Six-particle scattering in QCD

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The ‚most complicated terms‘ of scattering amplitudes are also the simplest!

Based on [CCHYZ, Bootstrapping Six-Gluon QCD Amplitudes, 2510.20565 hep-th]

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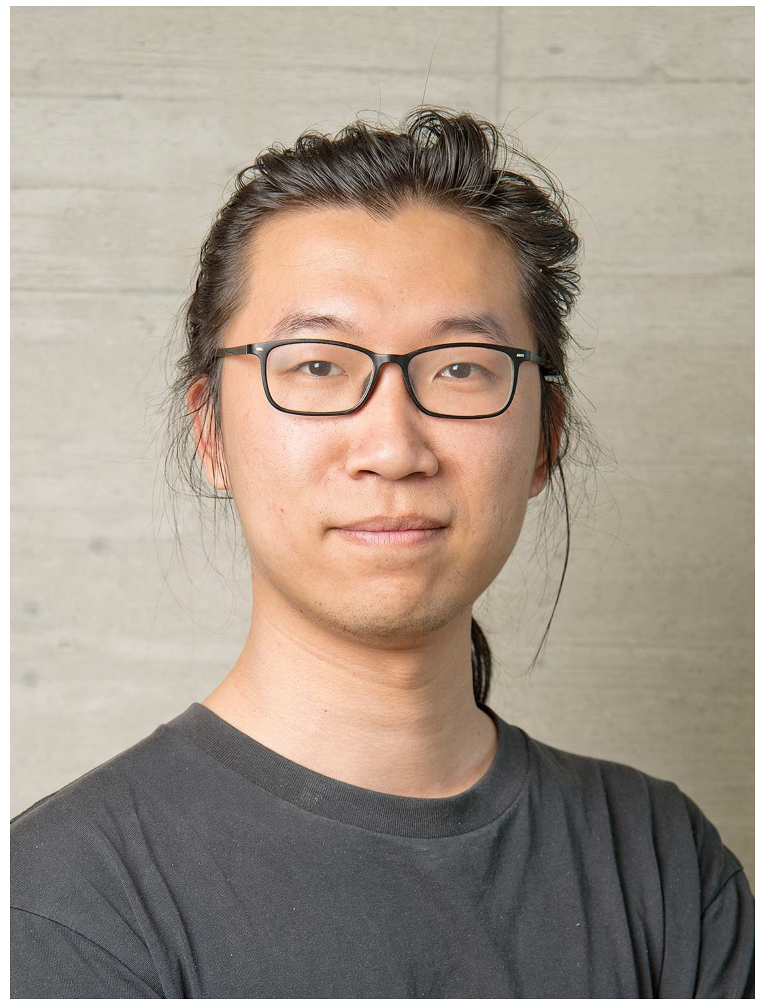

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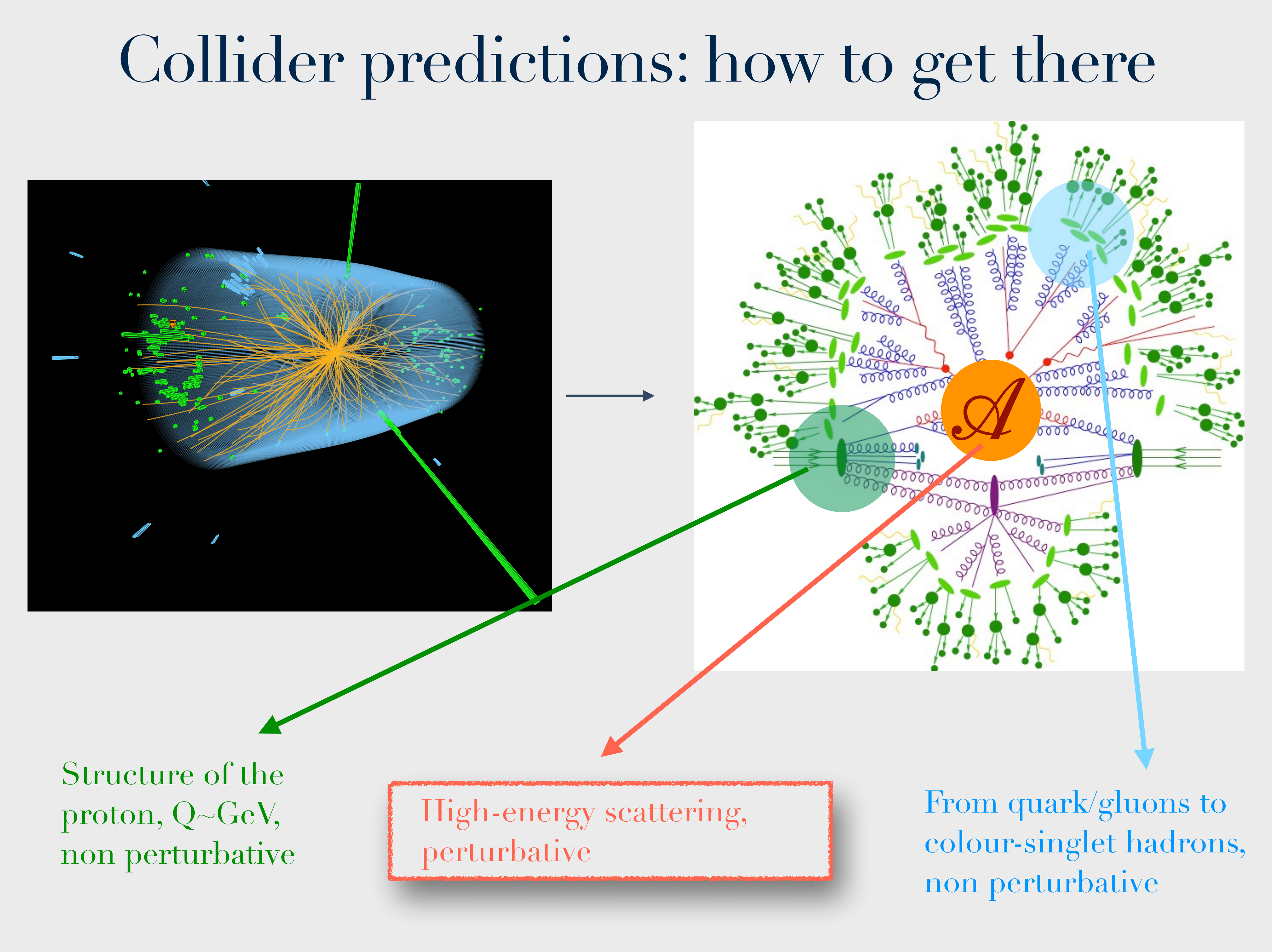
Scattering amplitudes connect theory and experiment



From Fabrizio Caola's talk at Amplitudes 2023, CERN]

‚Scattering Amplitudes: the most perfect microscopic structures in the Universe [Lance Dixon, arXiv:1105.0771]

‘Les Houches wishlist’ gives an idea of what is needed from an experimental viewpoint

NNLO QCD and NLO EW Les Houches Wishlist

Wishlist part 1 - Higgs (V=W,Z)

Process	known	desired	motivation
H	d\sigma @ NNLO QCD d\sigma @ NLO EW finite quark mass effects @ NLO	d\sigma @ NNNLO QCD + NLO EW MC@NNLO finite quark mass effects @ NNLO	H branching ratios and couplings
H+j	d\sigma @ NNLO QCD (g only) d\sigma @ NLO EW	d\sigma @ NNLO QCD + NLO EW finite quark mass effects @ NLO	H p_T
H+2j	\sigma_tot(VBF) @ NNLO(DIS) QCD d\sigma(gg) @ NLO QCD d\sigma(VBF) @ NLO EW	d\sigma @ NNLO QCD + NLO EW	H couplings
H+V	d\sigma(V decays) @ NNLO QCD d\sigma @ NLO EW	with H→bb @ same accuracy	H couplings
t\bar tH	d\sigma(stable tops) @ NLO QCD	d\sigma(NWA top decays) @ NLO QCD + NLO EW	top Yukawa coupling
HH	d\sigma @ LO QCD finite quark mass effects d\sigma @ NLO QCD large m_t limit	d\sigma @ NLO QCD finite quark mass effects d\sigma @ NNLO QCD	Higgs self coupling

Scattering amplitudes at next-to-next-to-leading-order (NNLO) and even beyond are needed to match the experimental precision.

State of the art Five-particle scattering at NNLO

Dramatic progress for massless 5-particle scattering

Analytic results for all Feynman integrals

[Gehrmann, Henn, Lo Presti 2015;
Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser 2018;
Abreu, Page, Zeng 2018; Chicherin, Henn, Mitev 2018;
Abreu, Dixon, Herrmann, Page, Zeng 2018;
Chicherin, Gehrmann, Henn, Wasser, Zhang, **SZ** 2018]

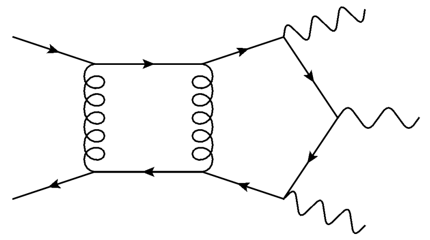
Special function basis

[Gehrmann, Henn, Lo Presti ’18; Chicherin, Sotnikov ’20]

[see M. Marcoli’s talk]

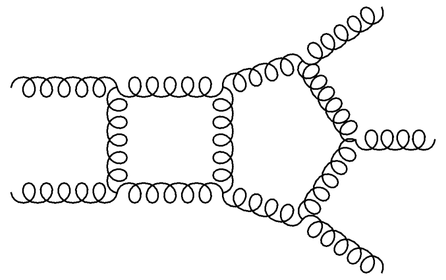
Analytic results for scattering amplitudes

3γ planar



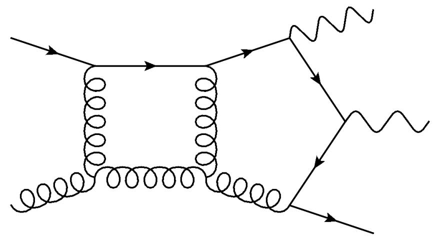
[Abreu, Page, Pascual, Sotnikov 2020; Chawdhry, Czakon, Mitov, Poncelet 2021]

$3j$ planar

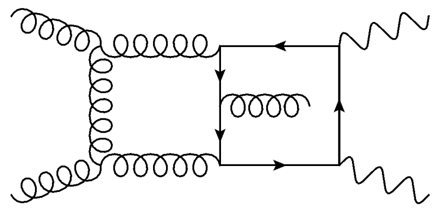


[Abreu, Febres-Cordero, Ita, Page, Sotnikov 2021; Badger, Brönnum-Hansen, Bayu Hartanto, Peraro, Moodie, **SZ**, *to appear*]

$2\gamma + j$ full colour



[Agarwal, Buccioni, von Manteuffel, Tancredi 2021 x2; Chawdhry, Czakon, Mitov, Poncelet 2021]



[Badger, Brönnum-Hansen, Chicherin, Gehrmann, B. Hartanto, Henn, Marcoli, Moodie, Peraro, **SZ** 2021]

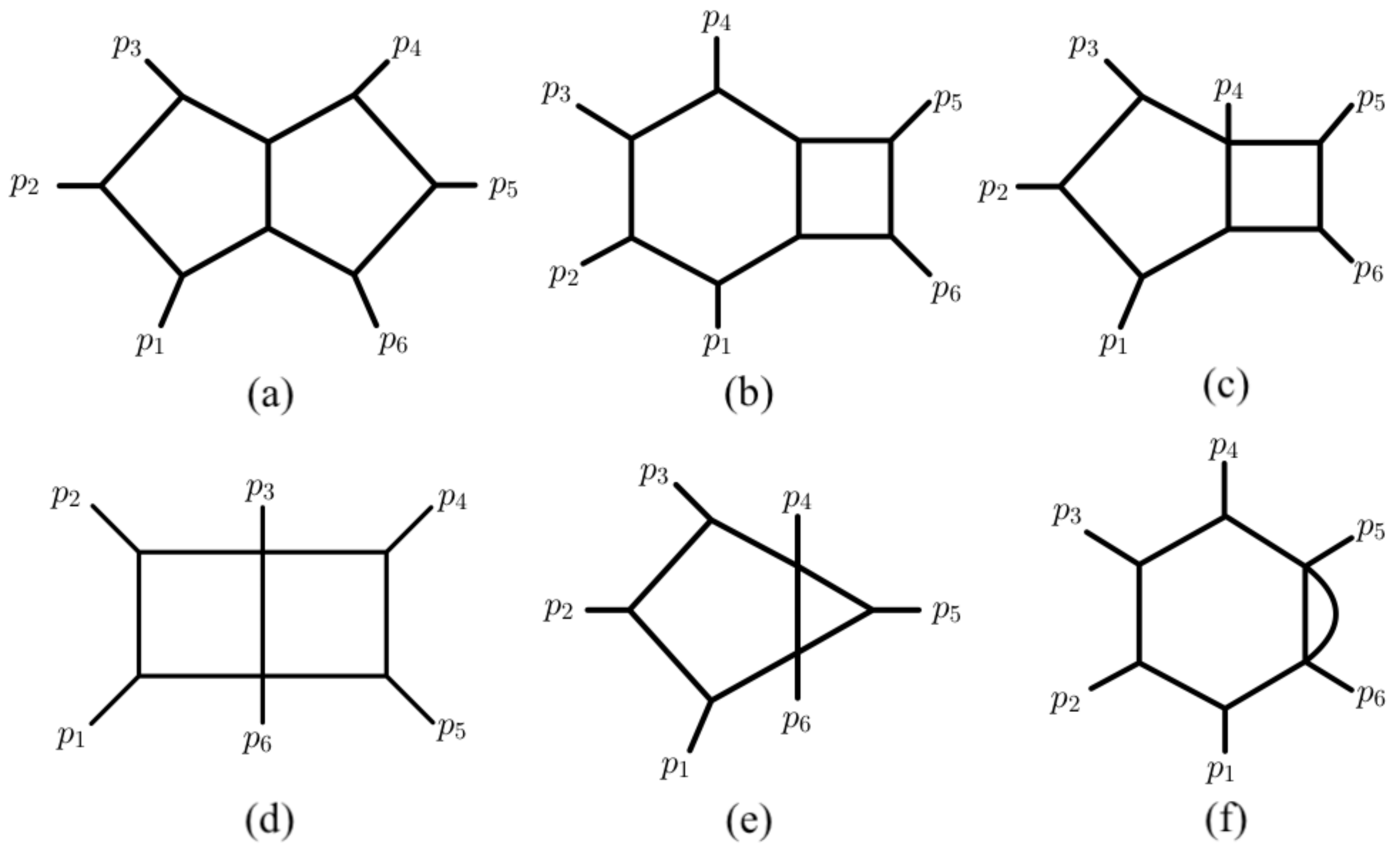
$d\sigma$ **@NNLO QCD:**

- $pp \to 3\gamma$ [Kallweit, Sotnikov, Wiesemann 2020; Chawdhry, Czakon, Mitov, Poncelet 2020]
- $pp \to 2\gamma + j$ [Chawdhry, Czakon, Mitov, Poncelet 2021; Badger, Gehrmann, Marcoli, Moodie 2021]
- $pp \to 3j$ [Czakon, Mitov, Poncelet 2021; Chen, Gehrmann, Glover, Huss, Marcoli 2022]

[slide from S. Zoia, LoopFest 2022]

All massless planar two-loop six-particle Feynman integrals computed

- Differential equations method used
- Most complicated integrals not needed in D=4
- Analytic result, proof-of-concept numerical evaluation



[JMH, Antonela Matijašić, Julian Miczajka, Tiziano Peraro, Yingxuan Xu, Yang Zhang, JHEP 01 (2023) 096, JHEP 08 (2024) 027, *Phys. Rev. Lett.* 135 (2025) 3, 031601. Samuel Abreu, Pier Monni, Johann Usovitsch, JHEP 03 (2025) 112]. See talk by Yang Zhang at this conference.

These results remove an important bottleneck for obtaining two-to-four scattering amplitudes at next-to-next-to-leading order.

Reducing a full-fledged QCD amplitude to these master integrals using state-of-the-art methods is still a formidable challenge.

Challenge: Proliferation of rational coefficients

Challenge

- Rationality of integral coefficients in momenta

$$\mathcal{N}_i(\vec{p}) = \sum_{\vec{\alpha}} n_{i,\vec{\alpha}} \left(s_{12}^{\alpha_1} s_{23}^{\alpha_2} \dots \right), \quad \mathcal{D}_i(\vec{p}) = \sum_{\vec{\alpha}} d_{i,\vec{\alpha}} \left(s_{12}^{\alpha_1} s_{23}^{\alpha_2} \dots \right)$$

$$r_i(\vec{p}) = \frac{\mathcal{N}_i(\vec{p})}{\mathcal{D}_i(\vec{p})}, \quad s_{ij} = (p_i + p_j)^2$$

↪ linear in numerical coefficients $n_{i,\vec{\alpha}} \in \mathbb{Q}$

- Linear systems constructed from multiple numerical computations of $r_i(\vec{p})$

$$\vec{p} \to \{\vec{p}_1, \vec{p}_2, \dots\} \in \mathbb{Q}$$

↪ linear system for required coefficients $n_{i,\vec{\alpha}}$ and $d_{i,\vec{\alpha}}$

- Bottleneck:
  - Run times
  - Complexity of coefficients = number of unknowns $n_{i,\vec{\alpha}}$

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Five-gluon amplitudes

[De Laurentis, Ita, Klinkert, Sotnikov ’23]

Helicity remainder	dim(basis)	ansatz size
$R^{(2),(2,0)}_{+++--}$	31	21,910
$R^{(2),(2,0)}_{++-+-}$	54	54,148
$R^{(2),(1,0)}_{+++--}$	274	163,635
$R^{(2),(1,0)}_{+-++-}$	270	241,156
$R^{(2),(1,0)}_{--+++}$	203	82,180
$R^{(2),(1,1)}_{+++--}$	31	21,910
$R^{(2),(1,1)}_{++-+-}$	54	54,148
$R^{(2),(0,1)}_{+++--}$	226	118,880
$R^{(2),(0,1)}_{+-++-}$	240	209,018
$R^{(2),(0,1)}_{--+++}$	157	76,845
$R^{(2),(-1,1)}_{+++--}$	25	5,320
$R^{(2),(-1,1)}_{++-+-}$	35	9,384

[slide from H. Ita, LoopFest 2025]

Perturbative structure in quantum field theory

$$f = \sum_{i,j} c_{i,j}\, r_{n,i}\, g_j$$

Constants (kinematic-independent)

Coefficients (rational, algebraic)

Special functions

Challenges:

- Complicated multivariate transcendental functions
- Proliferation of coefficients

$\longrightarrow$ both conceptual and practical advances needed

We’ll determine the ‚most complicated terms’ of the two-loop six-gluon split-helicity amplitude in QCD

Outline:

- Infrared-finite parts of the amplitude
- `Most complicated terms’
- Coefficients from on-shell diagrams, conformal symmetry
- Amplitude fixed from physical conditions
- (Bonus: double soft functions at NNLO)

Setup of our calculation

We calculate gluon helicity amplitudes in leading color.

$$\mathcal{A}(1_g, 2_g, 3_g, 4_g, 5_g, 6_g)|_{\text{leading color}} = \sum \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(6)}}) \times A(\sigma(1)_g, \cdots, \sigma(6)_g)$$

We include fermion loops (keeping the ratio N_f/N_c fixed):

$$A_6^{(2)} = A_6^{(2),0} + \frac{N_f}{N_c} A_6^{(2),1} + \left(\frac{N_f}{N_c}\right)^2 A_6^{(2),2}$$

Subtract the universal infrared divergences, directly compute the ‚hard' function.

$$I_n^{(1)} = -\frac{1}{\epsilon^2} \sum_{i=1}^{n} \left(\frac{-s_{i\,i+1}}{\mu^2}\right)^{-\epsilon}$$

$$H^{(1)} = A^{(1)} - I^{(1)} A^{(0)} ,$$
$$H^{(2)} = A^{(2)} - I^{(1)} A^{(1)} - I^{(2)} A^{(0)}$$

Simplifications in infrared-finite hard functions

Working in dimensional regularization, all infrared poles of amplitudes are predicted from information from lower loop orders, thanks to factorization and renormalization group equations.

We take this into account and compute directly infrared-finite hard functions. The latter have been observed to have better properties compared to the naive finite part of two loop amplitudes:

- In five-particle scattering, cancellations in the relevant function space have been observed, which is relevant for cluster algebra interpretations

[Chicherin, JMH, Papathanasiou (2020)]

- Four-dimensional methods can be employed for computing the Feynman integrals, as well as their coefficients.

Focus on the ‚most complicated terms‘

- Kotikov, Lipatov et *al* noticed that the ‚*most complicated terms*‘ in terms in QCD twist-two anomalous dimensions were the same as in maximally supersymmetric Yang-Mills. The argument is that gluons give the relevant dominant contribution to the BFKL equation.

[Kotikov, Lipatov, Nucl. Phys. B 661 19 (2003)]
[Kotikov, Lipatov, Onishchenko, Velizhanin, Phys. Lett. B 595, 521 (2004)]

- Similar observations hold for certain form factors and anomalous dimensions; but already one-loop, +-+- helicity amplitudes have a richer structure.

[Gehrmann, JMH, Huber, 1112.4524, JHEP 03 (2012)] [Dixon, 1712.07274, JHEP 01 (2018)]

- This corresponds to keeping the leading transcendental terms in QCD amplitudes only. In this talk, I show how to fix those ‚most complicated terms‘ for two-loop QCD amplitudes, working in the planar limit.

Maximal weight projection

Generalized unitarity cuts determine coefficients of an amplitude in a basis.

Conjecturally, the maximal weight part of the answer can be computed from cuts that fully localize the loop integration.

[Bobadilla, JMH, 2112.08900, JHEP 03 (2022)]

(Also: Arkani-Hamed et al (2010); JMH (2013))

$$\mathcal{A}^{(L)} = \int \omega^{(L)} .$$

integrand

$$\mathcal{I}_i^{(L)} = \sum_j b_{ij} \prod_{k=1}^{4L} d\log \alpha_{ijk} .$$

dlog-form integrand basis

$$\omega^{(L)} = \sum_{i=1}^{m} c_i \, \mathcal{I}_i^{(L)} + \ldots .$$

Maximal weight part + double-pole part in ⋯

$$\mathcal{P}\left(\omega^{(L)}\right) = \sum_{i=1}^{m} c_i \, \mathcal{I}_i^{(L)} .$$

Maximal weight projection

Caveat: works for suitably defined infrared-finite part (hard function).

Advantage: cuts can be computed in four dimensions, via on-shell diagrams.

Amplitude coefficents and on-shell diagrams

The contribution to the maximal weight part of appropriately renormalized amplitudes can be computed from maximal cuts in four dimensions. This implies that their coefficients are determined by the leading singularities of the amplitudes.

[Bobadilla, JMH, 2112.08900, JHEP 03 (2022)]

Here we take advantage that the latter can be expressed in terms of four-dimensional on-shell diagrams. The latter are known to be conformally invariant.

[JMH, Power, Zoia (2019)]

This predicts that the corresponding known coefficients in five-gluon amplitudes should have a simple interpretation!

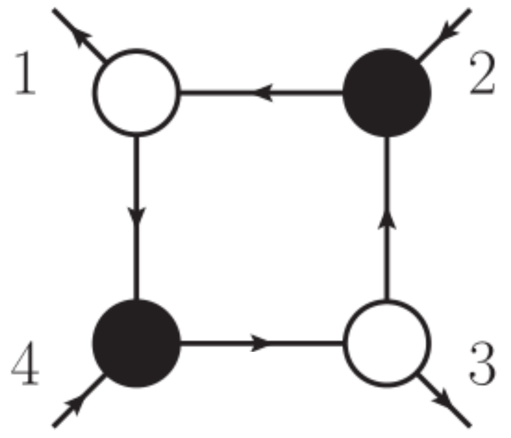
On-shell diagrams and leading singularities

All leading singularities can be calculated from on-shell diagrams.

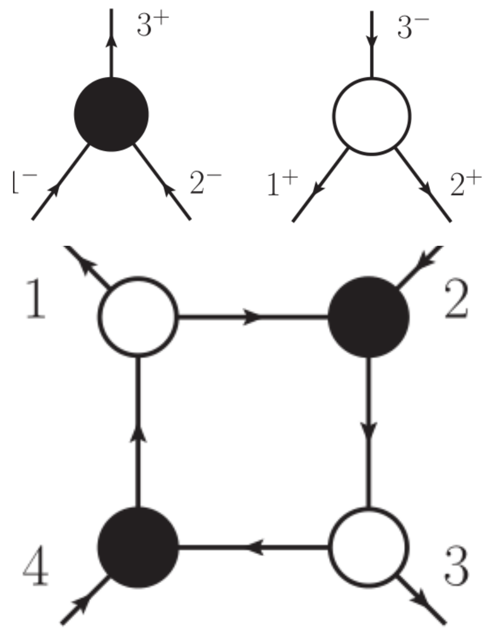
[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka (2012)]

They are built from three-point tree-level vertices.

At one loop, n=4:



$$= \frac{\langle 14\rangle^4 \langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 14\rangle\langle 13\rangle^4}$$



$$= \frac{\langle 12\rangle^4 \langle 34\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 14\rangle\langle 13\rangle^4}$$

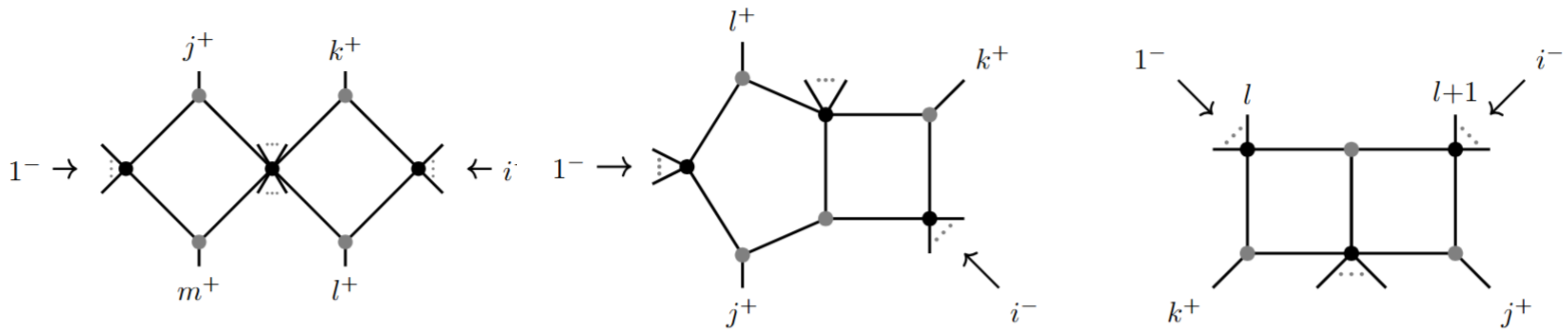
All MHV maximal weight coefficients, any n:

$$\mathrm{PT}_{n,ij} \times \left(\left(\frac{\langle ik\rangle\langle jl\rangle}{\langle kl\rangle\langle ij\rangle} \right)^4 + \left(\frac{\langle il\rangle\langle jk\rangle}{\langle kl\rangle\langle ij\rangle} \right)^4 \right), \quad 1 \leq i < k < j < l \leq n$$

Conformal invariance under action of $\sum_{i=1}^{n} \frac{\partial^2}{\partial \lambda^i_\alpha \partial \tilde{\lambda}^i_{\dot\alpha}}$. [Witten (2003); JMH, Power, Zoia (2019)]

Simplification of two-loop five-particle coefficients

Relevant two-loop MHV leading singularity configurations:



Natural simplification in spinor-helicity variables:

$$\{R[1] \to -\frac{5}{2},$$

$$R[2] \to -\frac{1}{4\, s12^3\, (s12 - s34 - s45)^3} \big(6\, s12^5\, s15 + 15\, s12^4\, s15^2 + 10\, s12^3\, s15^3 + 6\, s12^5\, s23 + 15\, s12^4\, s23^2 + 10\, s12^3\, s23^3 - 12\, s12^4\, s15\, s34 - 15\, s12^3\, s15^2\, s34 - 18\, s12^4\, s23\, s34 - 45\, s12^3\, s23^2\, s34 - 30\, s12^2\, s23^3\, s34 + 6\, s12^3\, s15\, s34^2 + 18\, s12^3\, s23\, s34^2 + 45\, s12^2\, s23^2\, s34^2 + 30\, s12\, s23^3\, s34^2 - 6\, s12^2\, s23\, s34^3 - 15\, s12\, s23^2\, s34^3 - 10\, s23^3\, s34^3 - 18\, s12^4\, s15\, s45 - 45\, s12^3\, s15^2\, s45 - 30\, s12^2\, s15^3\, s45 - 12\, s12^4\, s23\, s45 - 15\, s12^3\, s23^2\, s45 + 6\, s12^4\, s34\, s45 + 54\, s12^3\, s15\, s34\, s45 + 60\, s12^2\, s15^2\, s34\, s45 + 54\, s12^3\, s23\, s34\, s45 + 60\, s12^2\, s15\, s23\, s34\, s45 + 30\, s12\, s15^2\, s23\, s34\, s45 + 60\, s12^2\, s23^2\, s34\, s45 + 30\, s12\, s15\, s23^2\, s34\, s45 - 12\, s12^3\, s34^2\, s45 - 36\, s12^2\, s15\, s34^2\, s45 - 72\, s12^2\, s23\, s34^2\, s45 - 60\, s12\, s15\, s23\, s34^2\, s45 - 75\, s12\, s23^2\, s34^2\, s45 - 30\, s15\, s23^2\, s34^2\, s45 + 6\, s12^2\, s34^3\, s45 + 30\, s12\, s23\, s34^3\, s45 + 30\, s23^2\, s34^3\, s45 + 18\, s12^3\, s15\, s45^2 + 45\, s12^2\, s15^2\, s45^2 + 30\, s12\, s15^3\, s45^2 + 6\, s12^3\, s23\, s45^2 - 12\, s12^3\, s34\, s45^2 - 72\, s12^2\, s15\, s34\, s45^2 - 75\, s12\, s15^2\, s34\, s45^2 - 36\, s12^2\, s23\, s34\, s45^2 - 60\, s12\, s15\, s23\, s34\, s45^2 - 30\, s15^2\, s23\, s34\, s45^2 + 27\, s12^2\, s34^2\, s45^2 + 60\, s12\, s15\, s34^2\, s45^2 + 60\, s12\, s23\, s34^2\, s45^2 + 60\, s15\, s23\, s34^2\, s45^2 - 15\, s12\, s34^3\, s45^2 - 30\, s23\, s34^3\, s45^2 - 6\, s12^2\, s15\, s45^3 - 15\, s12\, s15^2\, s45^3 - 10\, s15^3\, s45^3 + 6\, s12^2\, s34\, s45^3 + 30\, s12\, s15\, s34\, s45^3 + 30\, s15^2\, s34\, s45^3 - 15\, s12\, s34^2\, s45^3 - 30\, s15\, s34^2\, s45^3 + 10\, s34^3\, s45^3 - 6\, s12^4\, tr5 - 15\, s12^3\, s15\, tr5 - 10\, s12^2\, s15^2\, tr5 - 15\, s12^3\, s23\, tr5 - 10\, s12^2\, s15\, s23\, tr5 - 10\, s12^2\, s23^2\, tr5 + 12\, s12^3\, s34\, tr5 + 15\, s12^2\, s15\, s34\, tr5 + 30\, s12^2\, s23\, s34\, tr5 + 10\, s12\, s15\, s23\, s34\, tr5 + 20\, s12\, s23^2\, s34\, tr5 - 6\, s12^2\, s34^2\, tr5 - 15\, s12\, s23\, s34^2\, tr5 - 10\, s23^2\, s34^2\, tr5 + 12\, s12^3\, s45\, tr5 + 30\, s12^2\, s15\, s45\, tr5 + 20\, s12\, s15^2\, s45\, tr5 + 15\, s12^2\, s23\, s45\, tr5 + 10\, s12\, s15\, s23\, s45\, tr5 - 27\, s12^2\, s34\, s45\, tr5 - 35\, s12\, s15\, s34\, s45\, tr5 - 35\, s12\, s23\, s34\, s45\, tr5 - 20\, s15\, s23\, s34\, s45\, tr5 + 15\, s12\, s34^2\, s45\, tr5 + 20\, s23\, s34^2\, s45\, tr5 - 6\, s12^2\, s45^2\, tr5 - 15\, s12\, s15\, s45^2\, tr5 - 10\, s15^2\, s45^2\, tr5 + 15\, s12\, s34\, s45^2\, tr5 + 20\, s15\, s34\, s45^2\, tr5 - 10\, s34^2\, s45^2\, tr5\big),$$

$$R[4] \to \frac{1}{6\, s12^3\, s34^3} \big(10\, s12^3\, s15^3 - 30\, s12^3\, s15^2\, s23 + 30\, s12^3\, s15\, s23^2 - 10\, s12^3\, s23^3 - 15\, s12^3\, s15^2\, s34 + 30\, s12^3\, s15\, s23\, s34 - 15\, s12^3\, s23^2\, s34 + 6\, s12^3\, s15\, s34^2 - 6\, s12^3\, s23\, s34^2 + 3\, s12^3\, s34^3 - 6\, s12^2\, s23\, s34^3 - 15\, s12\, s23^2\, s34^3 - 10\, s23^3\, s34^3 - 30\, s12^2\, s15^3\, s45 + 60\, s12^2\, s15^2\, s23\, s45 - 30\, s12^2\, s15\, s23^2\, s45 + 60\, s12^2\, s15^2\, s34\, s45 - 60\, s12^2\, s15\, s23\, s34\, s45 + 30\, s12\, s15^2\, s23\, s34\, s45 - 30\, s12\, s15\, s23^2\, s34\, s45 - 36\, s12^2\, s15\, s34^2\, s45 - 60\, s12\, s15\, s23\, s34^2\, s45 - 30\, s15\, s23^2\, s34^2\, s45 + 6\, s12^2\, s34^3\, s45 + 30\, s12\, s23\, s34^3\, s45 + 30\, s23^2\, s34^3\, s45 + 30\, s12\, s15^3\, s45^2 - 30\, s12\, s15^2\, s23\, s45^2 - 75\, s12\, s15^2\, s34\, s45^2 + 30\, s12\, s15\, s23\, s34\, s45^2 - 30\, s15^2\, s23\, s34\, s45^2 + 60\, s12\, s15\, s34^2\, s45^2 + 60\, s15\, s23\, s34^2\, s45^2 - 15\, s12\, s34^3\, s45^2 - 30\, s23\, s34^3\, s45^2 - 10\, s15^3\, s45^3 + 30\, s15^2\, s34\, s45^3 - 30\, s15\, s34^2\, s45^3 + 10\, s34^3\, s45^3 - 10\, s12^2\, s15^2\, tr5 + 20\, s12^2\, s15\, s23\, tr5 - 10\, s12^2\, s23^2\, tr5 + 15\, s12^2\, s15\, s34\, tr5 - 15\, s12^2\, s23\, s34\, tr5 + 10\, s12\, s15\, s23\, s34\, tr5 - 10\, s12\, s23^2\, s34\, tr5 - 6\, s12^2\, s34^2\, tr5 - 15\, s12\, s23\, s34^2\, tr5 - 10\, s23^2\, s34^2\, tr5 + 20\, s12\, s15^2\, s45\, tr5 - 20\, s12\, s15\, s23\, s45\, tr5 - 35\, s12\, s15\, s34\, s45\, tr5 + 10\, s12\, s23\, s34\, s45\, tr5 - 20\, s15\, s23\, s34\, s45\, tr5 + 15\, s12\, s34^2\, s45\, tr5 + 20\, s23\, s34^2\, s45\, tr5 - 10\, s15^2\, s45^2\, tr5 + 20\, s15\, s34\, s45^2\, tr5 - 10\, s34^2\, s45^2\, tr5\big),$$

$$R[5] \to -\frac{1}{s12^3\, s45^3} \big(10\, s12^3\, s15^3 - 30\, s12^3\, s15^2\, s23 + 30\, s12^3\, s15\, s23^2 - 10\, s12^3\, s23^3 + 30\, s12^2\, s15^2\, s23\, s34 - 60\, s12^2\, s15\, s23^2\, s34 + 30\, s12^2\, s23^3\, s34 + 30\, s12\, s15\, s23^2\, s34^2 - 30\, s12\, s23^3\, s34^2 + 10\, s23^3\, s34^3 + 15\, s12^3\, s15^2\, s45 - 30\, s12^3\, s15\, s23\, s45 + 15\, s12^3\, s23^2\, s45 + 60\, s12^2\, s15\, s23\, s34\, s45 + 30\, s12\, s15^2\, s23\, s34\, s45 - 60\, s12^2\, s23^2\, s34\, s45 - 30\, s12\, s15\, s23^2\, s34\, s45 - 30\, s12\, s15\, s23\, s34^2\, s45 + 75\, s12\, s23^2\, s34^2\, s45 + 30\, s15\, s23^2\, s34^2\, s45 - 30\, s23^2\, s34^3\, s45 + 6\, s12^3\, s15\, s45^2 - 6\, s12^3\, s23\, s45^2 + 36\, s12^2\, s23\, s34\, s45^2 + 60\, s12\, s15\, s23\, s34\, s45^2 + 30\, s15^2\, s23\, s34\, s45^2 - 60\, s12\, s23\, s34^2\, s45^2 - 60\, s15\, s23\, s34^2\, s45^2 + 30\, s23\, s34^3\, s45^2 + 6\, s12^2\, s15\, s45^3 + 15\, s12\, s15^2\, s45^3 + 10\, s15^3\, s45^3 - 6\, s12^2\, s34\, s45^3 - 30\, s12\, s15\, s34\, s45^3 - 30\, s15^2\, s34\, s45^3 + 15\, s12\, s34^2\, s45^3 + 30\, s15\, s34^2\, s45^3 - 10\, s34^3\, s45^3 + 10\, s12^2\, s15^2\, tr5 - 20\, s12^2\, s15\, s23\, tr5 + 10\, s12^2\, s23^2\, tr5 + 20\, s12\, s15\, s23\, s34\, tr5 - 20\, s12\, s23^2\, s34\, tr5 + 10\, s23^2\, s34^2\, tr5 + 15\, s12^2\, s15\, s45\, tr5 + 10\, s12\, s15^2\, s45\, tr5 - 15\, s12^2\, s23\, s45\, tr5 - 10\, s12\, s15\, s23\, s45\, tr5 - 10\, s12\, s15\, s34\, s45\, tr5 + 35\, s12\, s23\, s34\, s45\, tr5 + 20\, s15\, s23\, s34\, s45\, tr5 - 20\, s23\, s34^2\, s45\, tr5 + 6\, s12^2\, s45^2\, tr5 + 15\, s12\, s15\, s45^2\, tr5 + 10\, s15^2\, s45^2\, tr5 - 15\, s12\, s34\, s45^2\, tr5 - 20\, s15\, s34\, s45^2\, tr5 + 10\, s34^2\, s45^2\, tr5\big)\}$$

From [Abreu, Dorman's, Febres Cordero, Its, Page, Sotnikov (2019)]

Conformal invariance!

$$\Big\{1,\ -\frac{(-2\,ab[1,3]\times ab[2,4] + ab[1,2]\times ab[3,4])\,\big(10\,ab[1,3]^2\,ab[2,4]^2 - 10\,ab[1,2]\times ab[1,3]\times ab[2,4]\times ab[3,4] + ab[1,2]^2\,ab[3,4]^2\big)}{ab[1,2]^3\,ab[3,4]^3},$$

$$-\frac{(-2\,ab[1,3]\times ab[2,5] + ab[1,2]\times ab[3,5])\,\big(10\,ab[1,3]^2\,ab[2,5]^2 - 10\,ab[1,2]\times ab[1,3]\times ab[2,5]\times ab[3,5] + ab[1,2]^2\,ab[3,5]^2\big)}{ab[1,2]^3\,ab[3,5]^3},$$

$$-\frac{(-2\,ab[1,4]\times ab[2,5] + ab[1,2]\times ab[4,5])\,\big(10\,ab[1,4]^2\,ab[2,5]^2 - 10\,ab[1,2]\times ab[1,4]\times ab[2,5]\times ab[4,5] + ab[1,2]^2\,ab[4,5]^2\big)}{ab[1,2]^3\,ab[4,5]^3}\Big\}$$

[Carrôlo, Chicherin, JMH, Yang, Zhang, 2510.20565 hep-th]

This is for the maximal-weight coefficients. Other coefficients more complicated!

On-shell diagrams relevant to two-loop $- - + + + +$ amplitudes

$$R_1 = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}$$

$$R_{i,j}/R_1 = -1 + 12u_{i,j} - 30u_{i,j}^2 + 20u_{i,j}^3 \,,$$

$$u_{i,j} := \langle 1i\rangle\langle 2j\rangle/(\langle 12\rangle\langle ij\rangle)$$

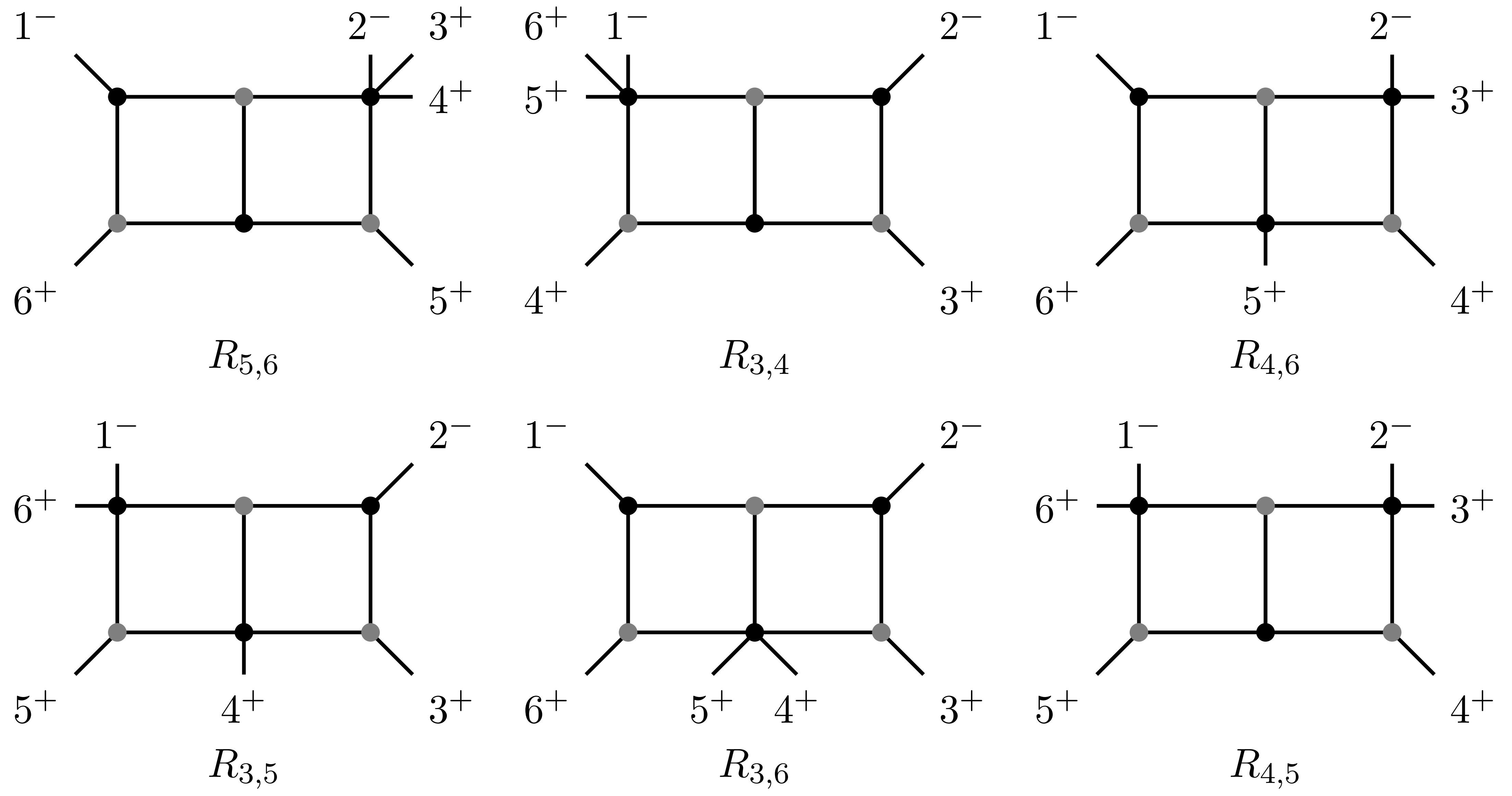


Figure 1. On-shell diagrams giving rise to leading singularities $R_{i,j}$, for the $--++++$ helicity configuration.

This predicts all prefactors of leading-weight contributions to these amplitudes.

Our prediction also applies to arbitrary multiplicity $- - + \ldots +$ amplitudes!

[Carrôlo, Chicherin, JMH, Yang, Zhang, 2510.20565 hep-th]

Ansatz for two-loop $- - + + + +$ six-gluon amplitudes

$$f = \sum_{i,j} c_{i,j}\, r_{n,i}\, g_j$$

Constants (kinematic-independent)

Coefficients (rational, algebraic)

Special functions

We make this ansatz directly for the hard function:

$$\mathcal{H}^{(2)}_{\mathrm{YM}} = R_1 G_1 + \sum_{2<i<j\leq 6} R_{i,j} G_{i,j}\,.$$

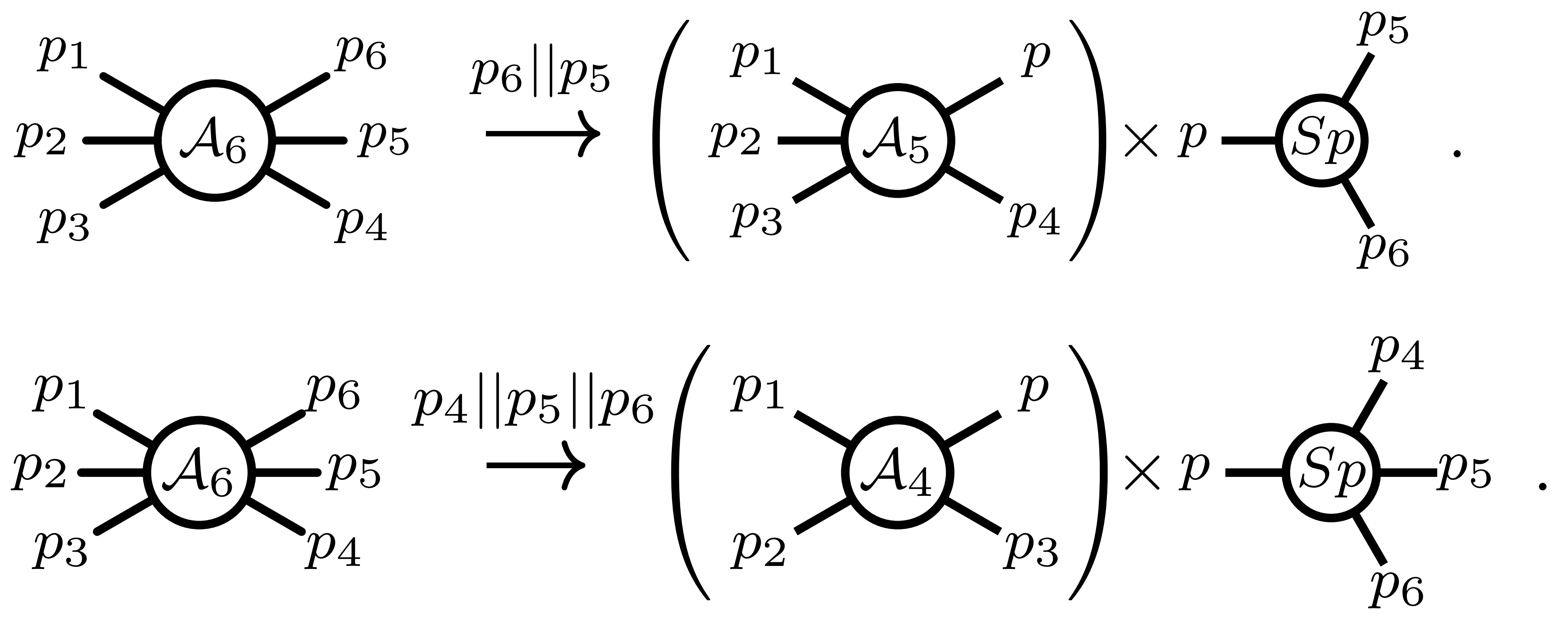
Six leading singularities, as computed from on-shell diagrams.

We use symbol technology to handle the ansatz for the functions space.

Answer for $- - + + + +$ fixed uniquely from physical properties

$$\mathcal{H}^{(2)}_{\text{YM}} = R_1 G_1 + \sum_{2<i<j\leq 6} R_{i,j} G_{i,j} \, .$$

We impose cancellations of spurious poles, as well as consistency with soft and collinear limits.



The answer is fixed uniquely!

Table I. Two-loop bootstrap constraints for $\mathcal{H}^{(2)}_{\text{YM}}$

Condition on $\mathcal{H}^{(2)}_{\text{YM}}$	No. of constraints
dimensionless symbols G	996
no spurious/high-order poles	
$\langle 36\rangle = 0$	333
$\langle 35\rangle = 0$	614
$\langle 34\rangle = 0$	629
$\langle 45\rangle = 0$	343
collinear limit $p_5\|\|p_6$	
$--++++$	1785
$+--+++$	1307
$++--++$	1785
$+++--+$	1646
$-++++-$	1646
triple collinear limit $p_4\|\|p_5\|\|p_6$	
$--++++$	1836
$++--++$	724
Total	2412

Fermions can be included analogously

Only linear term at maximal weight:

$$\mathcal{H}^{(2)}_{\text{QCD}} = \mathcal{H}^{(2)}_{\text{YM}} + \left(\frac{N_f}{N_c}\right)\mathcal{H}^{[1]}\,.$$

Leading singularities evaluated $S_{i,j}/R_1 = -2 + 12u_{i,j} - 21u^2_{i,j} + 11u^3_{i,j}\,,$

‚Effective supersymmetry' for some coefficients / functions

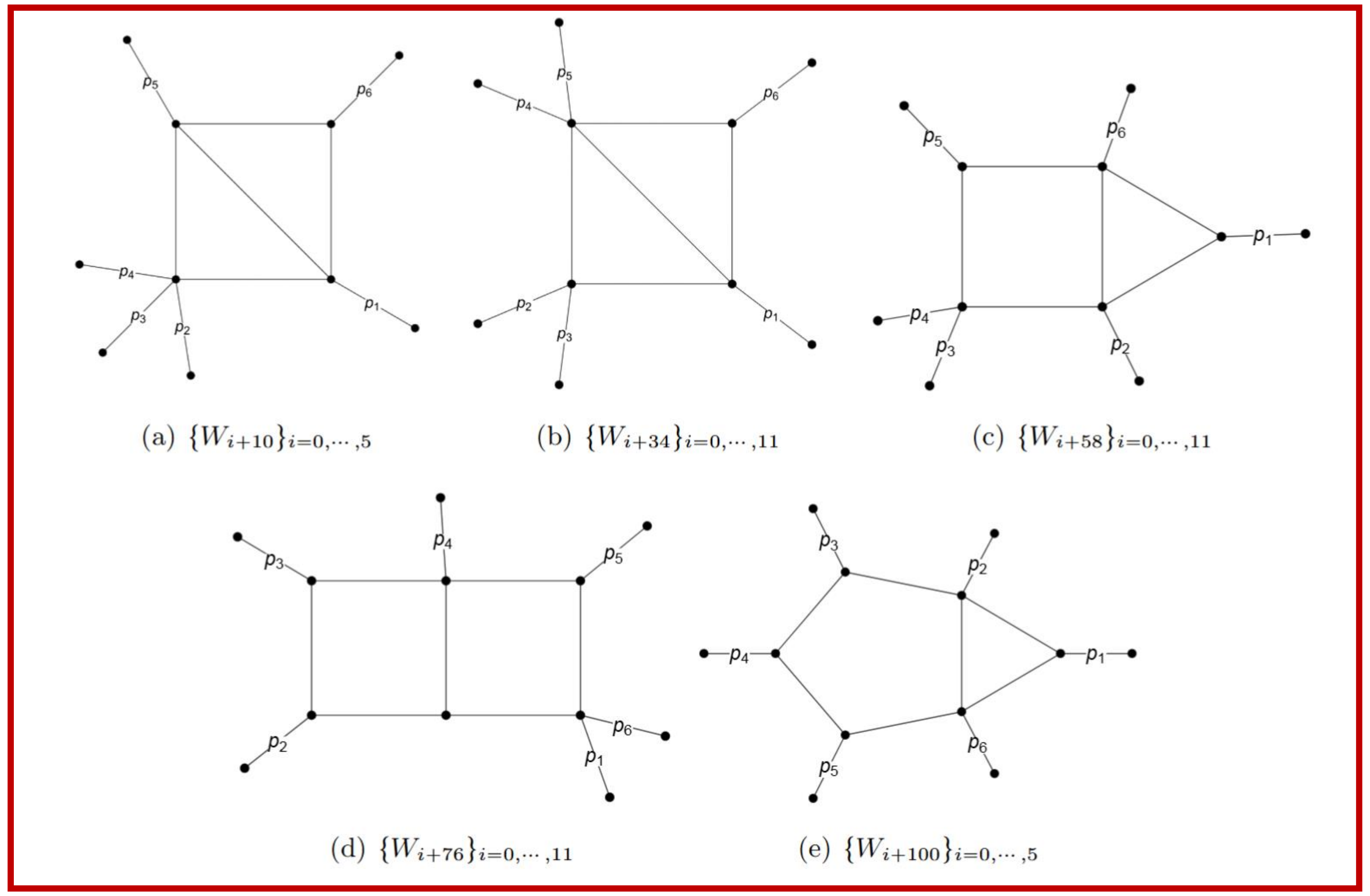
Answer is easier to fix compared to pure glue case!

Discussion of the result

Only 137 of the alphabet letters are needed, as in [Carrôlo, Chicherin, JMH, Yang, Zhang (2025)].

The appearing two-loop letters are associated to these integral sectors:

Sectors yielding contribution



Open question: cluster algebra interpretation?

[Bossinger, Drummond, Glew, Gürdoğan, Wright (2025); Pokraka, Spradlin, Volovich, Weng (2025)]

Additional letters expected in non-MHV configurations.

Take-home messages

- The coefficients of the maximal weight part of scattering amplitudes are given by on-shell diagrams that can be evaluated systematically.
- We used these insights, together with the symbol bootstrap, to determine planar six-gluon $--++++$ helicity amplitude at maximal weight. The function space depends on 137 of the possible alphabet letters only. Other MHV helicity configurations can be treated in a similar way. In the non-MHV case, further alphabet letters are expected.
- At lower weight, there are much fewer functions, but considerably more coefficients. Predicting the latter is an important open challenge!

The ‚most complicated terms‘ of scattering amplitudes are also the simplest!

Thank you!

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