High-Rank Structure Constants and Separation of Variables in planar N=4 SYM

Paul Ryan

Based on work to appear “soon”

Based on work to appear with

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Planar N=4 SYM is an integrable 4D CFT

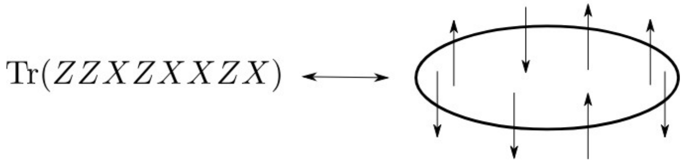
Integrability allows us to make testable predictions on both sides of the AdS/CFT correspondence

Highly constrains:

- CFT data - conformal dimensions & structure constants
- Scattering amplitudes (Yangian symmetry) [Drummond, Henn, Plefka] [...]
- Wilson loops [Drukker] [Klose, Loebbert, Münkler] [...]
- ...

<u>For the spectral problem</u>

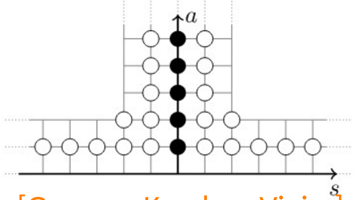
Weakly-coupled gauge theory → Asymptotic Bethe Ansatz → Thermodynamic Bethe Ansatz



[Minahin, Zarembo]

[Beisert, Staudacher] [...]

[Beisert, Eden, Staudacher] [...]



[Gromov, Kazakov, Vieira]

[Arutyunov, Frolov] [...]

Quantum Spectral Curve

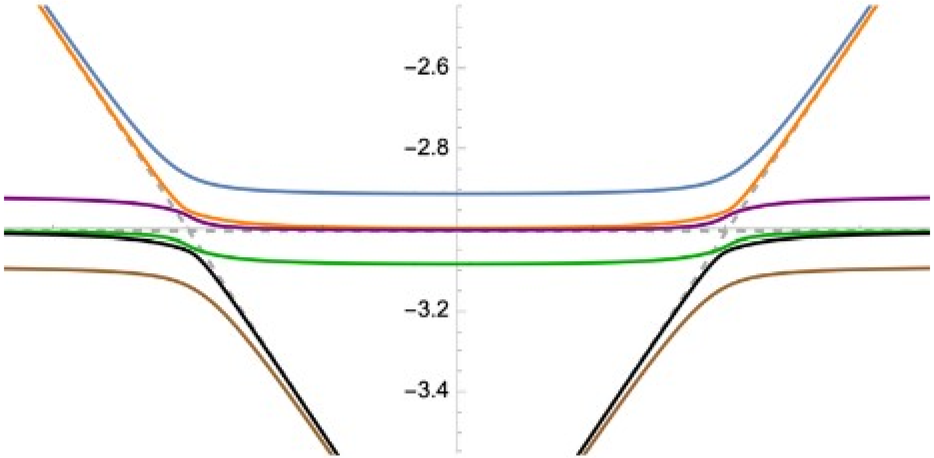
Exact solution of spectral problem of planar N=4 SYM

[Gromov, Kazakov, Leurent, Volin]

Interpolates between weakly-coupled gauge theory and string theory at strong coupling

Can explore analyticity in spin – Regge trajectories

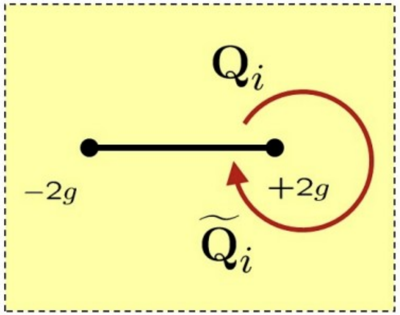
[Ekhammar, Gromov, Preti]



What is it?

- collection of functions Q(u) of a single complex variable u
- relations fixed by symmetry – psu(2,2|4)
- model-dependent analytic properties

Branch points related to ‘t Hooft coupling



$$g = \frac{\sqrt{\lambda}}{4\pi}$$

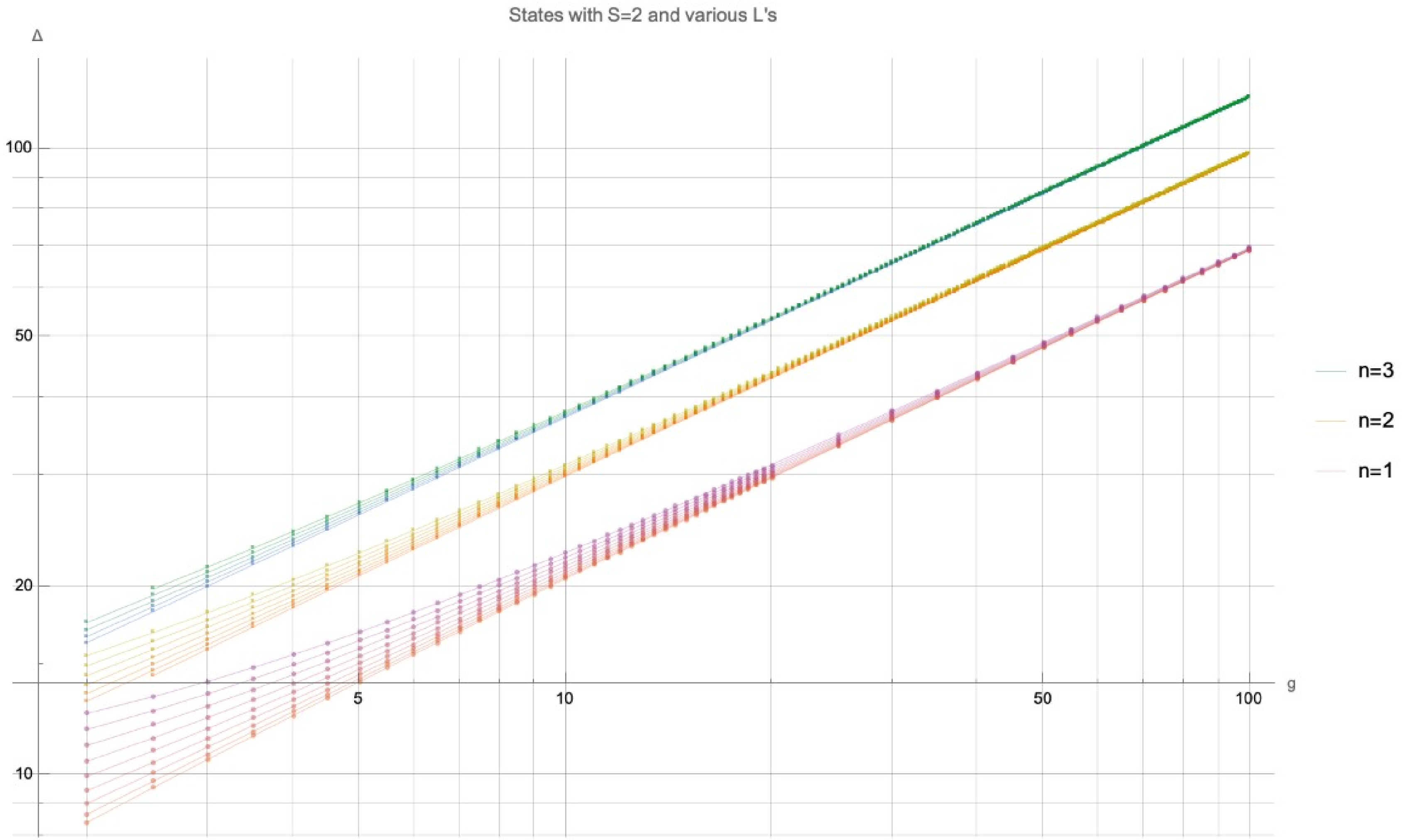
Q-functions encode conformal dimension

$$Q(u) \sim u^{\Delta}$$

QSC recently pushed to strong coupling

[Ekhammar, Gromov, PR]

$\lambda \sim 150000$



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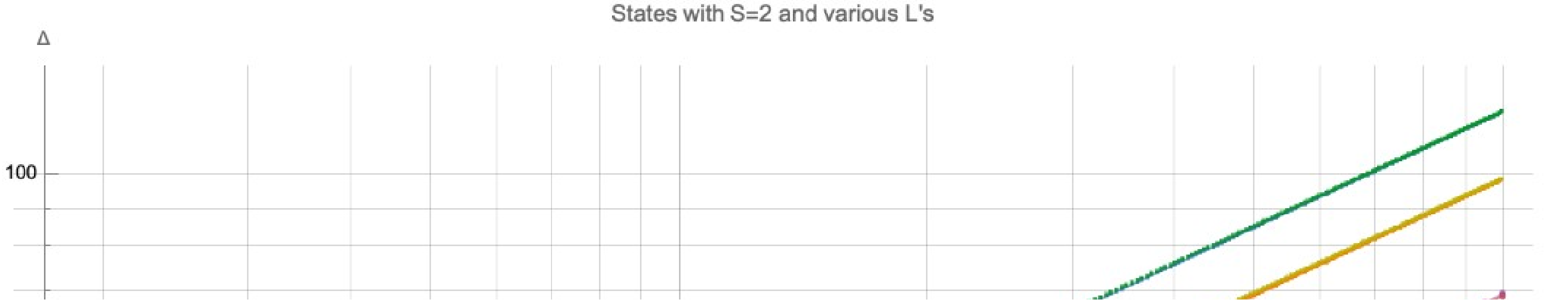
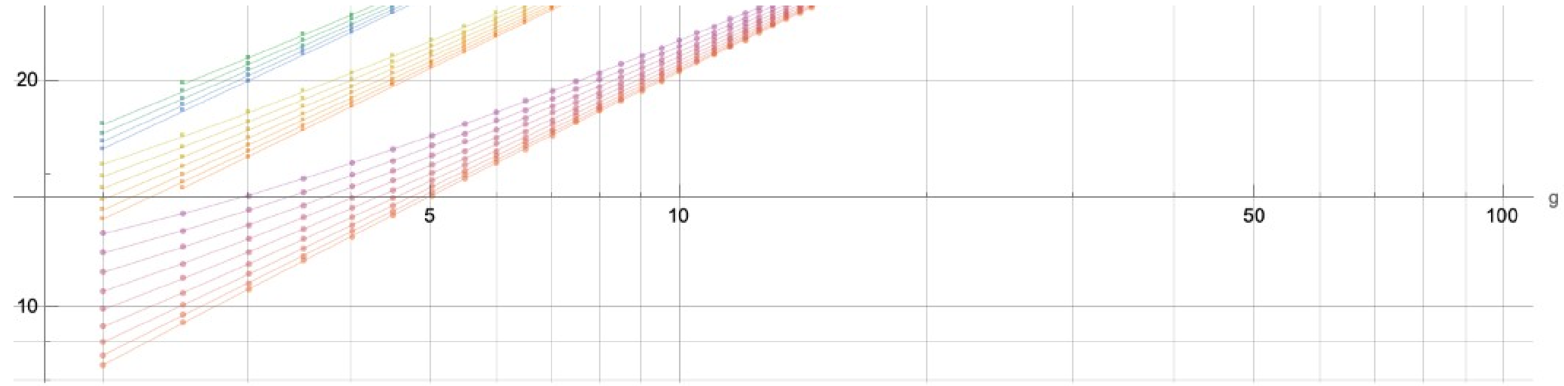


$$\Delta_{\text{Konishi}} = 2\sqrt[4]{\lambda} - 2 + 2\sqrt[4]{\frac{1}{\lambda}} + \left(\frac{1}{2} - 3\zeta_3\right)\left(\frac{1}{\lambda}\right)^{3/4} + \left(6\zeta_3 + \frac{15\zeta_5}{2} + \frac{1}{2}\right)\left(\frac{1}{\lambda}\right)^{5/4}$$
$$+ \left(-\frac{81\zeta_3^2}{4} + \frac{\zeta_3}{4} - 40\zeta_5 - \frac{315\zeta_7}{16} - \frac{27}{16}\right)\left(\frac{1}{\lambda}\right)^{7/4} + \ldots\,,$$



With Quantum Spectral Curve the planar N=4 SYM spectral problem is solved

Your favourite operator $\mathcal{O}$ $\longrightarrow$

$\longrightarrow$ Δ

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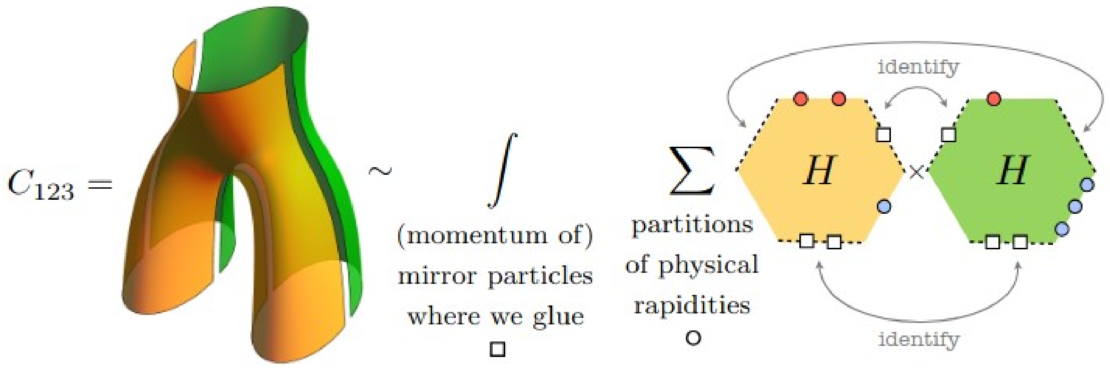
Your favourite operator $\mathcal{O}$ $\longrightarrow$

$\longrightarrow$ Δ

… but what about three-point structure constants?

Hexagon framework for structure constants

[Basso, Komatsu, Vieira]



Key results

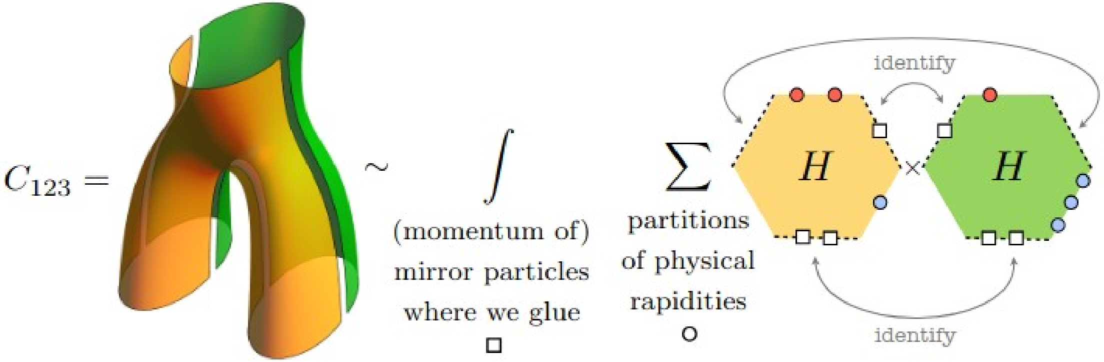
- Higher-point functions

  [Eden, Sfondrini] [Fleury, Komatsu]

- Non-planar corrections – not exclusively tied to integrability

  [Bargheer, Caetano, Fleury, Komatsu, Vieira]

Hexagon framework for structure constants [Basso, Komatsu, Vieira]

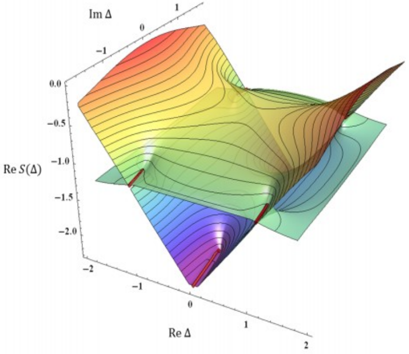


Some drawbacks

- Asymptotically large operators

- Analyticity in spin not manifest

Where are the beautiful Riemann surfaces?



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A Quantum Spectral Curve based approach to structure constants

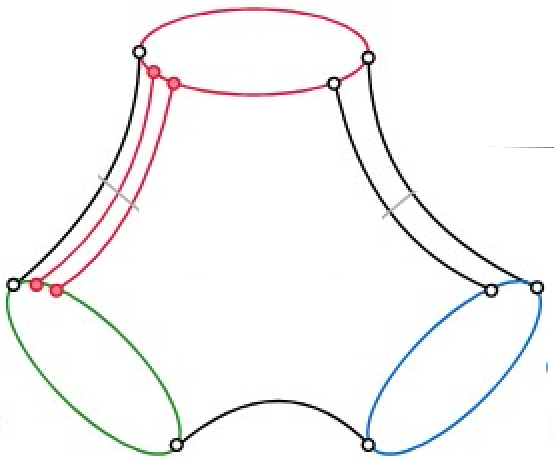
- any operator, any size
- strong coupling
- should naturally incorporate qualitative features like analyticity in spin

On-going program

[Basso, Cavaglià, Bargheer, Bercini, Ekhammar, Georgoudis, Giombi, Gromov, Homrich, Jiang, Komatsu, Kostov, Lai, Levkovich-Maslyuk, PR, Serban, Sueiro, Vieira, Volin ...]

Operators ⟷ Q-functions by QSC

$\mathcal{O}_1 \sim Q_1$

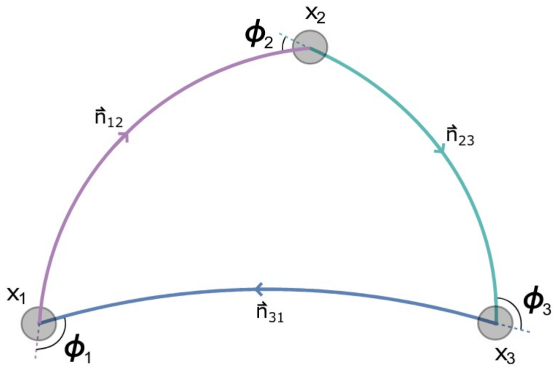


$\mathcal{O}_2 \sim Q_2$ $\quad$ $\mathcal{O}_3 \sim Q_3$

Some functional which eats Q-functions

$$\rightarrow \mathcal{F}(Q_1, Q_2, Q_3) \rightarrow C_{123}$$

Cusped Wilson loop



[Erickson, Semenoff, Szabo, Zarembo]

Ladders limit, all diagrams can be resummed

Cusp anomalous dimension can be computed by QSC [Gromov, Levkovich-Maslyuk]

Correlator can be computed explicitly
Massive simplification when expressed in terms of QSC Q-functions!

Fully non-perturbative

$$C_{123}^{\bullet\bullet\circ} = \frac{\langle q_1\, q_2\, e^{-\phi_3 u}\rangle}{\sqrt{\langle q_1^2\rangle\langle q_2^2\rangle}}$$

$$\langle f\rangle = \int_\gamma \mathrm{d}u\, \frac{f(u)}{2\pi i u}$$

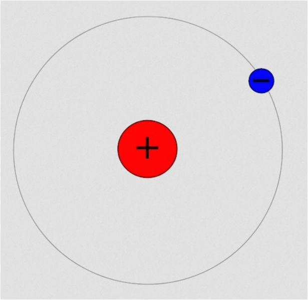
[Cavaglià, Gromov, Levkovich-Maslyuk]

Similar results [Giombi, Komatsu]

Strongly resembles correlators in integrable spin chains in **separated ``Sklyanin" variables** [Sklyanin] $\Psi(\mathrm{x}) \sim Q(x_1)Q(x_2)\ldots$ Q-functions play the role of exact wave functions

Some partial results in su(2) / sl(2) sector at low loops for correlators of local operators **We want to go beyond!**

Separation of Variables
SoV



Hydrogen atom

$$\psi_{nlm}(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

Wave function factorises in spherical coordinates

"N=4 SYM is the hydrogen atom of the 21st century"

-> we need the separated variables of the 21st century

Since 2016 - major advancements in separation of variables program for high-rank integrable spin chains

Operatorial separation of variables generalising Sklyanin:

- General recipe for separated coordinate bases

[Maillet, Niccoli] [Gromov, Levkovich-Maslyuk, Sizov] [PR, Volin]

- Relation to representation theory of quantum groups

[PR, Volin]

[Cavaglià, Gromov, Levkovich-Maslyuk]

Major development:
Functional Separation of Variables

- Extract all matrix elements of operators directly from QSC in spin chains -> complete solution

[Gromov, Primi, PR]

- Naturally complements operator formulation

[Gromov, Levkovich-Maslyuk, PR, Volin]

Goal of project: **systematically** move beyond rank one (SU(2) and SL(2))
and explore SoV for correlators in higher-rank SU(4) sector

Sketch of the Result

Planar N=4 SYM SU(4) sector: 6 complex scalar fields $\underbrace{Z, \bar{Z}, X, \bar{X}, Y, \bar{Y}}_{\Phi}$

Integrability:

Spin chain basis states

Vacuum state

$\mathrm{Tr}(Z^L)$ → Excited state

$\mathcal{O}_1 = \mathrm{Tr}\,(\Phi \ldots \Phi Z \ldots Z)$ → Excited spin chain state

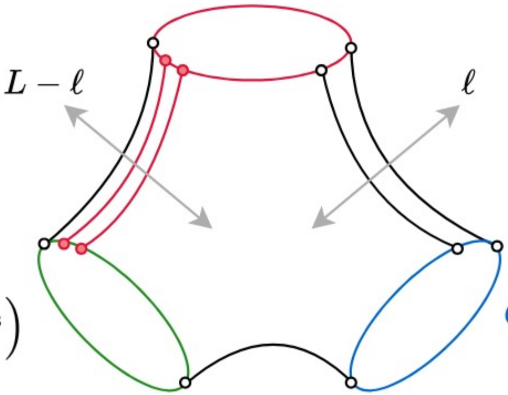
$|\Psi\rangle$

Non-trivial anomalous dimension

$\mathcal{O}_1 = \mathrm{Tr}\,(\Phi \ldots \Phi Z \ldots Z)$

Previously studied using hexagons

[Basso, Coronado, Komatsu, Tat Lam, Vieira, Zhong]



Rotated vacuum, "reservoir" state

[Drukker, Plefka]

$\tilde{Z} = Z + X + \bar{X} + \bar{Z}$

$\mathcal{O}_3 = \mathrm{Tr}\left(\tilde{Z}^{L_3}\right)$

Vacuum complex conjugate

$\mathcal{O}_2 = \mathrm{Tr}\left(\bar{Z}^{L_2}\right)$

C_{123} = Tree-level Wick contractions → Spin chain overlap

The result:

Compact determinant representation for structure constants in terms of a few key Q-functions

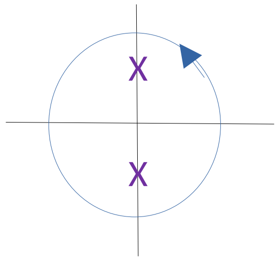
$$Q_1(u), Q_2(u), Q_{12}(u)$$

Q-functions found from Quantum Spectral Curve / Baxter TQ equations

$$Q_{12}(u) = Q_1(u+\tfrac{i}{2})Q_2(u-\tfrac{i}{2}) - Q_1(u-\tfrac{i}{2})Q_2(u+\tfrac{i}{2})$$

and a few simple integrals

$$\langle f \rangle_{l,\alpha} = \oint du \frac{e^{2\pi(\alpha-1)u}}{(u+\frac{i}{2})^l (u-\frac{i}{2})^l} f(u)$$



Main new result:

SU(4) sector: $$C_{123} \sim \det_{1\leqslant\alpha,\beta\leqslant l} \begin{pmatrix} \langle u^{\beta-1} Q_1(u+\frac{i}{2})\rangle_{l,\alpha} & \langle u^{\beta-1} Q_1(u-\frac{i}{2})\rangle_{l,\alpha} \\ \langle u^{\beta-1} Q_2(u+\frac{i}{2})\rangle_{l,\alpha} & \langle u^{\beta-1} Q_2(u-\frac{i}{2})\rangle_{l,\alpha} \end{pmatrix} \times \det_{1\leqslant\alpha,\beta\leqslant L} \langle u^{\beta-1} Q_{12}\rangle_{L,\alpha}$$

Where does the result come from?

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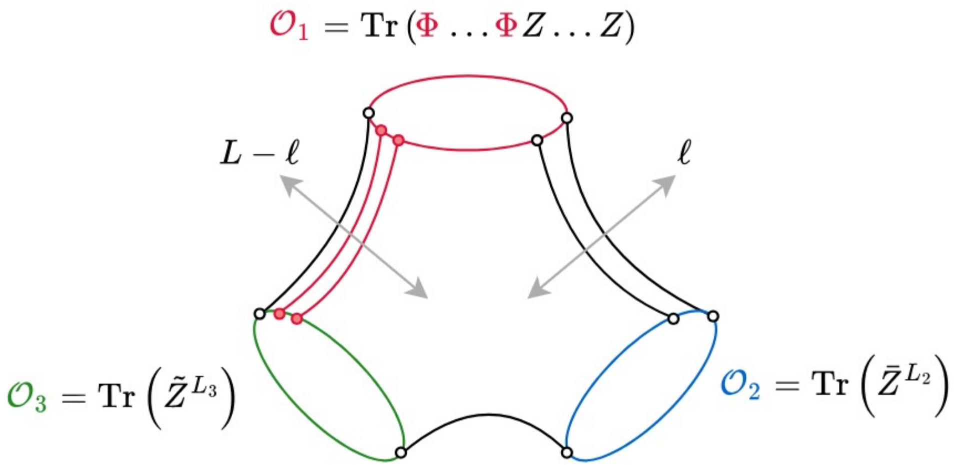
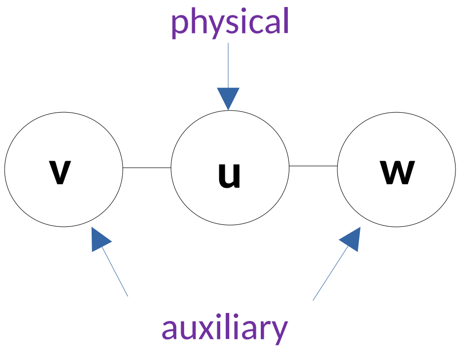
Structure constant is heavily constrained by **quantum symmetries**

SU(4) sector: 6 complex scalar fields $Z, \bar{Z}, X, \bar{X}, Y, \bar{Y}$



Integrability description of excited states



Bethe roots

$$\mathbf{v} = \{v_1, \ldots, v_K\}$$
$$\mathbf{u} = \{u_1, \ldots, u_M\}$$
$$\mathbf{w} = \{w_1, \ldots, w_{\bar{K}}\}$$

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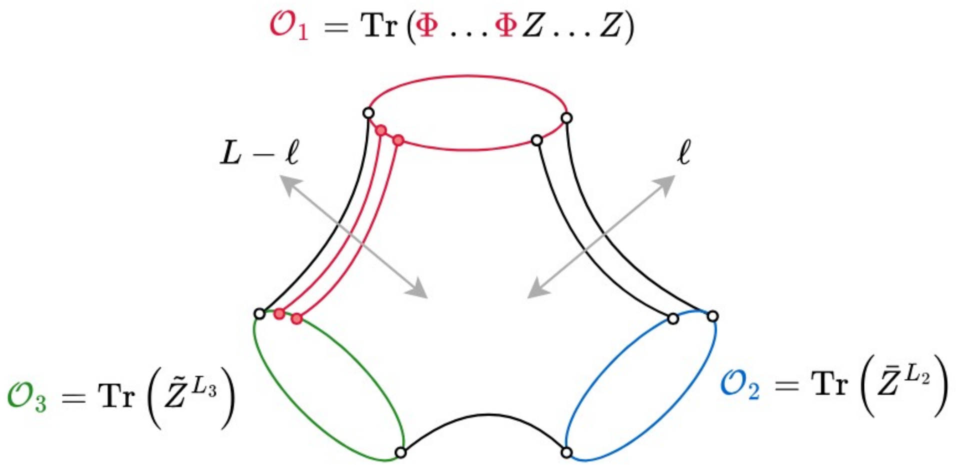

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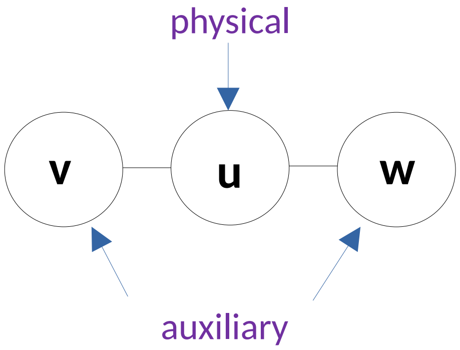

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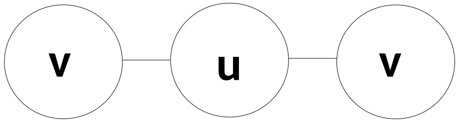
[Basso, Coronado, Komatsu, Tat Lam, Vieira]

Selection Rule

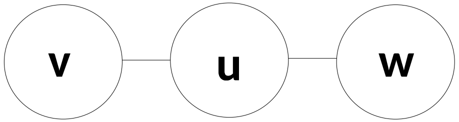
Structure constant vanishes unless excited state is Left-Right symmetric

Left-Right symmetric states:
Auxiliary roots equal

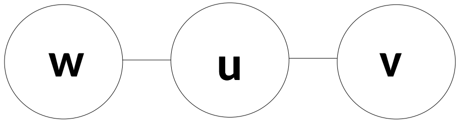


Can we make this selection rule manifest at the level of Q-functions?

Left-Right (LR) symmetry transformation



Flip auxiliary roots →



Integrals of motion in SU(4) spin chain with L sites

$\beta = 1, \ldots, L$

I_β^+, I_β^0, I_β^- — Left-right transform → I_β^-, I_β^0, I_β^+

In earlier work [Ekhammar, Gromov, PR]

a general method for constructing Q-function expressions that manifest a symmetry property was constructed

Based on integrable boundary states

[Caetano, Komatsu]

[Cavaglià, Gromov, Levkovich-Maslyuk]

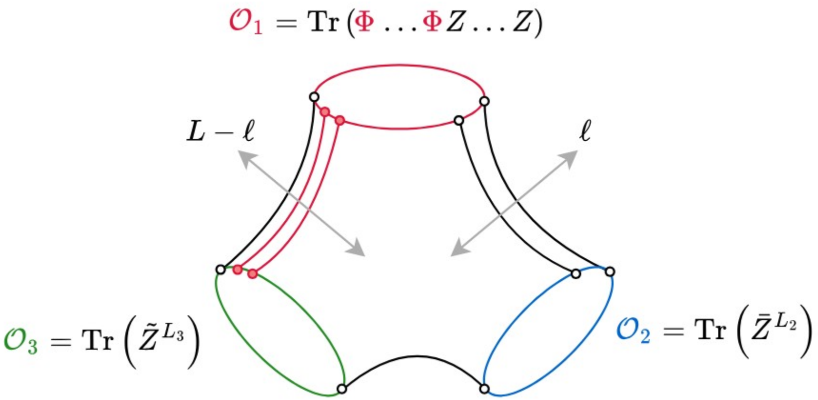
$$\langle f \rangle_{l,\alpha} = \oint \mathrm{d}u \frac{e^{2\pi(\alpha-1)u}}{(u+\frac{i}{2})^l (u-\frac{i}{2})^l} f(u)$$

At least some non-zero if state not LR symmetric:

$$\sum_{\beta=1}^{L} \langle u^{\beta-1} Q_{12} \rangle_{L,\alpha} (I_\beta^+ - I_\beta^-) = 0$$

$$\det_{1\leqslant\alpha,\beta\leqslant L} \langle u^{\beta-1} Q_{12} \rangle_{L,\alpha}$$

Vanishes unless state is LR symmetric



Structure constant vanishes unless state is left-right symmetric

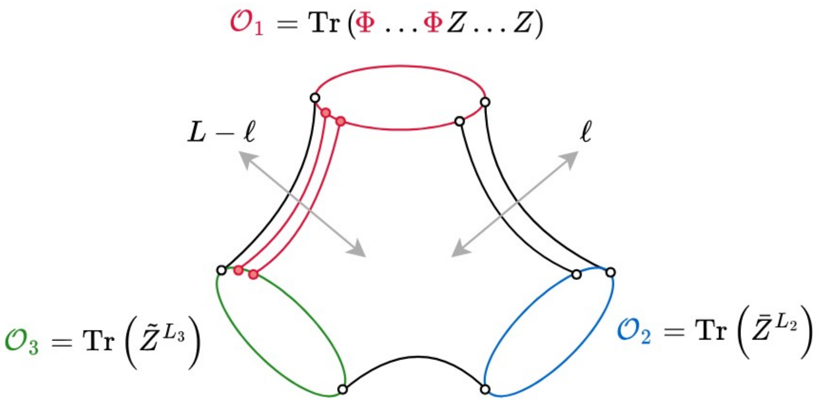
Determinant also vanishes unless state is left-right symmetric

$$\det_{1\leqslant\alpha,\beta\leqslant L} \langle u^{\beta-1} Q_{12} \rangle_{L,\alpha}$$

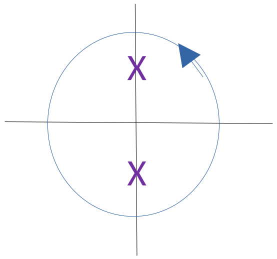
Natural building block of SoV expression for structure constant!

The other block is less obvious, but can be fixed by representation theory of separated variables

[PR, Volin]



$$\langle f\rangle_{l,\alpha} = \oint \mathrm{d}u \frac{e^{2\pi(\alpha-1)u}}{(u+\frac{i}{2})^l(u-\frac{i}{2})^l} f(u)$$



$$C_{123} \sim \det_{1\leqslant\alpha,\beta\leqslant l}\begin{pmatrix}\langle u^{\beta-1}Q_1(u+\frac{i}{2})\rangle_{l,\alpha} & \langle u^{\beta-1}Q_1(u-\frac{i}{2})\rangle_{l,\alpha}\\ \langle u^{\beta-1}Q_2(u+\frac{i}{2})\rangle_{l,\alpha} & \langle u^{\beta-1}Q_2(u-\frac{i}{2})\rangle_{l,\alpha}\end{pmatrix} \times \det_{1\leqslant\alpha,\beta\leqslant L}\langle u^{\beta-1}Q_{12}\rangle_{L,\alpha}$$

What does this buy us?

You shouldn’t try to reinvent the wheel

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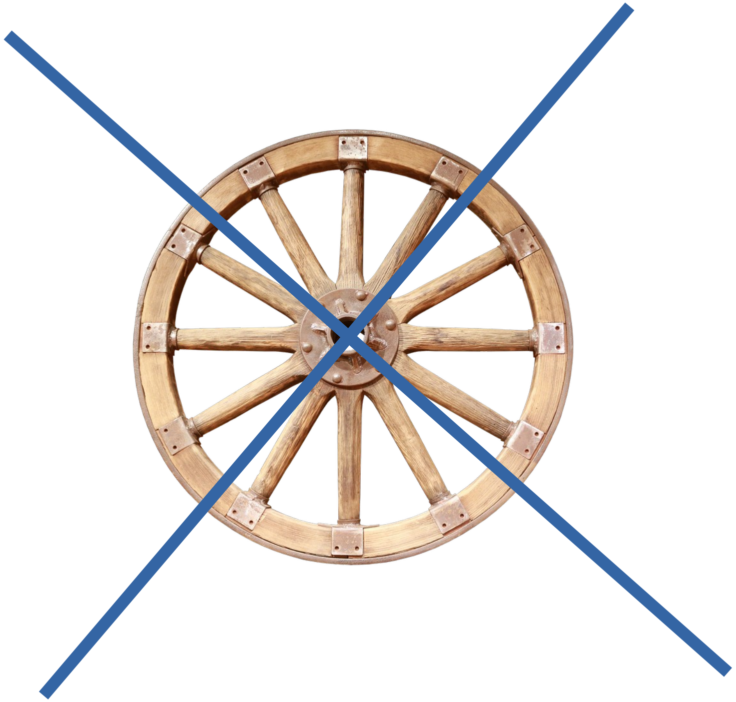

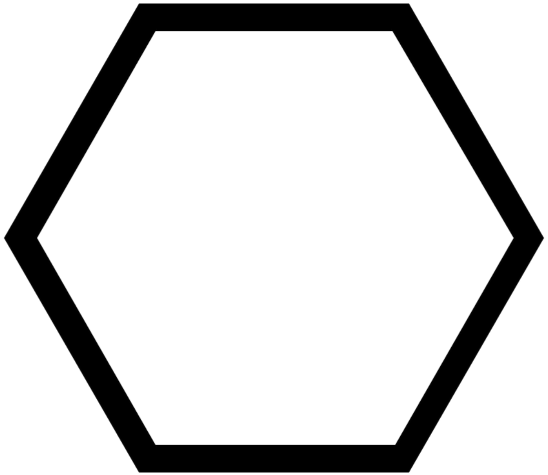

What does this buy us?

- Q-functions can be computed much faster than Bethe roots [Marboe, Volin]

- Q-functions should allow to analytically continue structure constants in spin – EEC applications?

- Q-functions natural objects at finite coupling:
  already examples in fishnet theory of structure constants in Q-function language [Cavaglia, Gromov, Levkovich-Maslyuk

- SoV / Q-function approach seems to work when Hexagons fails (ABJM, deformed N=4 SYM …) [to appear…]

Summary and Outlook

Summary

- Derived a compact determinant representation for tree-level structure constants in the SU(4) sector of planar N=4 SYM

- Main ingredients in the result:
    - Q-functions from Quantum Spectral Curve
    - simple state-independent measures and integrals
    - Quantum symmetries (left-right symmetry)

Outlook

- Q-function expressions for structure constants in su(2) sector known up to two loops, but results not enlightening. Useful to revisit using what we learned here, may help to extend to all loops

[Bercini, Homrich, Vieria]

- All-loop proposal for sl(2) sector structure constants using QSC objects – how does it relate to SoV expressions?

[Basso, Georgoudis, Sueiro]

- Extend to psu(2,2|4) – supersymmetric SoV still a major open problem

- Fishnet theory in 2d: full operatorial +functional sov control at all loops, Quantum Spectral Curve to appear soon. Can we use it to systematically compute correlators at all loops?

[Ekhammar, Gromov, Levkovich-Maslyuk, PR]

- Structure constants in deformations of N=4 SYM, orbifolds, beta- or gamma-deformation
  - initial results from Hexagons but complicated due to twist, but twist is natural in SoV, so may be a natural approach to try

[Eden, le Plat, Spiering]

[le Plat, Skrzypek]

- More speculatively: clear Separation of Variables can be used to extract far more than just the spectrum from Quantum Spectral Curve. What about the finite-coupling quantum algebra?

Thank you!