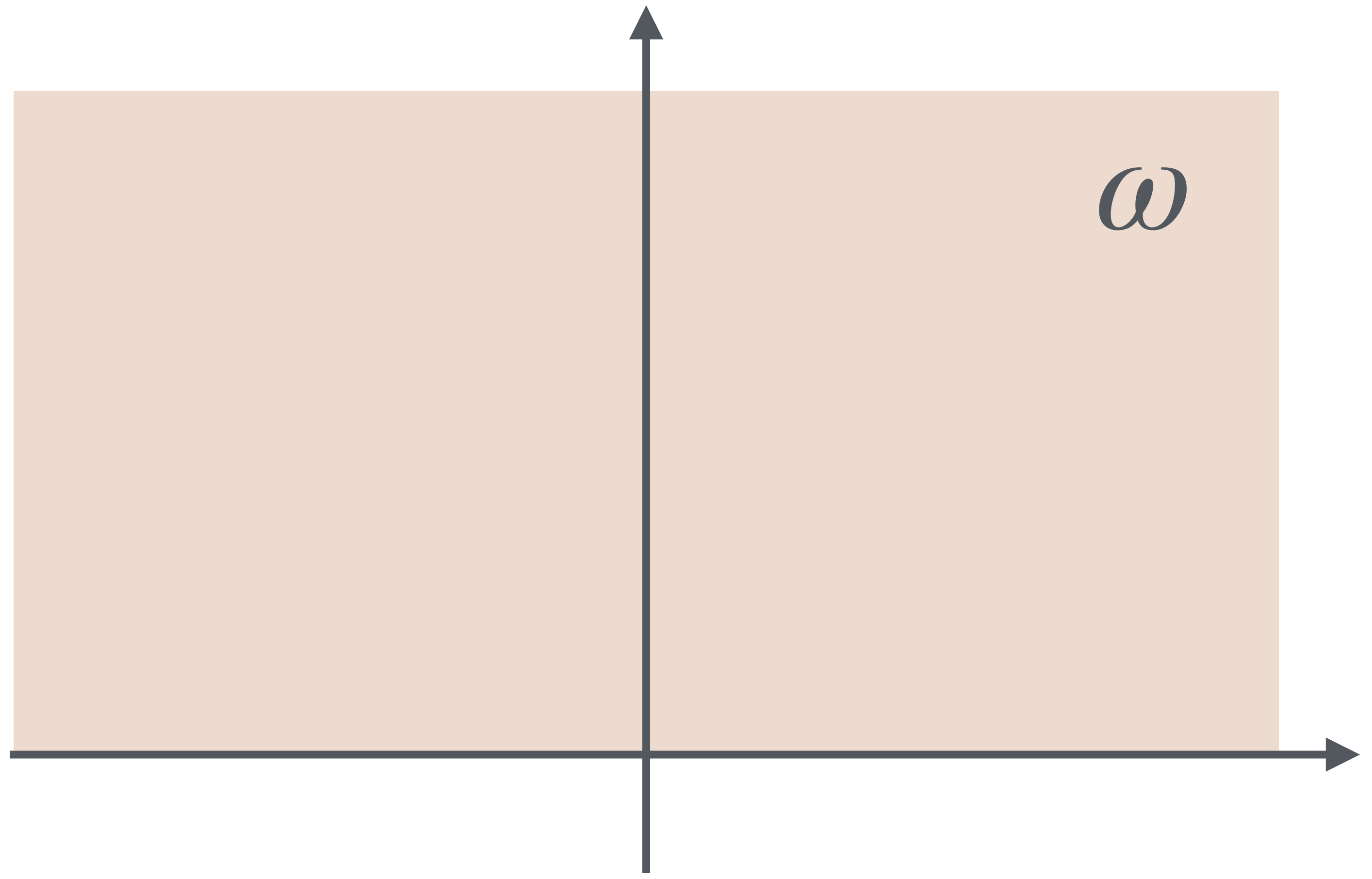


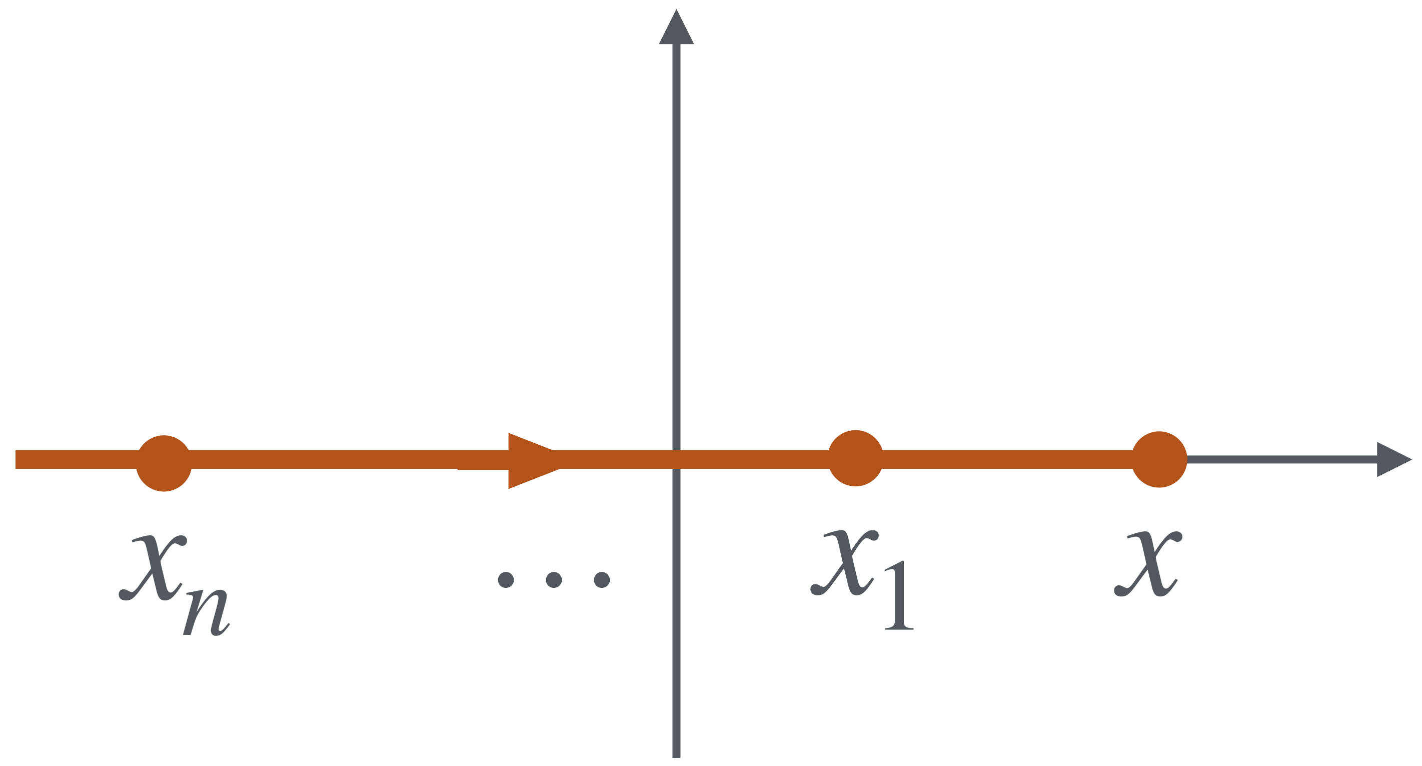


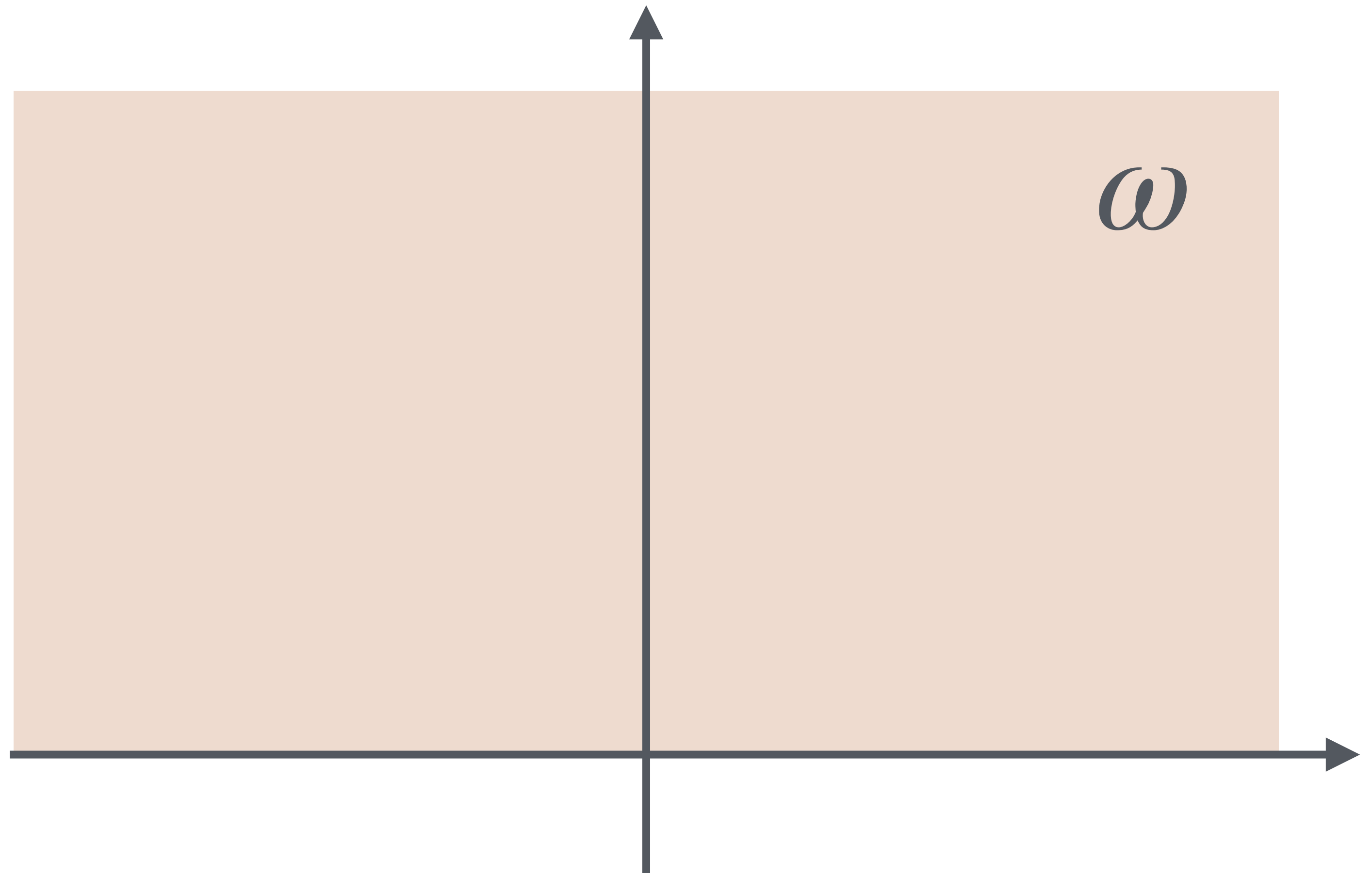
uniform convergence

$$|\chi_L(\omega, x)| = |\sum_n \chi_L^{(n)}(\omega, x)| \leq e^{\frac{1}{|\omega|}\int_{-\infty}^{x} |V(x')| dx'}$$

$\Rightarrow$ $\chi_L(\omega, x)$ is analytic in the UHP

Warmup: Volterra equations for the wavefunction





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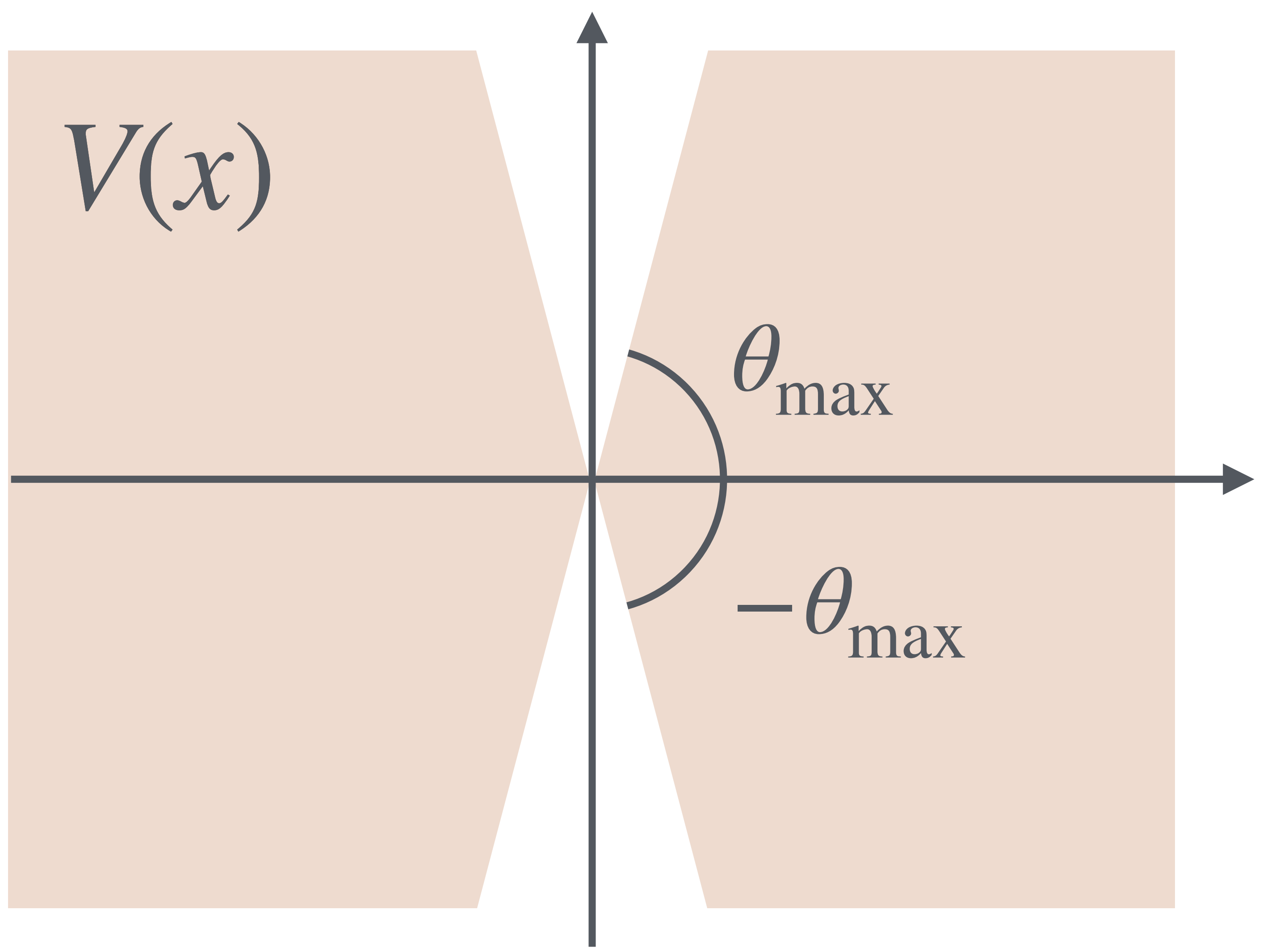
$G_R(\omega,x)$ is analytic in the UHP

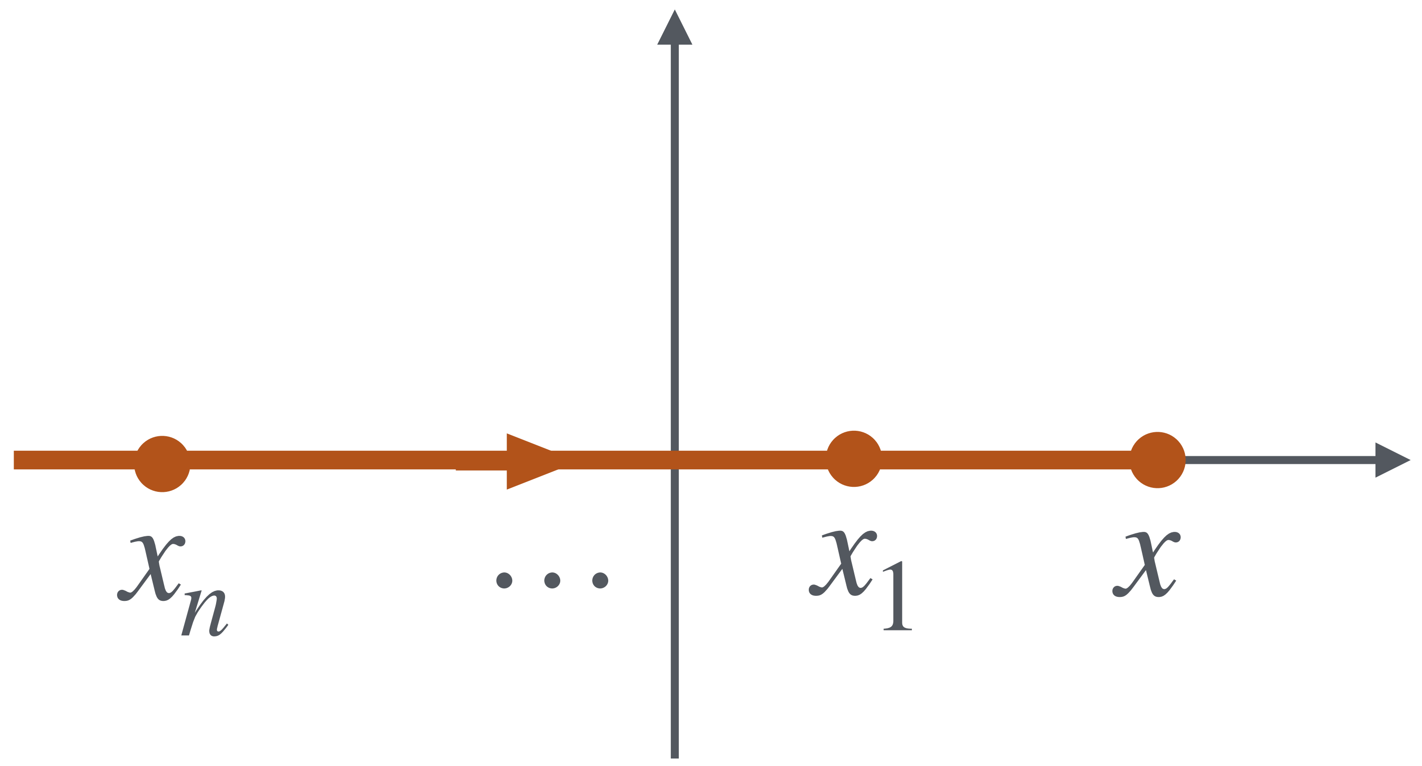
Simplest incarnation of causality

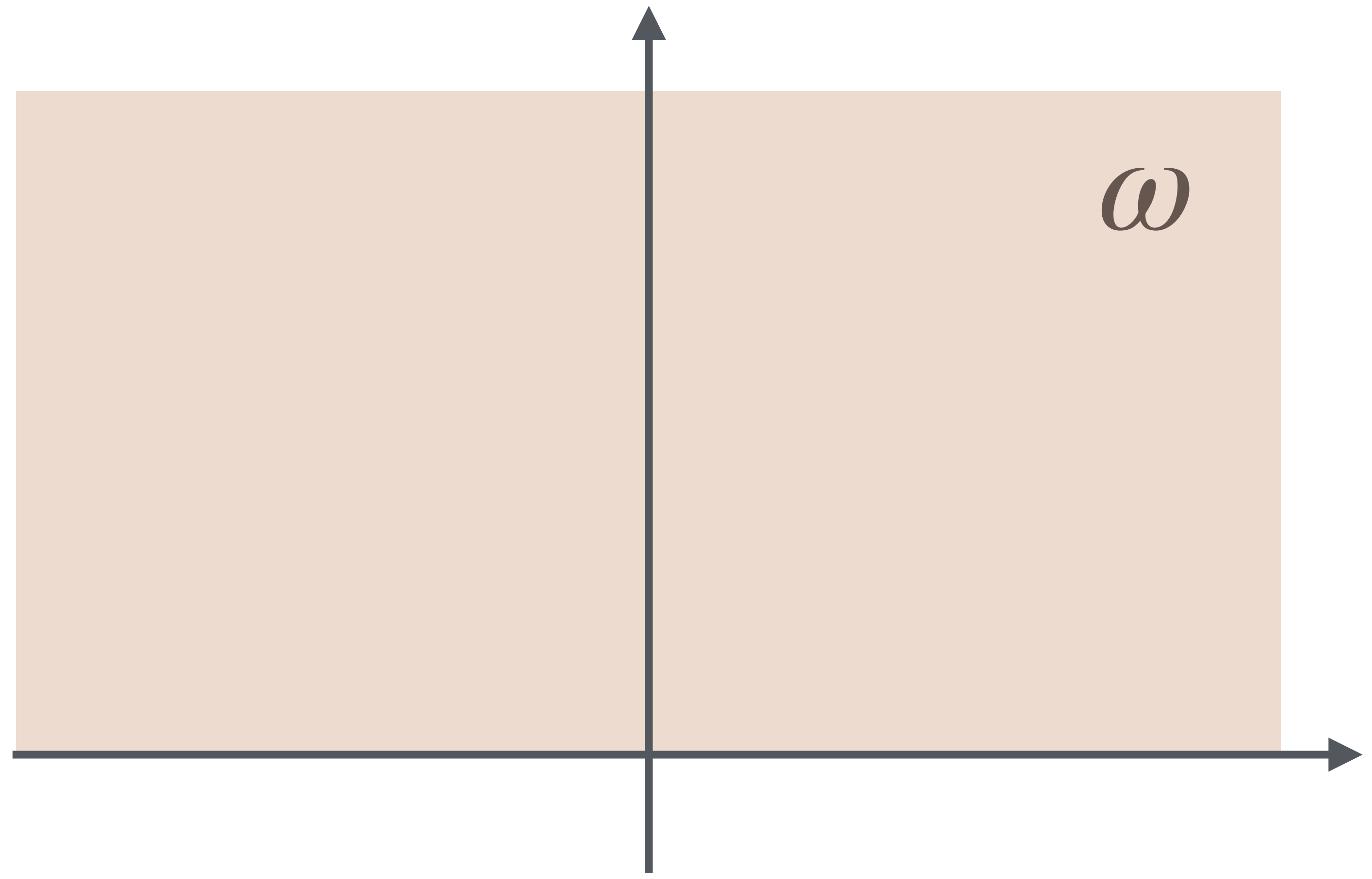
$$G_R(t-t',x)=0\,,\qquad t-t'<0$$

More analyticity from the potential



$$V(x \to \infty) \lesssim \frac{1}{|x|}$$





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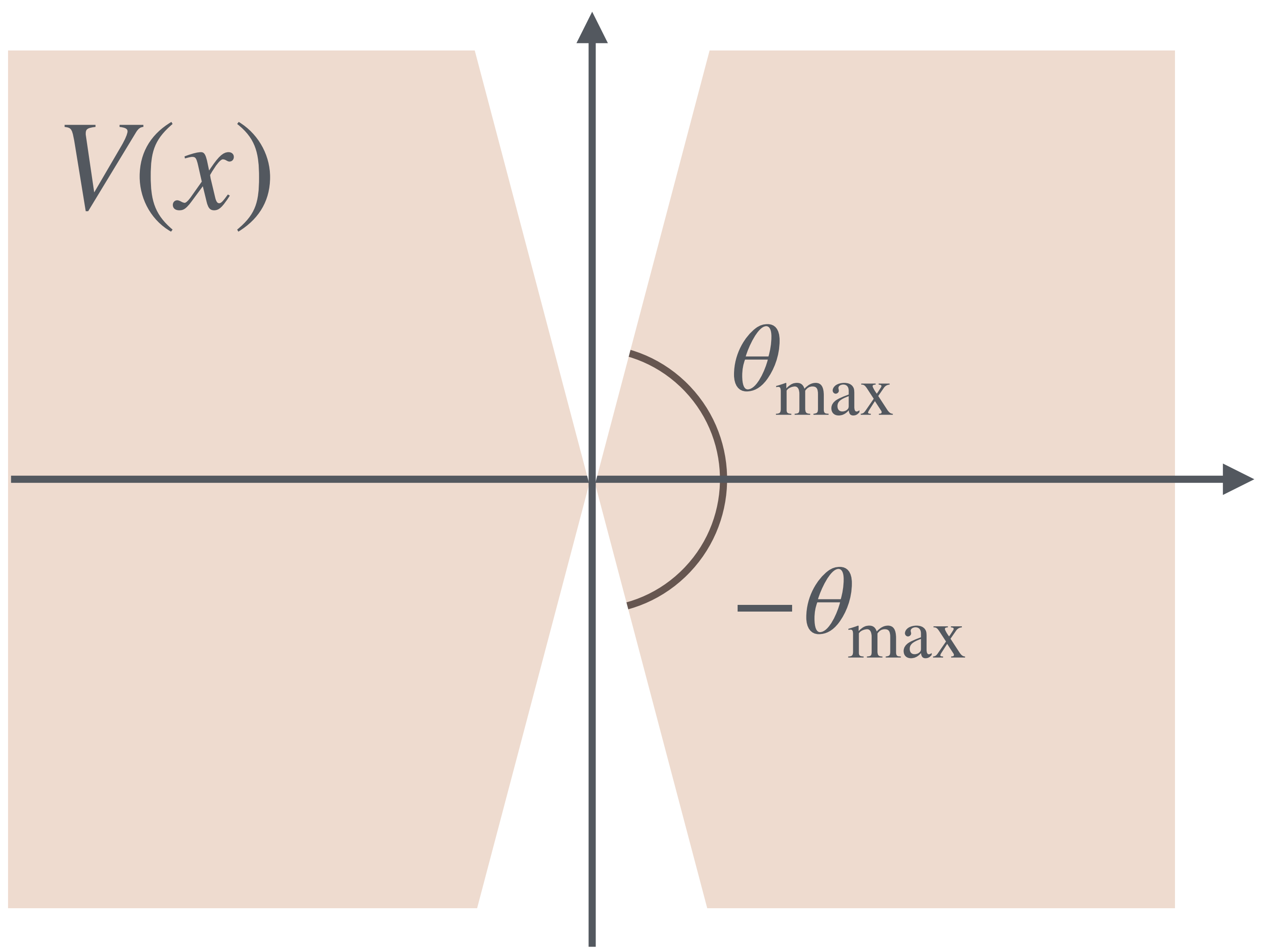

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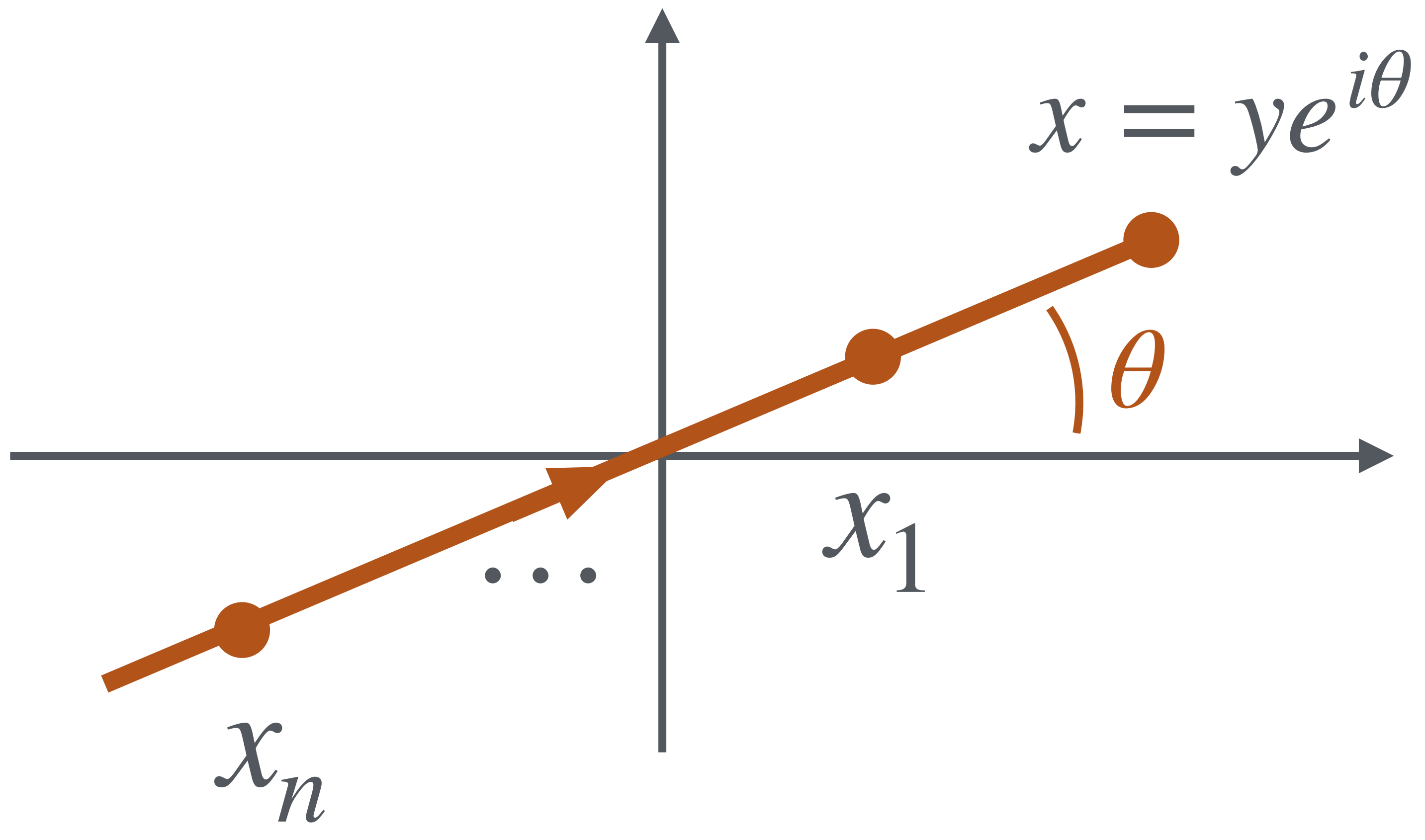

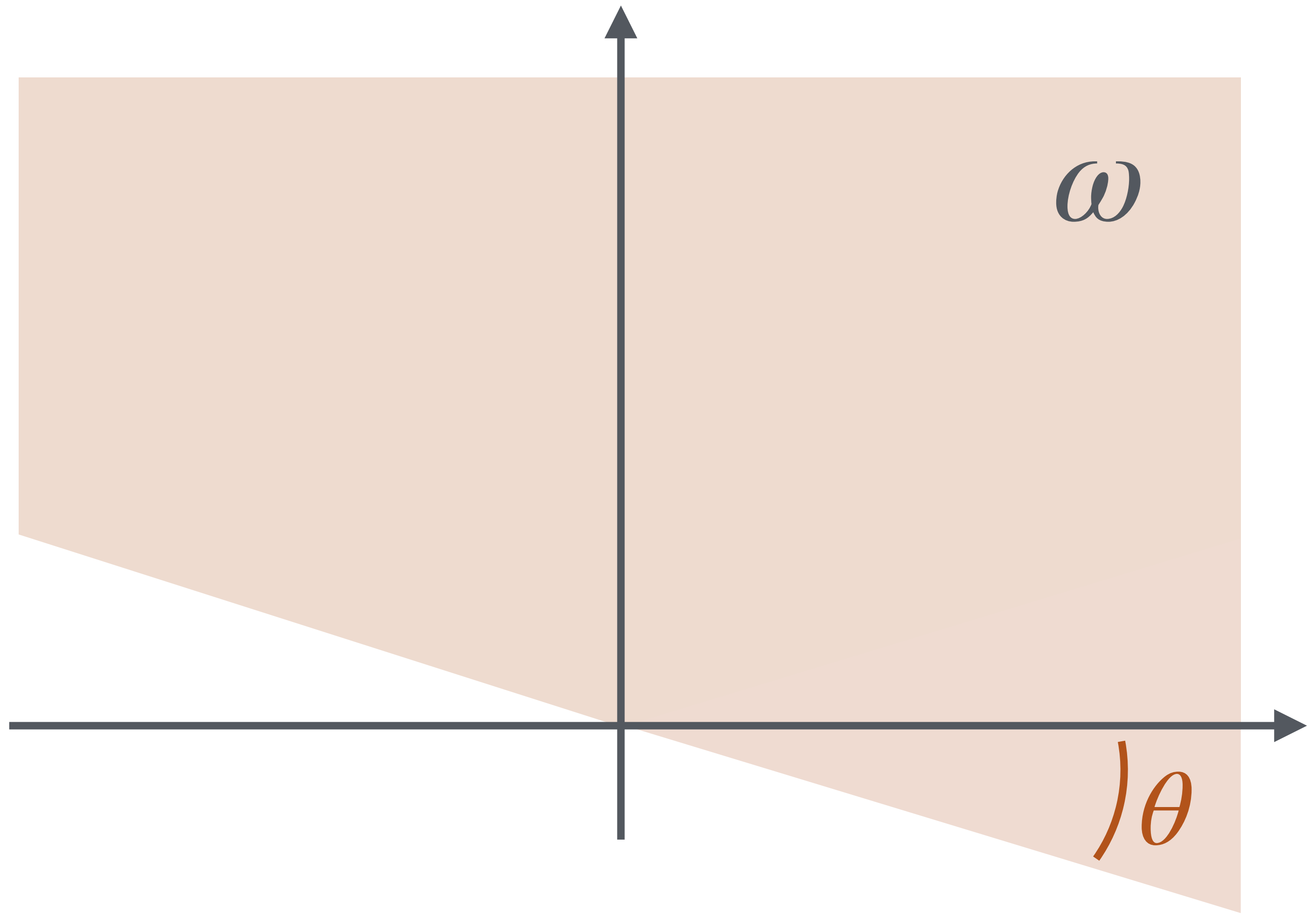

$$|\chi_L(\omega, ye^{i\theta})| = |\sum_n \chi_L^{(n)}(\omega, ye^{i\theta})| \leq e^{\frac{1}{|\omega|}\int_{-\infty}^{y} |V(e^{i\theta}y')|dy'}$$

$$\mathrm{Im}\, e^{i\theta}\omega > 0$$

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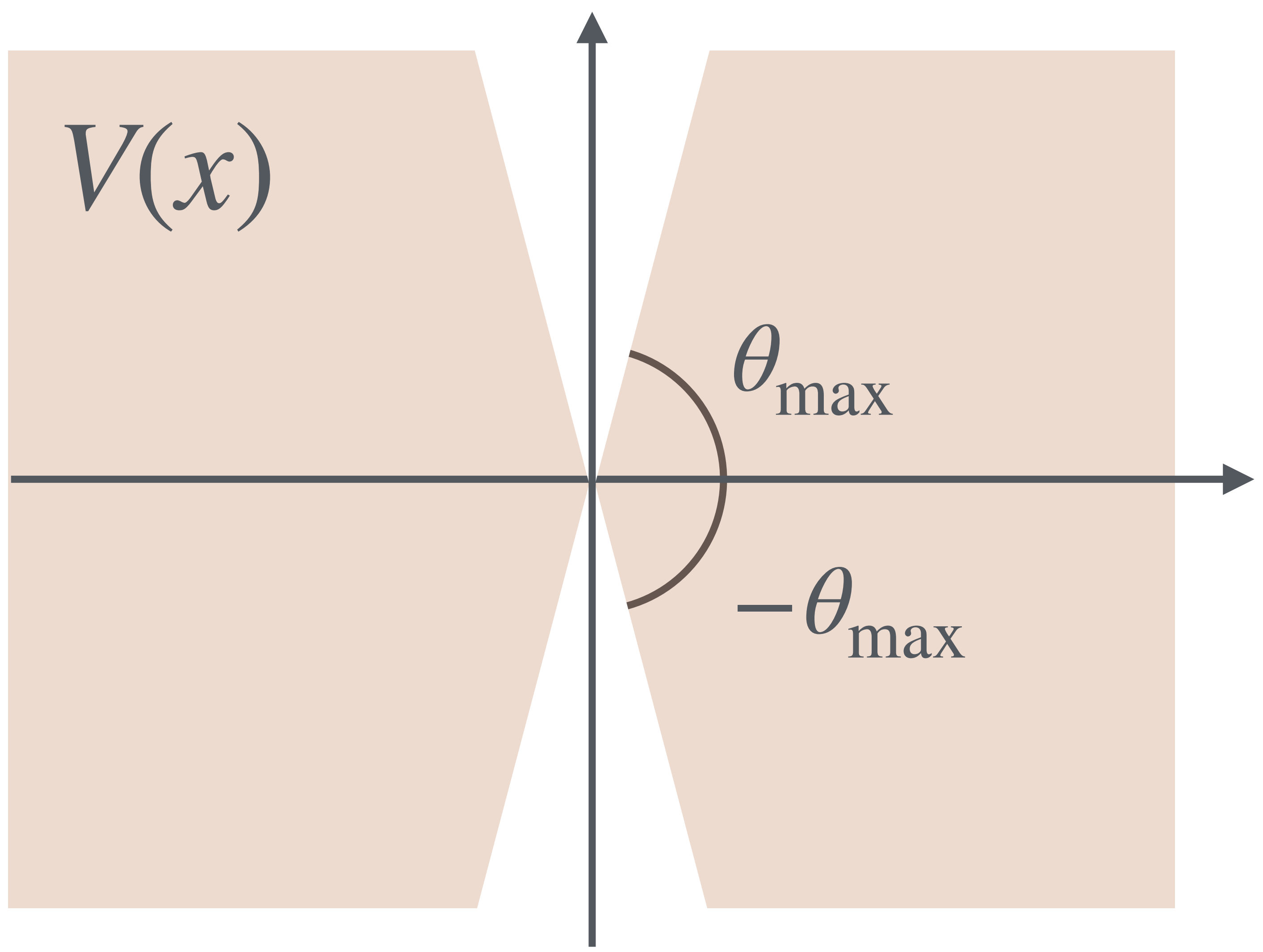

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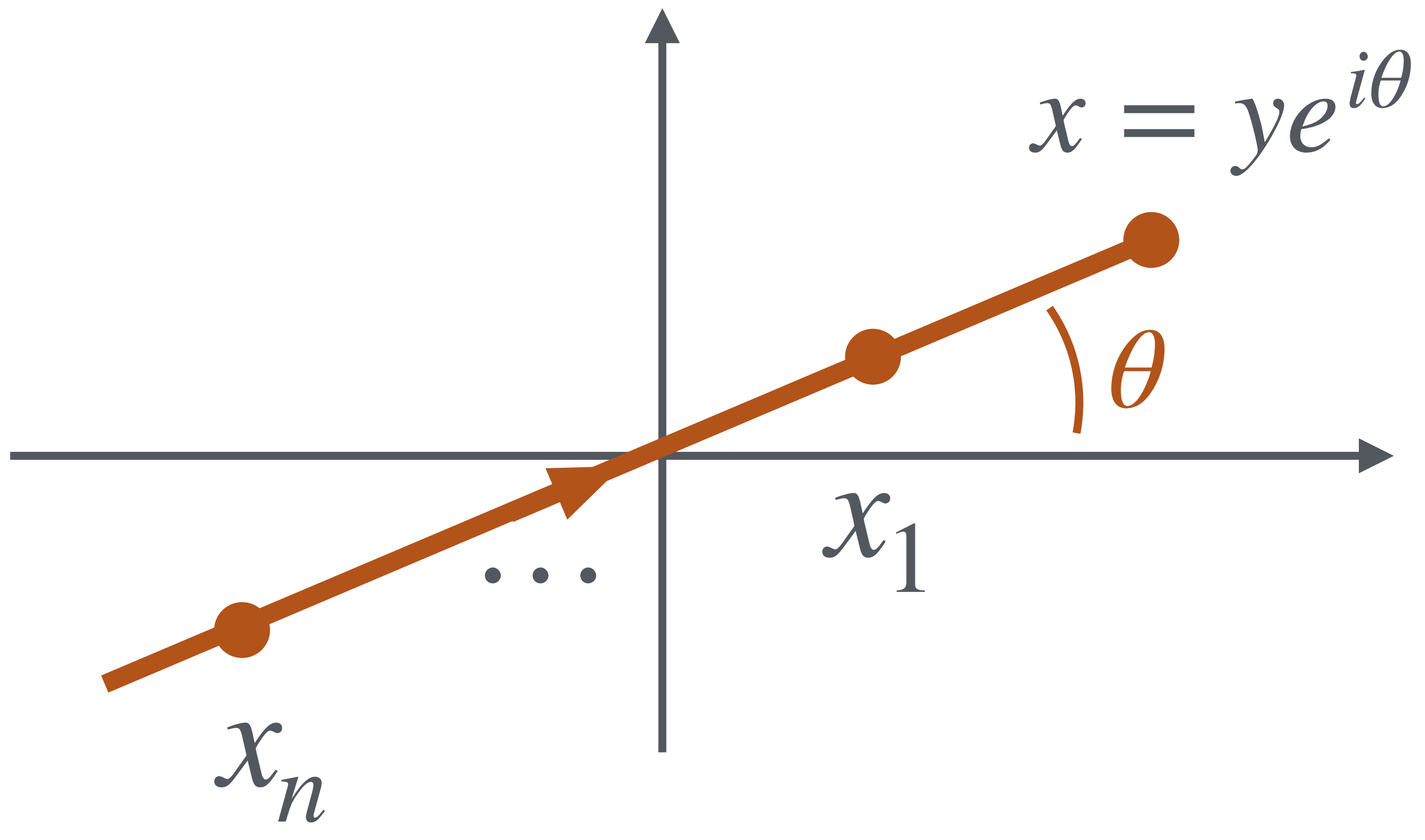

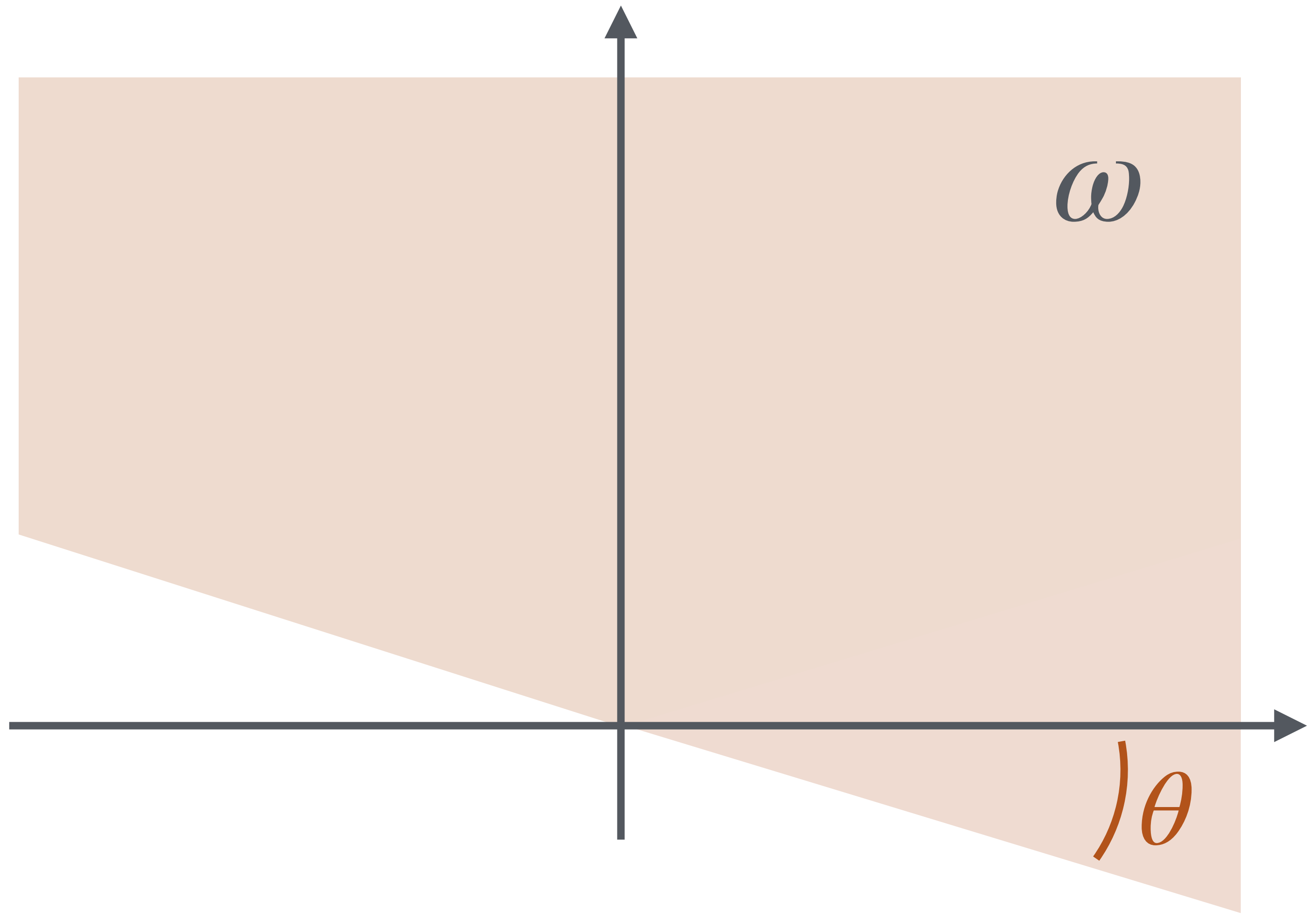

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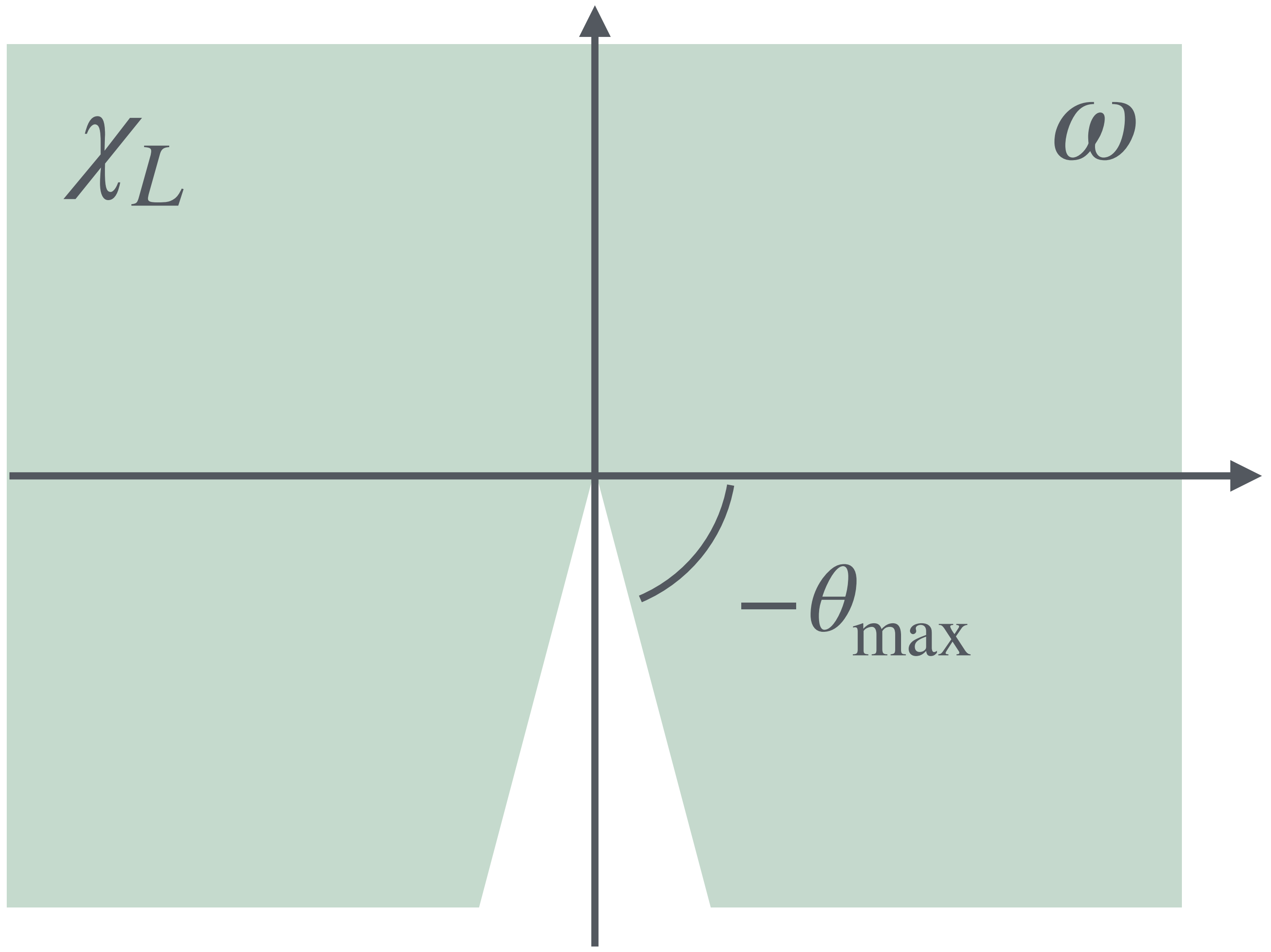

Analyticity of connection coefficients

$$\phi_L(\omega, x) = e^{-i\omega x} + \frac{1}{2i\omega}\int_{-\infty}^{x} \left[e^{i\omega(x-x')} - e^{i\omega(x'-x)}\right] V(x')\,\phi_L(\omega, x')\,dx'$$

$$x \to \infty$$

$$A_{\text{in}}(\omega) = 1 - \frac{1}{2i\omega}\int_{-\infty}^{\infty} V(x')\,\chi_L(\omega, x')\,dx'$$

$$A_{\text{out}}(\omega) = \frac{1}{2i\omega}\int_{-\infty}^{\infty} e^{-2i\omega x'} V(x')\,\chi_L(\omega, x')\,dx'$$

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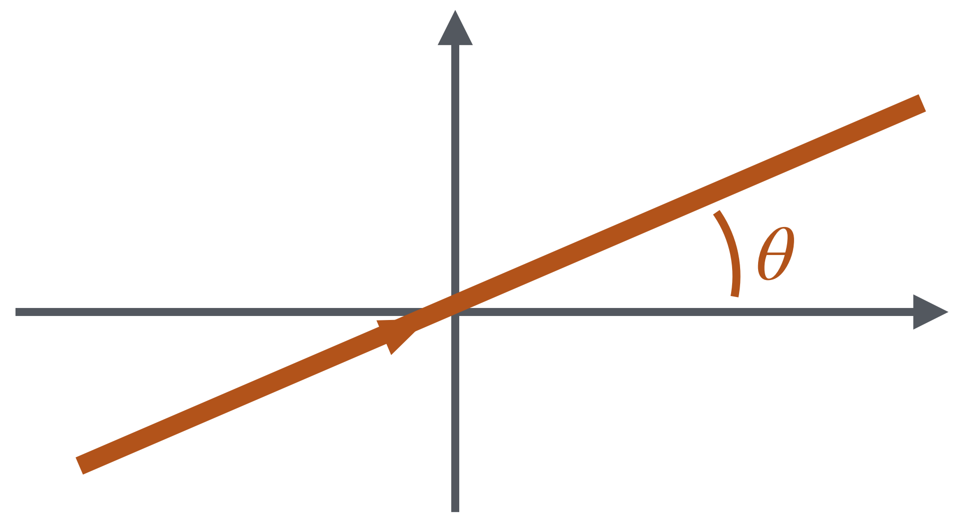

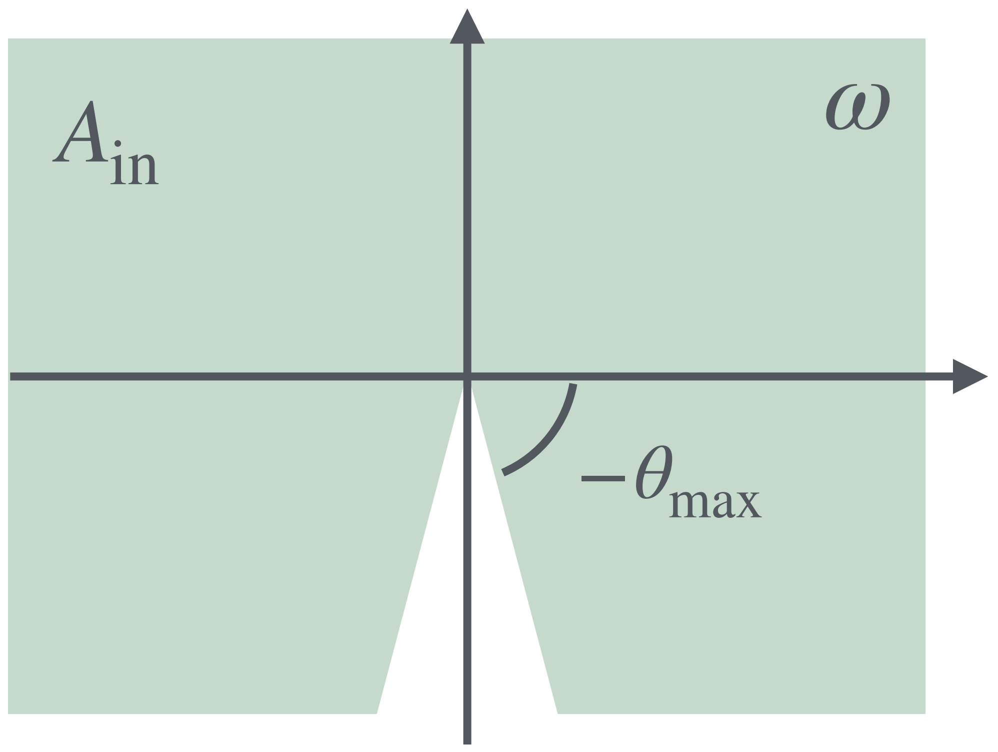

$$A_{\text{out}}(\omega) = \frac{1}{2i\omega}\int_{-\infty}^{\infty} e^{-2i\omega x'} V(x')\,\chi_L(\omega, x')\,dx'$$

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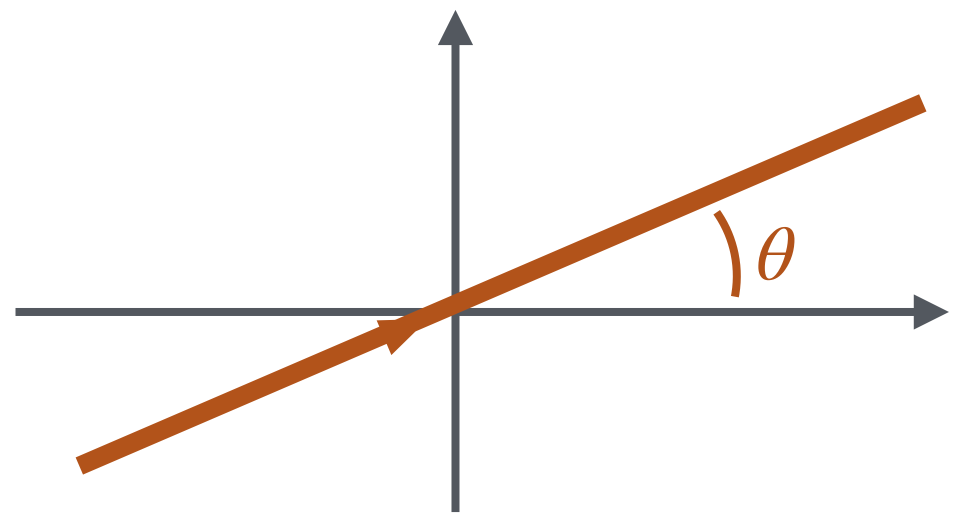
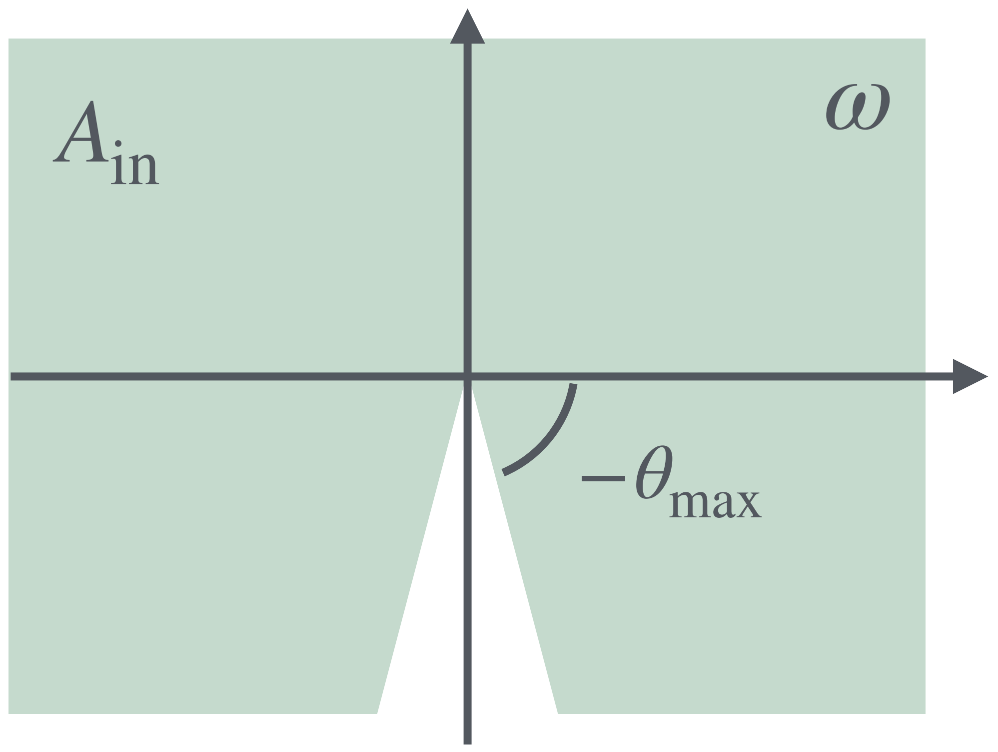
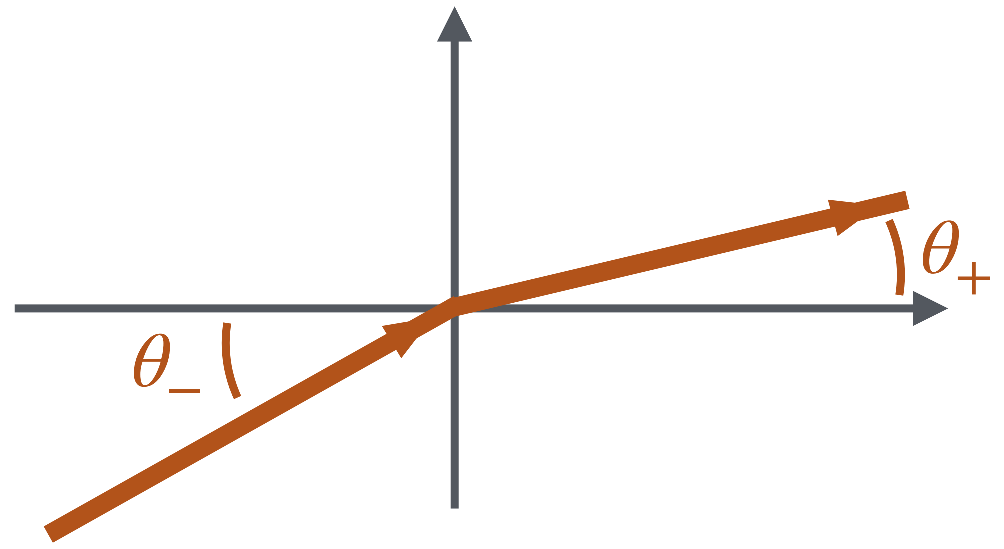


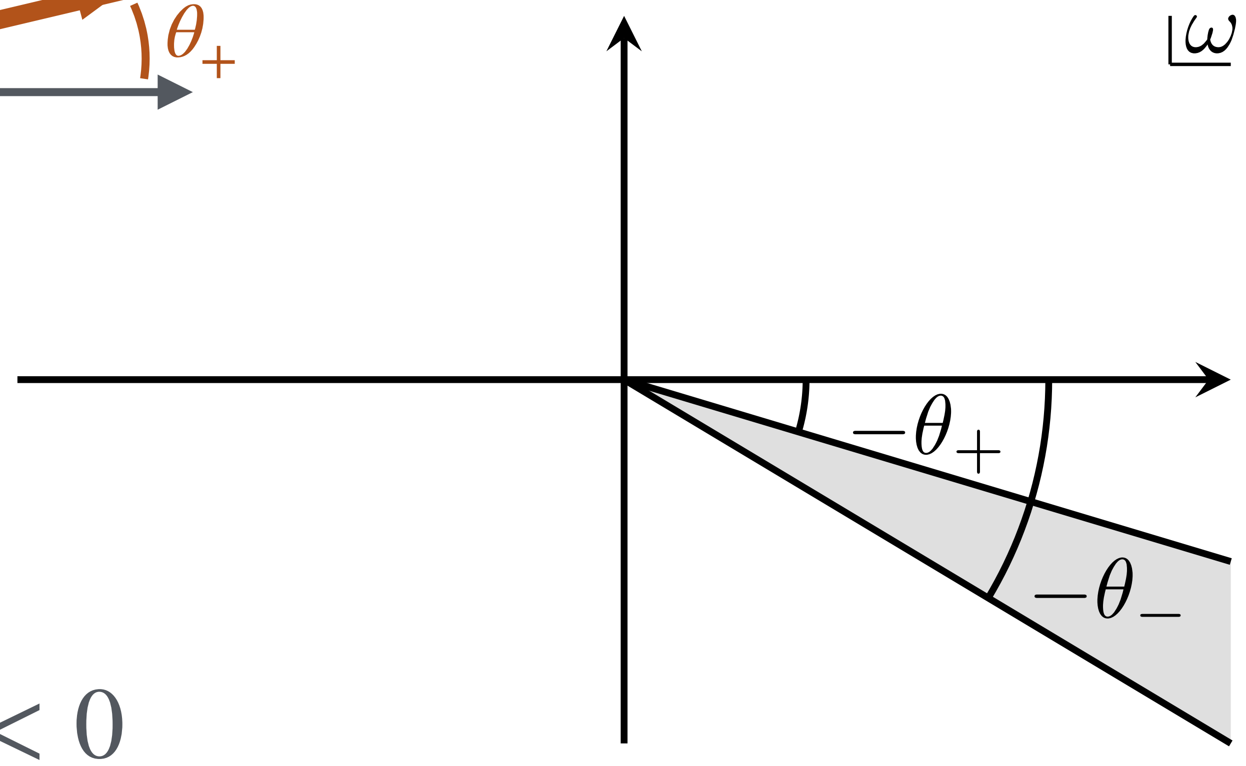


$$A_{\rm out}(\omega) = \frac{1}{2i\omega}\int_{-\infty}^{\infty} e^{-2i\omega x'} V(x')\,\chi_L(\omega,x')\,dx'$$



$\mathrm{Im}(e^{i\theta_-}\omega) > 0 \;\cap$

$\mathrm{Im}(e^{i\theta_+}\omega) < 0$



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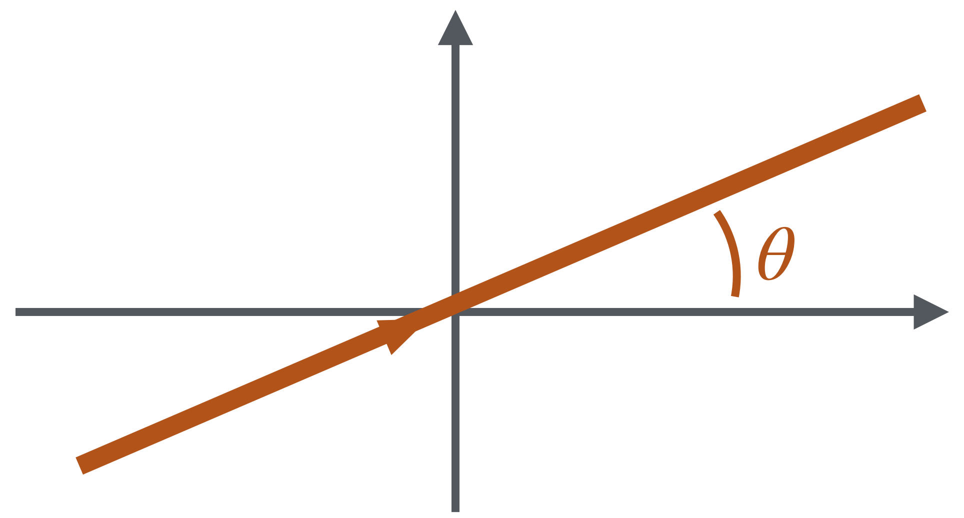

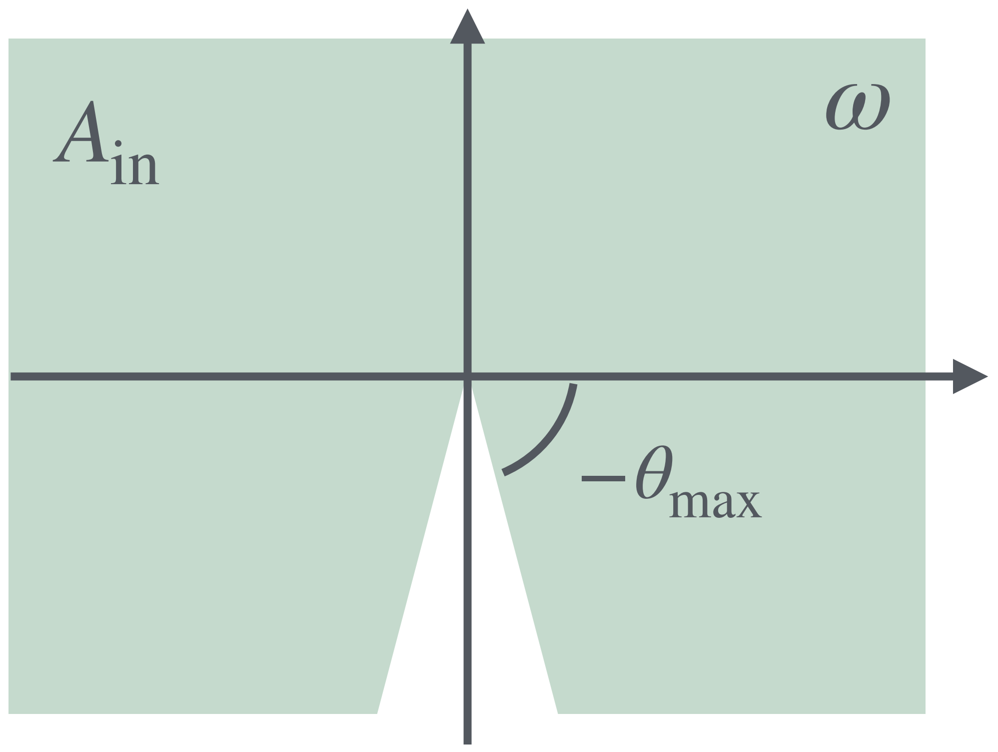

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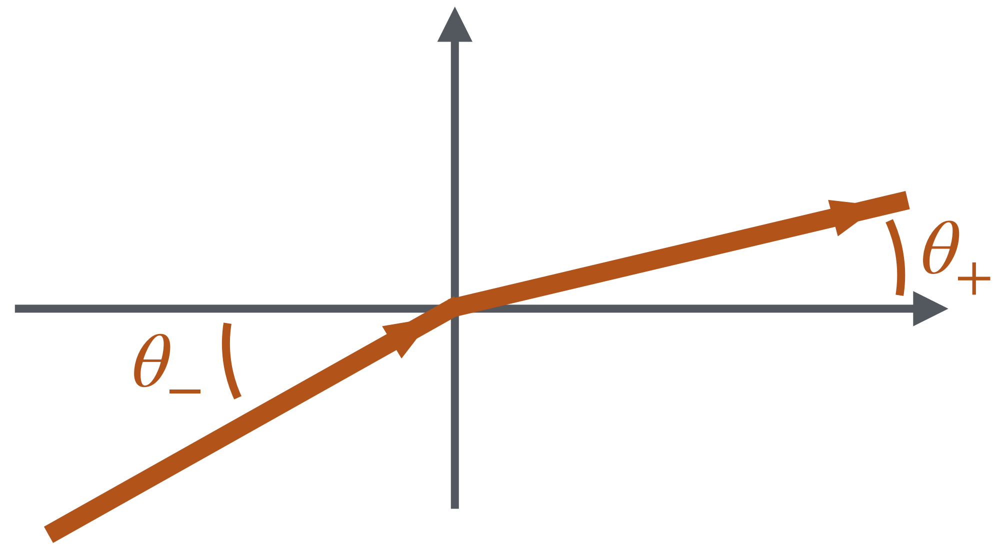
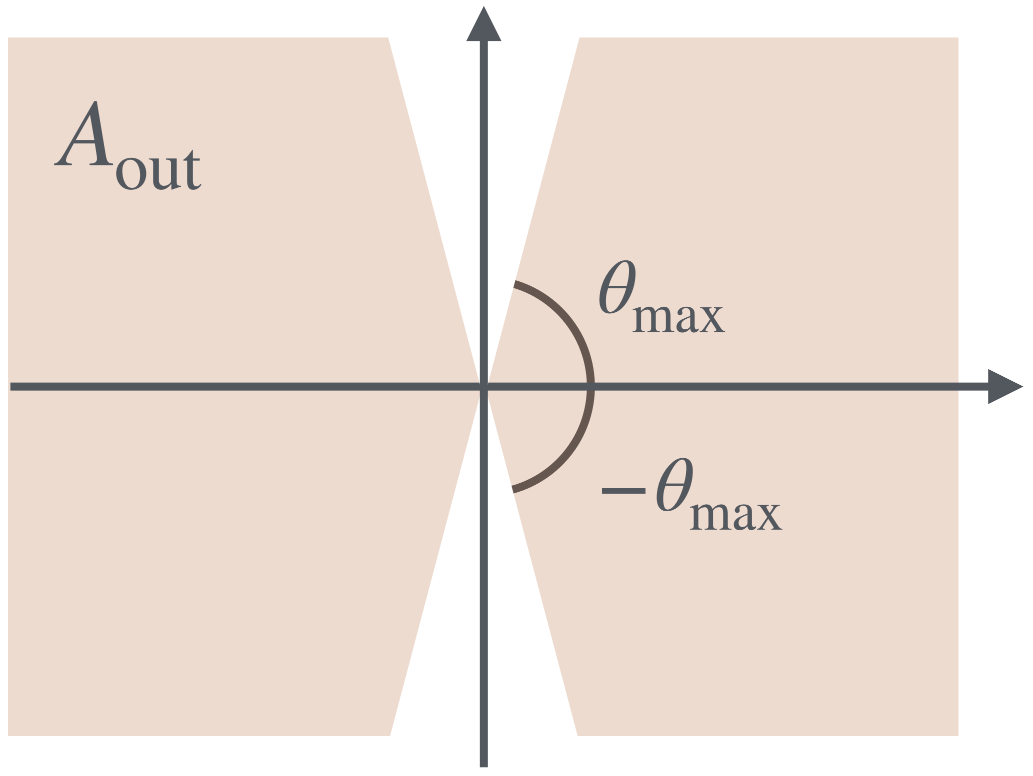
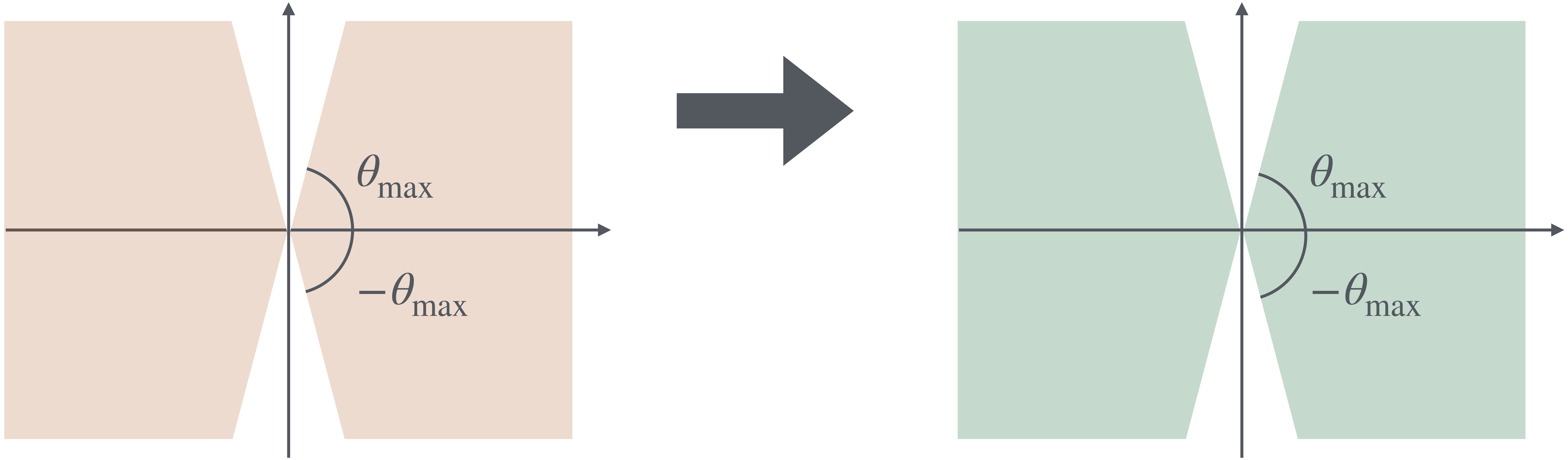




Analyticity of the S-matrix

$$V(x \to \infty) \lesssim \frac{1}{|x|}$$

$$S_\ell(\omega) = -\frac{A_{\text{out}}(\omega)}{A_{\text{in}}(\omega)}$$

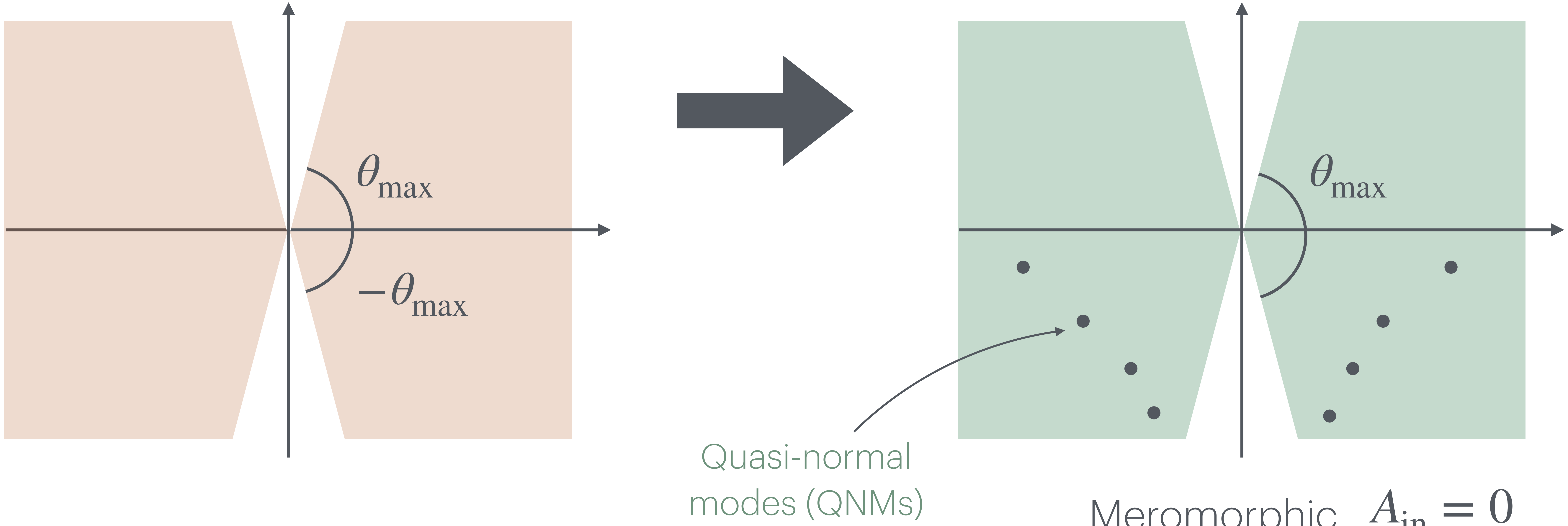


Meromorphic

Analyticity of the S-matrix

$$V(x \to \infty) \lesssim \frac{1}{|x|}$$

$$S_\ell(\omega) = -\frac{A_{\text{out}}(\omega)}{A_{\text{in}}(\omega)}$$



Meromorphic $A_{\text{in}} = 0$

Analyticity of the Black Hole S-matrix

The Schwarzschild potential

Regge-Wheeler : $\partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) = 0$

Tortoise coordinates: $x = r + R_s \log \left(\frac{r}{R_s} - 1 \right)$

$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x) \right] \phi(\omega, x) = 0$$

$$V\left(r(x)\right) = \left(1 - \frac{R_s}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} + \frac{R_s}{r^3}(1 - s^2) \right)$$

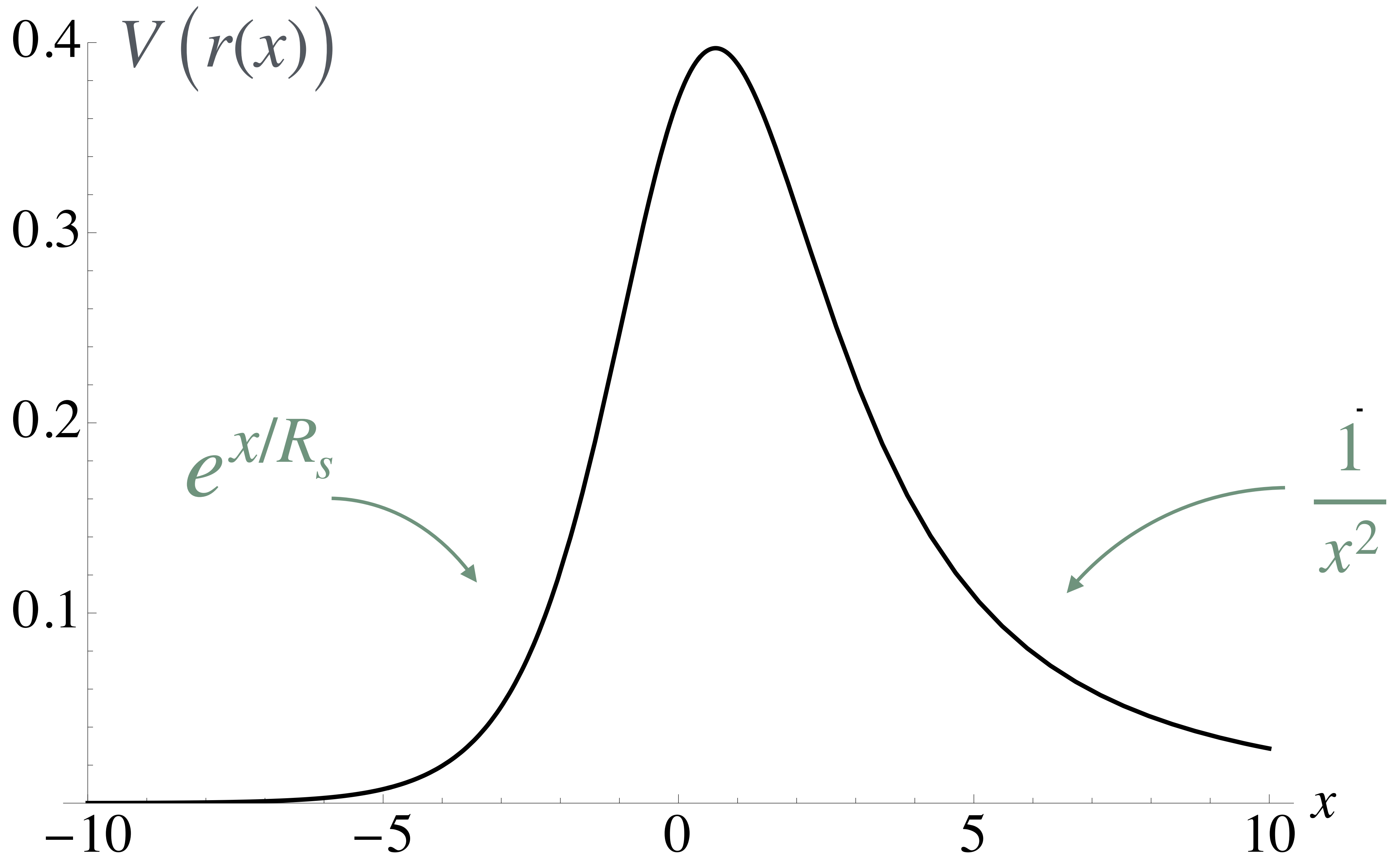
Angular momentum

Spin of the scattered wave

The Schwarzschild potential

$$V\left(r(x)\right) = \left(1 - \frac{R_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{R_s}{r^3}(1 - s^2)\right)$$

$$x = r + R_s \log\left(\frac{r}{R_s} - 1\right)$$



The Schwarzschild potential

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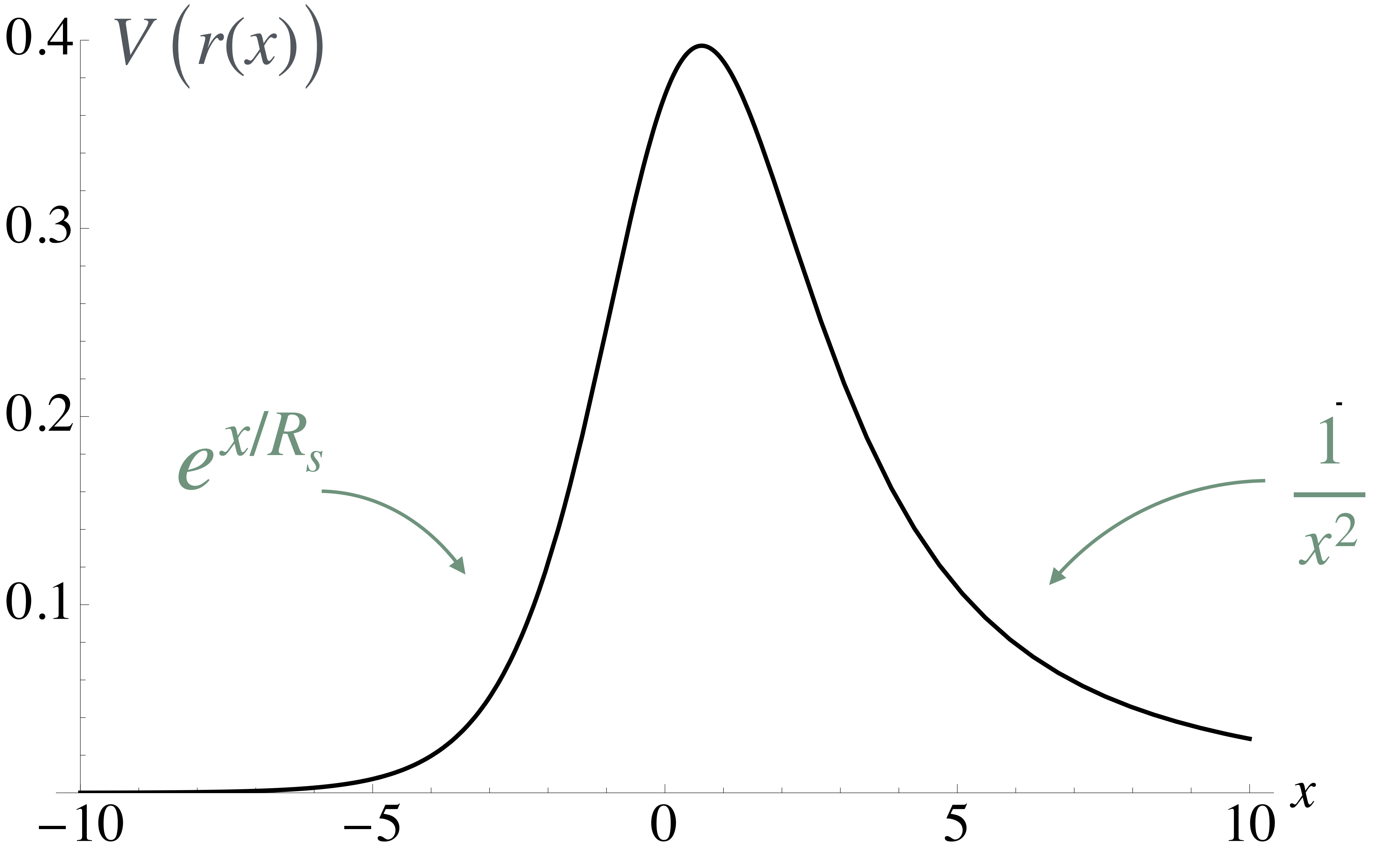

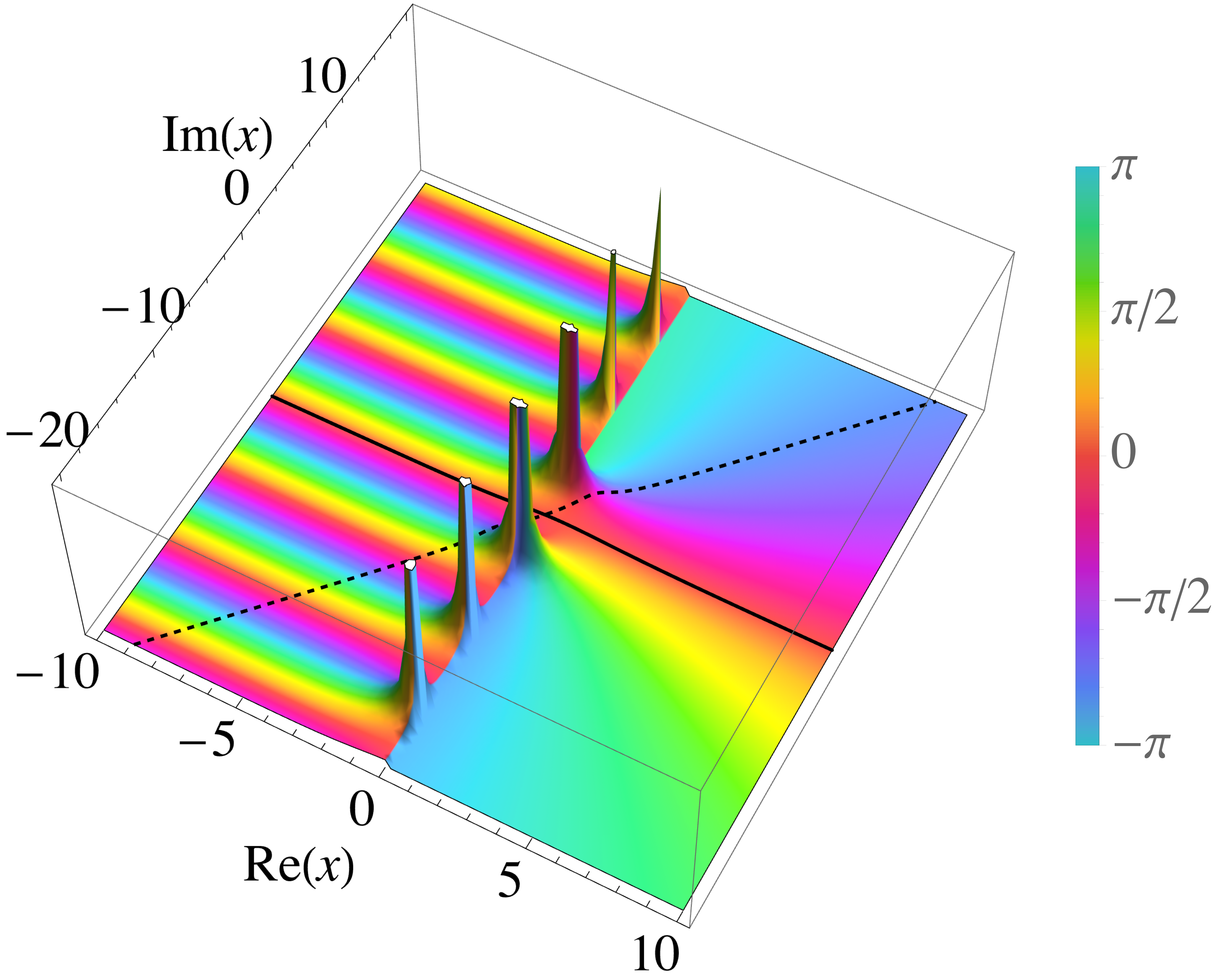

The Schwarzschild potential

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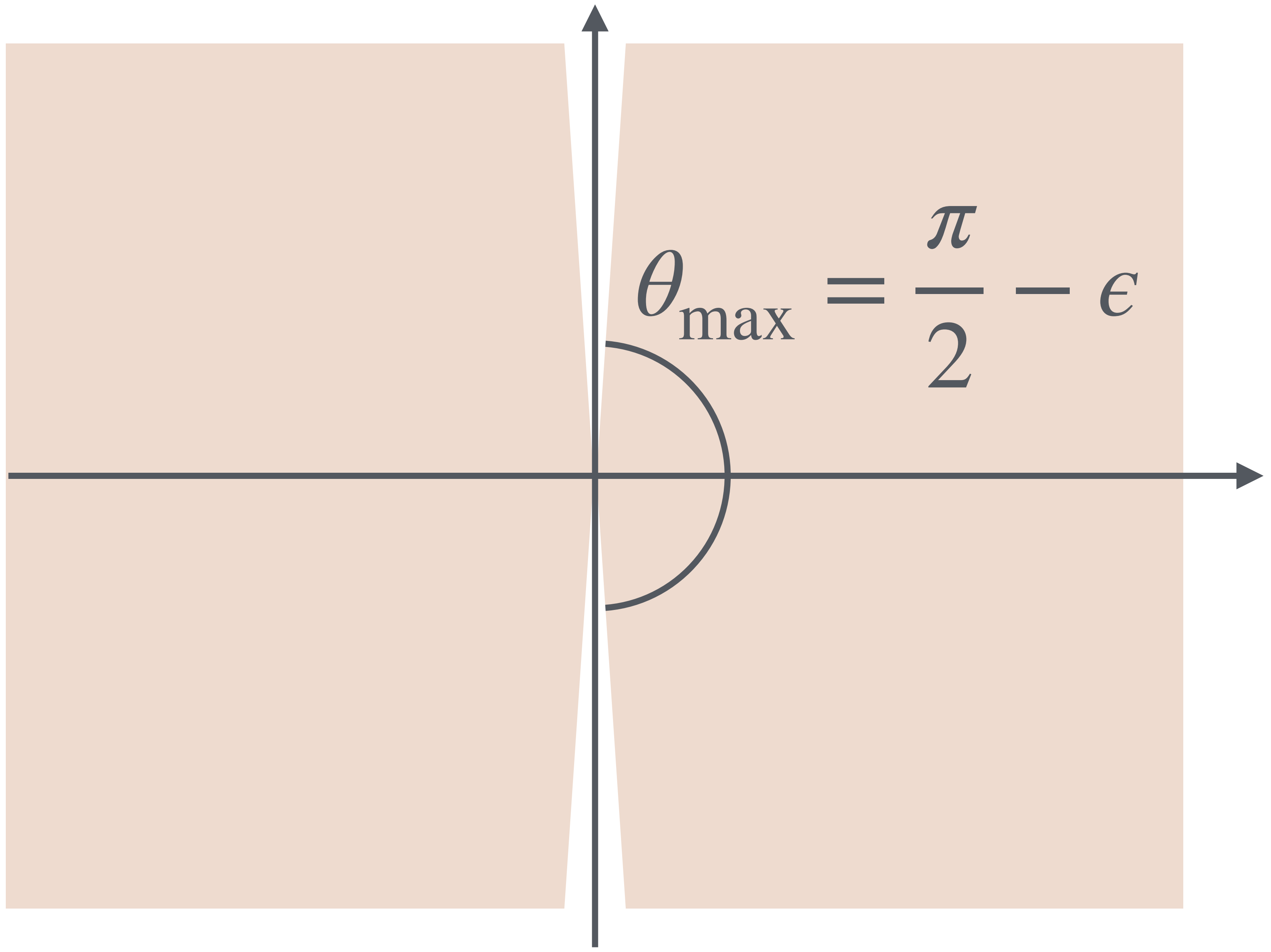
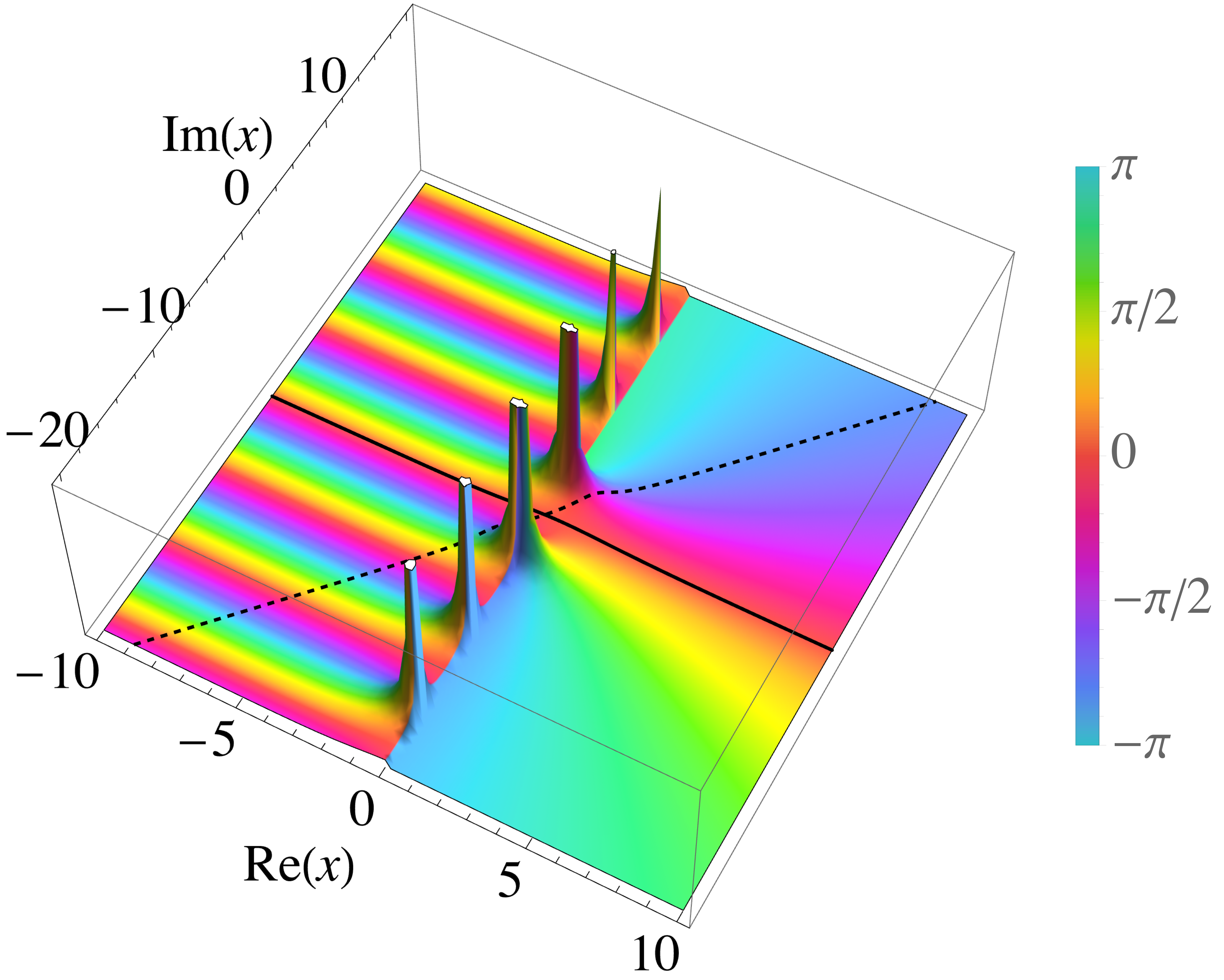
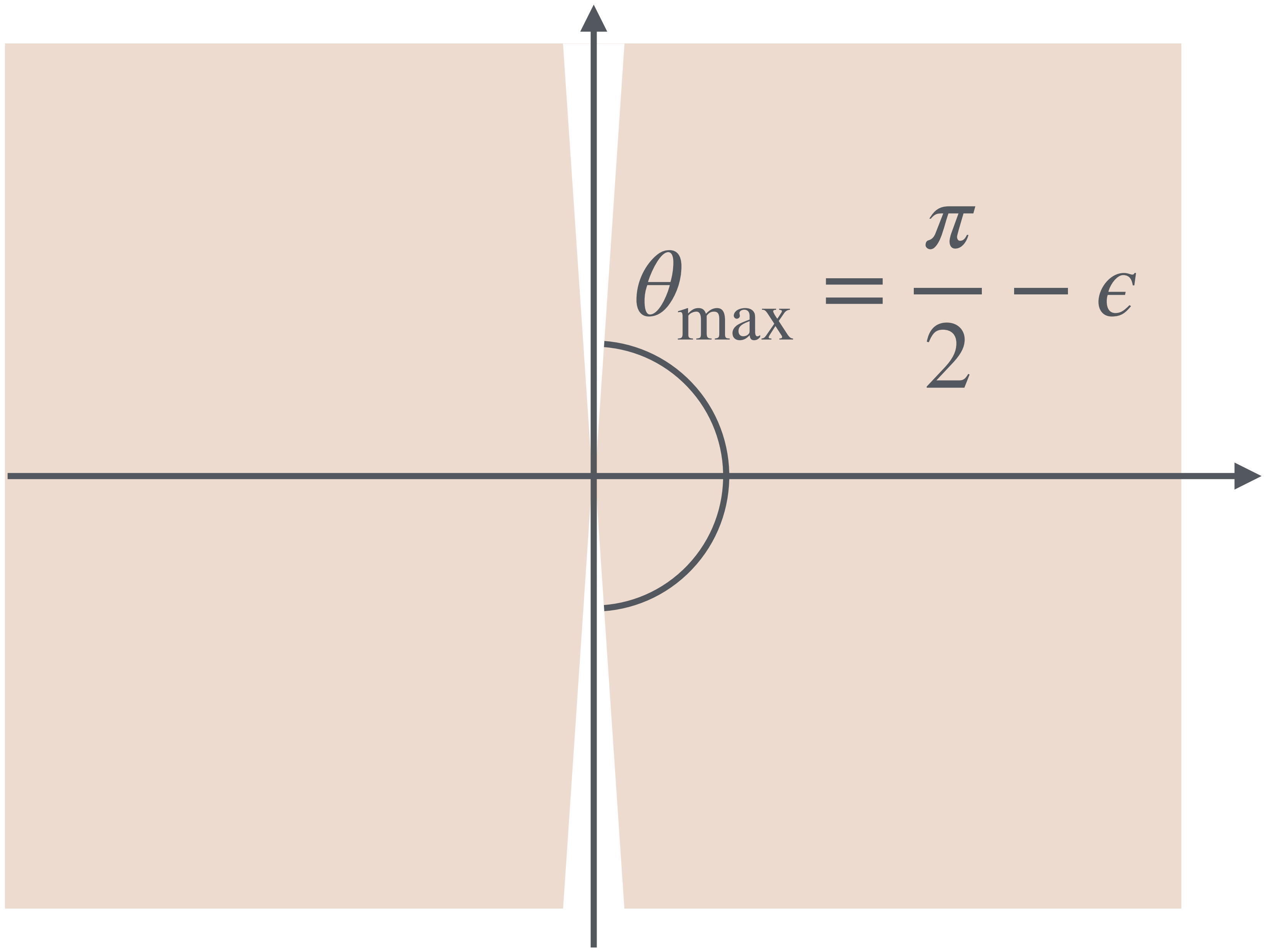
$$V(x \to \infty) \lesssim \frac{1}{|x|}$$

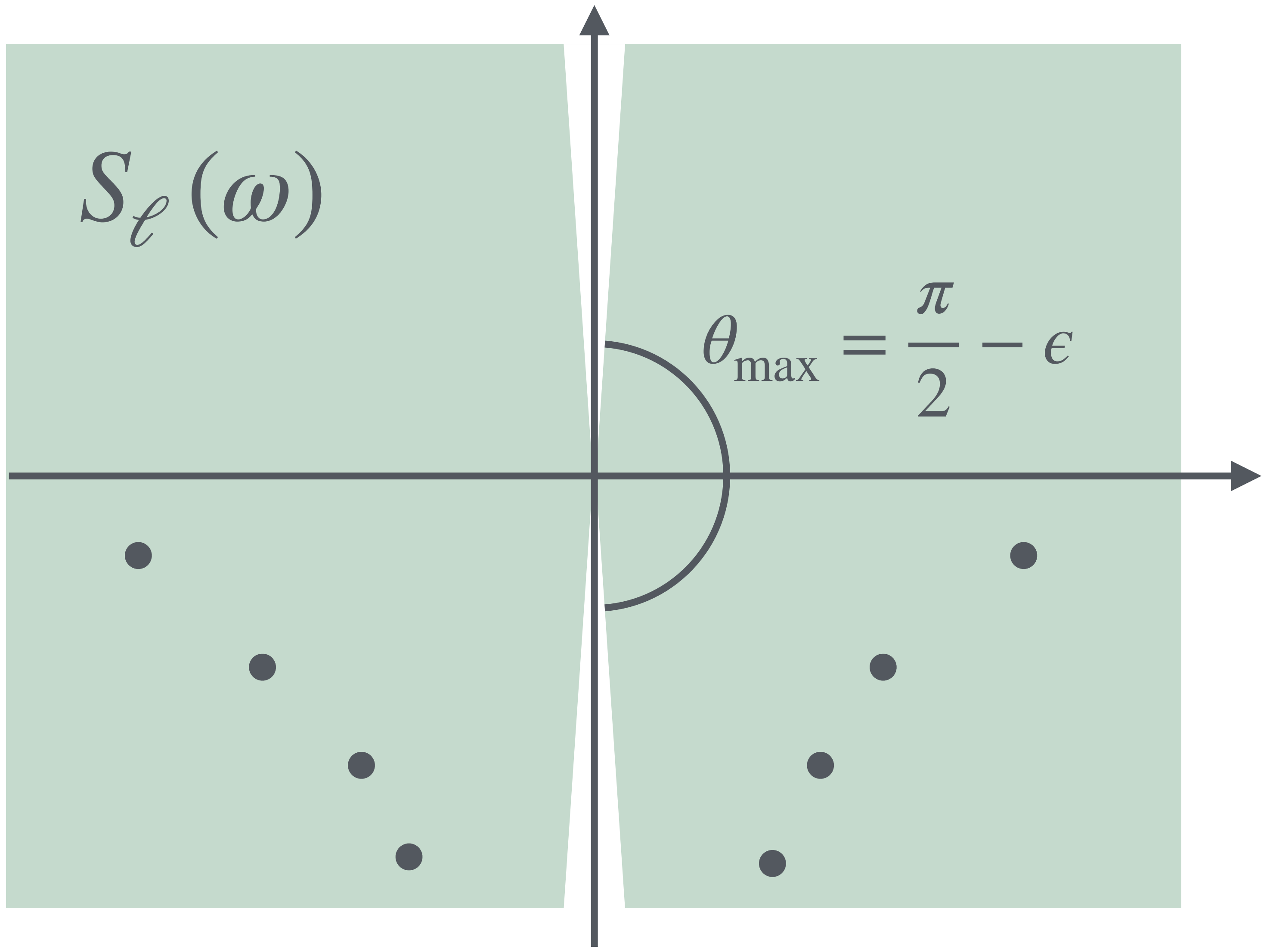


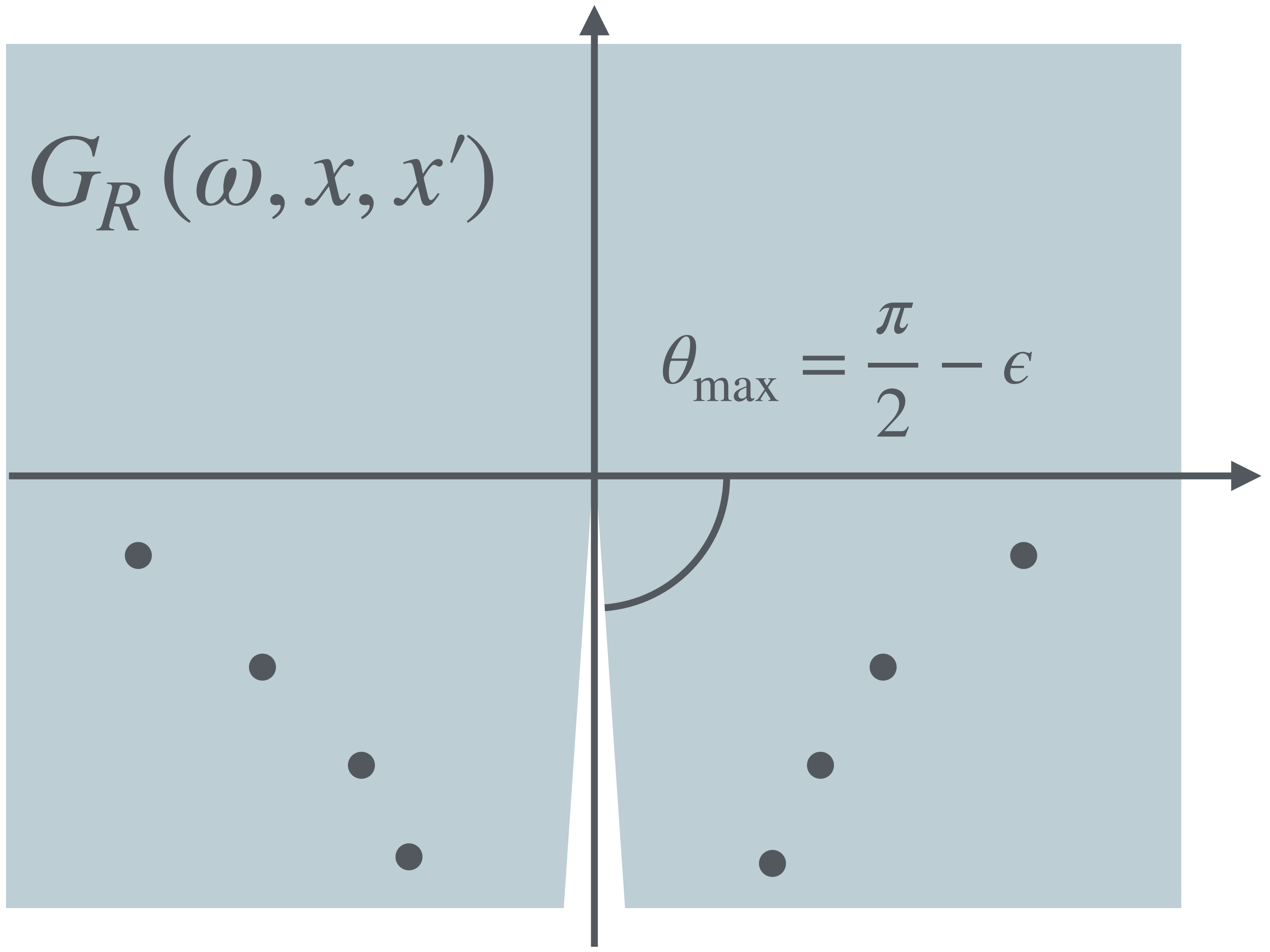


Proven analyticity of the Black Hole amplitude

$$V(x \to \infty) \lesssim \frac{1}{|x|}$$



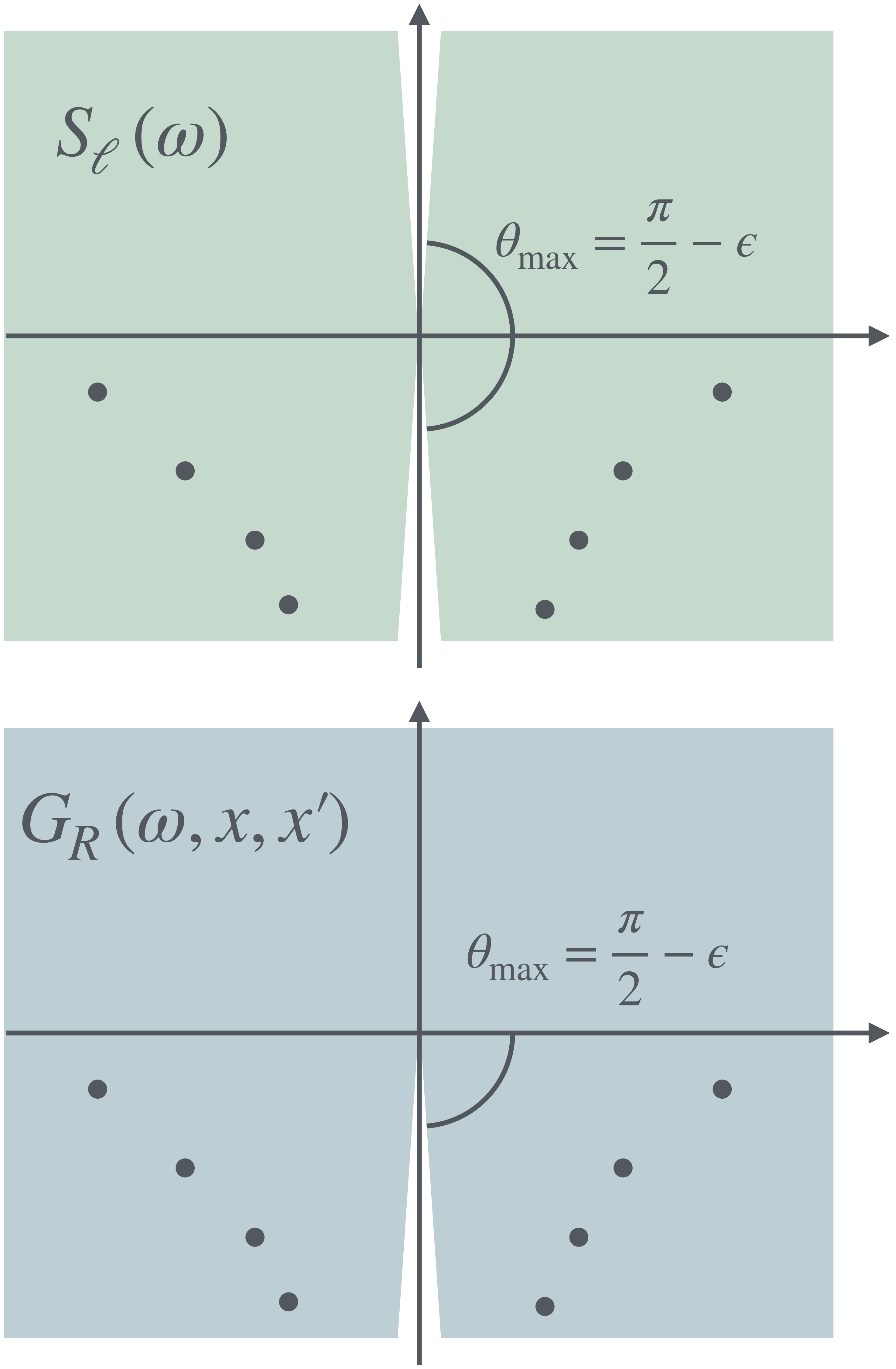




Proven analyticity of the Black Hole amplitude

- Explore solvable regimes
  - Small frequency expansion
  - High frequency limit
  - Newtonian potential
- Consistency with

$$G_R(\omega, x, x') \xrightarrow{x, x' \to \infty} -\frac{1}{2i\omega}\left[e^{i\omega(x-x')} - S_\ell(\omega)\, e^{i\omega(x+x')}\right]$$



Explicit solutions

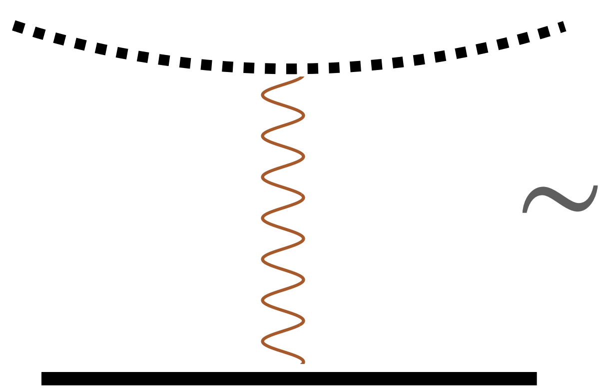
Small frequency expansion $S_\ell = e^{2i\delta_\ell}$

[MST, 1976]
[Dodelson,Grassi,Iossa,Lichtig,Zhiboedov, 2022]

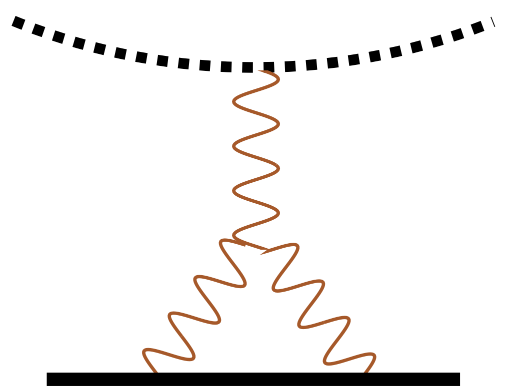
$\ell = 0$

$V \sim \frac{GM}{r}$

$$\delta_0(\omega) = R_s\,\omega\left(\frac{1}{2}\log(4R_s^2\omega^2) + \gamma_E - \frac{1}{2}\right) + R_s^2\,\omega\left(\frac{11\pi}{12}\sqrt{\omega^2} + i\omega\right)$$



$\sim \log\omega$



$\sim \sqrt{\omega^2}$

Small frequency expansion

$S_\ell = e^{2i\delta_\ell}$

[MST, 1976]
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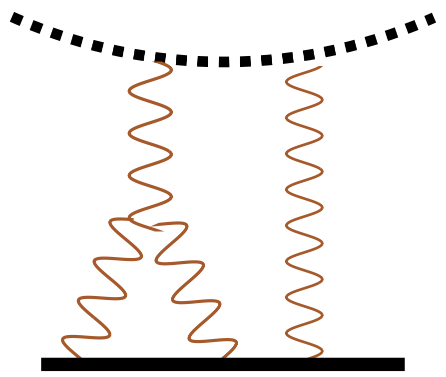
$$+R_s^3\,\omega^2\left[\left(\frac{1}{2} - \frac{1}{2}\log(4R_s^2\omega^2) + \frac{11}{6}\zeta_2 - \frac{1}{3}\zeta_3 - \gamma_E\right)\omega + i\pi\sqrt{\omega^2}\right]$$



UV divergence running

Small frequency expansion $S_\ell = e^{2i\delta_\ell}$

$\ell = 0$

[MST, 1976]
[Dodelson,Grassi,Iossa,Lichtig,Zhiboedov, 2022]

$$\delta_0(\omega) = R_s\,\omega\left(\frac{1}{2}\log(4R_s^2\omega^2) + \gamma_E - \frac{1}{2}\right) + R_s^2\,\omega\left(\frac{11\pi}{12}\sqrt{\omega^2} + i\omega\right)$$

$$+R_s^3\,\omega^2\left[\left(\frac{1}{2} - \frac{1}{2}\log(4R_s^2\omega^2) + \frac{11}{6}\zeta_2 - \frac{1}{3}\zeta_3 - \gamma_E\right)\omega + i\pi\sqrt{\omega^2}\right]$$

$$+R_s^4\omega^3\left[\left(-\frac{\pi}{2}\left[\log(4R_s^2\,\omega^2) + 2\gamma_E\right] + \frac{2299\pi}{2160}\right)\sqrt{\omega^2}\right.$$

$$\left. + \left(\frac{11}{6}\left[\log(4R_s^2\,\omega^2) + 2\gamma_E\right] + 4\zeta_2 + \frac{227}{36}\right) i\omega\right]$$

Fixed by:

- Unitarity

  $$|S_\ell(\omega)| \leq 1$$

- Crossing

  $$S_\ell^*(\omega) = S_\ell(-\omega^*)$$

Small frequency expansion

$S_\ell = e^{2i\delta_\ell}$

$\ell = 0$

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$$\delta_0(\omega) = R_s\,\omega\left(\frac{1}{2}\log(4R_s^2\omega^2) + \gamma_E - \frac{1}{2}\right) + R_s^2\,\omega\left(\frac{11\pi}{12}\sqrt{\omega^2} + i\omega\right)$$
$$+R_s^3\,\omega^2\left[\left(\frac{1}{2} - \frac{1}{2}\log(4R_s^2\omega^2) + \frac{11}{6}\zeta_2 - \frac{1}{3}\zeta_3 - \gamma_E\right)\omega + i\pi\sqrt{\omega^2}\right]$$
$$+R_s^4\omega^3\left[\left(-\frac{\pi}{2}\left[\log(4R_s^2\,\omega^2) + 2\gamma_E\right] + \frac{2299\pi}{2160}\right)\sqrt{\omega^2}\right.$$
$$\left.+\left(\frac{11}{6}\left[\log(4R_s^2\,\omega^2) + 2\gamma_E\right] + 4\zeta_2 + \frac{227}{36}\right)i\omega\right]$$

Fixed by:

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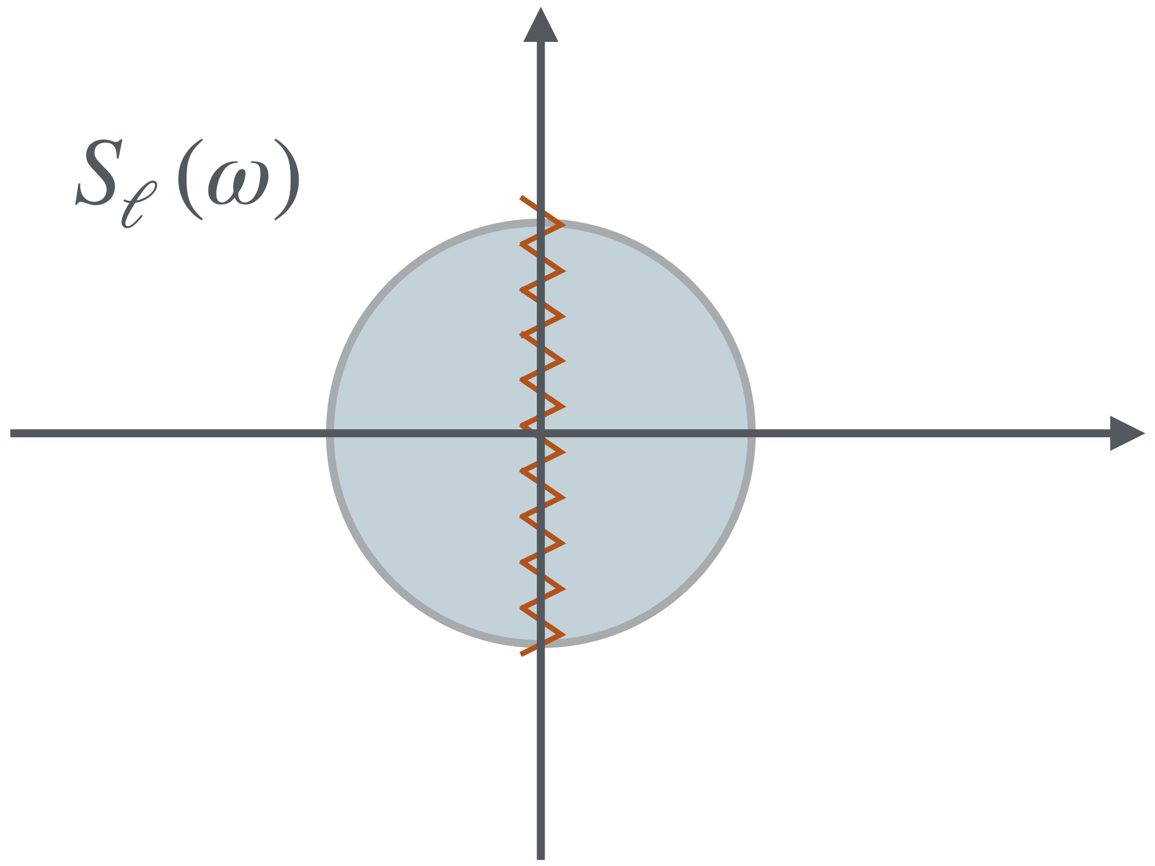


Large frequency limit of Regge-Wheeler $\quad |GM\omega| \gg 1$

gr-qc/9607064 [N. Andersson]

$$\phi_L = \left(\frac{r}{2GM}\right)^{1/2}\left(\frac{r}{2GM}-1\right)^{-2i\omega GM} M\left(1/2-4i\omega GM, 1-4i\omega GM, -2i\omega(r-2GM)\right) e^{i\omega(r-4GM)}$$

$$\phi_R = (-4iGM\omega)^{1/2-4iGM\omega}\left(\frac{r}{2M}\right)^{1/2}\left(\frac{r}{2GM}-1\right)^{-4iGM\omega} U\left(1/2-4iGM\omega, 1-4iGM\omega, -2i\omega(r-2GM)\right) e^{i\omega x}$$

Confluent Hypergeometric functions

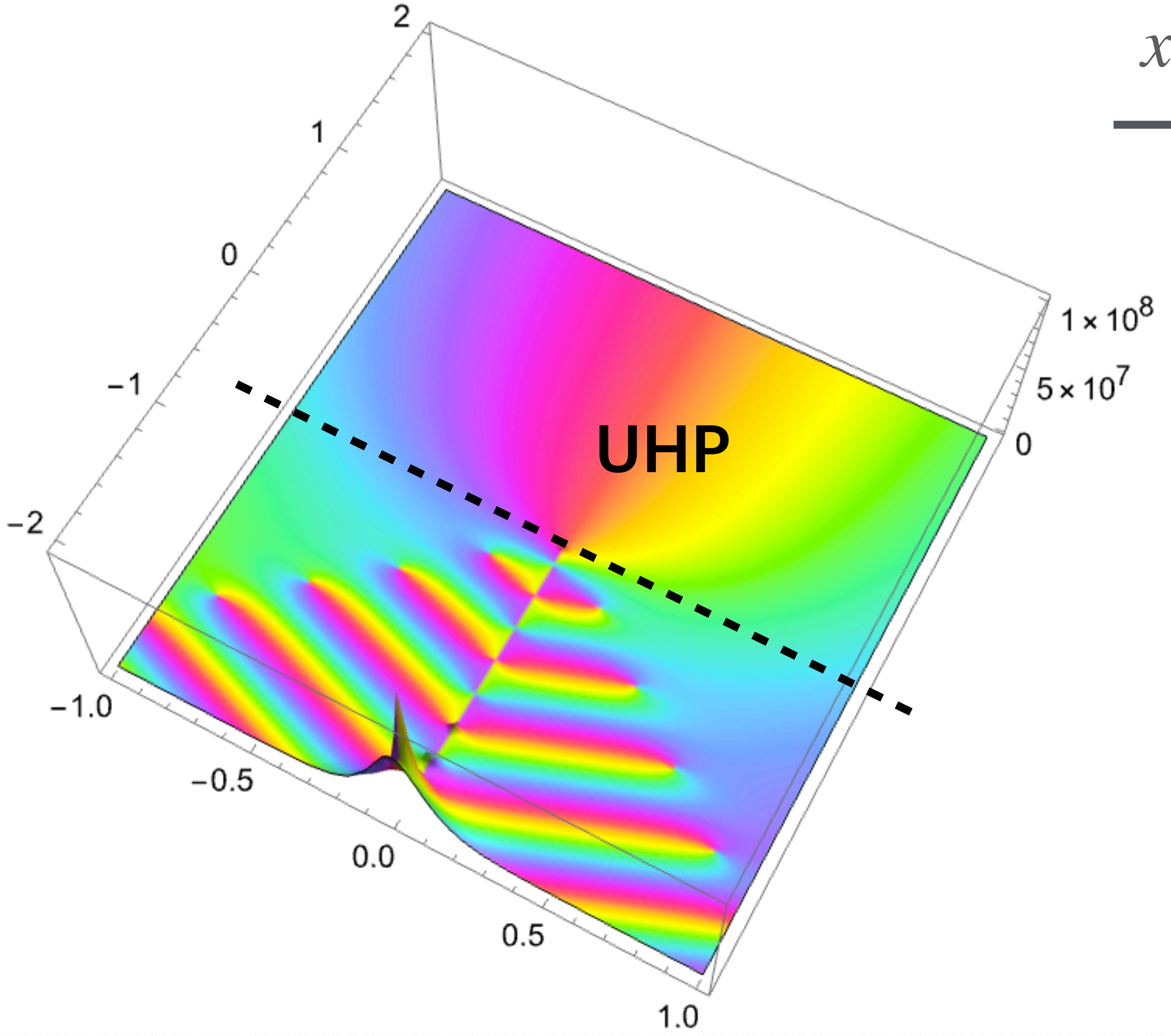
$$G_R(\omega, x, x') \xrightarrow{x, x' \to \infty} S_\ell(\omega) = -\frac{2^{8iGM\omega} e^{-4iGM\omega}\sqrt{-iGM\omega}\,(iGM\omega)^{-1/2+4iGM\omega}\,\Gamma(1/2-4iGM\omega)}{\sqrt{\pi}}$$

Large frequency limit of Regge-Wheeler $|GM\omega| \gg 1$

gr-qc/9607064 [N. Andersson]

Retarded Green's function

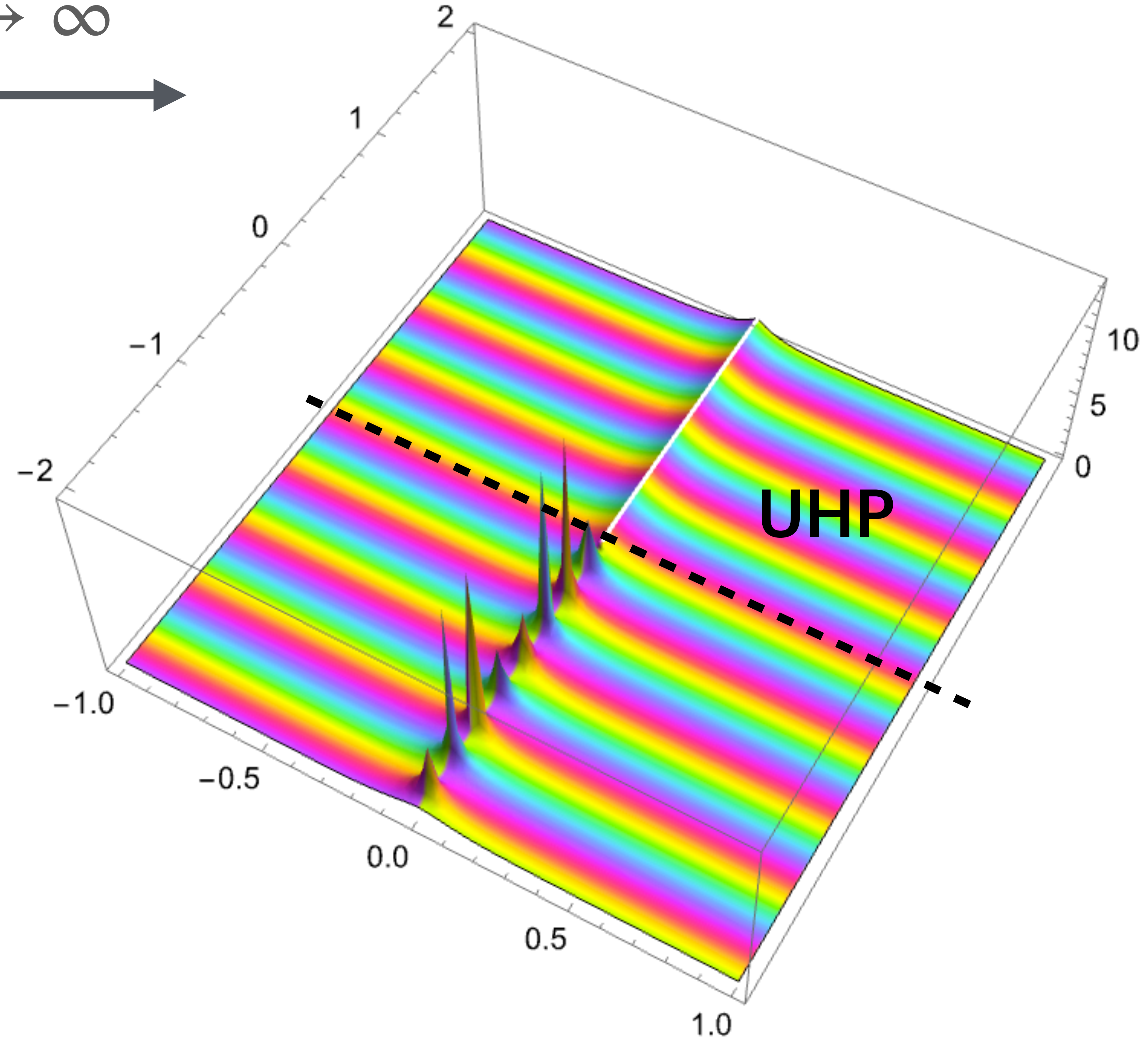
$G_R(\omega, x, x')$



$x, x' \to \infty$

S-matrix

$S_\ell(\omega)$



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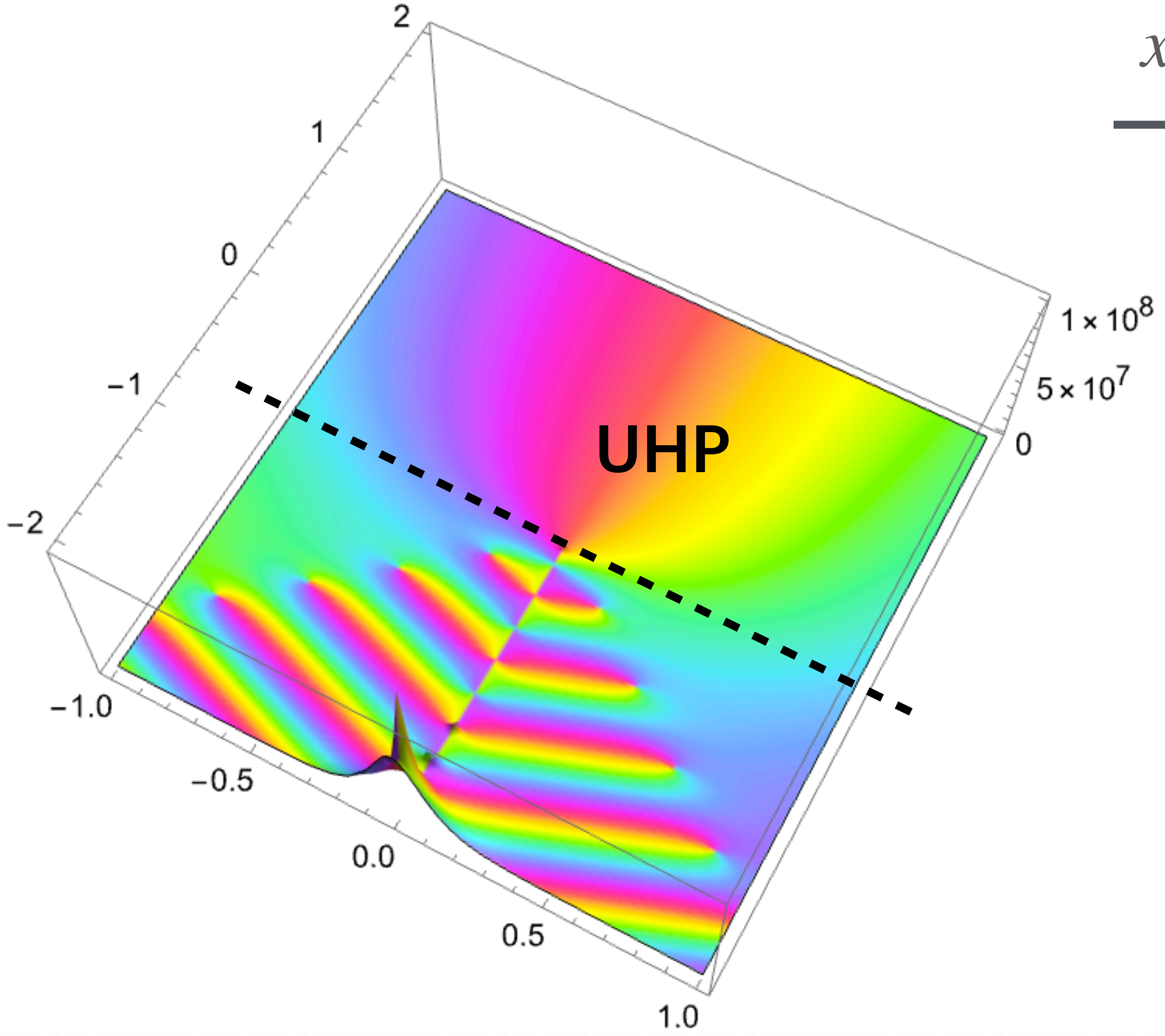

$x, x' \to \infty$

S-matrix

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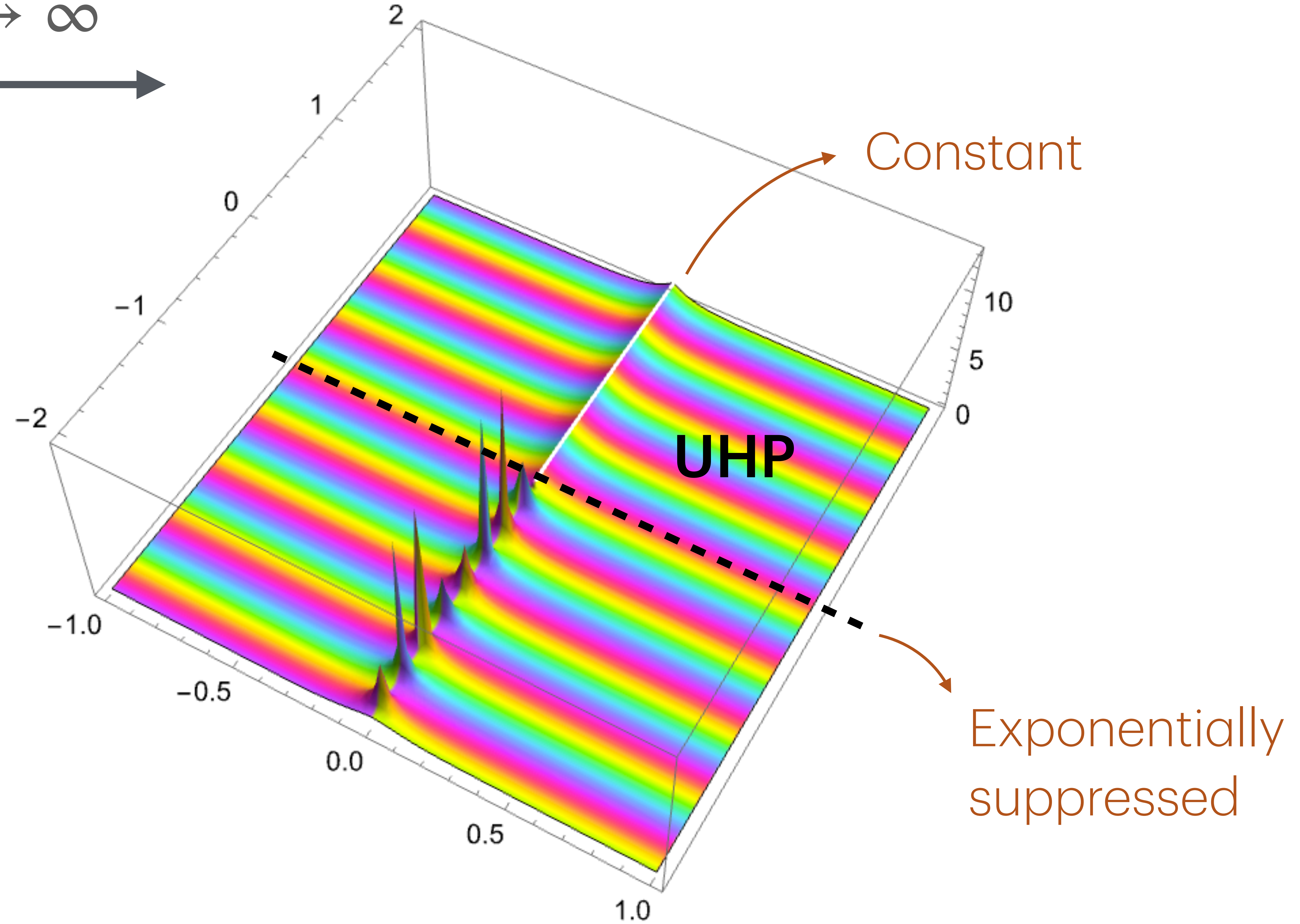


Newtonian potential $V \sim \frac{4GM\omega^2}{r}$

$$\phi_L \sim \left(\frac{r}{GM}\right)^{\ell+1} e^{i\omega r} M\left(\ell + 1 - 2iGM\omega, 2\ell + 2, -2i\omega r\right)$$

$$\phi_R \sim \left(\frac{r}{GM}\right)^{\ell+1} e^{i\omega r} U\left(\ell + 1 - 2iGM\omega, 2\ell + 2, -2i\omega r\right)$$

Confluent Hypergeometric functions

$x > x'$

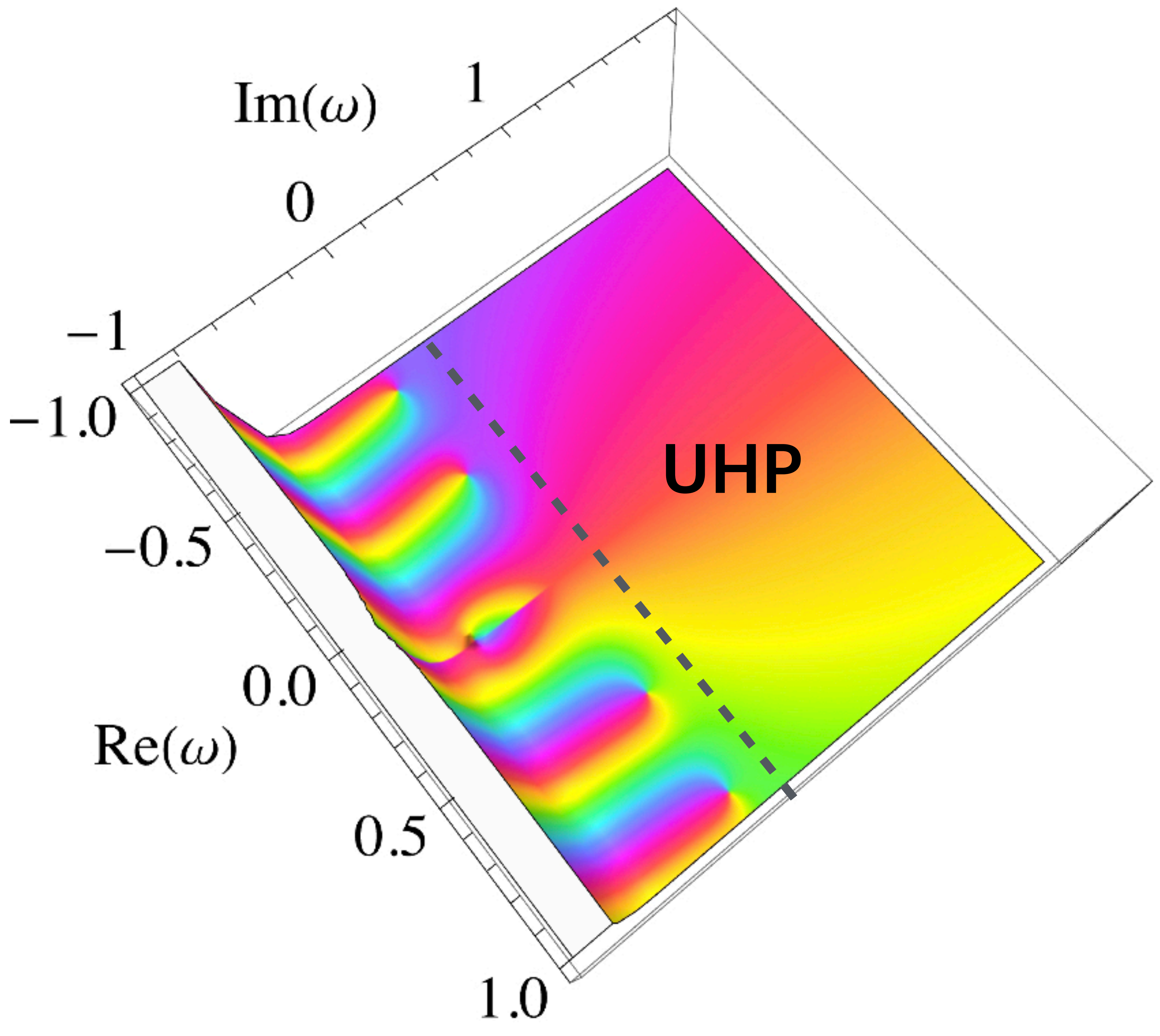
$$G_R(x, x', \omega) = -\frac{1}{2i\omega A_{\text{in}}} \phi_L(x, \omega)\, \phi_R(x', \omega)$$

$$G_R(\omega, x, x') \xrightarrow{x, x' \to \infty} S_\ell(\omega) = \frac{\left[(2GM\omega)^2\right]^{2iGM\omega} \Gamma\left(1 + \ell - 2iGM\omega\right)}{\Gamma\left(1 + \ell + 2iGM\omega\right)}$$

Newtonian potential $V \sim \frac{4GM\omega^2}{r}$

Retarded Green's function

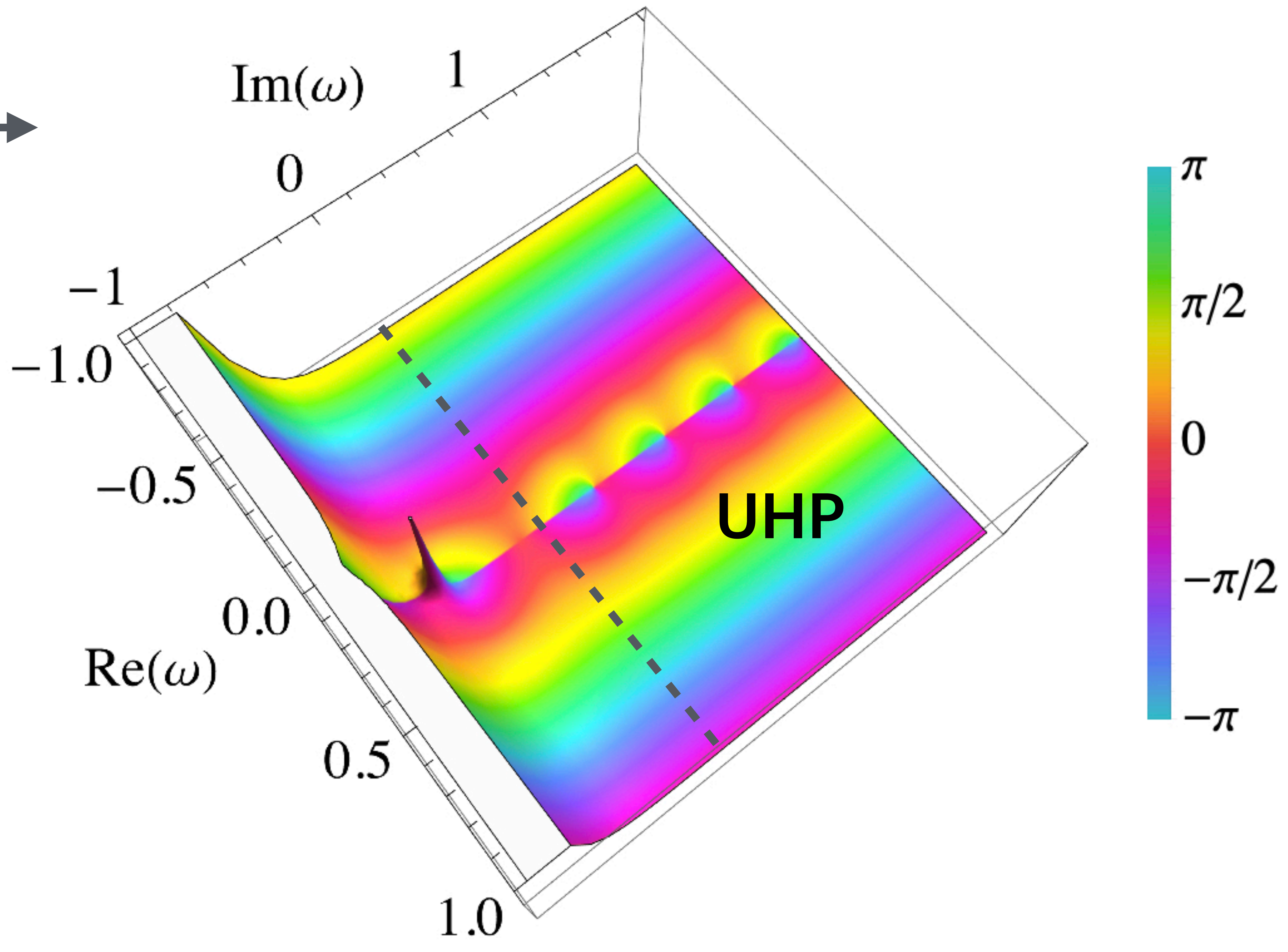
$G_R(\omega, x, x')$



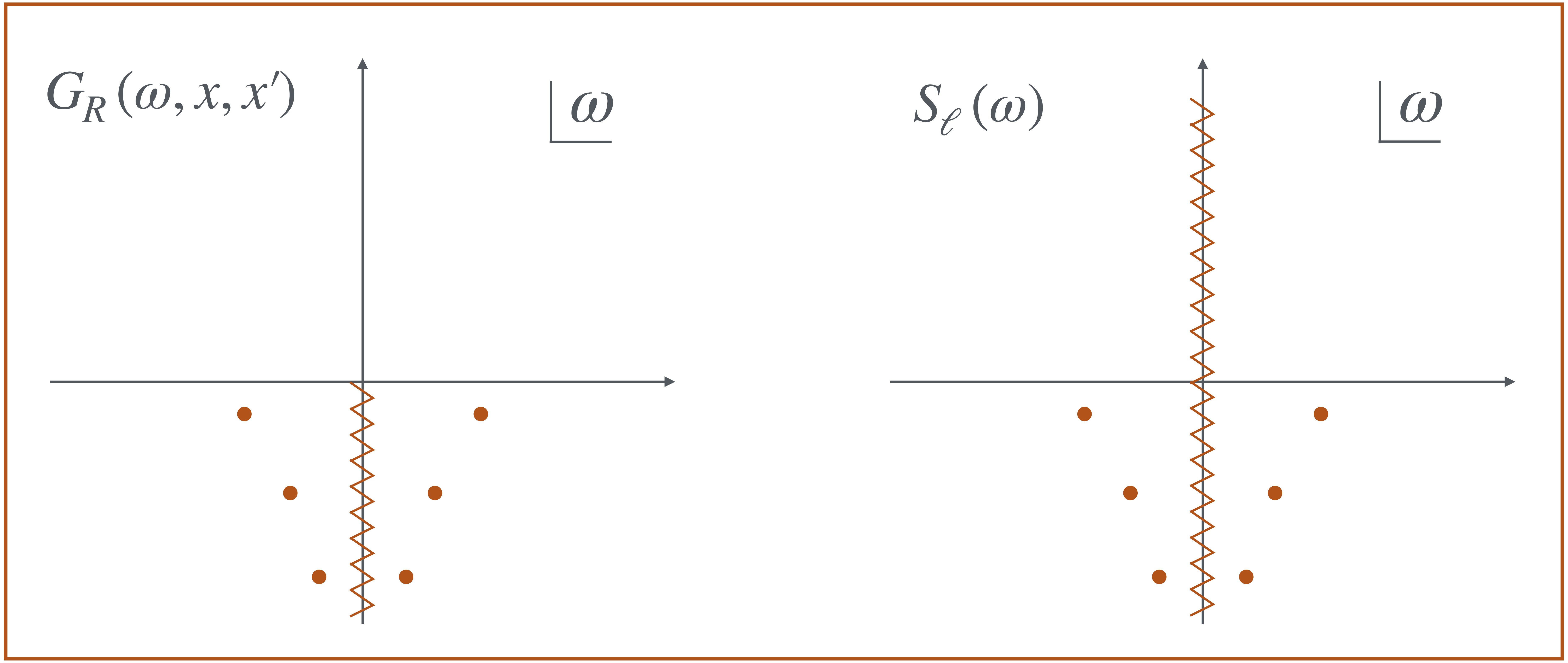
$x, x' \to \infty$

S-matrix

$S_\ell(\omega)$



So what is happening?



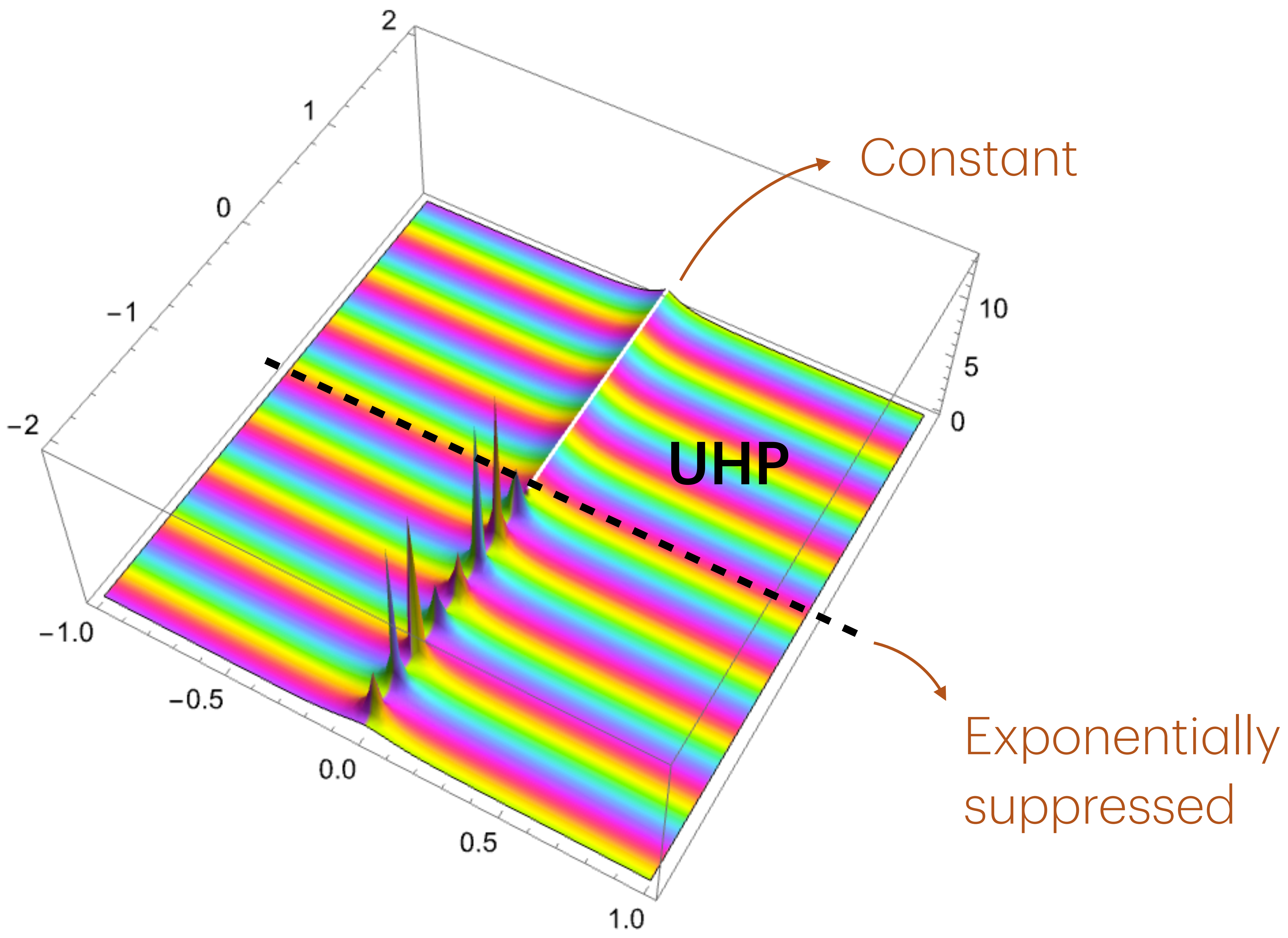
Stokes phenomena

$$G_R(\omega, x, x') \xrightarrow{x, x' \to \infty} -\frac{1}{2i\omega}\left[e^{i\omega(x-x')} - S_\ell(\omega)\, e^{i\omega(x+x')}\right]$$

IR effects

S-matrix at large frequencies in tortoise coordinates

$S_\ell(\omega)$



S-matrix with hard IR cutoff

$$S_\ell^{\text{hard}}(\omega) = S_\ell(\omega)\left(\frac{R_{\text{IR}}}{R_s}\right)^{2iR_s\omega}$$

$R_{\text{IR}} < R_s$: Exponential growth in the complex plane

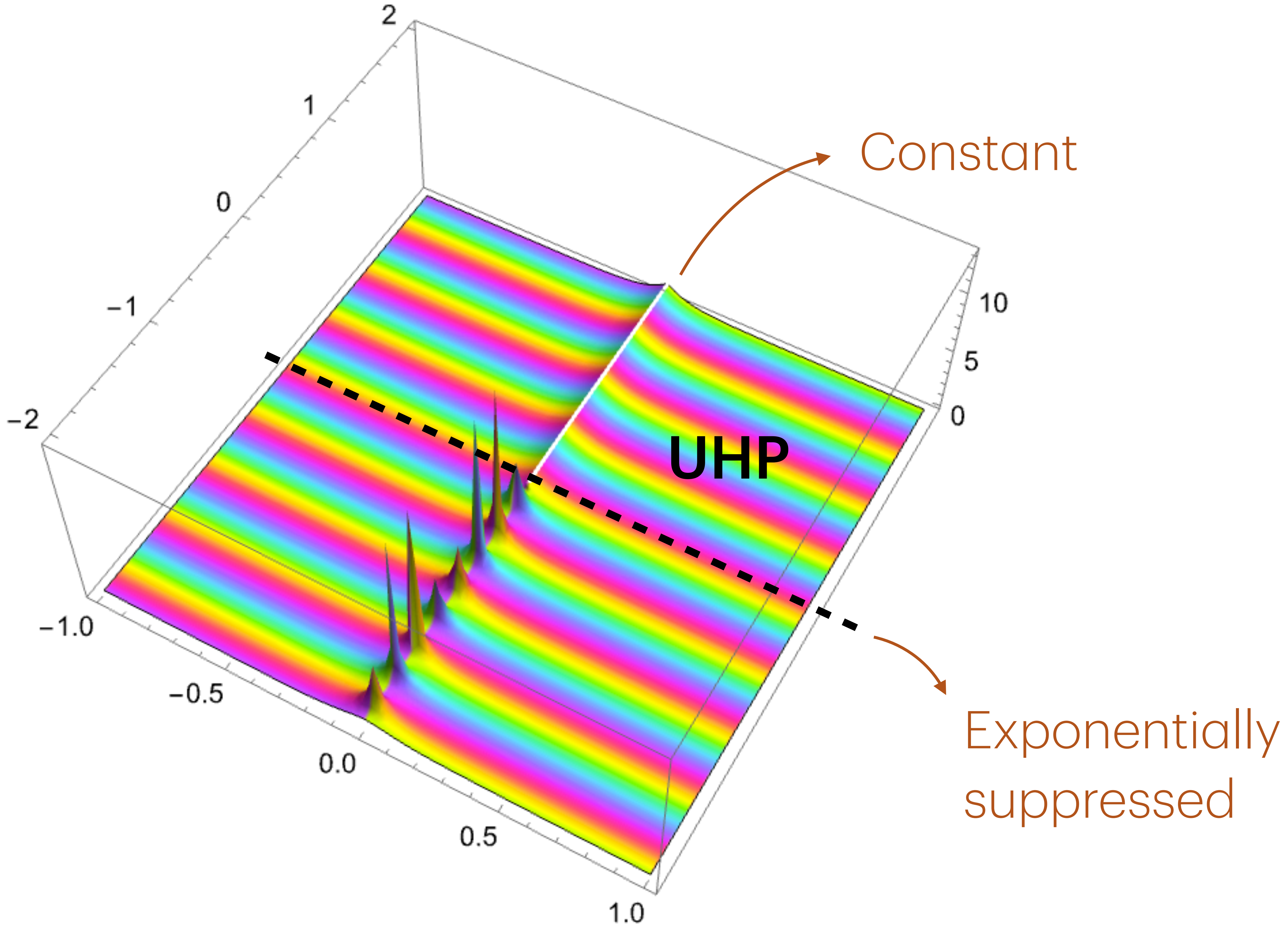
⇝

Polynomial boundedness

IR effects

S-matrix at large frequencies in tortoise coordinates

$S_\ell(\omega)$



S-matrix with hard IR cutoff

$$S_\ell^{\text{hard}}(\omega) = S_\ell(\omega)\left(\frac{R_{\text{IR}}}{R_s}\right)^{2iR_s\omega}$$

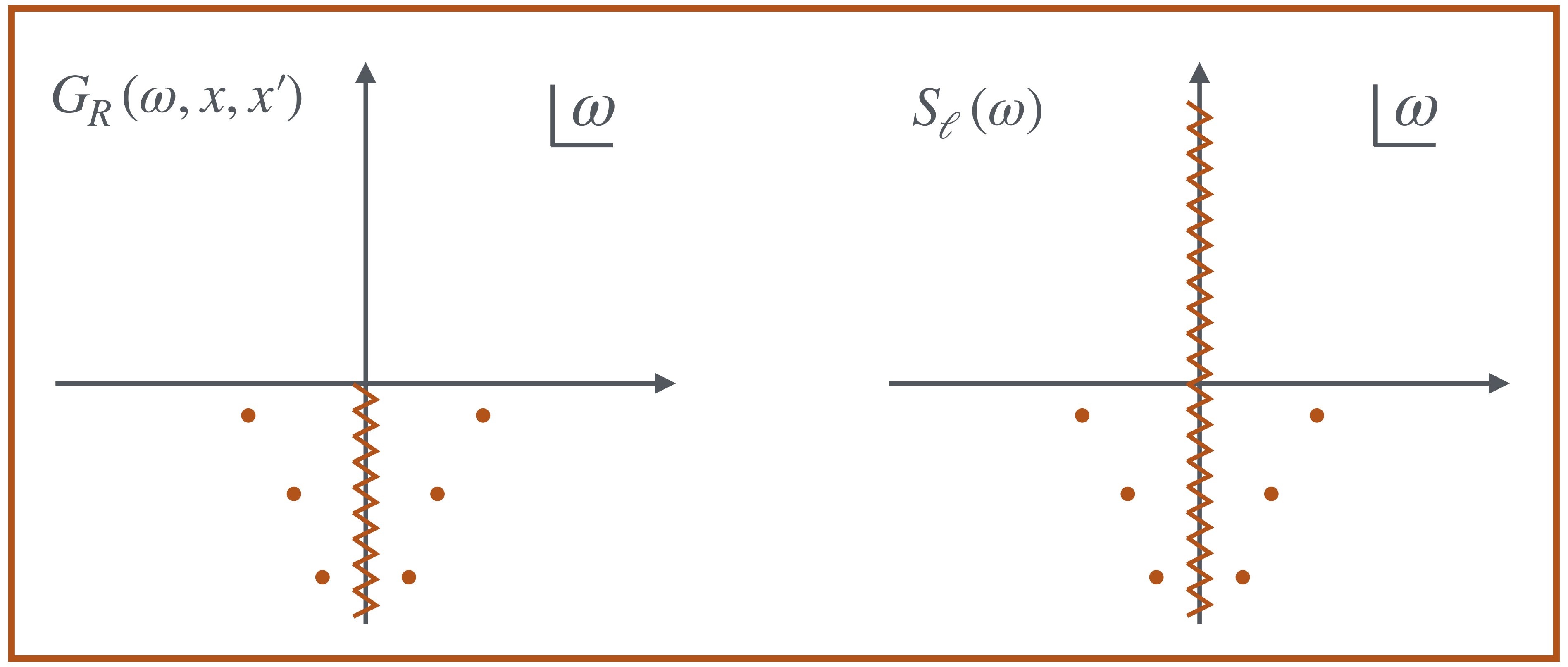
Polynomial boundedness implies a lower bound on the IR cutoff!

$$R_{\text{IR}} \geq R_s$$

➡ **Francesco's talk**

Conclusion

Established the analytic structure of classical wave scattering on a black hole background



- General proof of analyticity
- Verification through exact solutions of Regge-Wheeler equations in different regimes

Conclusion

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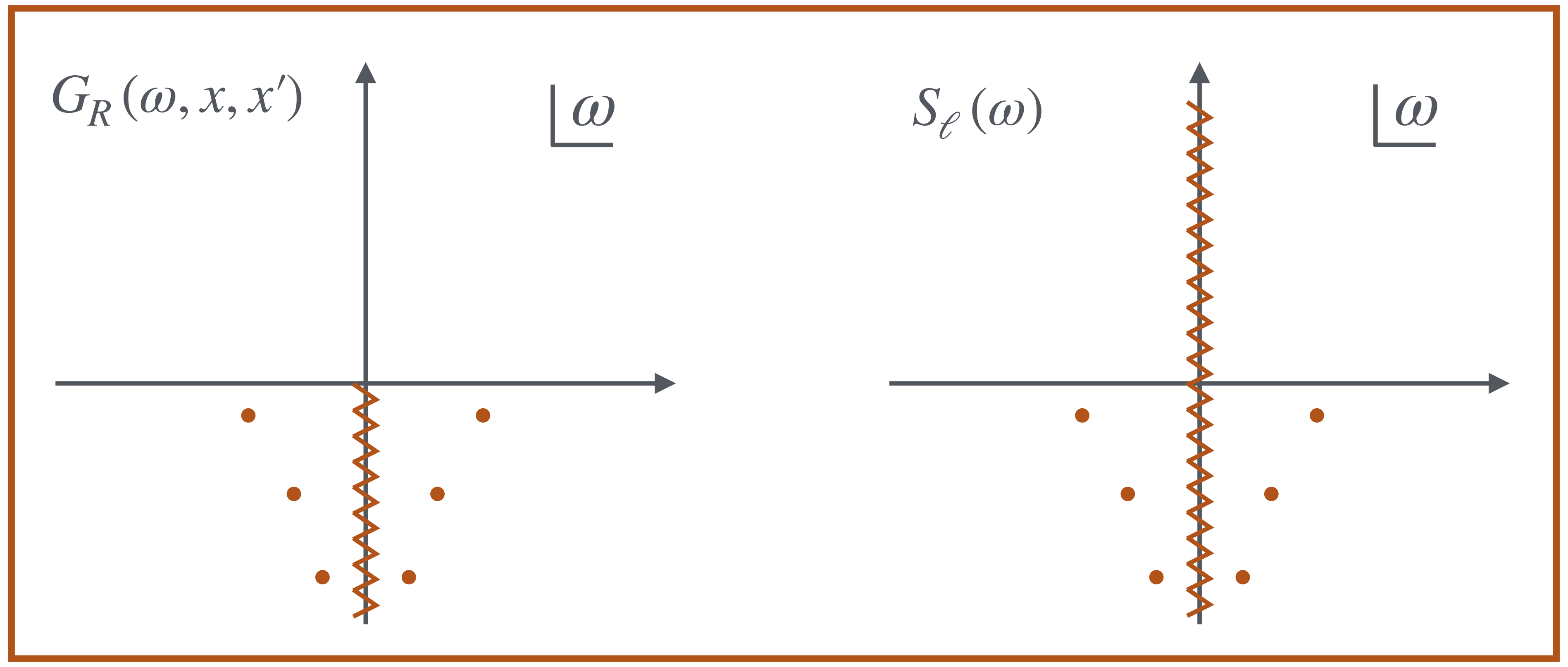

- General proof of analyticity
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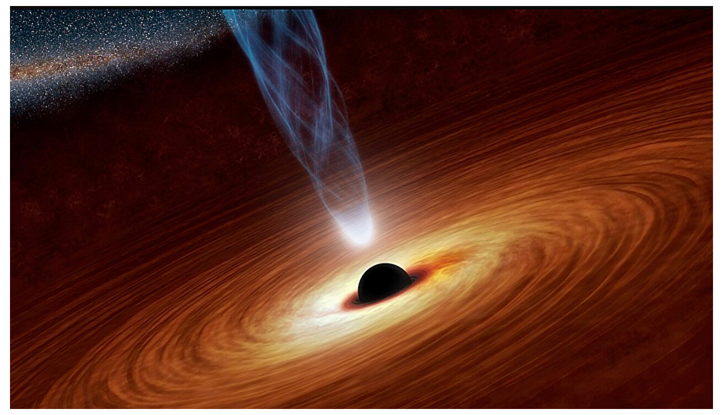
Future directions

- Analyticity of the S-matrix on spinning black holes



Conclusion

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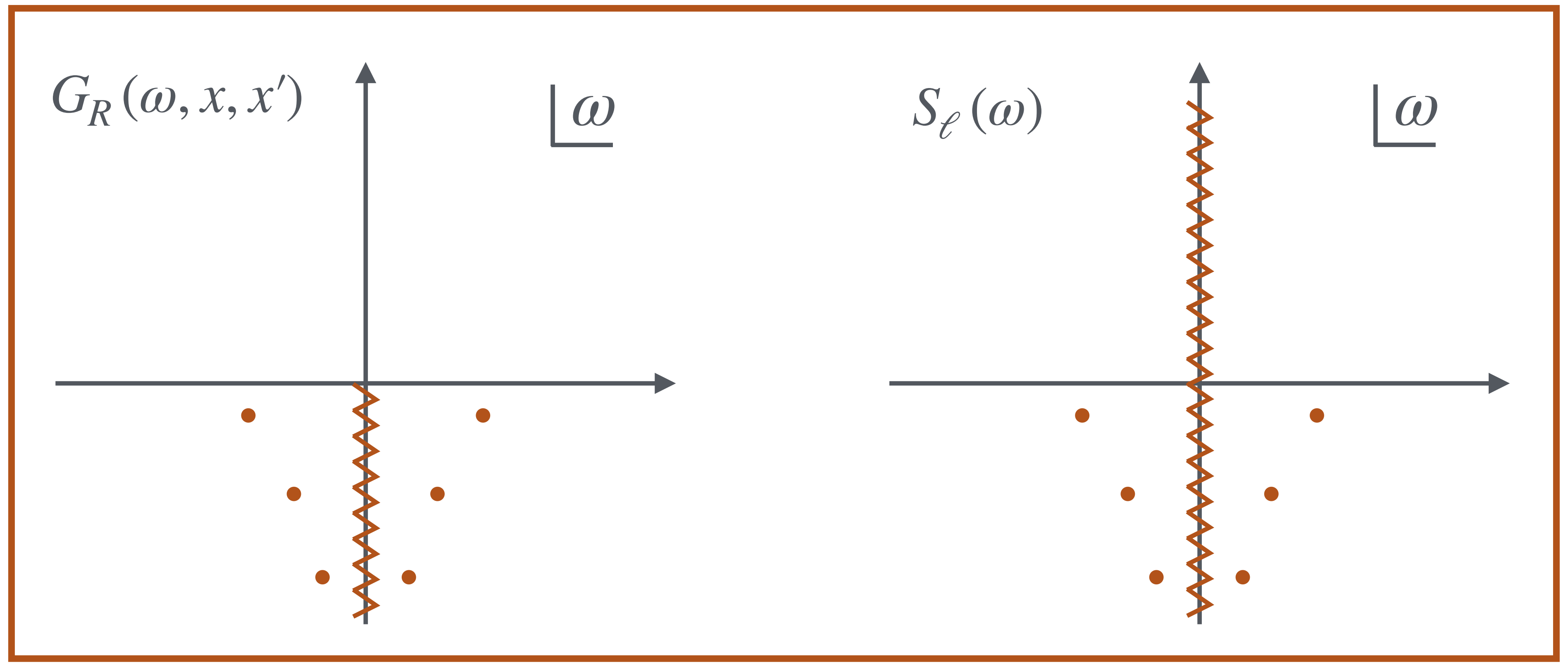

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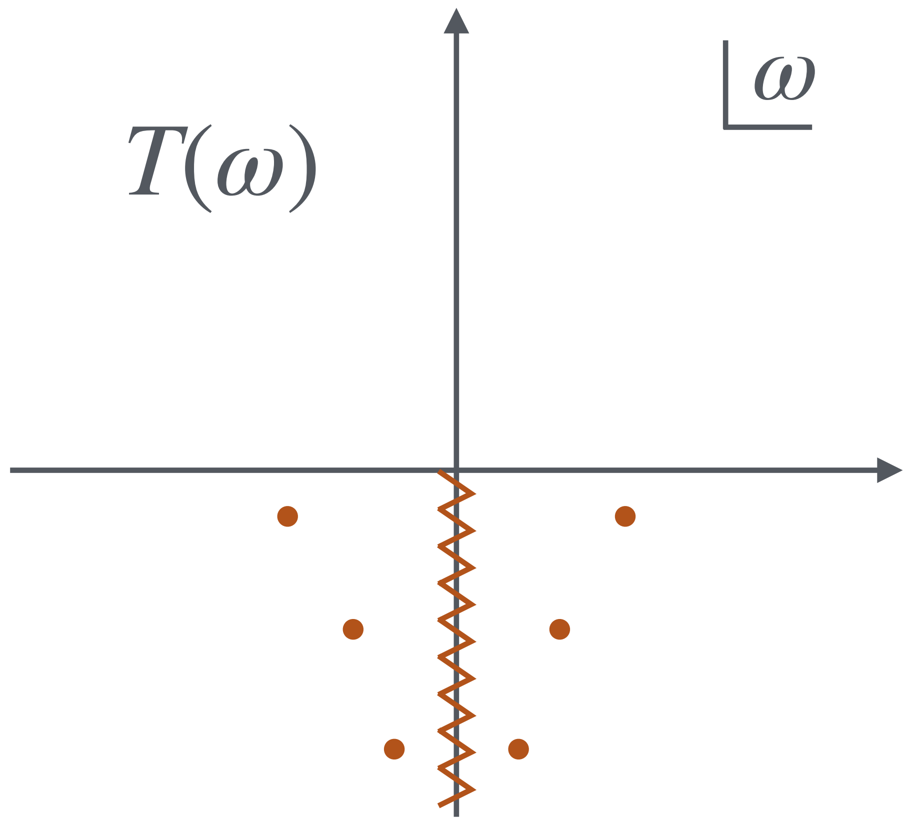
Future directions

- Analyticity of the S-matrix on spinning black holes
- Incorporating this structure on a bootstrap approach to bound Love numbers



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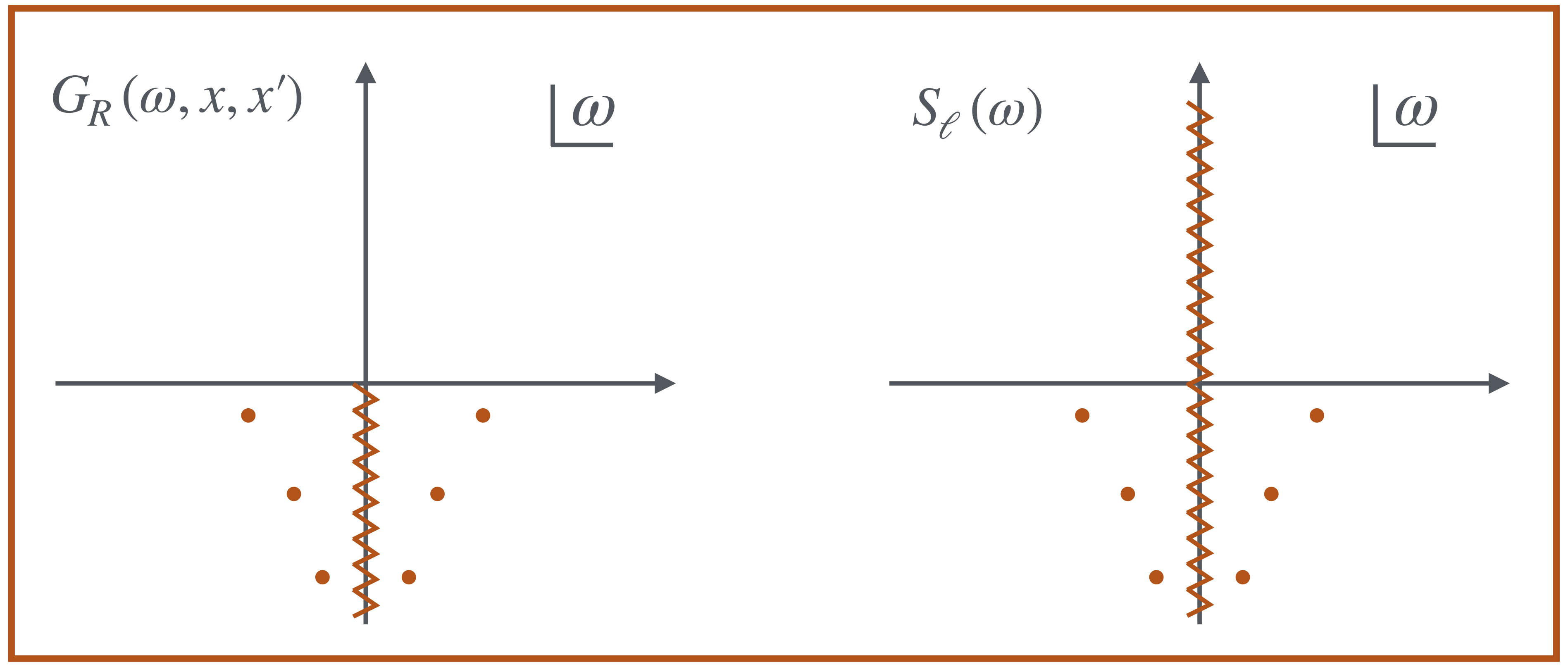

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- Analyticity of the S-matrix on spinning black holes
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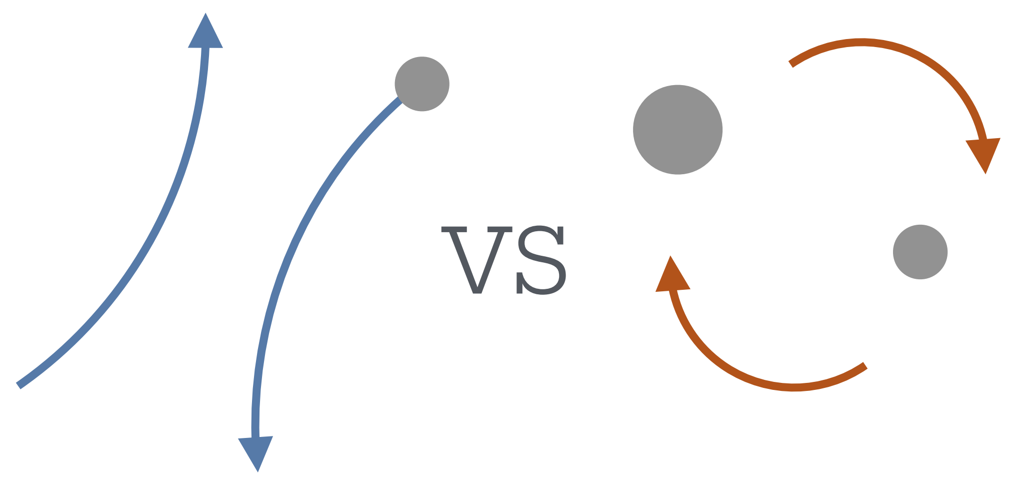
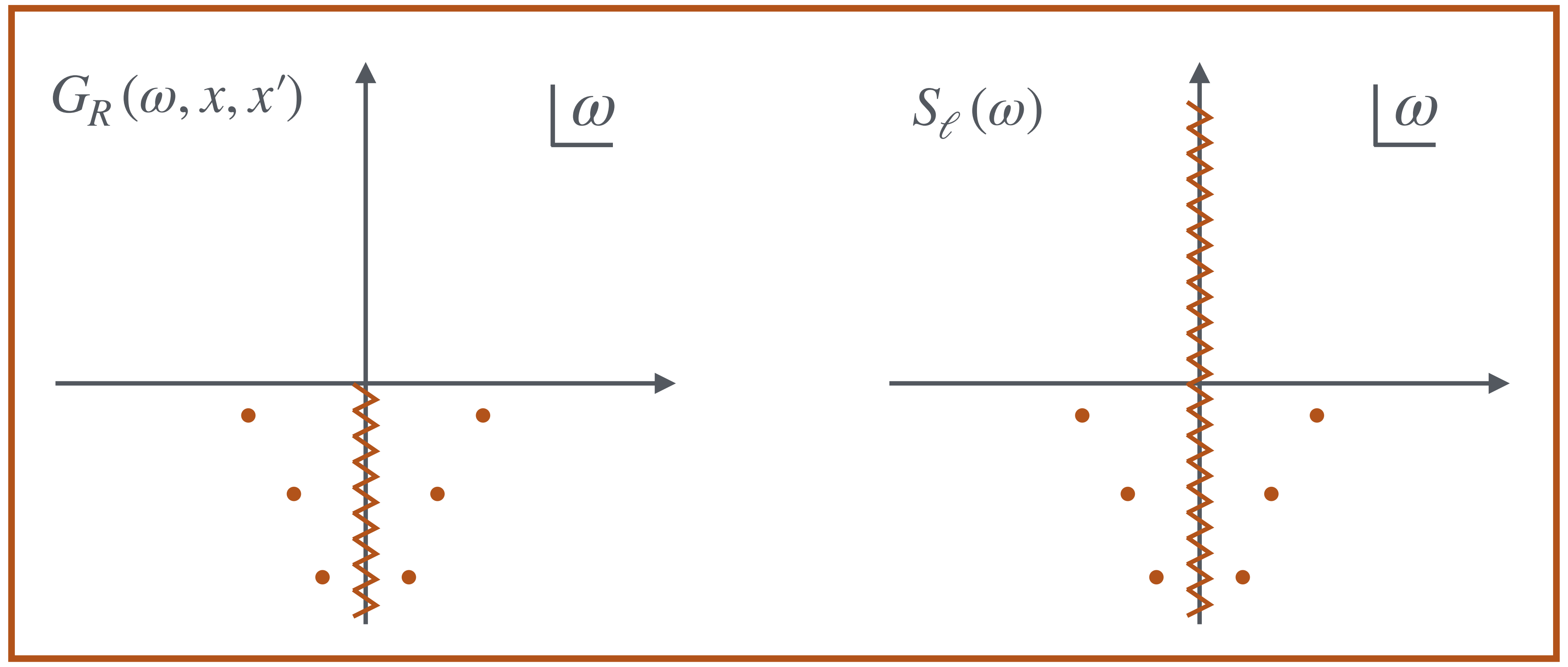


Conclusion

Established the analytic structure of classical wave scattering on a black hole background



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- Analyticity of the S-matrix on spinning black holes
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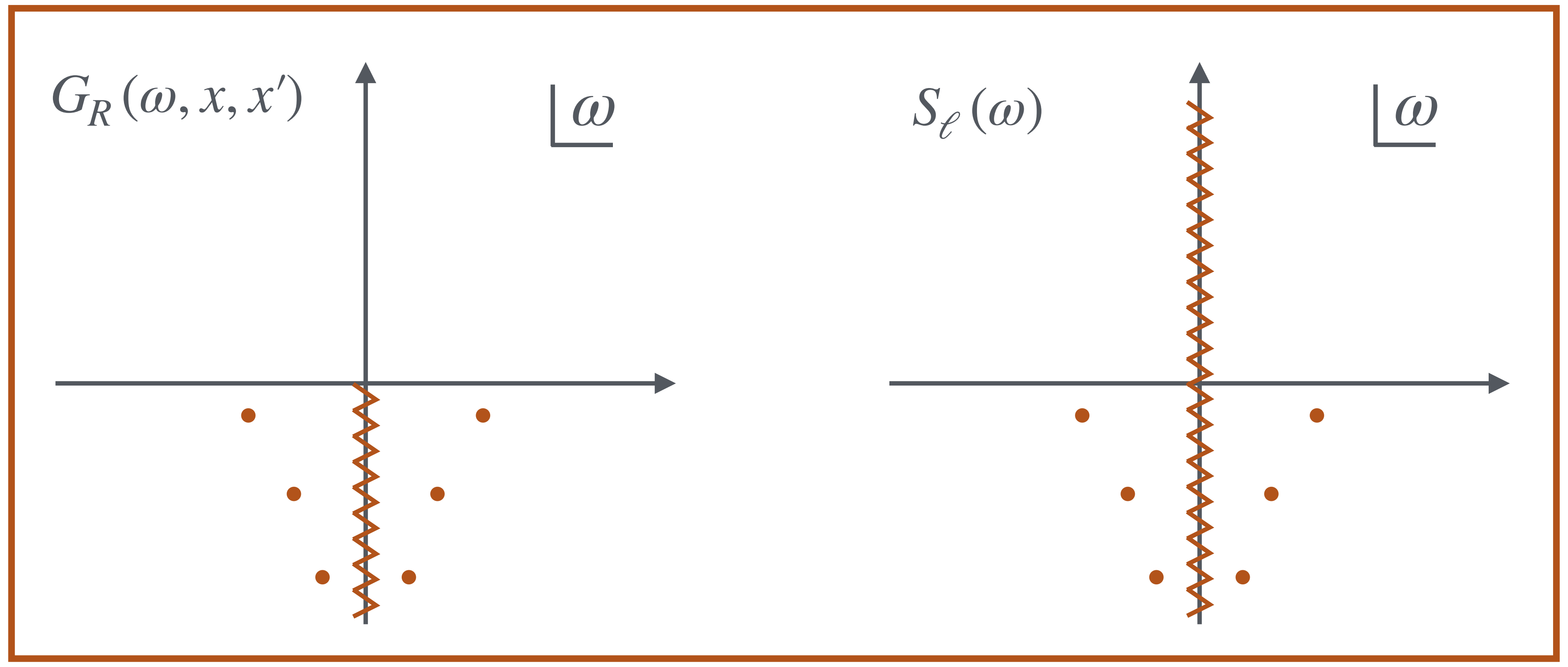
- Bound-to-boundary map in the 2-body problem
- Extension to AdS and non-flat backgrounds

Conclusion

Established the analytic structure of classical wave scattering on a black hole background



- General proof of analyticity
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Future directions

- Analyticity of the S-matrix on spinning black holes
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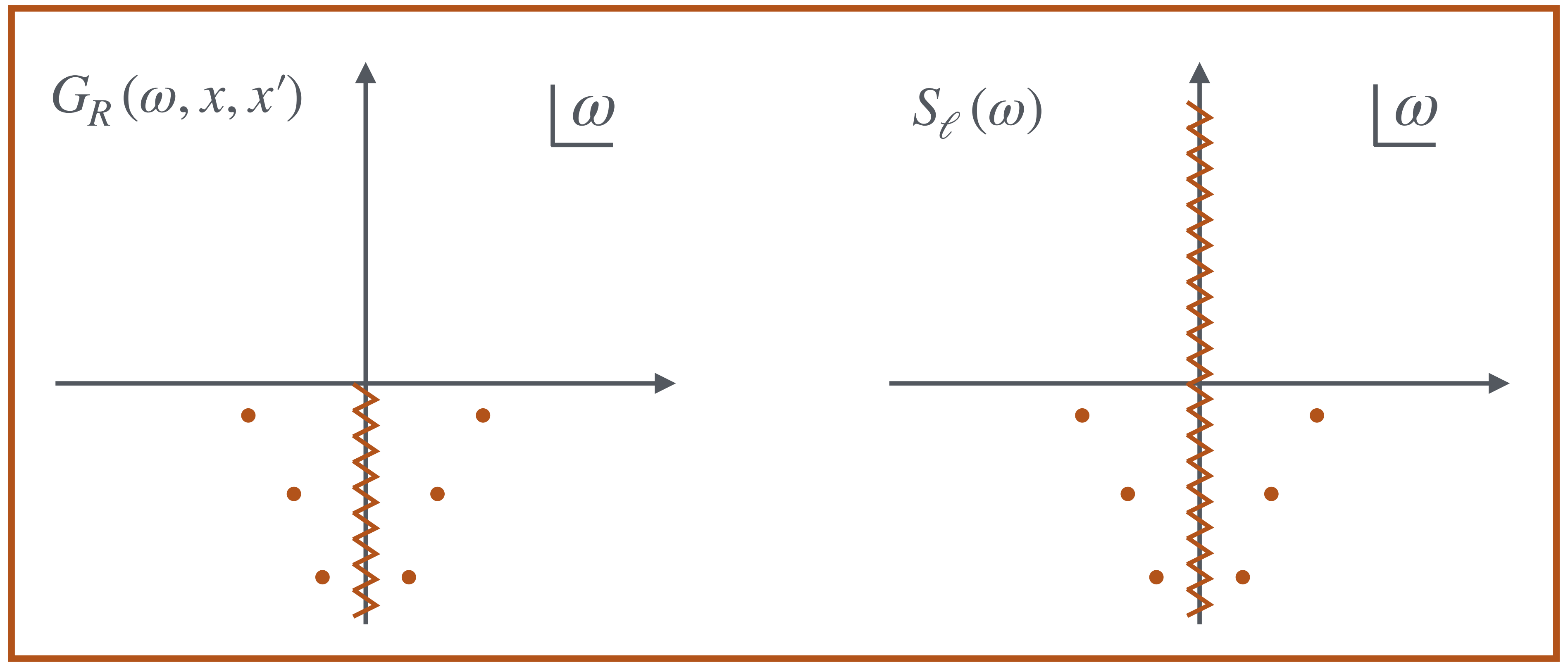
- Stokes phenomena in QFT?
  Alternative path to prove analyticity in QFT?

Conclusion

Established the analytic structure of classical wave scattering on a black hole background



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  Alternative path to prove analyticity in QFT?
- Lower bounds on IR cutoffs in QFT?