Applying Color-Kinematics duality in pure YM theory at three loops

Gang Yang

Institute of Theoretical Physics,
Chinese Academy of Sciences

Based on: Zeyu Li and GY, JHEP 02 (2024) 199; Zeyu Li, GY, Guorui Zhu JHEP 02 (2025) 011

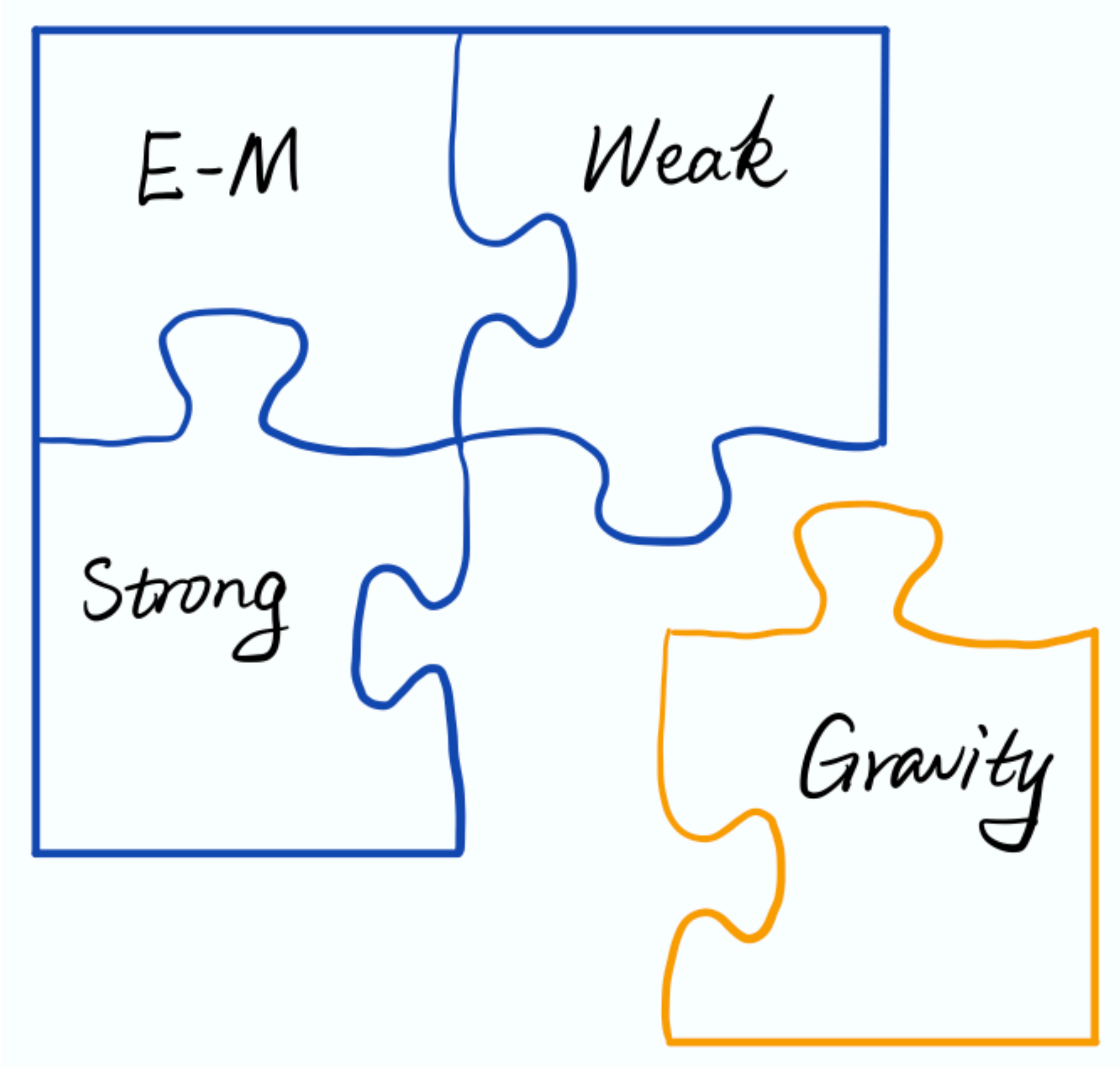
Zeyu Li (李泽宇)

Guorui Zhu (朱国瑞)

Outline

- Introduction
- Constructing CK-dual numerators
- New strategy of deformation
- Summary and outlook

Gauge and gravity theories



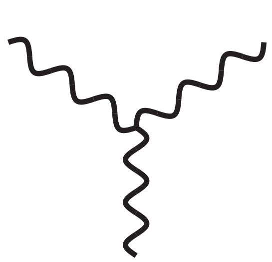
There is clearly a **big difference** between gauge theory and gravity

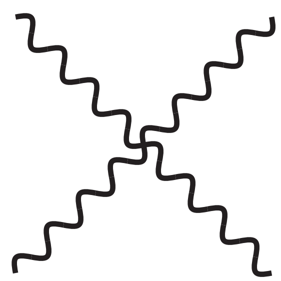
Gauge and Gravity

Yang-Mills action

$$S_{\rm YM} = -\frac{1}{4g^2}\int d^4x F^a_{\mu\nu}F^{a,\mu\nu}$$

Only three and four-point vertices



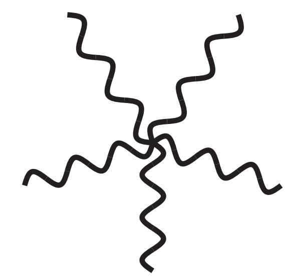


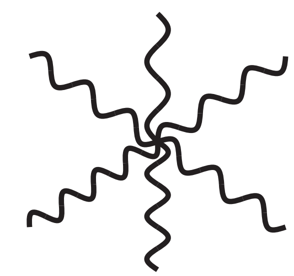
Renormalizable

Einstein-Hilbert action

$$S_{\rm gravity} = -\frac{2}{\kappa^2}\int d^4x\sqrt{g}R$$

Infinite number of high-point vertices



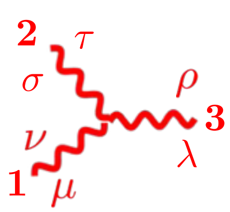


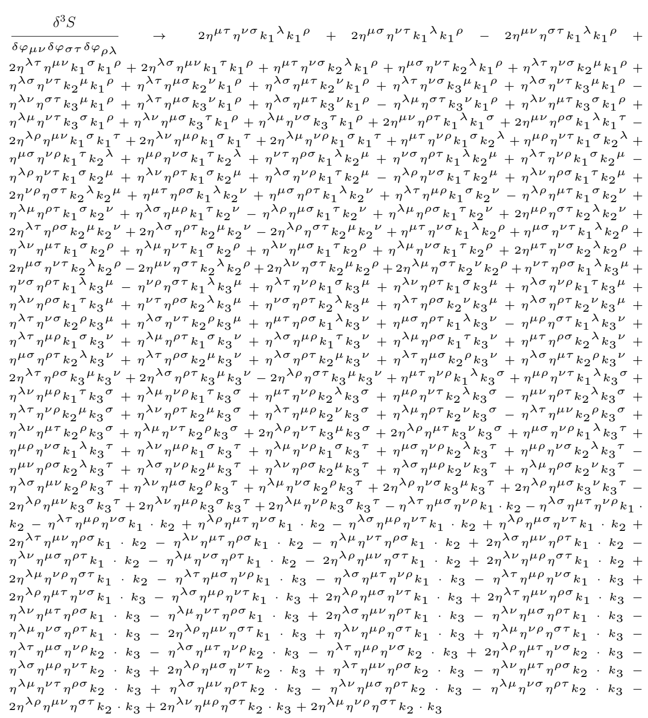
...

Not renormalizable

Higher loop gravity

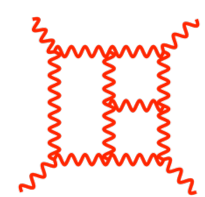
High-loop gravity can be very difficult using Feynman diagram:





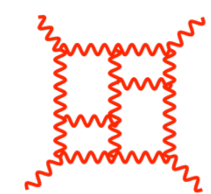
3-vertex
more than
100 terms

3−loop



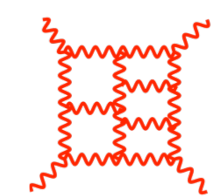
$\sim 10^{20}$ terms

4−loop

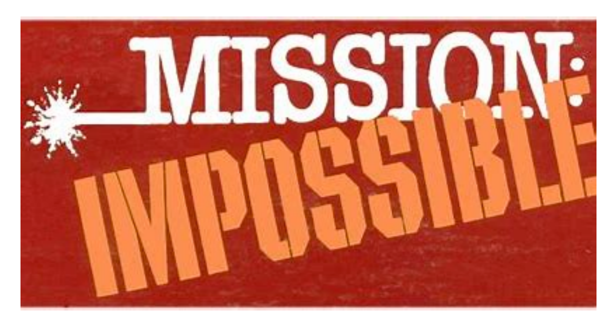


$\sim 10^{26}$ terms

5−loop

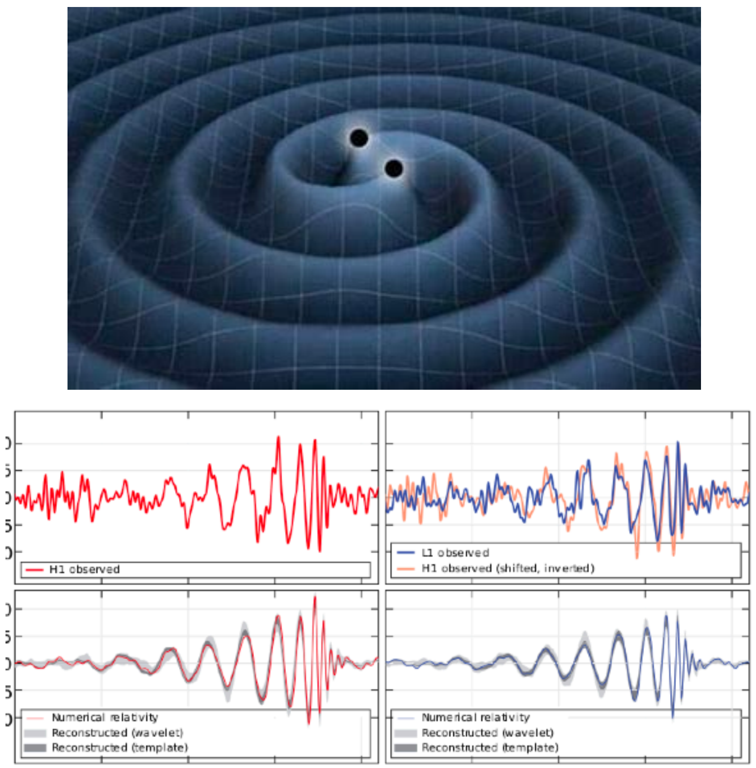


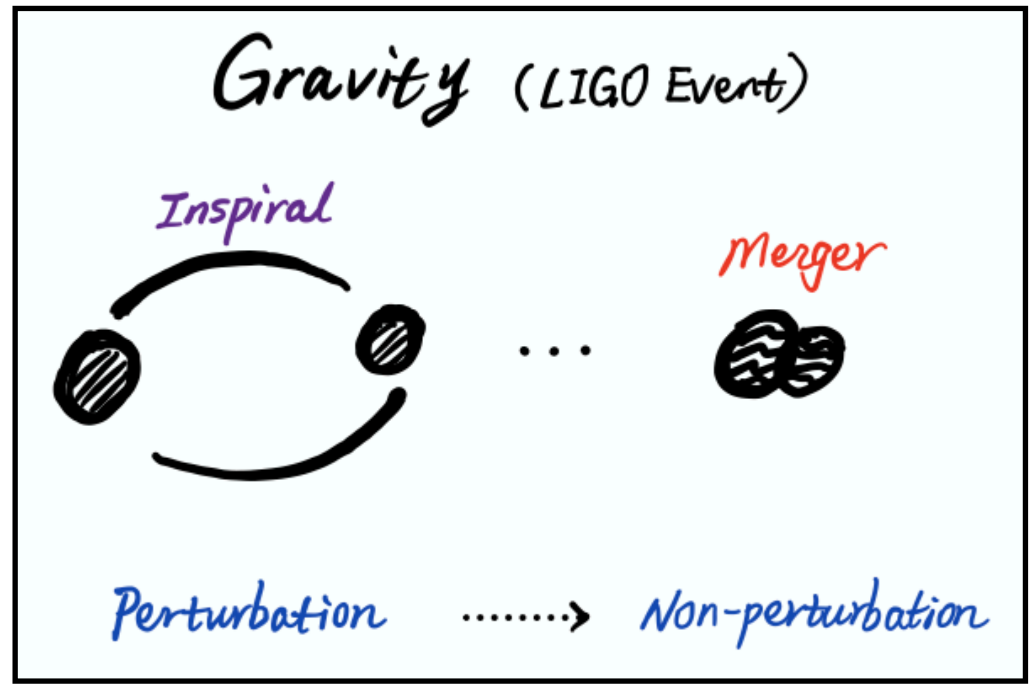
$\sim 10^{31}$ terms



Higher loop perturbation is important

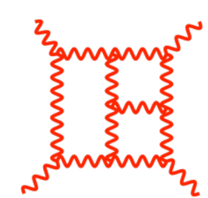
- Gravitational wave related computation



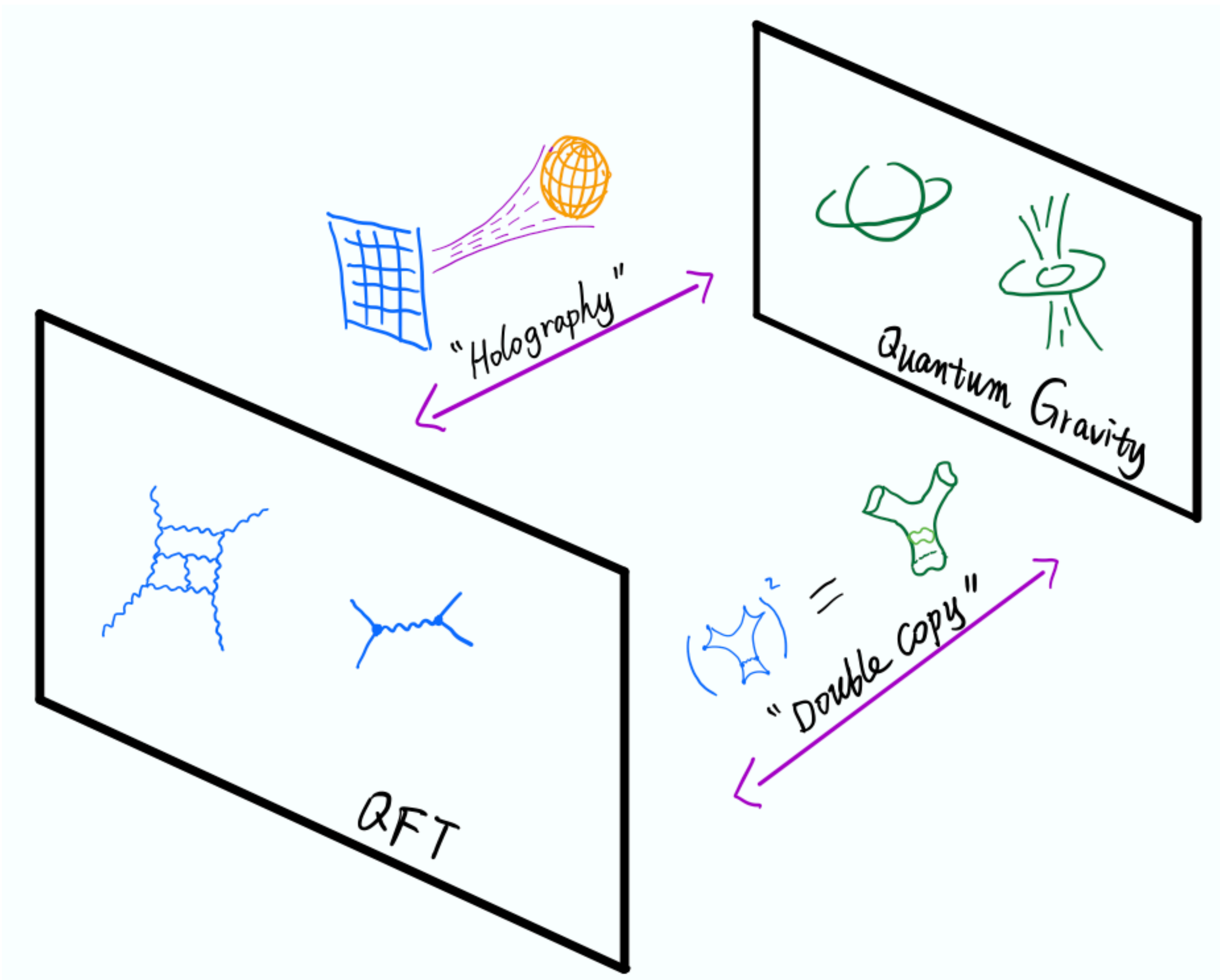


- Understanding the structure of ultraviolet divergences

$$\mathscr{L} = \sqrt{-g}(R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \cdots)$$

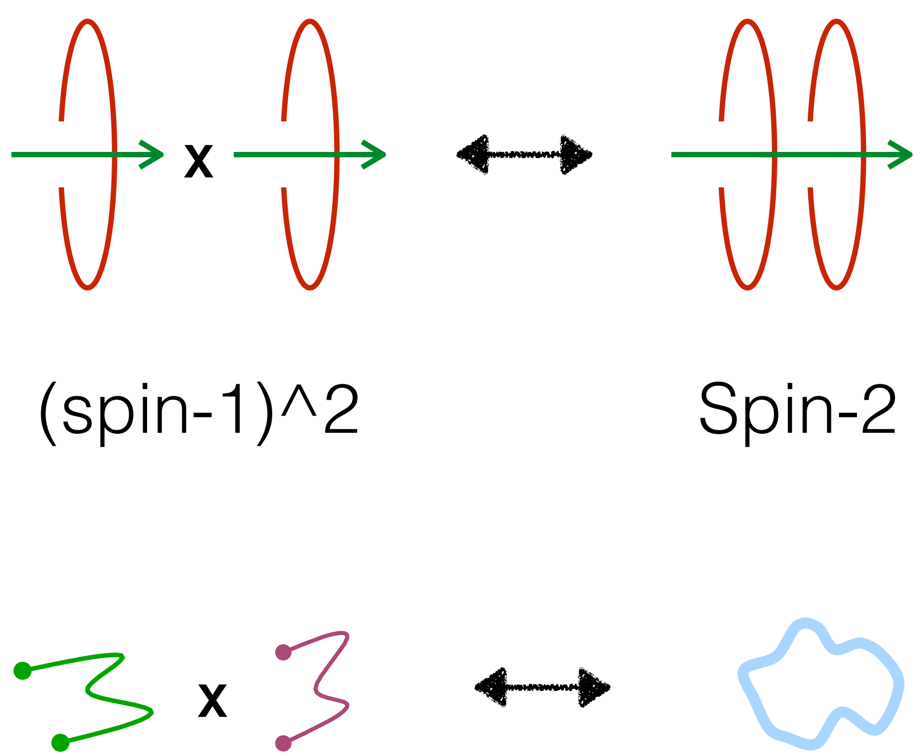


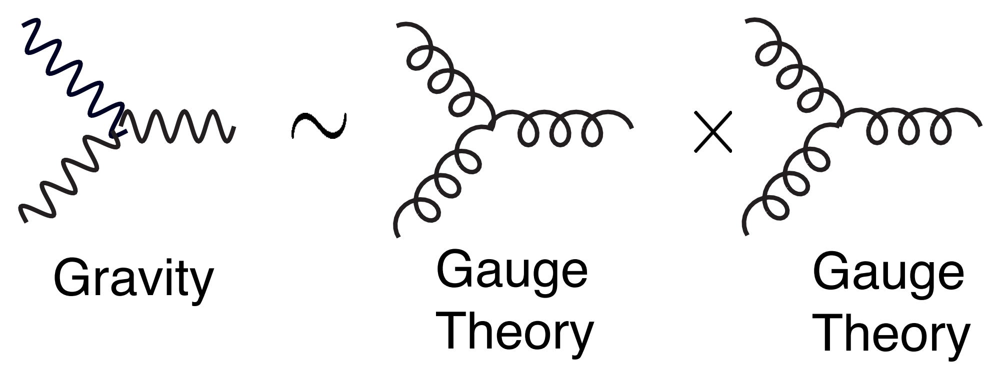
From gauge theory to gravity



Double copy

Gravity ⟷ **(Yang-Mills)^2**





[Kawai, Lewellen, Tye 1986]

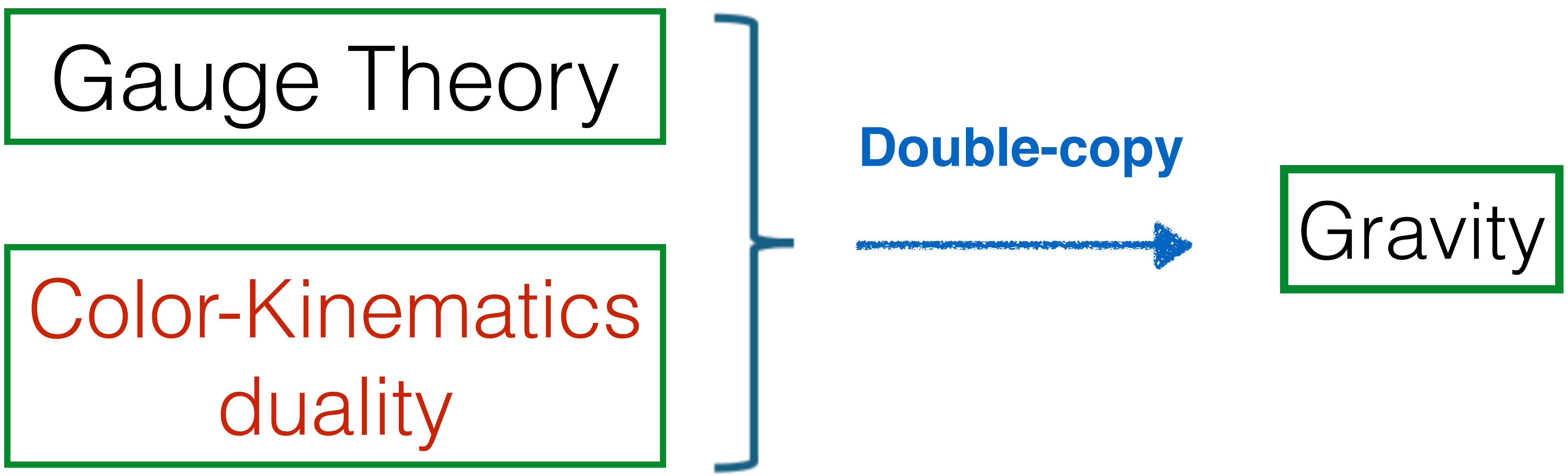
[Bern, Carrasco, Johansson 2008]

[Cachazo, He, Yuan 2013]

Color-kinematics duality

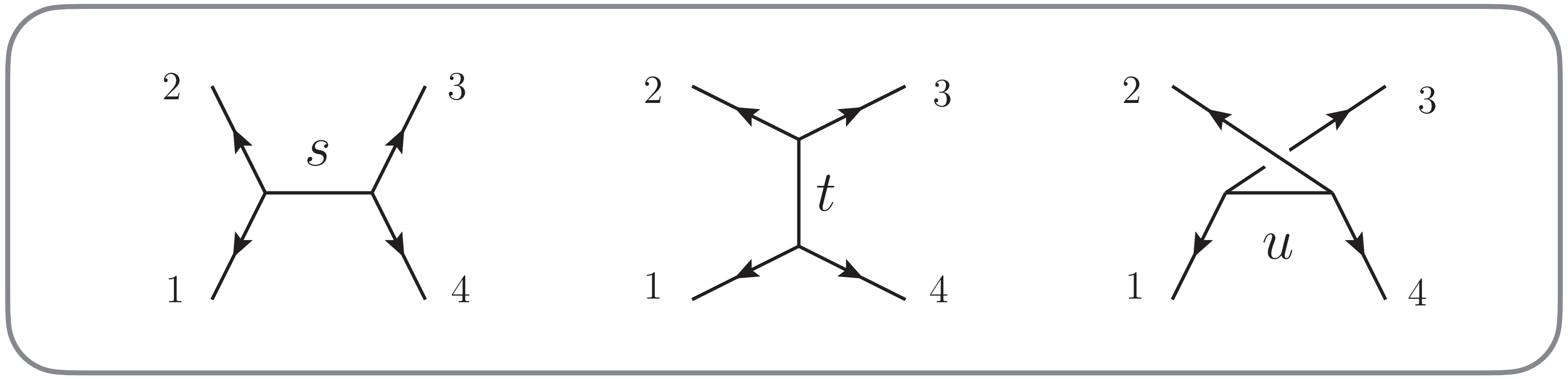
Color-kinematics duality was introduced in 2008 by Bern, Carrasco, and Johansson.

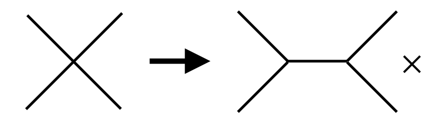
Bern, Carrasco, Johansson 2008



Generalizing double-copy to quantum (loop) level.

Example: 4-pt amplitude





$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

$$c_s = \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}, \quad c_t = \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}, \quad c_u = \tilde{f}^{a_1 a_3 b}\tilde{f}^{b a_2 a_4}$$

$$c_s = c_t + c_u \quad \Rightarrow \quad n_s = n_t + n_u$$

Jacobi identity

dual Jacobi relation

in general a non-trivial requirement

Double copy

If the gauge amplitude satisfies CK duality, one can directly construct gravity amplitude：

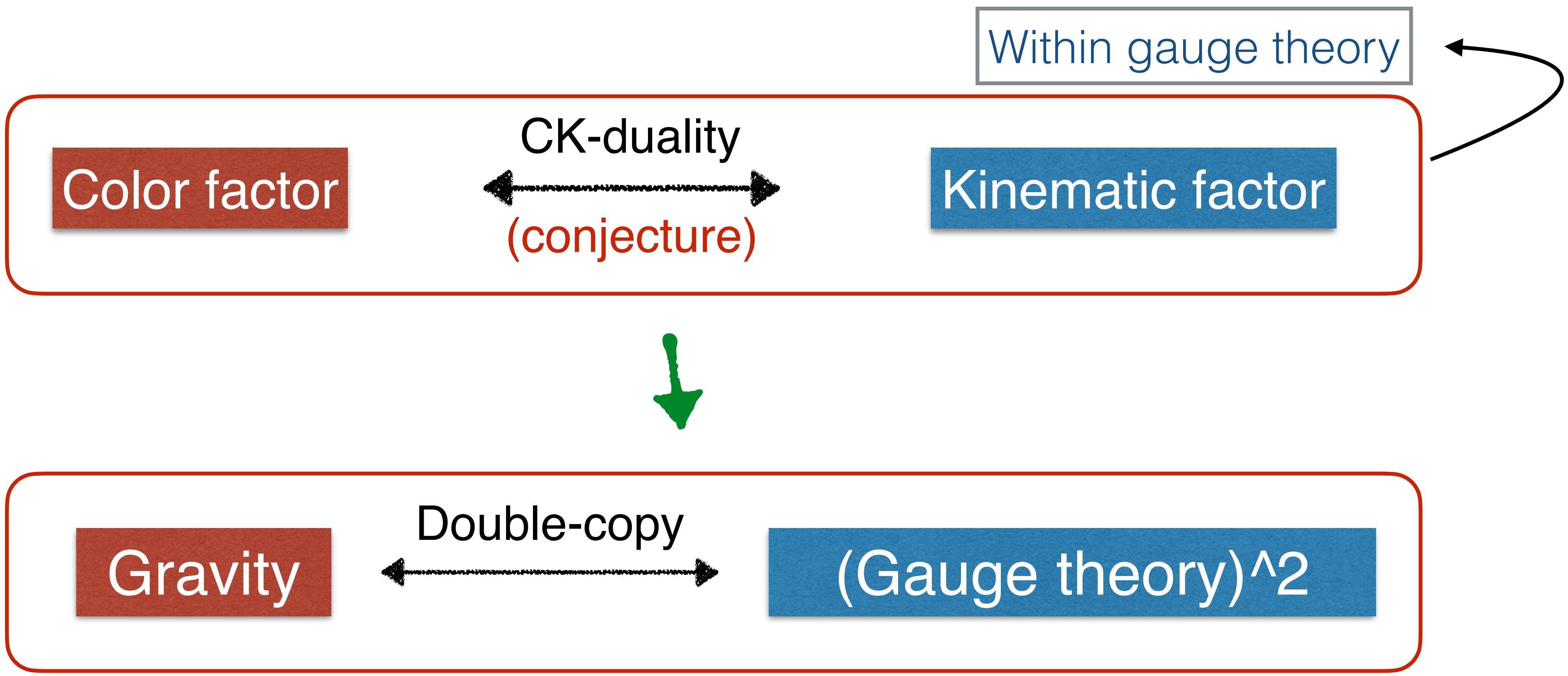
$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \longrightarrow M_4(1,2,3,4) = \frac{n_s n_s}{s} + \frac{n_t n_t}{t} + \frac{n_u n_u}{u}$$

"double-copy" can be generalized to **high loops**:

$$A^{(\ell)} \sim \sum_i \int \frac{C_i \times N_i}{\prod D} \longrightarrow M^{(\ell)} \sim \sum_i \int \frac{N_i \times N_i}{\prod D}$$

Gauge x Gauge ⟵ CK-duality ⟶ Gravity

CK-duality v.s. Double-copy



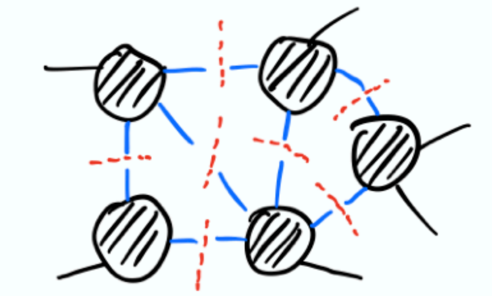
By studying the simpler gauge theory, one may understand the far more complicated gravity theory.

It is usually non-trivial to find CK-dual solutions at high loops.

Outline

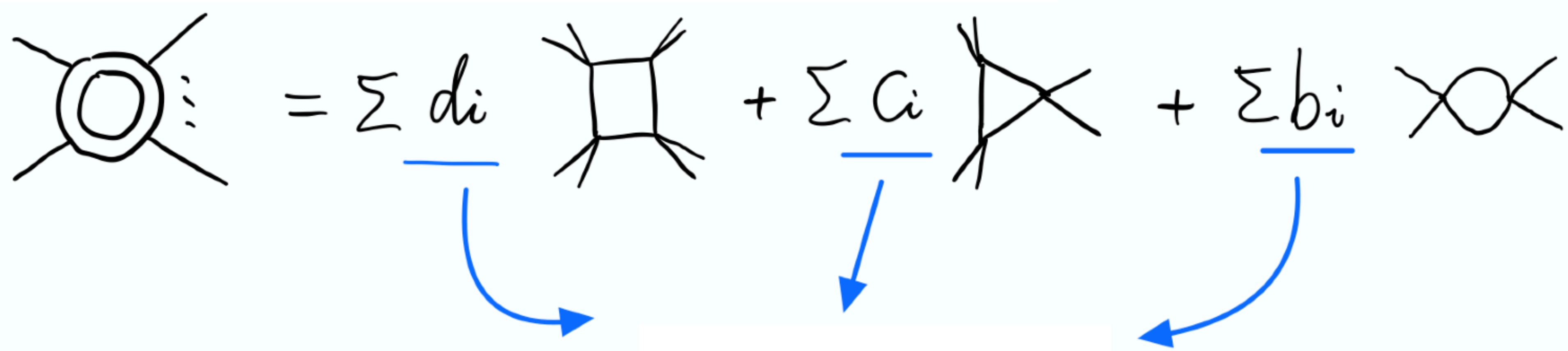
- Introduction
- Constructing CK-dual numerators
- New strategy of deformation
- Summary and outlook

On-shell methods



One-loop structure

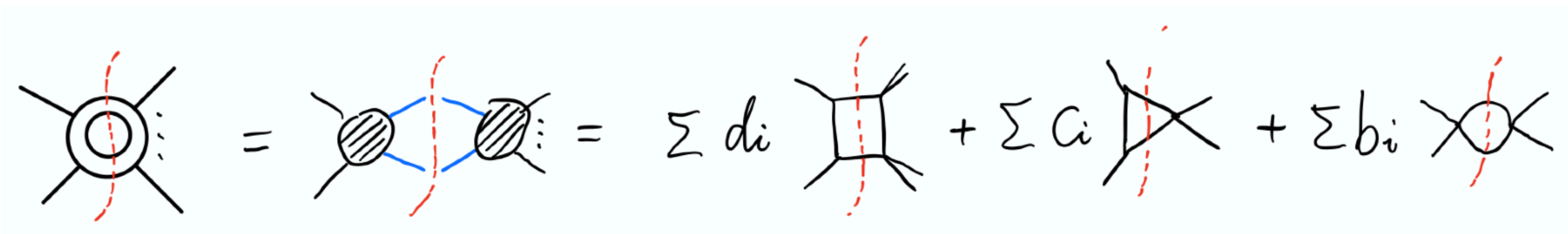
Consider one-loop amplitudes:



What we really want

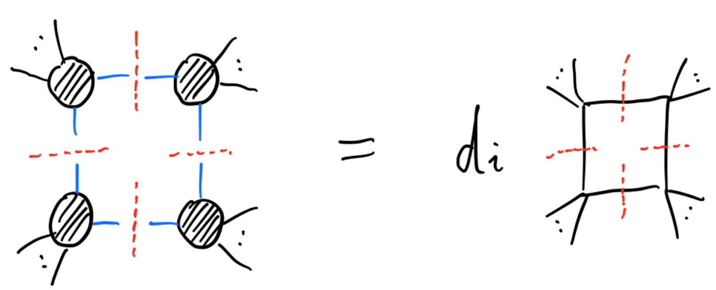
Unitarity cuts

Using simpler tree-level blocks, one can derive the coefficients more efficiently:



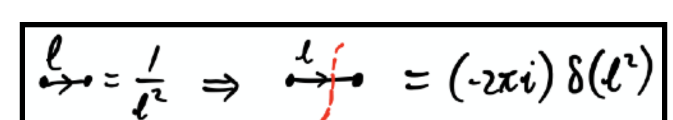
[Bern, Dixon, Dunbar, Kosower 1994]

Generalized multiple cuts



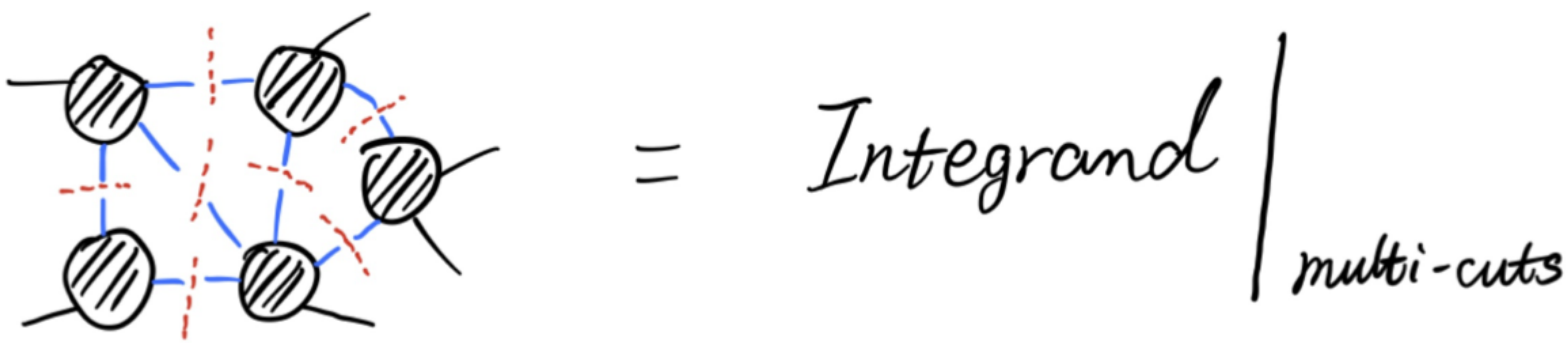
[Britto, Cachazo, Feng 2004]

Cutkosky cutting rule:



Loop integrands

This strategy applies at general loop orders:



“Theorem”:

A loop integrand is physically correct and complete, once it is consistent with all possible unitarity cuts.

Constructing CK-dual integrand

CK-duality

$c_s = c_t + c_u \quad \Rightarrow \quad n_s = n_t + n_u$

Conjecture

Compact ansatz of the loop integrand

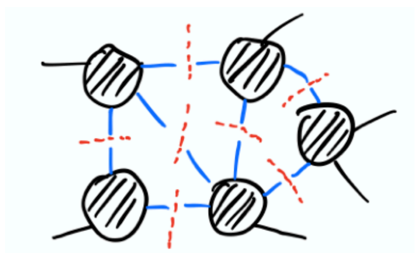
$$A^{(\ell)} \sim \sum_i \int \frac{C_i \times N_i}{\prod D}$$

Constructing CK-dual integrand

CK-duality $c_s = c_t + c_u \quad \Rightarrow \quad n_s = n_t + n_u$

Conjecture

Compact ansatz of the loop integrand



Unitarity cuts

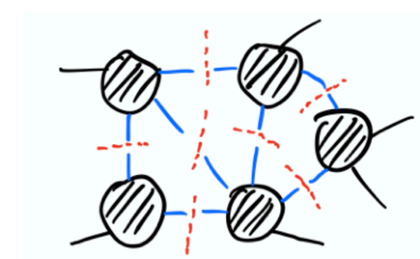
$$\text{Loop-ansatz}\,|_{\text{cut}} = \prod \text{tree-blocks}$$

Solving linear equations

Constructing CK-dual integrand

CK-duality $c_s = c_t + c_u \quad \Rightarrow \quad n_s = n_t + n_u$

Conjecture → **Compact ansatz of the loop integrand**



↔ **Unitarity cuts**

$$\text{Loop-ansatz}\,|_{\text{cut}} = \prod \text{tree-blocks}$$

Solving linear equations

Main challenge: it is a priori not known whether the solution exists

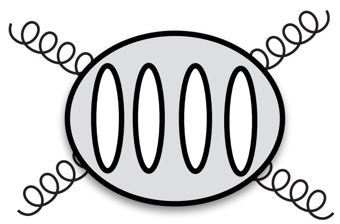
Loop-level CK duality

For N=4 SYM, there are high loop examples that manifest **global CK-dual Jacobi relations**:

Amplitudes:

- 4-loop 4-point amplitude in N=4

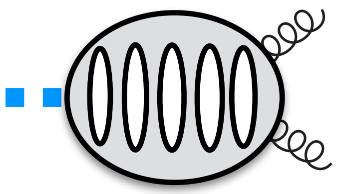
  Bern, Carrasco, Dixon, Johansson, Roiban, 2012



Form factor with stress-tensor operator (Higgs+gluon amplitudes):

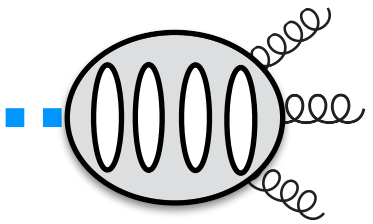
- 5-loop Sudakov form factor in N=4

  GY, 2016

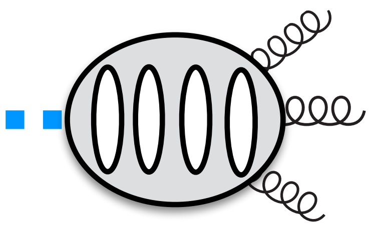


- 4-loop three-point form factor in N=4

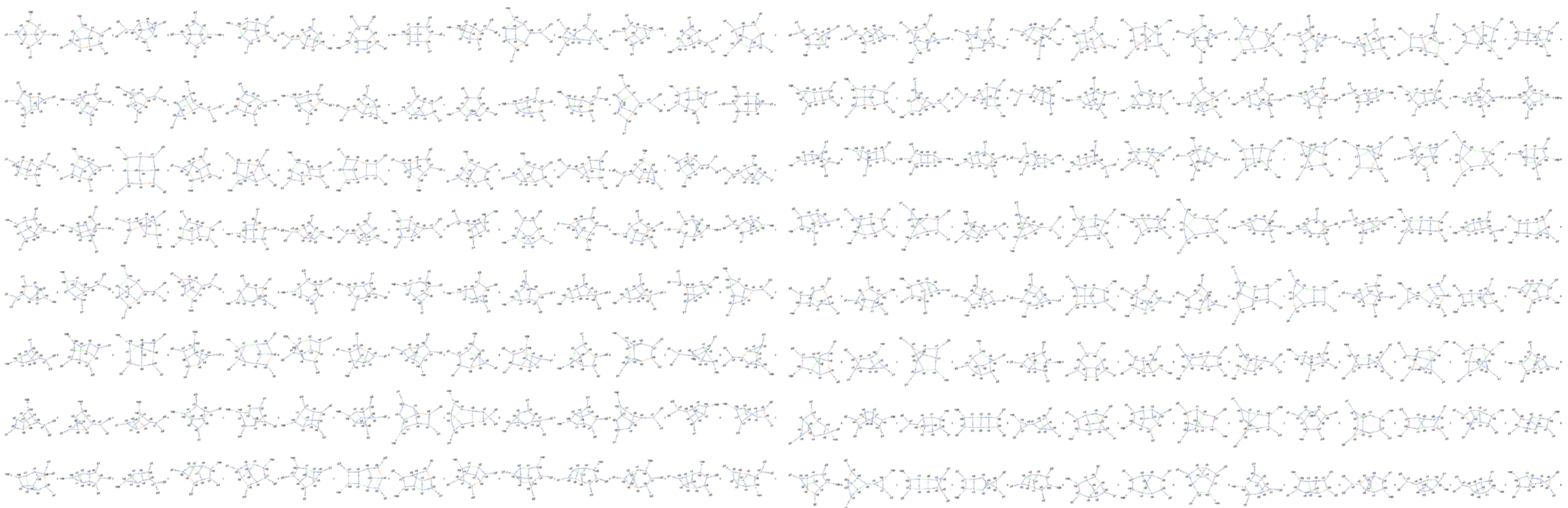
  Lin, GY, Zhang, 2021



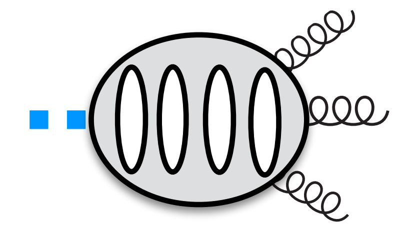
4-loop 3-point form factor

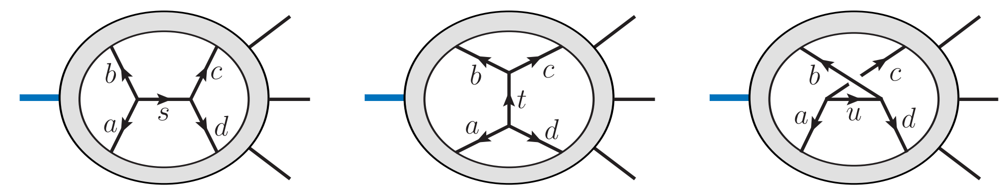


229 trivalent graphs



4-loop 3-point form factor

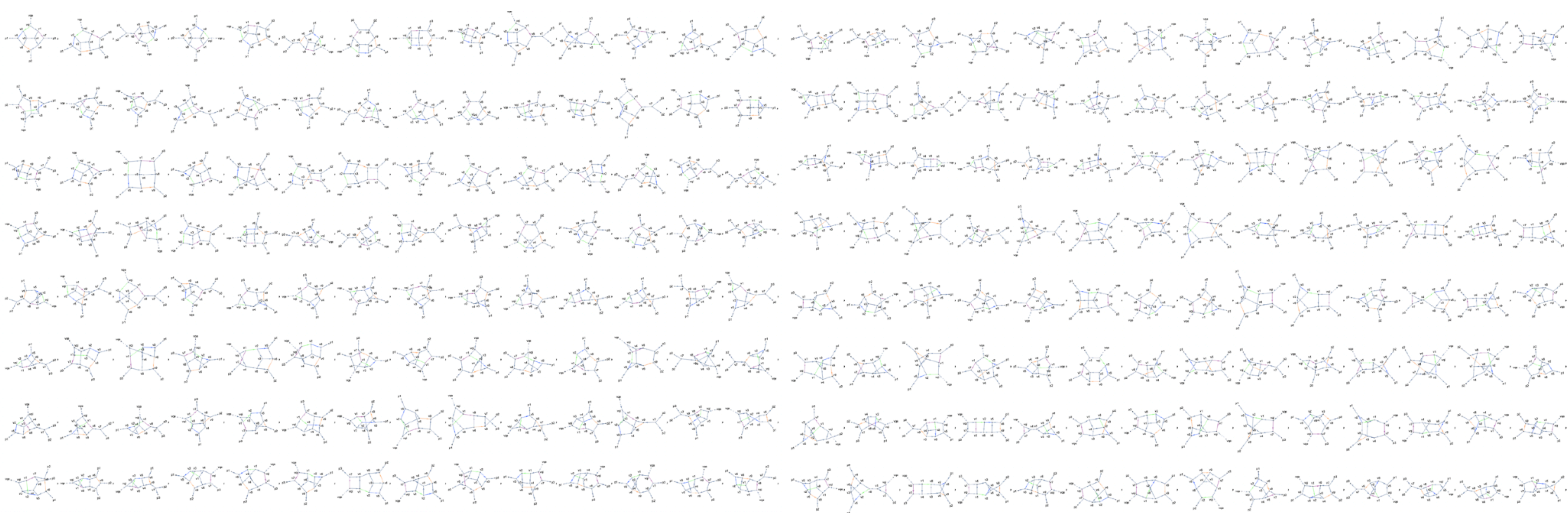




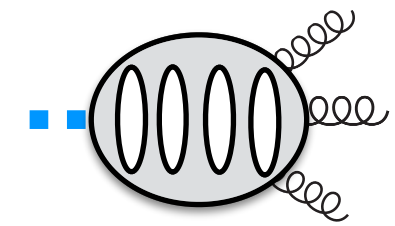
229 trivalent graphs

$$N_s + N_t + N_u = 0$$

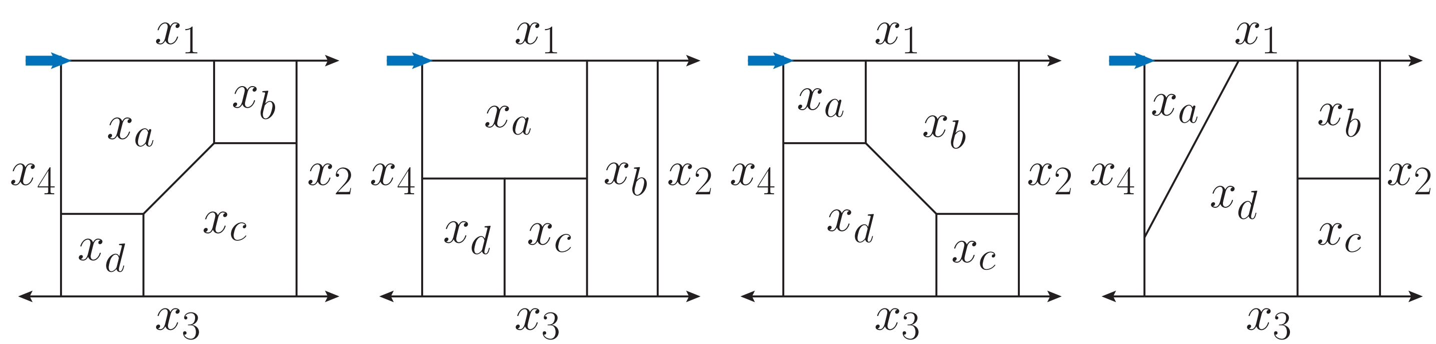
dual Jacobi relations



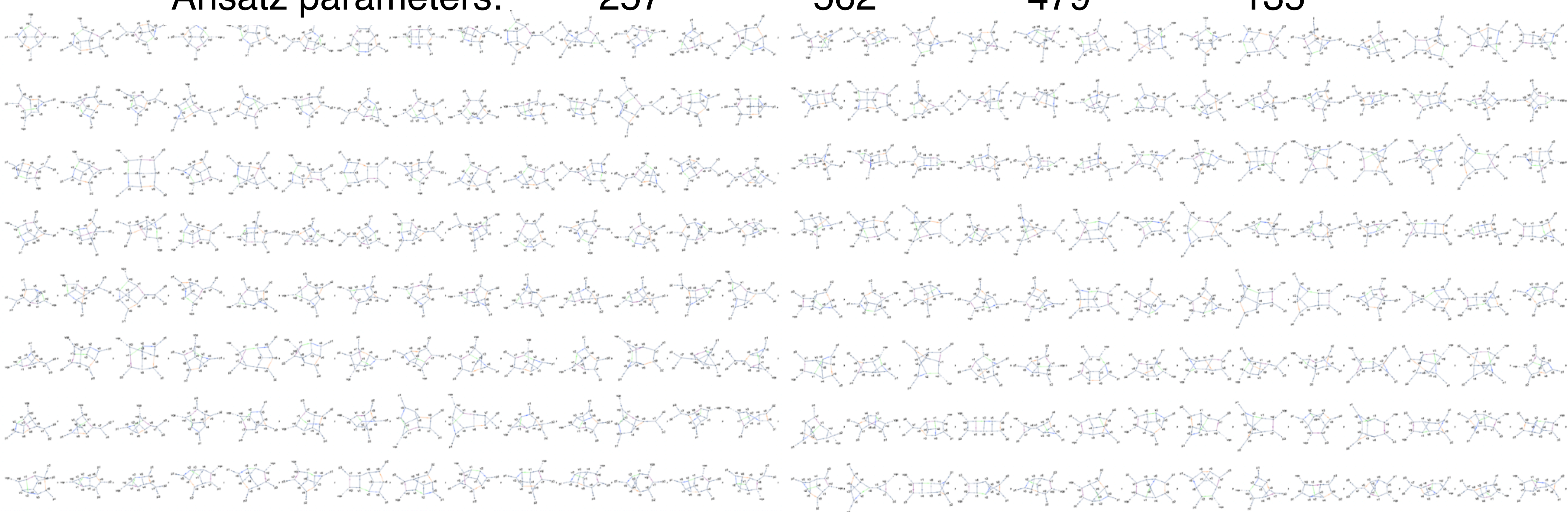
4-loop 3-point form factor



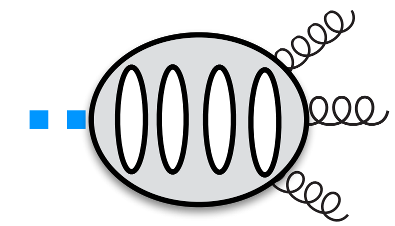
Master graphs



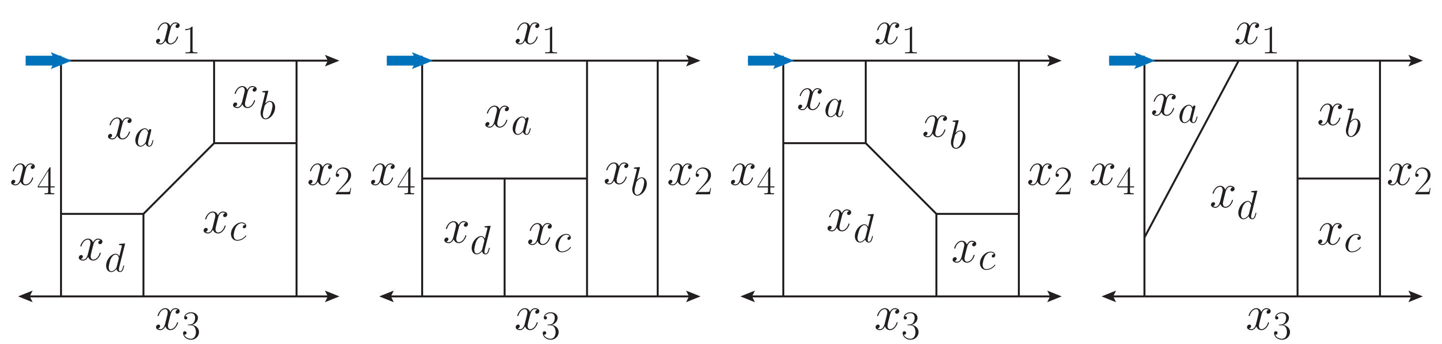
Ansatz parameters: 257 562 479 135



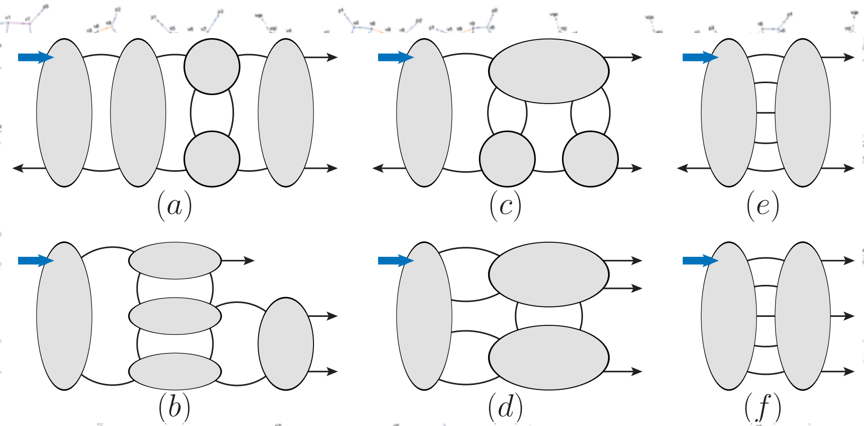
4-loop 3-point form factor



Master graphs

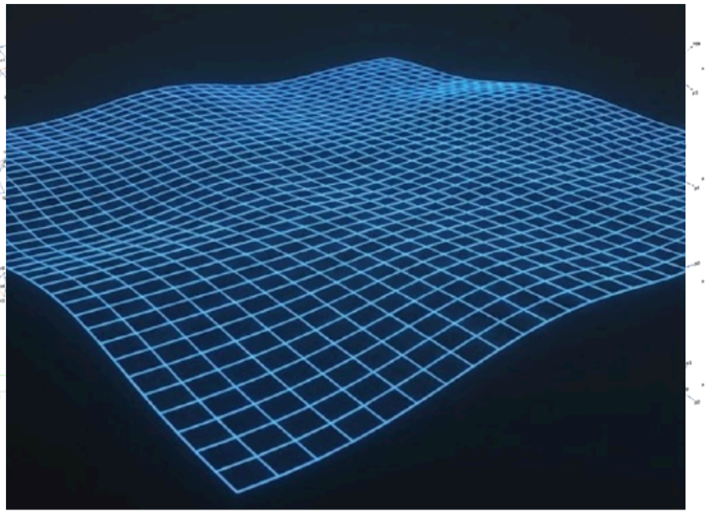


Unitarity cuts



$$\boldsymbol{F}_3^{(4)} = \sum_{\sigma_3}\sum_{i=1}^{229}\int\prod_{j=1}^{4} d^D\ell_j \frac{1}{S_i}\,\sigma_3\cdot\frac{\mathcal{F}_3^{(0)}\, C_i\, N_i}{\prod_{\alpha_i} P_{\alpha_i}^2}$$

All cuts are satisfied !



Non-supersymmetric Yang-Mills

For non-supersymmetric **pure YM**, even two-loop is challenging:

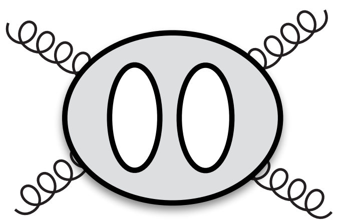
- 2-loop 4-gluon all-plus-helicity amplitude in pure YM

$A_4^{(2)}(1^+,2^+,3^+,4^+)$ Bern, Davies, Dennen, Huang, Nohle 2013

- 2-loop 5-gluon all-plus-helicity amplitude in pure YM

$A_5^{(2)}(1^+,2^+,3^+,4^+,5^+)$ O'Connell and Mogull 2015 $n^{\text{CK}} \sim \ell^{12}$

No global CK-dual solution is known for generic helicity configurations at two loops.



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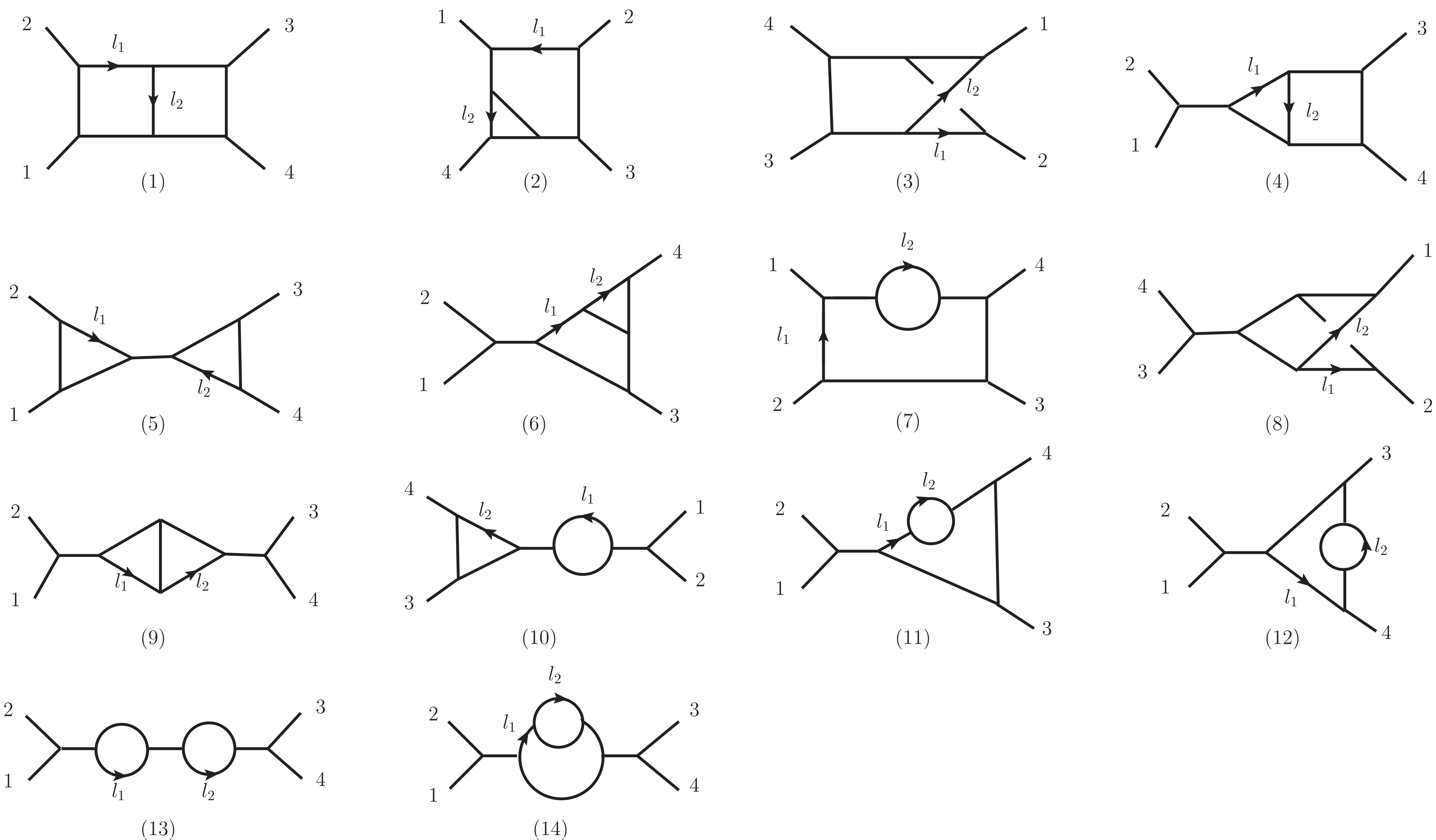
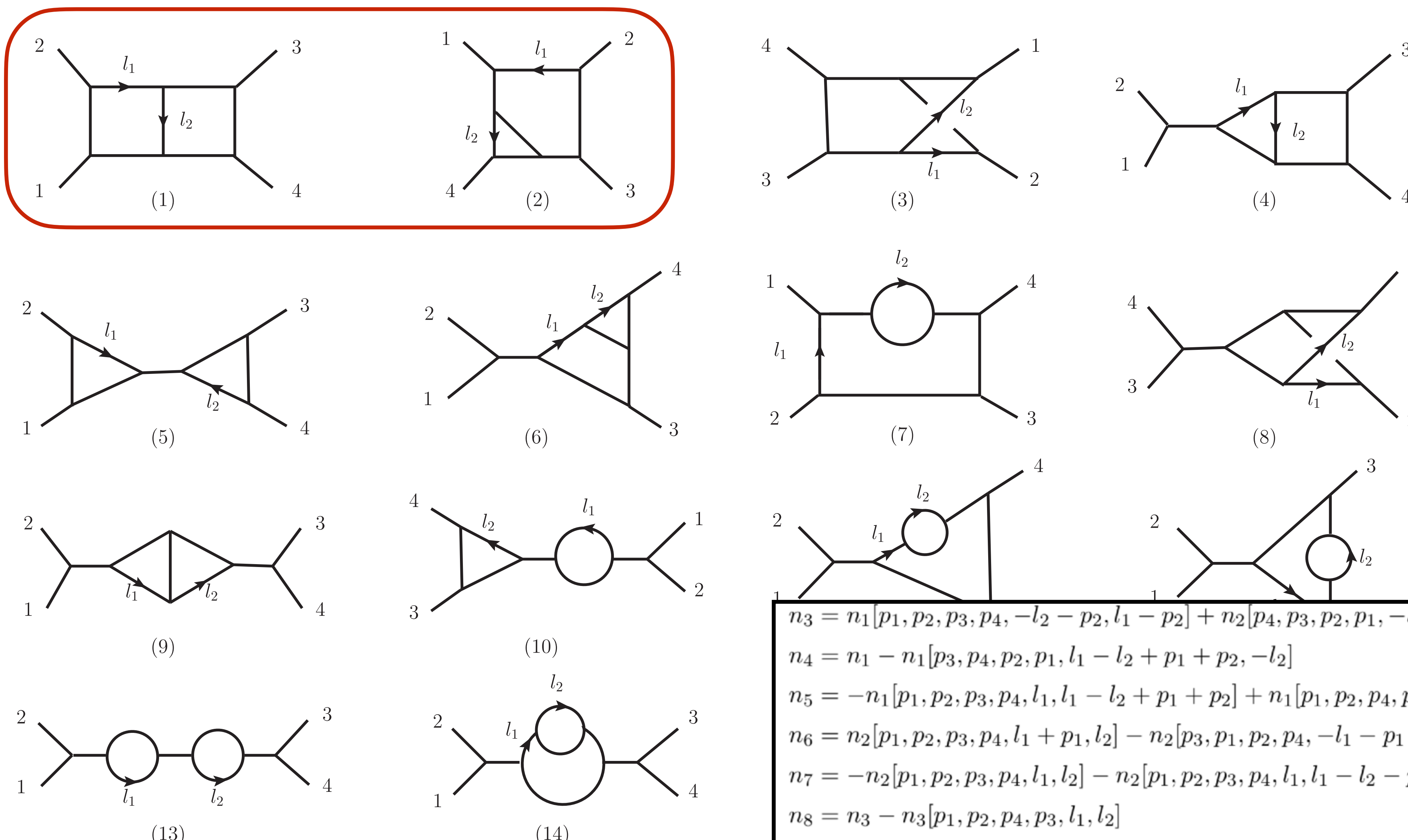
Two-loop trivalent diagrams

Let us first review the standard construction.



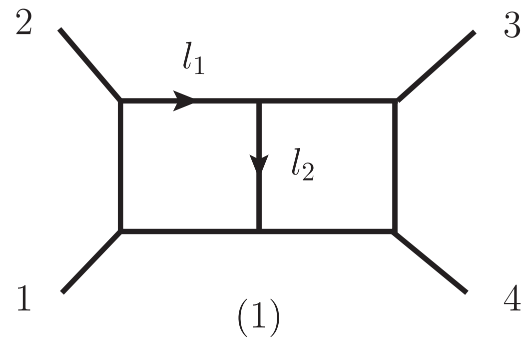
Two-loop trivalent diagrams

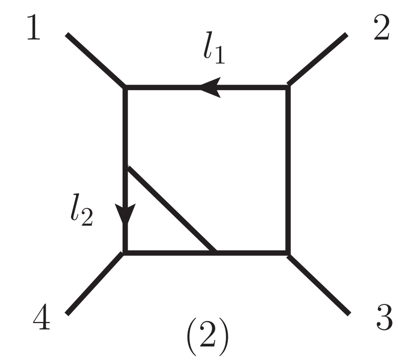
Master topologies



$$n_3 = n_1[p_1,p_2,p_3,p_4,-l_2-p_2,l_1-p_2] + n_2[p_4,p_3,p_2,p_1,-l_1-l_2-p_3,-l_2+p_1]$$
$$n_4 = n_1 - n_1[p_3,p_4,p_2,p_1,l_1-l_2+p_1+p_2,-l_2]$$
$$n_5 = -n_1[p_1,p_2,p_3,p_4,l_1,l_1-l_2+p_1+p_2] + n_1[p_1,p_2,p_4,p_3,l_1,l_1+l_2]$$
$$n_6 = n_2[p_1,p_2,p_3,p_4,l_1+p_1,l_2] - n_2[p_3,p_1,p_2,p_4,-l_1-p_1-p_2,-l_2-p_1-p_2-p_3]$$
$$n_7 = -n_2[p_1,p_2,p_3,p_4,l_1,l_2] - n_2[p_1,p_2,p_3,p_4,l_1,l_1-l_2-p_1]$$
$$n_8 = n_3 - n_3[p_1,p_2,p_4,p_3,l_1,l_2]$$
$$n_9 = -n_4[p_1,p_2,p_3,p_4,l_1,l_1-l_2] + n_4[p_1,p_2,p_4,p_3,l_1,l_1-l_2]$$
$$n_{10} = -n_4[p_1,p_2,p_3,p_4,l_1,l_1+l_2+p_1+p_2] - n_4[p_1,p_2,p_3,p_4,-l_1-p_1-p_2,-l_1+l_2]$$
$$n_{11} = -n_4[p_1,p_2,p_3,p_4,-l_2-p_1-p_2,l_1-l_2] - n_4[p_1,p_2,p_3,p_4,-l_1+l_2-p_1-p_2,l_2]$$
$$n_{12} = -n_6[p_1,p_2,p_3,p_4,l_1,l_1-l_2] - n_6[p_1,p_2,p_3,p_4,l_1,l_2-p_1-p_2-p_3]$$
$$n_{13} = n_9 + n_9[p_1,p_2,p_3,p_4,-l_1-p_1-p_2,l_2]$$
$$n_{14} = n_9[p_1,p_2,p_3,p_4,l_1-l_2,l_1] + n_9[p_1,p_2,p_3,p_4,-l_2-p_1-p_2,-l_1-p_1-p_2]\,,$$

Ansatz for the master numerators





Polynomials in D-dim kinematics:

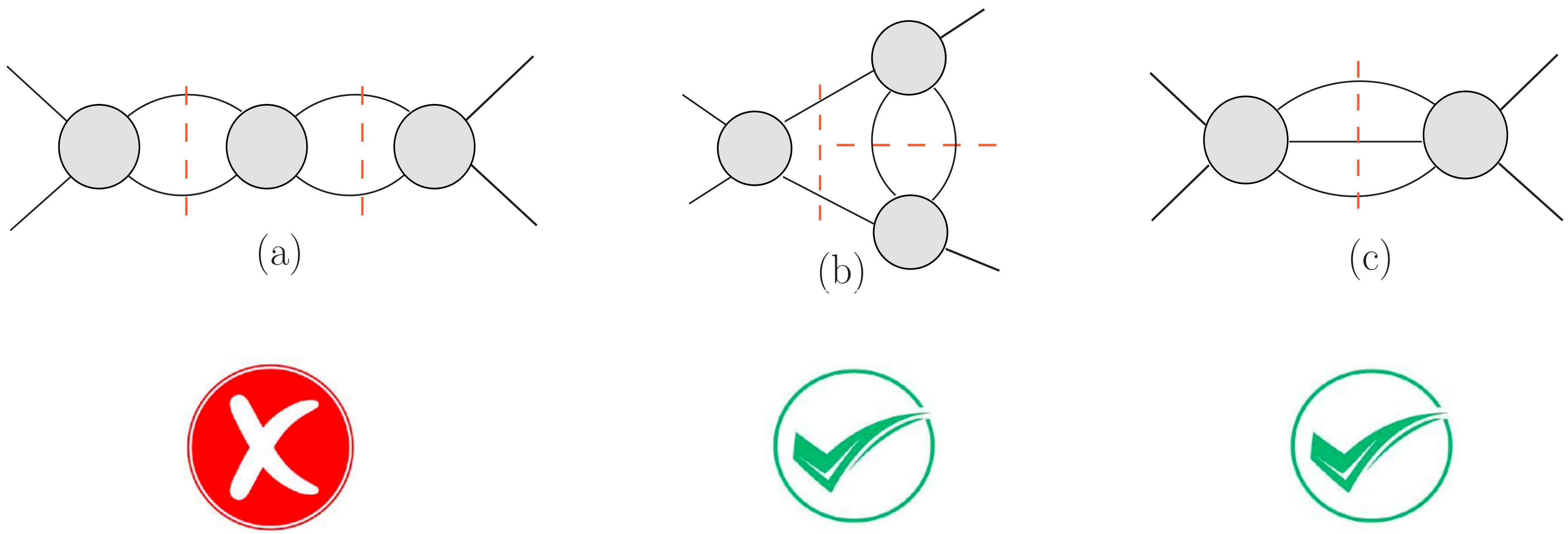
$$\boxed{n_m = \sum_k a_{mk} M_k \,, \qquad m = 1, 2 \,,}$$

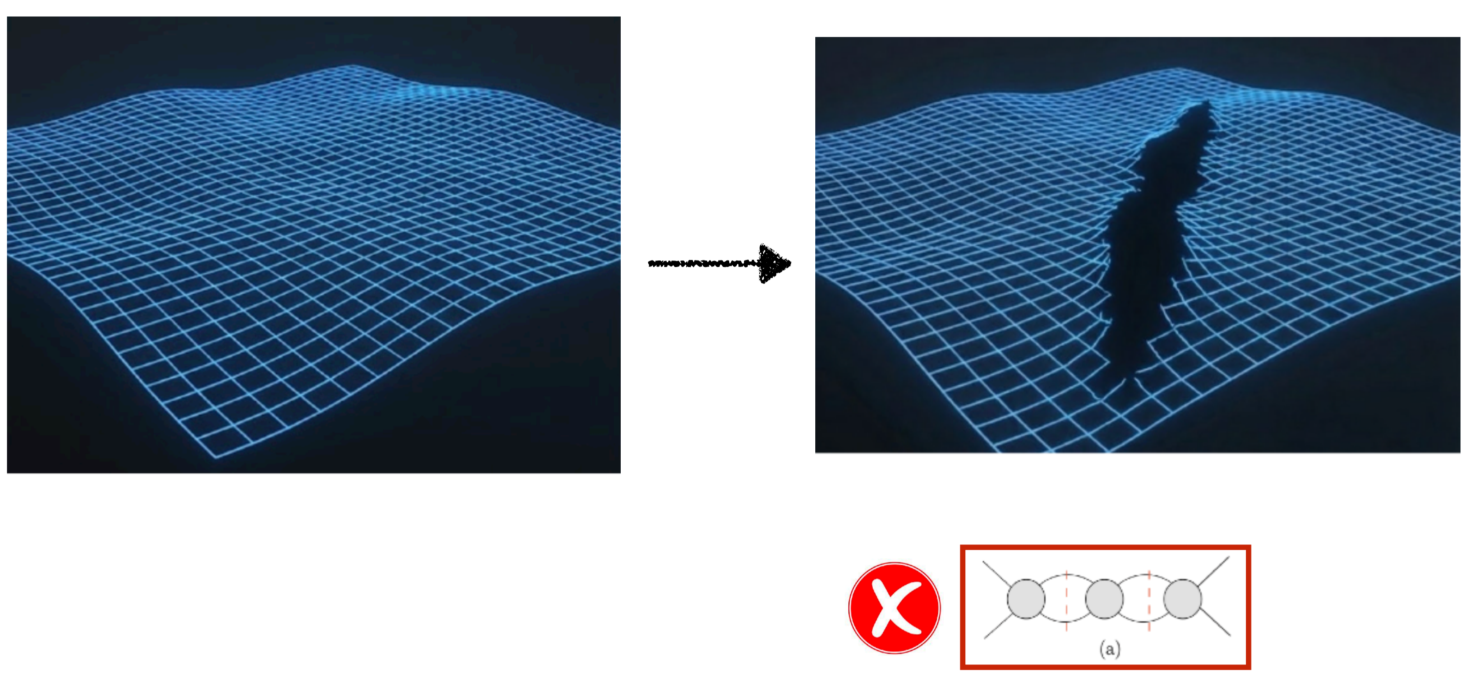
$$\{\varepsilon_i \cdot \varepsilon_j,\ \varepsilon_i \cdot p_j,\ \varepsilon_i \cdot l_\alpha,\ p_i \cdot l_\alpha,\ l_\alpha \cdot l_\beta\ , p_1 \cdot p_2\ , p_2 \cdot p_3\}$$

e.g.: $(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4)(p_1 \cdot p_2)(p_1 \cdot l_1)(p_1 \cdot l_2)$

Parameters: ~ 20,000 ⟶ (Symmetry) ~1400

Unitarity cuts

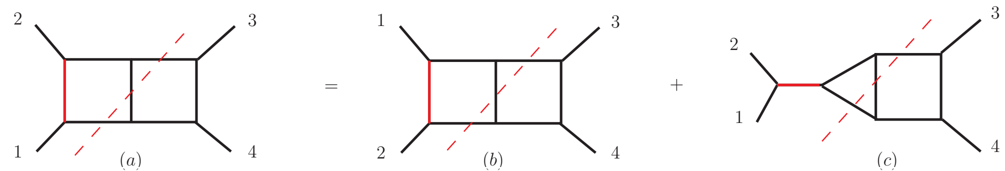




Give up global CK duality

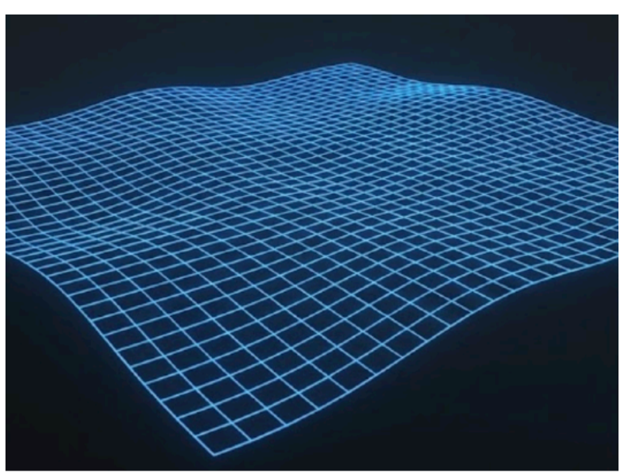
One may give up global CK duality, and make ansatz for all topologies separately, while only imposing CK-duality on cuts.

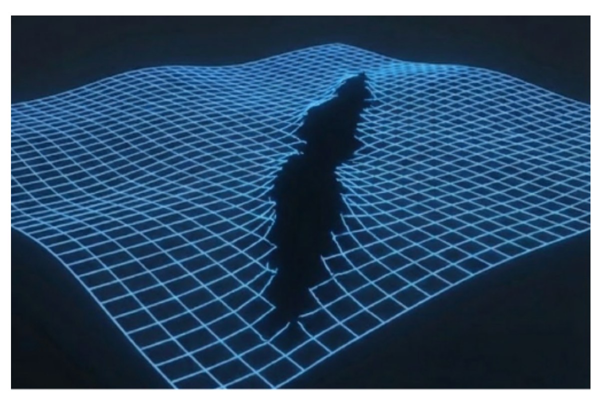
Bern, Davies and Nohle 2015

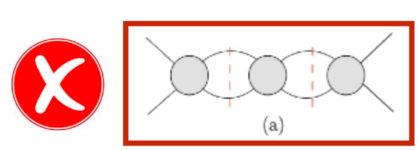


$$(n_a - n_b - n_c)\big|_{\text{cut}} = 0$$

CK-duality only on cuts
initial parameters: ∼120,000
after symmetry constraints: ∼28,000
after cut and CK constraints: ∼6,300

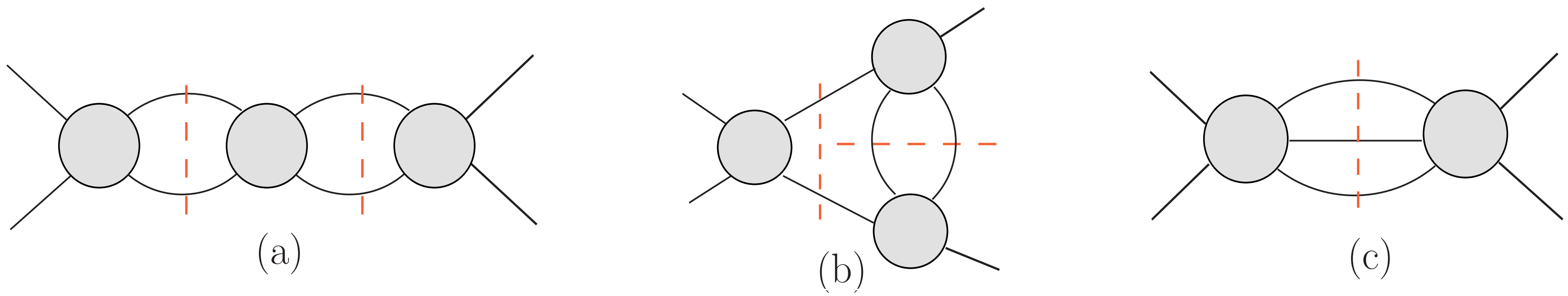






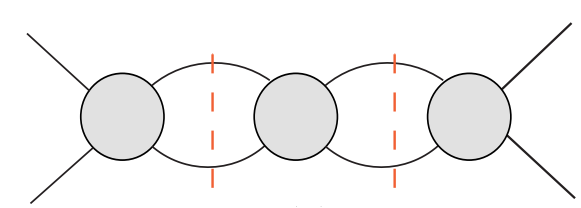
better way?

Unitarity cuts

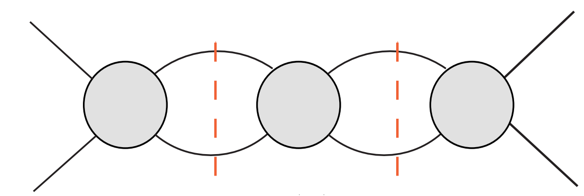


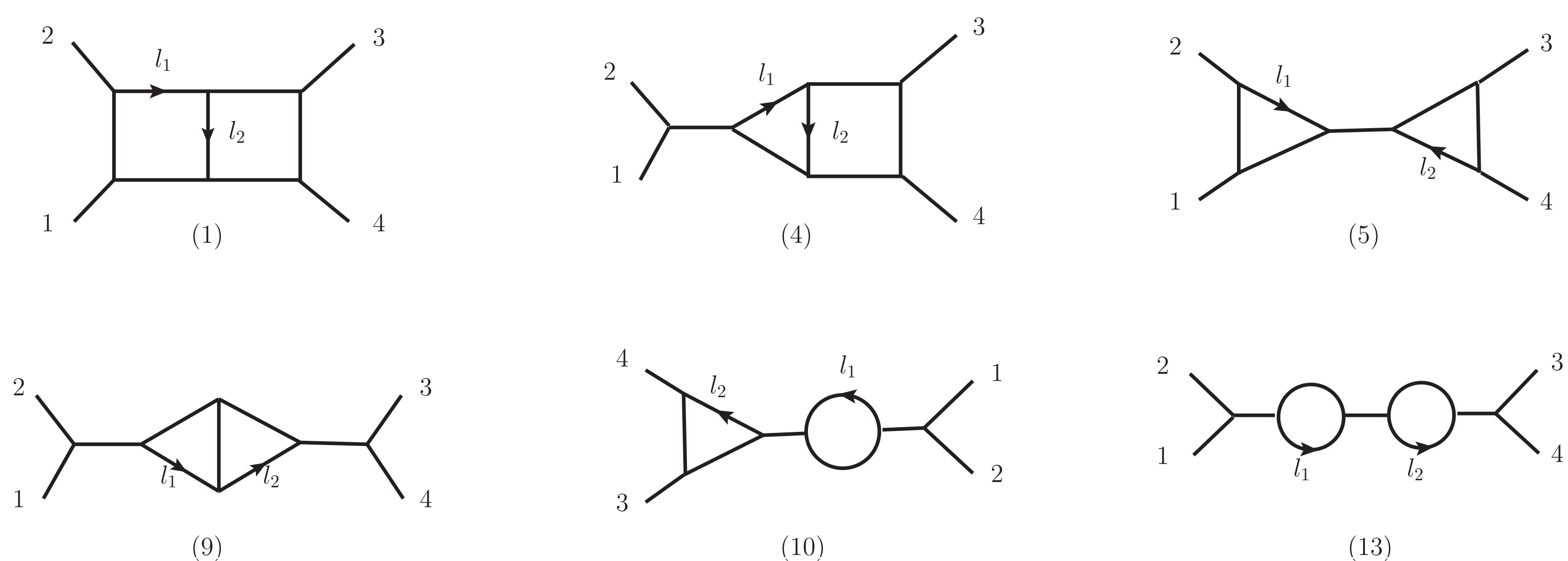
We would like to “keep” the advantage of the global CK-dual relations

$$N_i = n_i + \Delta_i$$ Deformation →



Deformed trivalent diagrams

Topologies that affect the ladder-double-cut: 



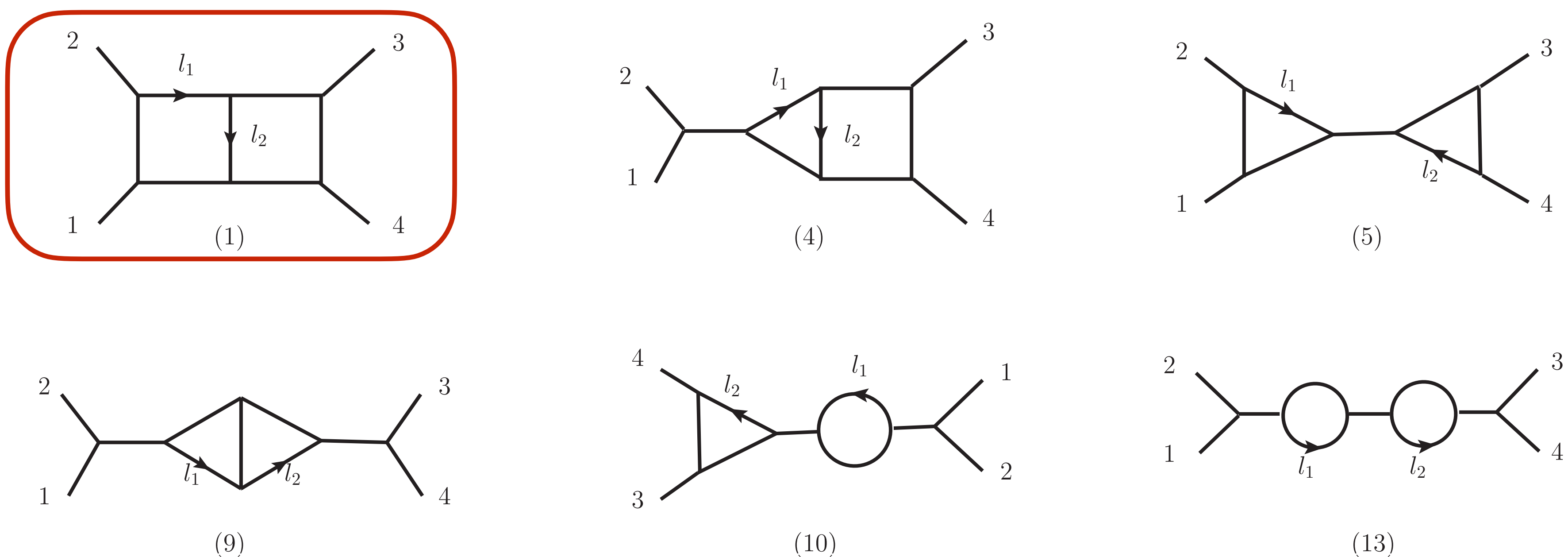
Deformation

$$N_i = \begin{cases} n_i + \Delta_i, & i = 1,4,5,9,10,13, \\ n_i, & \text{others.} \end{cases}$$

Deformed numerators

We ask that deformation satisfies a sub-set of dual Jacobi relations.

Master topologies for deformation



$$\Delta_4 = \Delta_1 - \Delta_1[p_3, p_4, p_2, p_1, l_1 - l_2 + p_1 + p_2, -l_2]$$

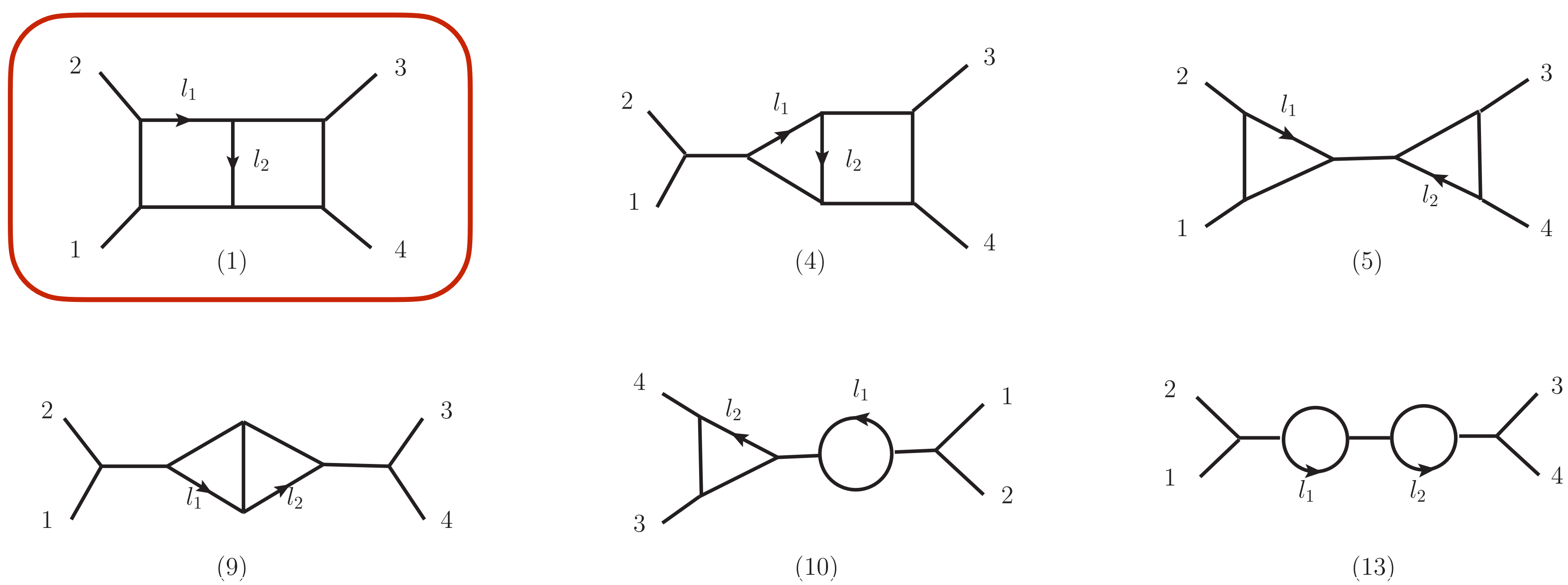
$$\Delta_5 = -\Delta_1[p_1, p_2, p_3, p_4, l_1, l_1 - l_2 + p_1 + p_2] + \Delta_1[p_1, p_2, p_4, p_3, l_1, l_1 + l_2]$$

$$\Delta_9 = -\Delta_4[p_1, p_2, p_3, p_4, l_1, l_1 - l_2] + \Delta_4[p_1, p_2, p_4, p_3, l_1, l_1 - l_2]$$

$$\Delta_{10} = -\Delta_4[p_1, p_2, p_3, p_4, l_1, l_1 + l_2 + p_1 + p_2] - \Delta_4[p_1, p_2, p_3, p_4, -l_1 - p_1 - p_2, -l_1 + l_2]$$

$$\Delta_{13} = \Delta_9 + \Delta_9[p_1, p_2, p_3, p_4, -l_1 - p_1 - p_2, l_2]\,.$$

Ansatz of the master numerator



Consider different Lorentz structure separately:

$$\Delta_i = \Delta_i^{[1]} + \Delta_i^{[2]} + \Delta_i^{[3]}.$$

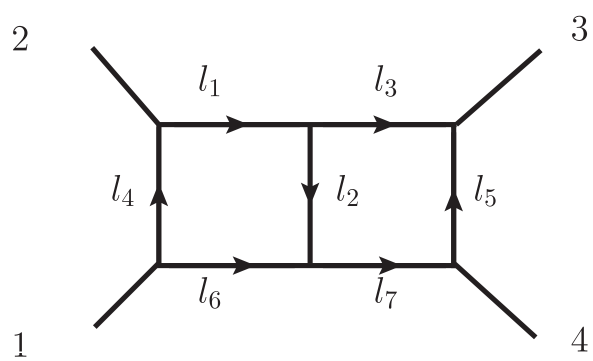
$$\Delta_1^{[1]} = (\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4)(\sum_k c_k^{[1]} M_k^{[1]})\, l_2^2$$

$$\Delta_1^{[2]} = \Big[(\varepsilon_1 \cdot \varepsilon_2)(\sum_a c_a^{[2]} M_{1,a}^{[2]}) + (\varepsilon_3 \cdot \varepsilon_4)(\sum_b c_b^{[2]} M_{2,b}^{[2]})\Big] l_2^2$$

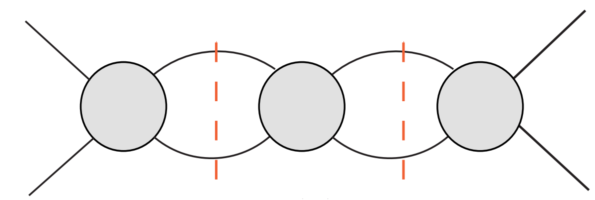
$$\Delta_1^{[3]} = (\sum_k c_k^{[3]} M_k^{[3]}) l_2^2$$

Some requirement:
1) do not affect other cuts,
2) double copy still applicable.

Solving the master numerator



$$N_i = n_i + \Delta_i$$



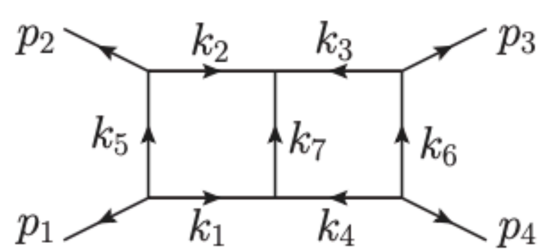
$$\Delta_1^{[1]} = (d-2)^2(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) l_4^2\, l_2^2\, l_5^2$$

$$\Delta_1^{[2]} = -4(d-2)^2 \Big[(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot l_5)(\varepsilon_4 \cdot l_5) l_4^2 + (\varepsilon_3 \cdot \varepsilon_4)(\varepsilon_1 \cdot l_4)(\varepsilon_2 \cdot l_4) l_5^2\Big] l_2^2$$

Deformation

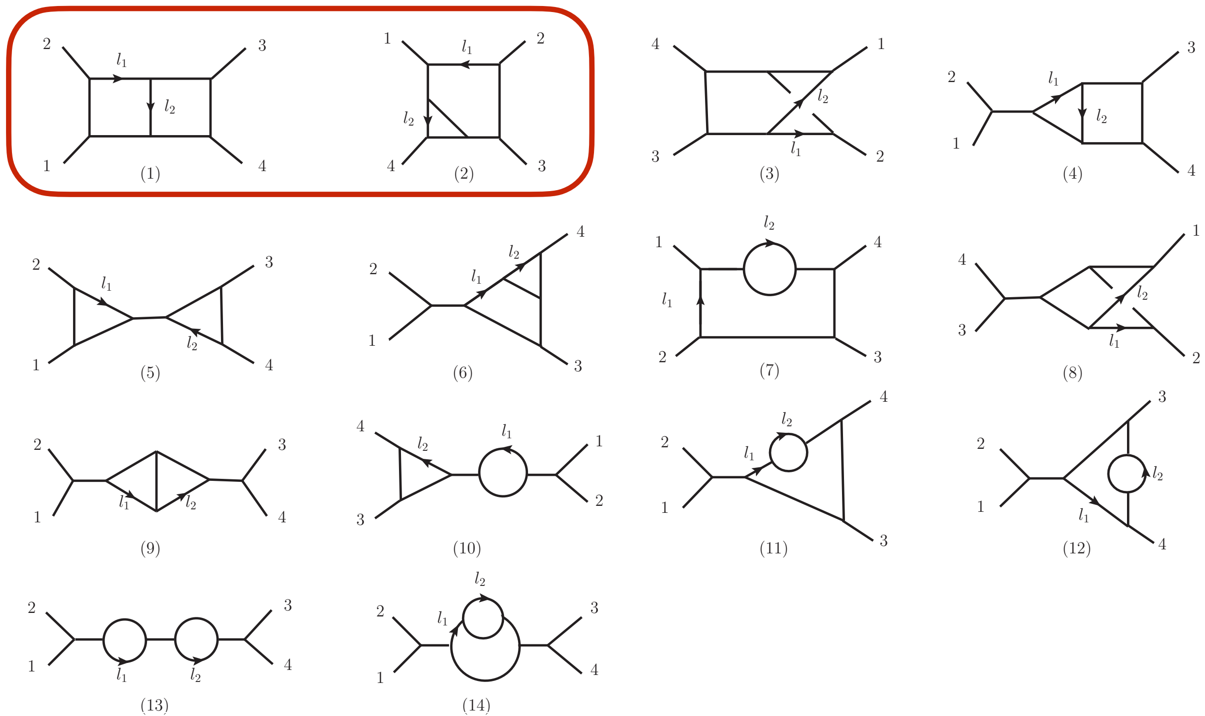
Final solution for the master numerator:

(There is a solution space with free parameters, here is a special simple choice.)



$$
\begin{aligned}
\Delta\left[\;\cdots\;\right]\Big/ k_7^2 =\\
&+(d-2)^2\Big\{(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_4)k_5^2\,k_6^2+16(\varepsilon_1\cdot k_5)(\varepsilon_2\cdot k_5)(\varepsilon_3\cdot k_6)(\varepsilon_4\cdot k_6)\\
&\qquad -4\Big[(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot k_6)(\varepsilon_4\cdot k_6)k_5^2+(\varepsilon_3\cdot\varepsilon_4)(\varepsilon_1\cdot k_5)(\varepsilon_2\cdot k_5)k_6^2\Big]\Big\}\\
&+(d-2)4\Big\{-10\Big[(\varepsilon_1\cdot k_6)(\varepsilon_2\cdot k_6)(\varepsilon_3\cdot k_5)(\varepsilon_4\cdot k_5)+(\varepsilon_1\cdot k_2)(\varepsilon_2\cdot k_1)(\varepsilon_3\cdot k_4)(\varepsilon_4\cdot k_3)\Big]\\
&\qquad +20\Big[(\varepsilon_1\cdot k_6)(\varepsilon_2\cdot k_1)(\varepsilon_3\cdot k_5)(\varepsilon_4\cdot k_3)+(\varepsilon_1\cdot k_2)(\varepsilon_2\cdot k_6)(\varepsilon_3\cdot k_4)(\varepsilon_4\cdot k_5)\Big]\\
&\qquad +32\Big[(\varepsilon_1\cdot k_5)(\varepsilon_2\cdot k_5)(\varepsilon_3\cdot p_1)(\varepsilon_4\cdot p_2)+(\varepsilon_1\cdot p_3)(\varepsilon_2\cdot p_4)(\varepsilon_3\cdot k_6)(\varepsilon_4\cdot k_6)\Big]\\
&\qquad +47\Big[(\varepsilon_1\cdot k_4)(\varepsilon_2\cdot k_3)(\varepsilon_3\cdot k_4)(\varepsilon_4\cdot k_3)+(\varepsilon_1\cdot k_2)(\varepsilon_2\cdot k_1)(\varepsilon_3\cdot k_2)(\varepsilon_4\cdot k_1)\Big]\Big\},
\end{aligned}
$$

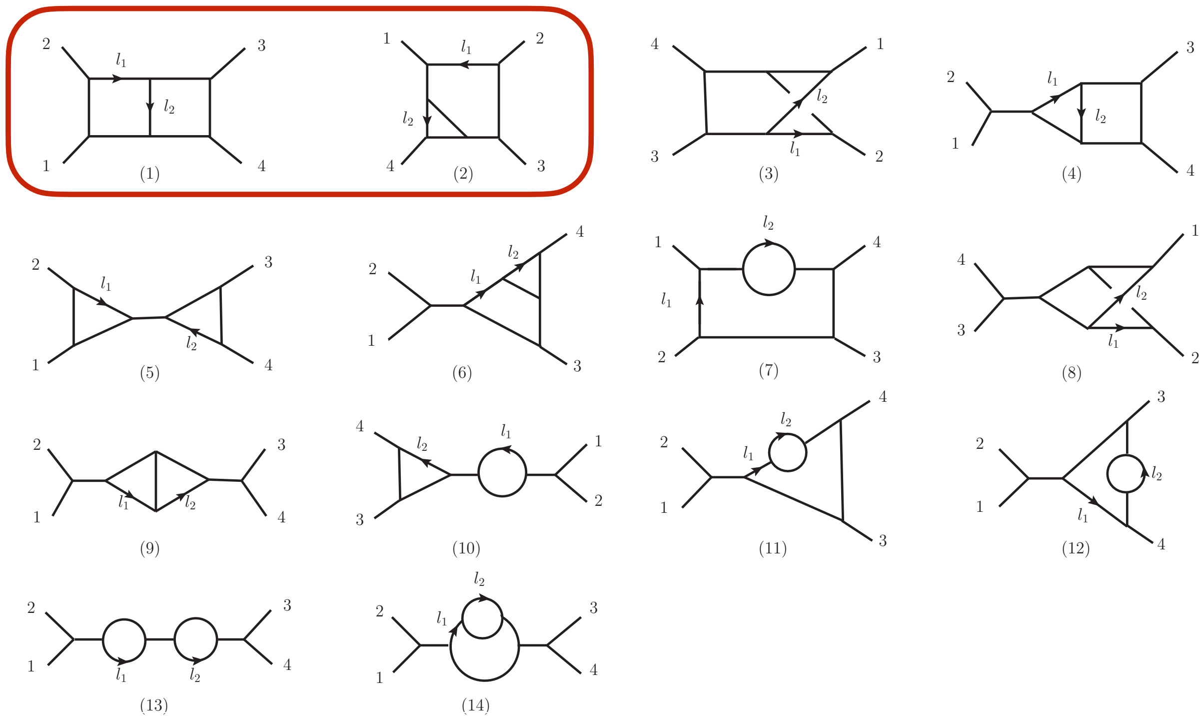
Master topologies



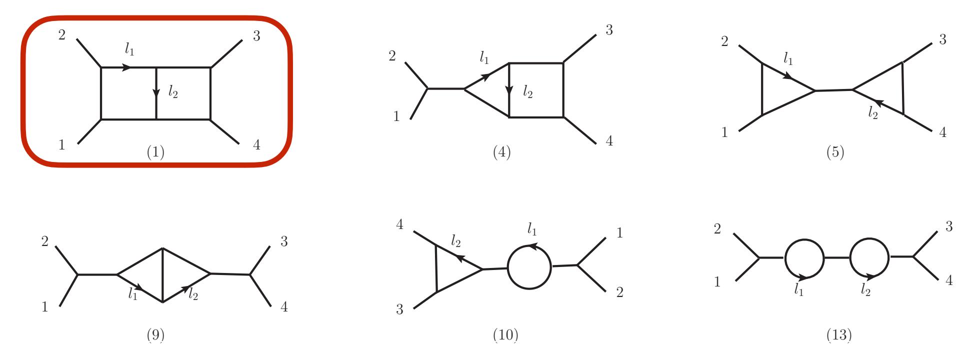
$$n_3 = n_1[p_1, p_2, p_3, p_4, -l_2 - p_2, l_1 - p_2] + n_2[p_4, p_3, p_2, p_1, -l_1 - l_2 - p_3, -l_2 + p_1]$$
$$n_4 = n_1 - n_1[p_3, p_4, p_2, p_1, l_1 - l_2 + p_1 + p_2, -l_2]$$
$$n_5 = -n_1[p_1, p_2, p_3, p_4, l_1, l_1 - l_2 + p_1 + p_2] + n_1[p_1, p_2, p_4, p_3, l_1, l_1 + l_2]$$
$$n_6 = n_2[p_1, p_2, p_3, p_4, l_1 + p_1, l_2] - n_2[p_3, p_1, p_2, p_4, -l_1 - p_1 - p_2, -l_2 - p_1 - p_2 - p_3]$$
$$n_7 = -n_2[p_1, p_2, p_3, p_4, l_1, l_2] - n_2[p_1, p_2, p_3, p_4, l_1, l_1 - l_2 - p_1]$$
$$n_8 = n_3 - n_3[p_1, p_2, p_4, p_3, l_1, l_2]$$
$$n_9 = -n_4[p_1, p_2, p_3, p_4, l_1, l_1 - l_2] + n_4[p_1, p_2, p_4, p_3, l_1, l_1 - l_2]$$
$$n_{10} = -n_4[p_1, p_2, p_3, p_4, l_1, l_1 + l_2 + p_1 + p_2] - n_4[p_1, p_2, p_3, p_4, -l_1 - p_1 - p_2, -l_1 + l_2]$$
$$n_{11} = -n_4[p_1, p_2, p_3, p_4, -l_2 - p_1 - p_2, l_1 - l_2] - n_4[p_1, p_2, p_3, p_4, -l_1 + l_2 - p_1 - p_2, l_2]$$
$$n_{12} = -n_6[p_1, p_2, p_3, p_4, l_1, l_1 - l_2] - n_6[p_1, p_2, p_3, p_4, l_1, l_2 - p_1 - p_2 - p_3]$$
$$n_{13} = n_9 + n_9[p_1, p_2, p_3, p_4, -l_1 - p_1 - p_2, l_2]$$
$$n_{14} = n_9[p_1, p_2, p_3, p_4, l_1 - l_2, l_1] + n_9[p_1, p_2, p_3, p_4, -l_2 - p_1 - p_2, -l_1 - p_1 - p_2]\,,$$

$$N_i = n_i$$

Master topologies



$$N_i = n_i + \Delta_i$$



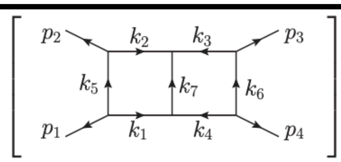
$$
\begin{aligned}
n_3 &= n_1[p_1,p_2,p_3,p_4,-l_2-p_2,l_1-p_2] + n_2[p_4,p_3,p_2,p_1,-l_1-l_2-p_3,-l_2+p_1]\\
n_4 &= n_1 - n_1[p_3,p_4,p_2,p_1,l_1-l_2+p_1+p_2,-l_2]\\
n_5 &= -n_1[p_1,p_2,p_3,p_4,l_1,l_1-l_2+p_1+p_2] + n_1[p_1,p_2,p_4,p_3,l_1,l_1+l_2]\\
n_6 &= n_2[p_1,p_2,p_3,p_4,l_1+p_1,l_2] - n_2[p_3,p_1,p_2,p_4,-l_1-p_1-p_2,-l_2-p_1-p_2-p_3]\\
n_7 &= -n_2[p_1,p_2,p_3,p_4,l_1,l_2] - n_2[p_1,p_2,p_3,p_4,l_1,l_1-l_2-p_1]\\
n_8 &= n_3 - n_3[p_1,p_2,p_4,p_3,l_1,l_2]\\
n_9 &= -n_4[p_1,p_2,p_3,p_4,l_1,l_1-l_2] + n_4[p_1,p_2,p_4,p_3,l_1,l_1-l_2]\\
n_{10} &= -n_4[p_1,p_2,p_3,p_4,l_1,l_1+l_2+p_1+p_2] - n_4[p_1,p_2,p_3,p_4,-l_1-p_1-p_2,-l_1+l_2]\\
n_{11} &= -n_4[p_1,p_2,p_3,p_4,-l_2-p_1-p_2,l_1-l_2] - n_4[p_1,p_2,p_3,p_4,-l_1+l_2-p_1-p_2,l_2]\\
n_{12} &= -n_6[p_1,p_2,p_3,p_4,l_1,l_1-l_2] - n_6[p_1,p_2,p_3,p_4,l_1,l_2-p_1-p_2-p_3]\\
n_{13} &= n_9 + n_9[p_1,p_2,p_3,p_4,-l_1-p_1-p_2,l_2]\\
n_{14} &= n_9[p_1,p_2,p_3,p_4,l_1-l_2,l_1] + n_9[p_1,p_2,p_3,p_4,-l_2-p_1-p_2,-l_1-p_1-p_2]\,,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_4 &= \Delta_1 - \Delta_1[p_3,p_4,p_2,p_1,l_1-l_2+p_1+p_2,-l_2]\\
\Delta_5 &= -\Delta_1[p_1,p_2,p_3,p_4,l_1,l_1-l_2+p_1+p_2] + \Delta_1[p_1,p_2,p_4,p_3,l_1,l_1+l_2]\\
\Delta_9 &= -\Delta_4[p_1,p_2,p_3,p_4,l_1,l_1-l_2] + \Delta_4[p_1,p_2,p_4,p_3,l_1,l_1-l_2]\\
\Delta_{10} &= -\Delta_4[p_1,p_2,p_3,p_4,l_1,l_1+l_2+p_1+p_2] - \Delta_4[p_1,p_2,p_3,p_4,-l_1-p_1-p_2,-l_1+l_2]\\
\Delta_{13} &= \Delta_9 + \Delta_9[p_1,p_2,p_3,p_4,-l_1-p_1-p_2,l_2]\,.
\end{aligned}
$$

Simplicity of deformation

$$N_1 = n_1 + \Delta_1$$

Deformation

Δ 

$\Big/ k_7^2 =$ (5.2)

$$+(d-2)^2\Big\{(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_4)k_5^2k_6^2+16(\varepsilon_1\cdot k_5)(\varepsilon_2\cdot k_5)(\varepsilon_3\cdot k_6)(\varepsilon_4\cdot k_6)$$
$$-4\Big[(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot k_6)(\varepsilon_4\cdot k_6)k_5^2+(\varepsilon_3\cdot\varepsilon_4)(\varepsilon_1\cdot k_5)(\varepsilon_2\cdot k_5)k_6^2\Big]\Big\}$$
$$+(d-2)4\Big\{-10\Big[(\varepsilon_1\cdot k_6)(\varepsilon_2\cdot k_6)(\varepsilon_3\cdot k_5)(\varepsilon_4\cdot k_5)+(\varepsilon_1\cdot k_2)(\varepsilon_2\cdot k_1)(\varepsilon_3\cdot k_4)(\varepsilon_4\cdot k_3)\Big]$$
$$+20\Big[(\varepsilon_1\cdot k_6)(\varepsilon_2\cdot k_1)(\varepsilon_3\cdot k_5)(\varepsilon_4\cdot k_3)+(\varepsilon_1\cdot k_2)(\varepsilon_2\cdot k_6)(\varepsilon_3\cdot k_4)(\varepsilon_4\cdot k_5)\Big]$$
$$+32\Big[(\varepsilon_1\cdot k_5)(\varepsilon_2\cdot k_5)(\varepsilon_3\cdot p_1)(\varepsilon_4\cdot p_2)+(\varepsilon_1\cdot p_3)(\varepsilon_2\cdot p_4)(\varepsilon_3\cdot k_6)(\varepsilon_4\cdot k_6)\Big]$$
$$+47\Big[(\varepsilon_1\cdot k_4)(\varepsilon_2\cdot k_3)(\varepsilon_3\cdot k_4)(\varepsilon_4\cdot k_3)+(\varepsilon_1\cdot k_2)(\varepsilon_2\cdot k_1)(\varepsilon_3\cdot k_2)(\varepsilon_4\cdot k_1)\Big]\Big\},$$

Undeformed part

1-Numerators.m
2-SymmetyBasis.m
n1.txt
n2.txt

```
(256 - 128*d)*ep[p1,
l1]*ep[p2, l1]*ep[p3,
l1]*ep[p4, l1]*pp[l1, l1] +
(-128 + 64*d)*ep[p1,
l2]*ep[p2, l1]*ep[p3,
l1]*ep[p4, l1]*pp[l1, l1] +
(-9049/16 + (47261*d)/
160)*ep[p1, p2]*ep[p2,
l1]*ep[p3, l1]*ep[p4,
l1]*pp[l1, l1] + (-3615/8 +
(18363*d)/80)*ep[p1,
p3]*ep[p2, l1]*ep[p3,
l1]*ep[p4, l1]*pp[l1, l1] +
(-128 + 64*d)*ep[p1,
l1]*ep[p2, l2]*ep[p3,
l1]*ep[p4, l1]*pp[l1, l1] +
(128 - 64*d)*ep[p1,
l2]*ep[p2, l2]*ep[p3,
l1]*ep[p4, l1]*pp[l1, l1] +
```

n1.txt
Plain Text Document - 1.4 MB

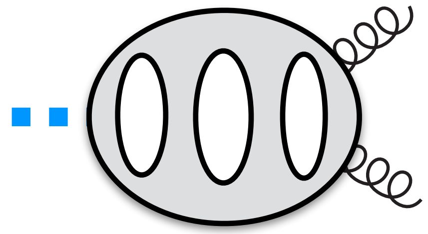
Bern, Davies and Nohle 2015

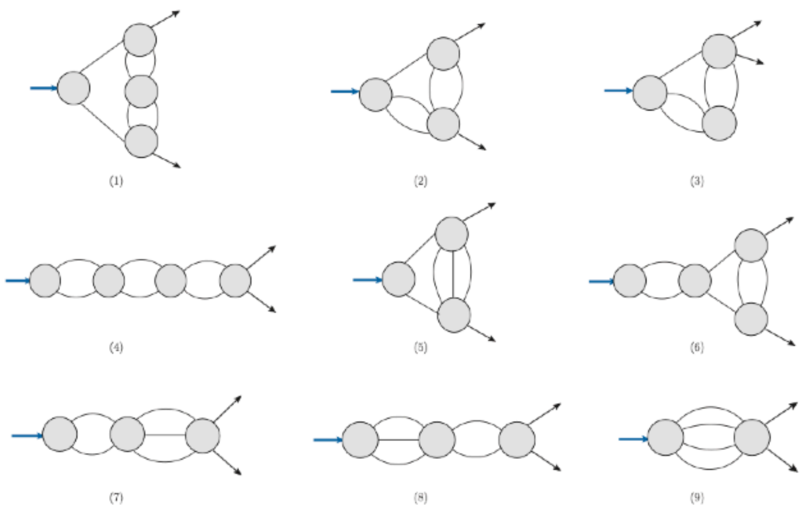
CK-duality only on cuts
initial parameters: ∼120,000
after symmetry constraints: ∼28,000
after cut and CK constraints: ∼6,300

→

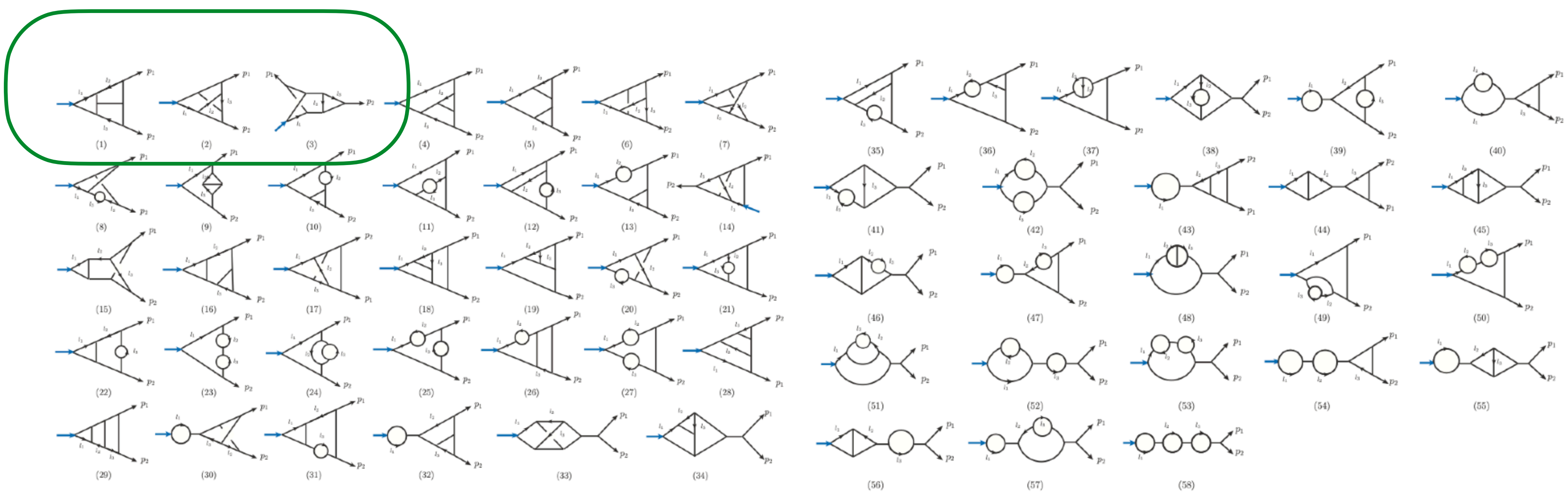
Duality with deformaltion
initial parameters: ∼20,000
after CK and symmetry constraints: ∼1400
with partial cuts + deformation : ∼500
with remaining cut: ∼200

More examples: higher loop cases?

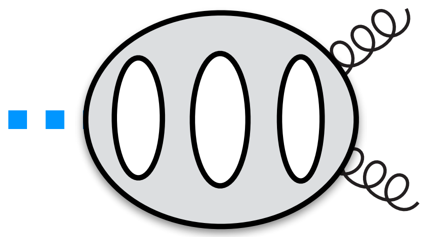




Zeyu Li, GY, Guorui Zhu 2025

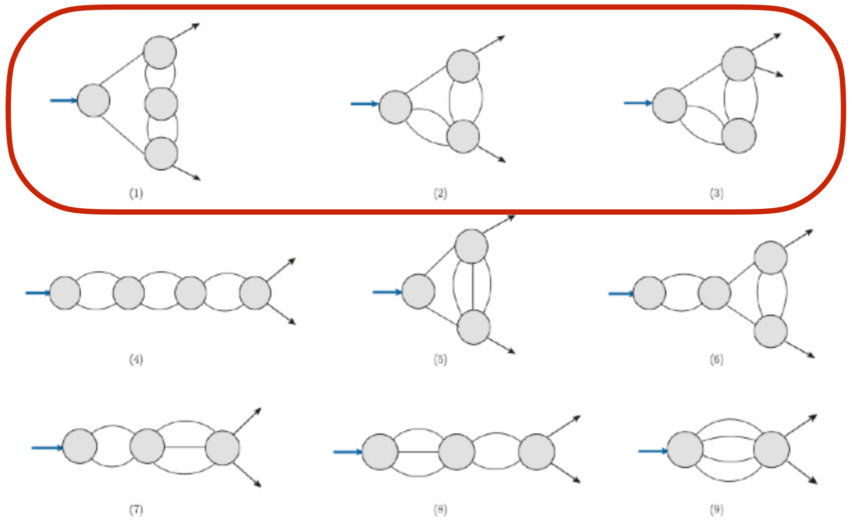


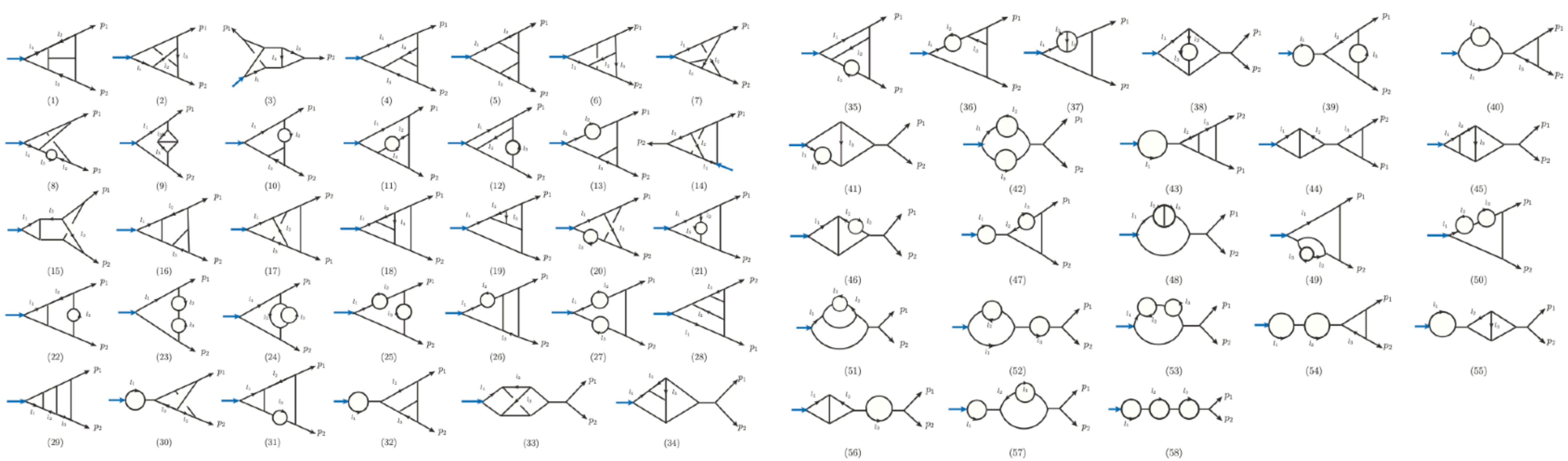
Three-loop Sudakov form factor



Zeyu Li, GY, Guorui Zhu 2025

Failed cuts for global CK ansatz

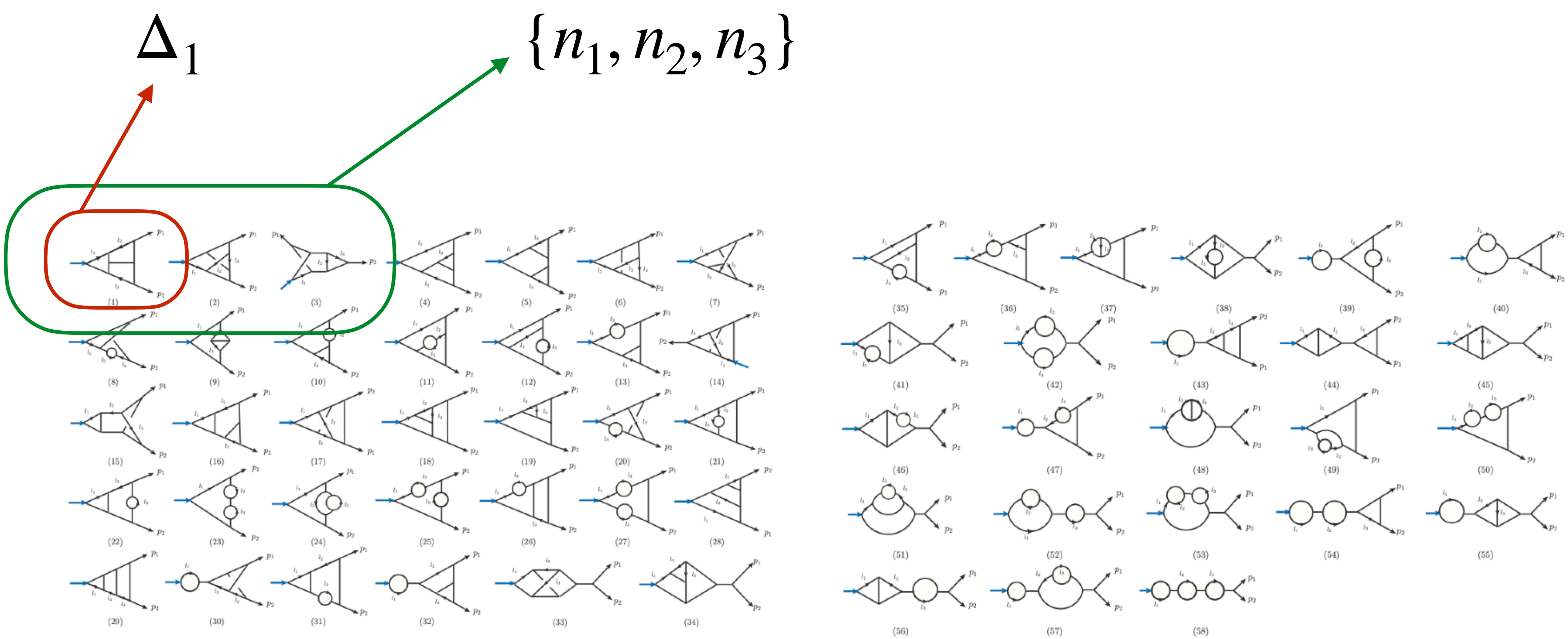




Three-loop Sudakov form factor

Zeyu Li, GY, Guorui Zhu 2025

$$N_i = \begin{cases} n_i + \Delta_i, \\ n_i, \end{cases}$$



Three-loop Sudakov form factor

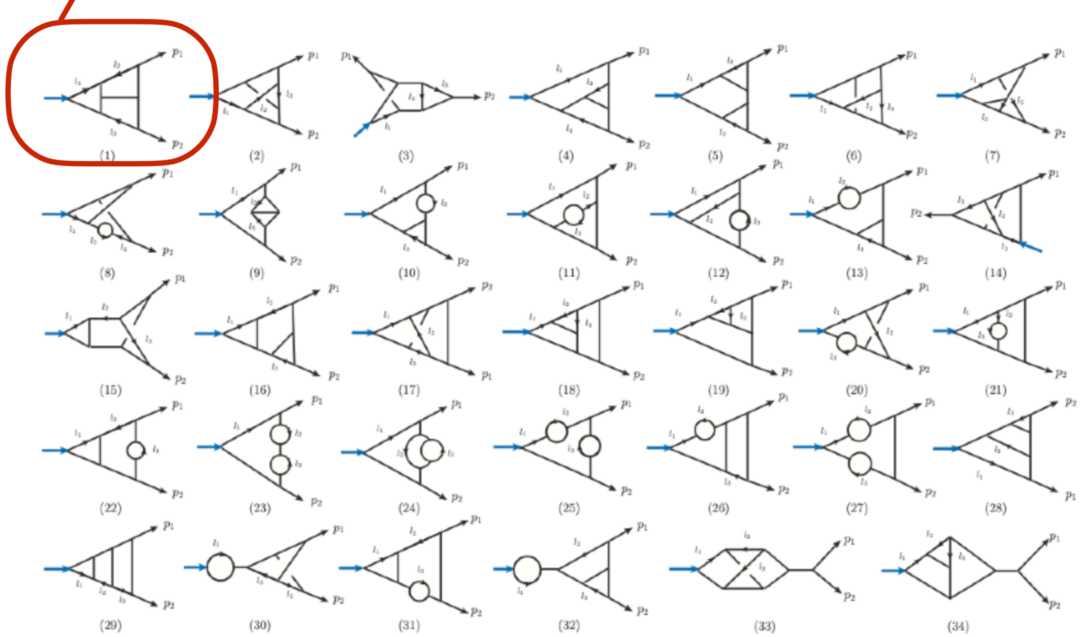
Zeyu Li, GY, Guorui Zhu 2025

$$N_i = \begin{cases} n_i + \Delta_i, \\ n_i, \end{cases}$$

$$\Delta_1 = \Delta_1^{[1]} + \Delta_1^{[2]} \,,$$

$$\Delta_1^{[1]} = 2(d-2)^2(\varepsilon_1\cdot\varepsilon_2)(p_1\cdot p_2)\, l_2^2\, l_3^2\, l_4^2 \,,$$

$$\begin{aligned}\Delta_1^{[2]} = -2(d-2)^2 l_4^2 \big(&24S_1^{[2]} + S_2^{[2]} - 5S_3^{[2]} + 4S_4^{[2]} - 2S_5^{[2]} - 9S_6^{[2]} - 15S_7^{[2]} - 4S_8^{[2]} - 20S_9^{[2]} \\ &-4S_{10}^{[2]} - 20S_{11}^{[2]} + 4S_{12}^{[2]} + 12S_{13}^{[2]} + 4S_{14}^{[2]} + 8S_{15}^{[2]} - 24S_{16}^{[2]} - 4S_{17}^{[2]} + 4S_{18}^{[2]} + 28S_{19}^{[2]} + 4S_{20}^{[2]} \\ &-20S_{21}^{[2]} + 8S_{22}^{[2]} - 4S_{23}^{[2]} - 28S_{24}^{[2]} - 40S_{25}^{[2]} - 4S_{26}^{[2]} - 8S_{27}^{[2]} + 4S_{28}^{[2]} - 8S_{29}^{[2]} + 8S_{30}^{[2]} + 8S_{31}^{[2]}\big)\end{aligned}$$



$$S_1^{[2]} = (\varepsilon_1\cdot l_8)(\varepsilon_2\cdot l_7) l_2^2 l_3^2 \,, \qquad S_2^{[2]} = (\varepsilon_1\cdot p_2)(\varepsilon_2\cdot p_1) l_2^2 l_3^2$$

$$S_3^{[2]} = (\varepsilon_1\cdot l_3)(\varepsilon_2\cdot l_3)(l_2^2)^2 + (\varepsilon_1\cdot l_2)(\varepsilon_2\cdot l_2)(l_3^2)^2$$

$$S_4^{[2]} = (\varepsilon_1\cdot l_3)(\varepsilon_2\cdot l_2)(l_2^2)^2 + (\varepsilon_1\cdot l_3)(\varepsilon_2\cdot l_2)(l_3^2)^2$$

$$S_5^{[2]} = (\varepsilon_1\cdot l_3)(\varepsilon_2\cdot l_4)(l_2^2)^2 + (\varepsilon_1\cdot l_4)(\varepsilon_2\cdot l_2)(l_3^2)^2$$

$$S_6^{[2]} = (\varepsilon_1\cdot l_2)(\varepsilon_2\cdot l_3)(l_2^2)^2 + (\varepsilon_1\cdot l_2)(\varepsilon_2\cdot l_3)(l_3^2)^2$$

$$S_7^{[2]} = (\varepsilon_1\cdot l_4)(\varepsilon_2\cdot l_3)(l_2^2)^2 + (\varepsilon_1\cdot l_2)(\varepsilon_2\cdot l_4)(l_3^2)^2$$

$\cdots$

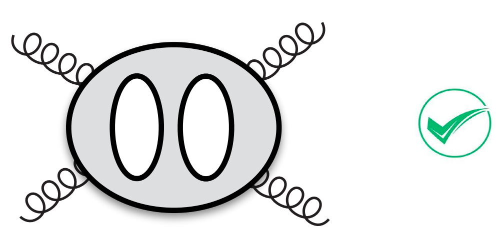
Outline

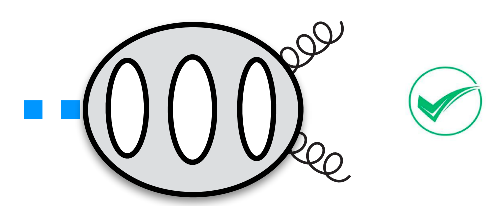
- Introduction
- Constructing CK-dual numerators
- New strategy of deformation
- Summary and outlook

Summary and outlook

- Gauge and gravity theories are related by double copy.
- The key of double copy is to achieve "color-kinematics duality".
- Finding CK-dual numerators is generally difficult, and introducing "a simple deformation" may solve it.

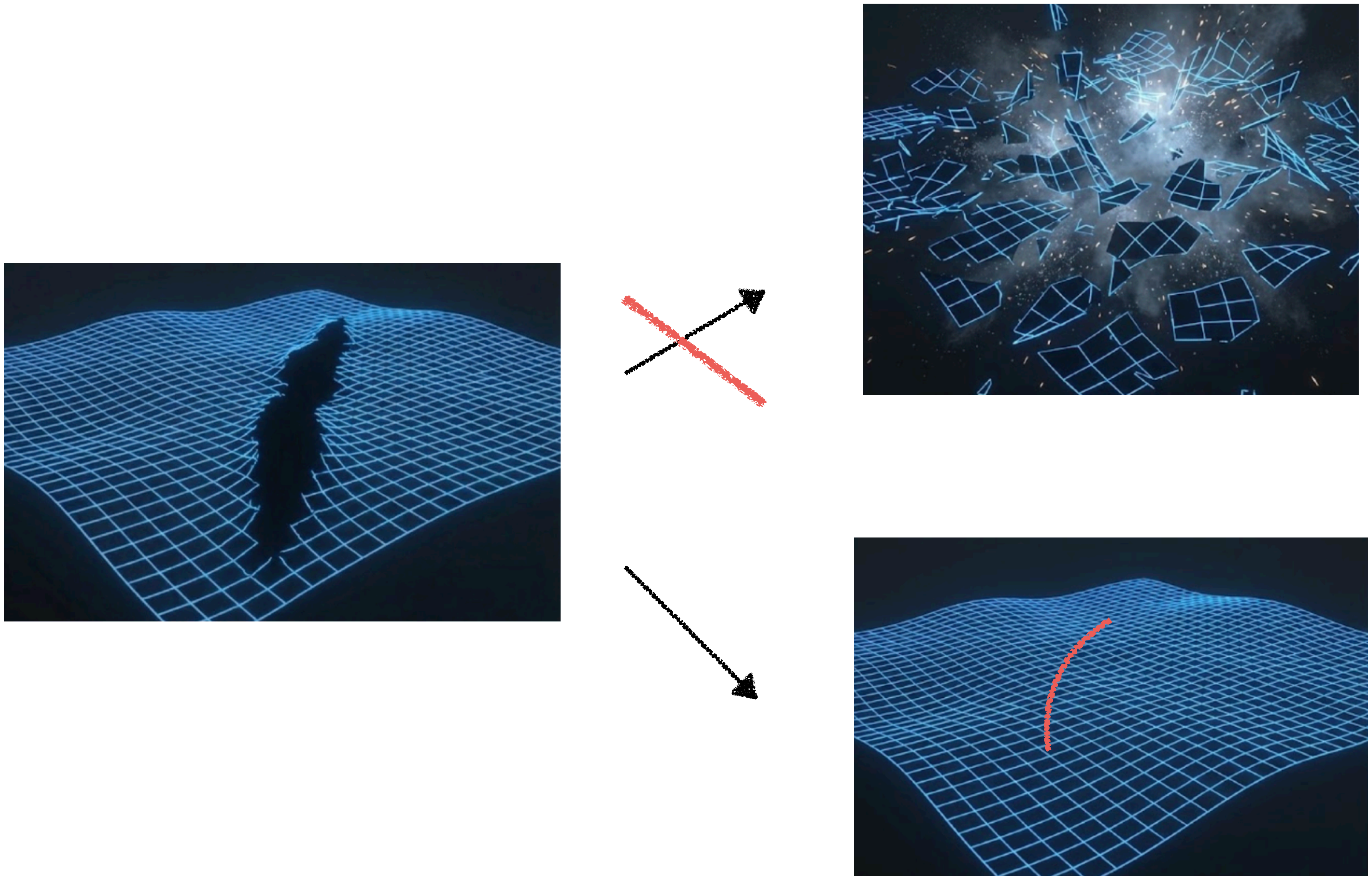
We provide a two-loop and a three-loop example:





The deformation part is surprisingly simple

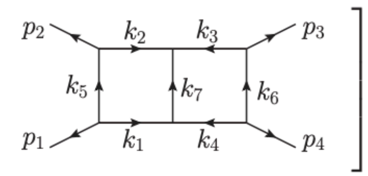
Summary and outlook



A very small patch could be sufficient
to “amend” the global structure !

Summary and outlook

- Why so simple?

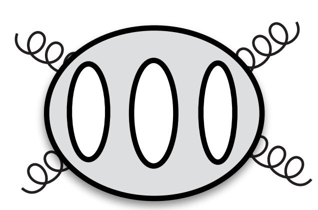
Δ 

$$\Big/ k_7^2 =$$

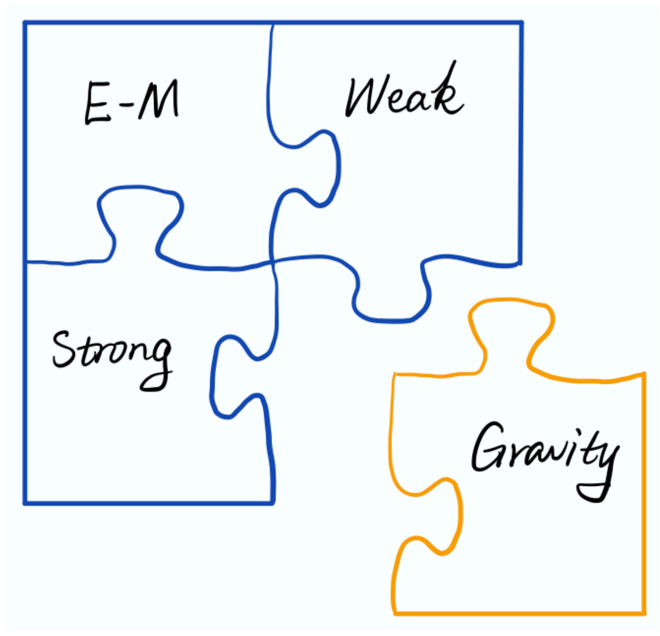
$$\begin{aligned}
&+ (d-2)^2 \Big\{ (\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) k_5^2\, k_6^2 + 16 (\varepsilon_1 \cdot k_5)(\varepsilon_2 \cdot k_5)(\varepsilon_3 \cdot k_6)(\varepsilon_4 \cdot k_6) \\
&\qquad - 4 \Big[(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot k_6)(\varepsilon_4 \cdot k_6) k_5^2 + (\varepsilon_3 \cdot \varepsilon_4)(\varepsilon_1 \cdot k_5)(\varepsilon_2 \cdot k_5) k_6^2 \Big] \Big\} \\
&+ (d-2) 4 \Big\{ -10 \Big[(\varepsilon_1 \cdot k_6)(\varepsilon_2 \cdot k_6)(\varepsilon_3 \cdot k_5)(\varepsilon_4 \cdot k_5) + (\varepsilon_1 \cdot k_2)(\varepsilon_2 \cdot k_1)(\varepsilon_3 \cdot k_4)(\varepsilon_4 \cdot k_3) \Big] \\
&\qquad + 20 \Big[(\varepsilon_1 \cdot k_6)(\varepsilon_2 \cdot k_1)(\varepsilon_3 \cdot k_5)(\varepsilon_4 \cdot k_3) + (\varepsilon_1 \cdot k_2)(\varepsilon_2 \cdot k_6)(\varepsilon_3 \cdot k_4)(\varepsilon_4 \cdot k_5) \Big] \\
&\qquad + 32 \Big[(\varepsilon_1 \cdot k_5)(\varepsilon_2 \cdot k_5)(\varepsilon_3 \cdot p_1)(\varepsilon_4 \cdot p_2) + (\varepsilon_1 \cdot p_3)(\varepsilon_2 \cdot p_4)(\varepsilon_3 \cdot k_6)(\varepsilon_4 \cdot k_6) \Big] \\
&\qquad + 47 \Big[(\varepsilon_1 \cdot k_4)(\varepsilon_2 \cdot k_3)(\varepsilon_3 \cdot k_4)(\varepsilon_4 \cdot k_3) + (\varepsilon_1 \cdot k_2)(\varepsilon_2 \cdot k_1)(\varepsilon_3 \cdot k_2)(\varepsilon_4 \cdot k_1) \Big] \Big\},
\end{aligned}$$

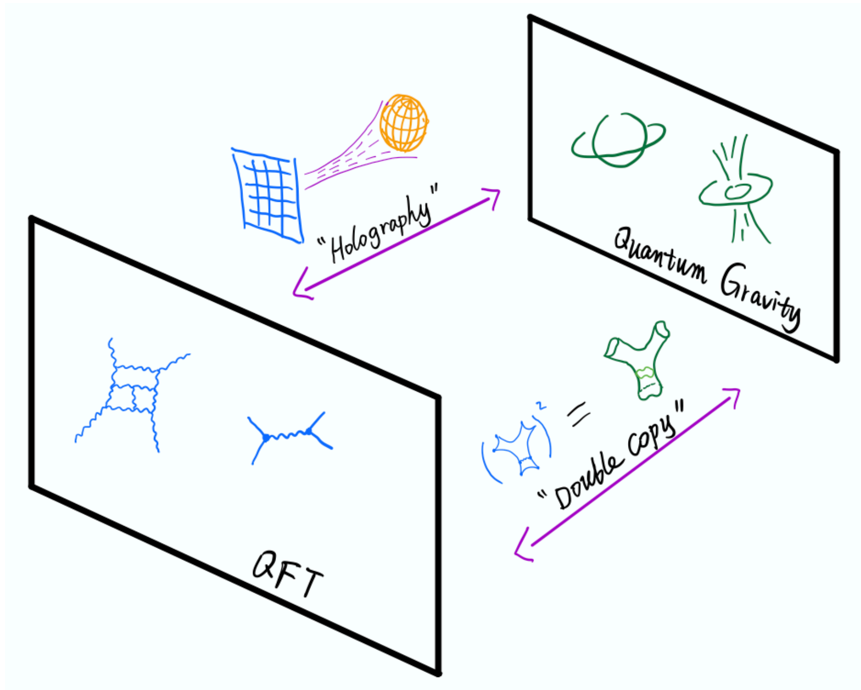
- Are there underlying structures for the deformation?

- More examples:



?

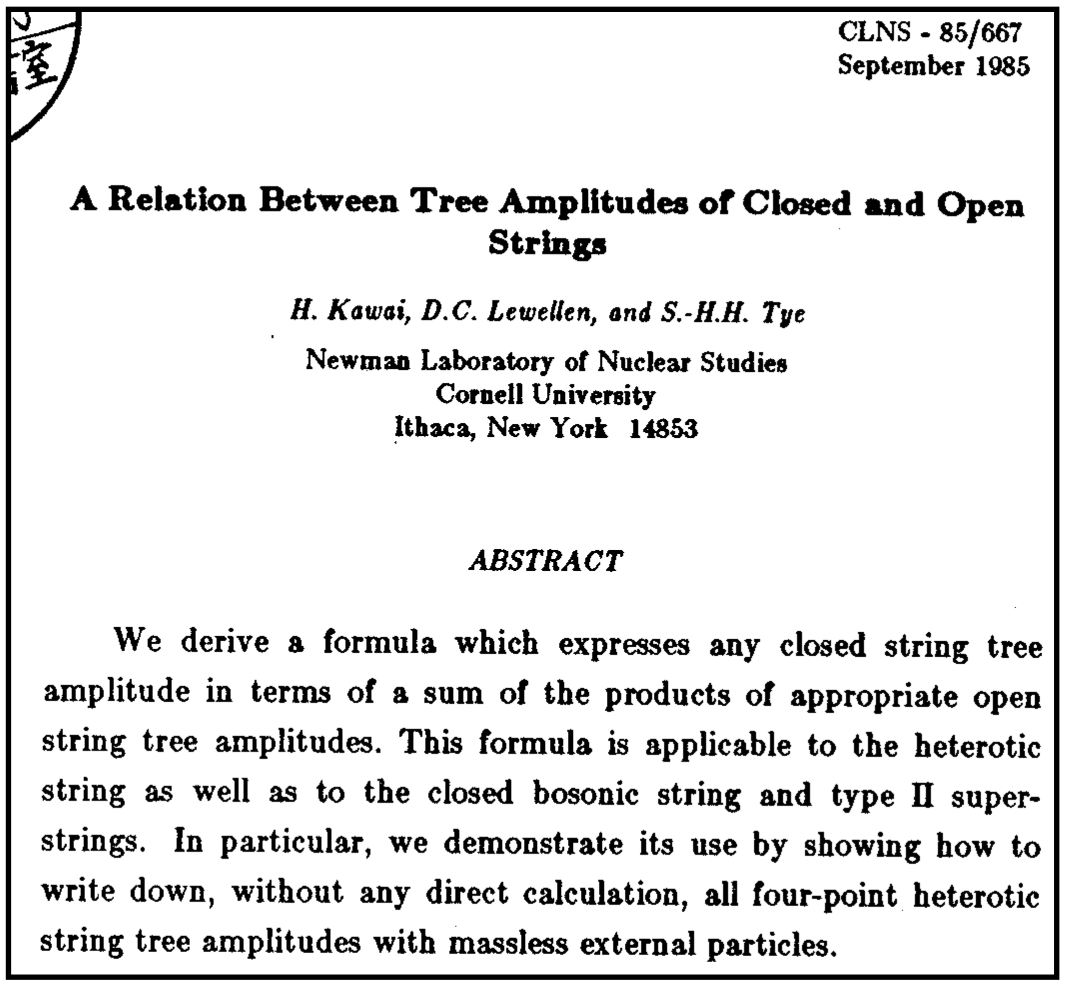




Thank you for your attention!

Back up slides

String theory



CLNS - 85/667
September 1985

A Relation Between Tree Amplitudes of Closed and Open Strings

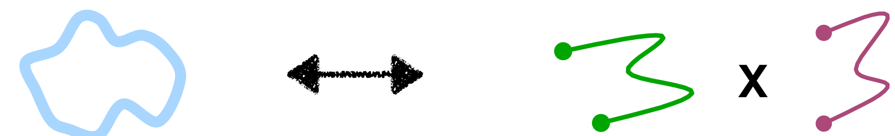
H. Kawai, D.C. Lewellen, and S.-H.H. Tye

Newman Laboratory of Nuclear Studies
Cornell University
Ithaca, New York 14853

ABSTRACT

We derive a formula which expresses any closed string tree amplitude in terms of a sum of the products of appropriate open string tree amplitudes. This formula is applicable to the heterotic string as well as to the closed bosonic string and type II superstrings. In particular, we demonstrate its use by showing how to write down, without any direct calculation, all four-point heterotic string tree amplitudes with massless external particles.

KLT relation:



$$A^{(4)}_{closed} = -\pi\kappa^2 \sin(\pi\kappa_1\cdot\kappa_2) A^{(4)}_{open}(s,t)\ \bar{A}^{(4)}_{open}(s,u)$$

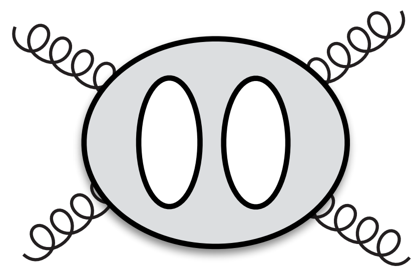
$$A^{(5)}_{closed} = \pi\kappa^3 A^{(5)}_{open}(12345)\bar{A}^{(5)}_{open}(21435)\sin(\pi\kappa_1\cdot\kappa_2)\sin(\pi\kappa_3\cdot\kappa_4) + \pi\kappa^3\ A^{(5)}_{open}(13245)\bar{A}^{(5)}_{open}(31425)\sin(\pi k_1\cdot k_3)\sin(\pi k_2\cdot k_4).$$

Field theory limit

$$M_4^{\rm tree}(1,2,3,4) = -i s_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

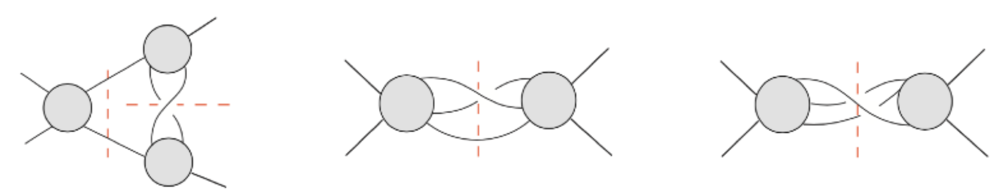
$$M_5^{\rm tree}(1,2,3,4,5) = i s_{12} s_{34} A_5^{\rm tree}(1,2,3,4,5) A_5^{\rm tree}(2,1,4,3,5) + i s_{13} s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

Checks of the solution



$$N_i = \begin{cases} n_i + \Delta_i, & i = 1, 4, 5, 9, 10, 13, \\ n_i, & \text{others.} \end{cases}$$

- Pass the full set of D-dimensional planar and non-planar cuts



- Satisfy all CK-dual relations on cuts, so double-copy applies
- Free parameters cancel after the integral IBP reduction
- Integrated result satisfies the Catani IR formula

From YM to gravity

If the gauge amplitude satisfies CK duality, one can directly construct gravity amplitude：

$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \longrightarrow M_4(1,2,3,4) = \frac{n_s n_s}{s} + \frac{n_t n_t}{t} + \frac{n_u n_u}{u}$$

Gauge invariance $\xrightarrow{\text{CK-duality}}$ Diffeomorphism invariance

$$\varepsilon_i^\mu \to \varepsilon_i^\mu + p_i^\mu \qquad \varepsilon_i^{\mu\nu} \to \varepsilon_i^{\mu\nu} + p_i^{(\mu} q^{\nu)}$$

$$n_i \to n_i + \delta_i\,,$$

$$\delta_i = n_i|_{\varepsilon_j \to p_j}$$

$$\sum_i \frac{c_i \delta_i}{D_i} = 0 \longrightarrow \sum_i \frac{n_i \delta_i}{D_i} = 0$$

$$c_i = c_j + c_k \qquad n_i = n_j + n_k$$

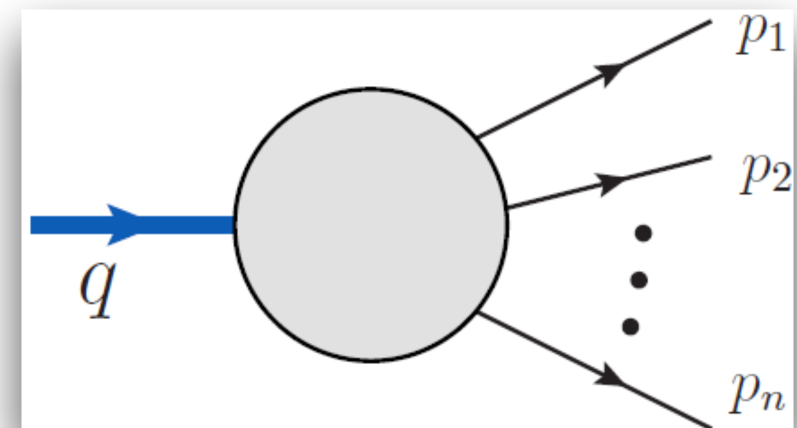
Form Factors

Matrix element of on-shell states and a local operators:

$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x} \langle p_1 \ldots p_n | \mathcal{O}(x) | 0 \rangle$$

$$= \delta^{(4)}(\sum_{i=1}^{n} p_i - q) \langle p_1 \ldots p_n | \mathcal{O}(0) | 0 \rangle$$

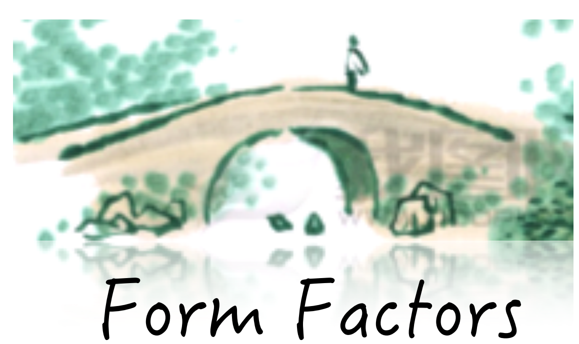
(work in momentum space)



$$q = \sum_i p_i, \qquad q^2 \neq 0$$

$$\langle p_1 p_2 \ldots p_n | 0 \rangle$$

Amplitudes



$$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n \rangle$$

Correlation functions

Double copy of form factor

Gauge x Gauge —Double-copy→ Gravity

- An surprising new mechanism for form factors:

Spurious poles (Gauge theory) —Double-copy→ Real propagators (Gravity theory)

- Hidden "factorization" relations of gauge form factors

$$\vec{v} \cdot \vec{\mathcal{F}}_n \Big|_{\text{spurious pole}} = \mathcal{F}_m \times \mathcal{A}_{n+2-m}$$

2111.12719, 2211.01386, 2306.04672 with Guanda Lin

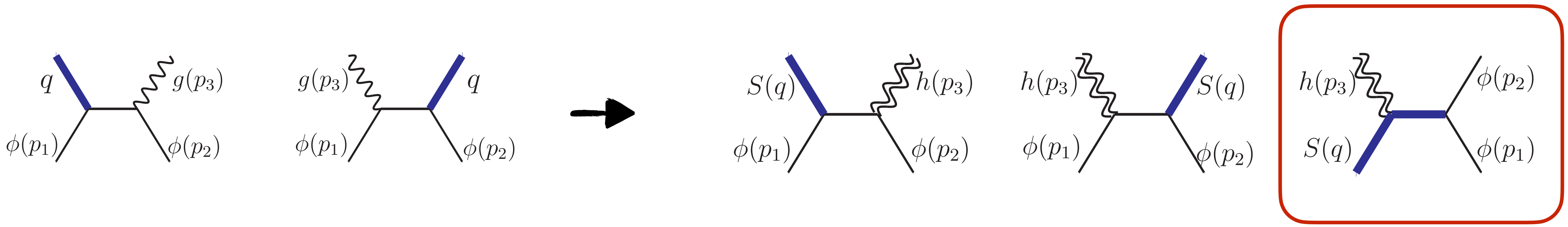
Example: 3-point form factor

$$\mathcal{G}_3 = \frac{(N_1^{\rm CK})^2}{s_{23}} + \frac{(N_2^{\rm CK})^2}{s_{13}} = \frac{s_{13}s_{23}}{s_{13}+s_{23}}\Big(\mathcal{F}_3(1^\phi, 3^g, 2^\phi)\Big)^2$$

There is a nice factorization behavior at the new pole:

$$s_{13} + s_{23} = q^2 - s_{12} = 0$$

$$\text{Res}\left[\mathcal{G}_3\right]_{s_{12}=q^2} = (\epsilon_3 \cdot q)^2 = \Big(\mathcal{F}_2(1^\phi, 2^\phi)\Big)^2 \times \left(\mathcal{A}_3(\mathbf{q}_2^S, 3^g, -q^S)\right)^2$$



A new graph
in gravity

A new type of hidden relations

For gauge-theory form factors:

$$\vec{v} \cdot \vec{\mathcal{F}}_n \Big|_{\text{spurious pole}} = \mathcal{F}_m \times \mathcal{A}_{n+2-m}$$

"Factorization" at spurious poles

$$\left[s_{42}\mathcal{F}_4(1,3,4,2) + (s_{42}+s_{43})\mathcal{F}_4(1,4,3,2) \right]\Big|_{s_{123}=q^2} = \mathcal{F}_3(1^\phi, 3^g, 2^\phi)\ \mathcal{A}_3(\mathbf{q}_3^S, 4^g, -q^S)$$

The relation is reminiscent of the BCJ relation for amplitudes:

$$s_{42}\mathcal{A}_4(1,3,4,2) + (s_{42}+s_{43})\mathcal{A}_4(1,4,3,2) = 0$$