Virasoro-Shapiro Amplitude in AdS_3

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with Shai Chester

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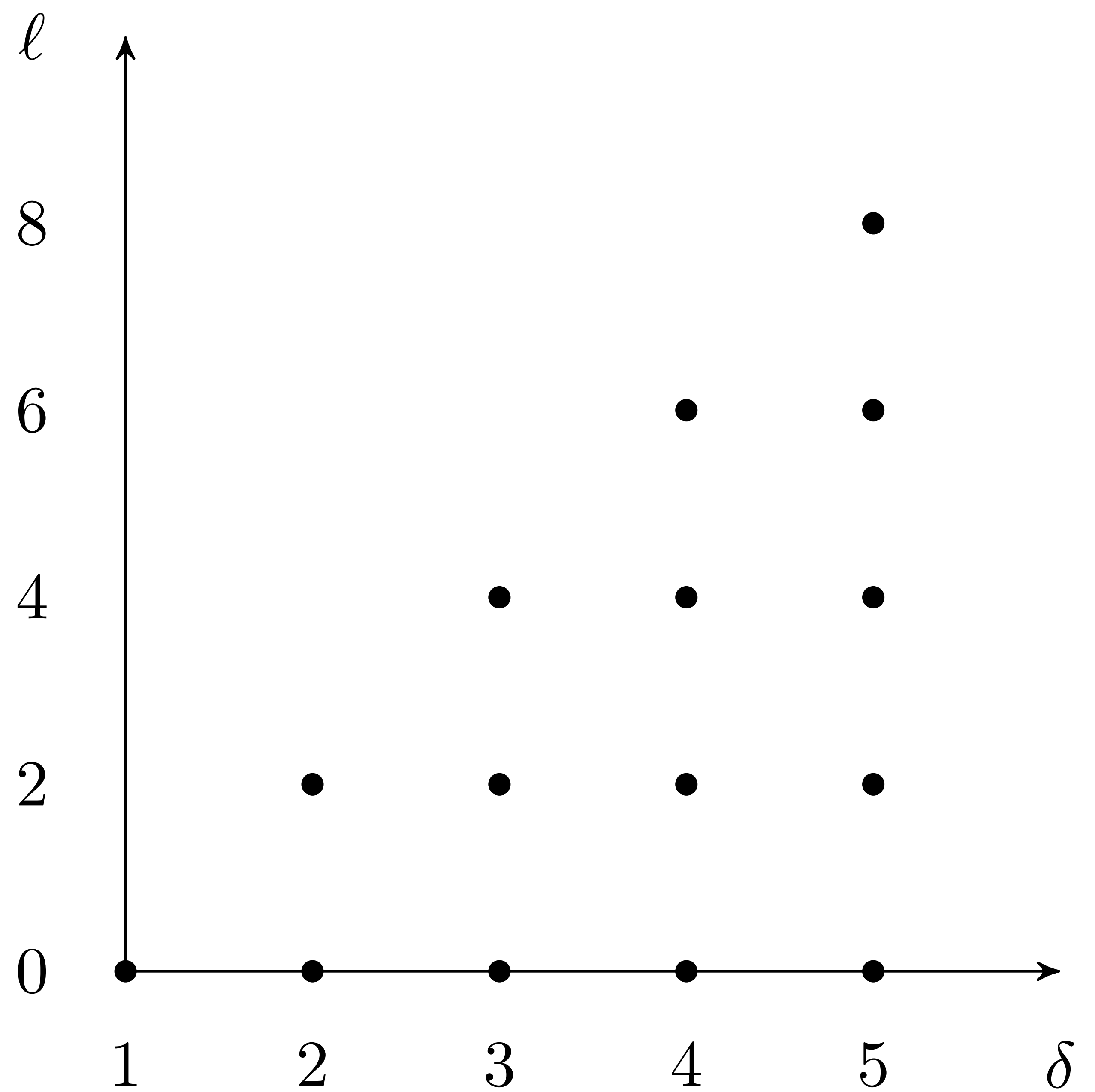
with Hongliang Jiang

History

❖ **Veneziano amplitude:** meson scattering @ high energy

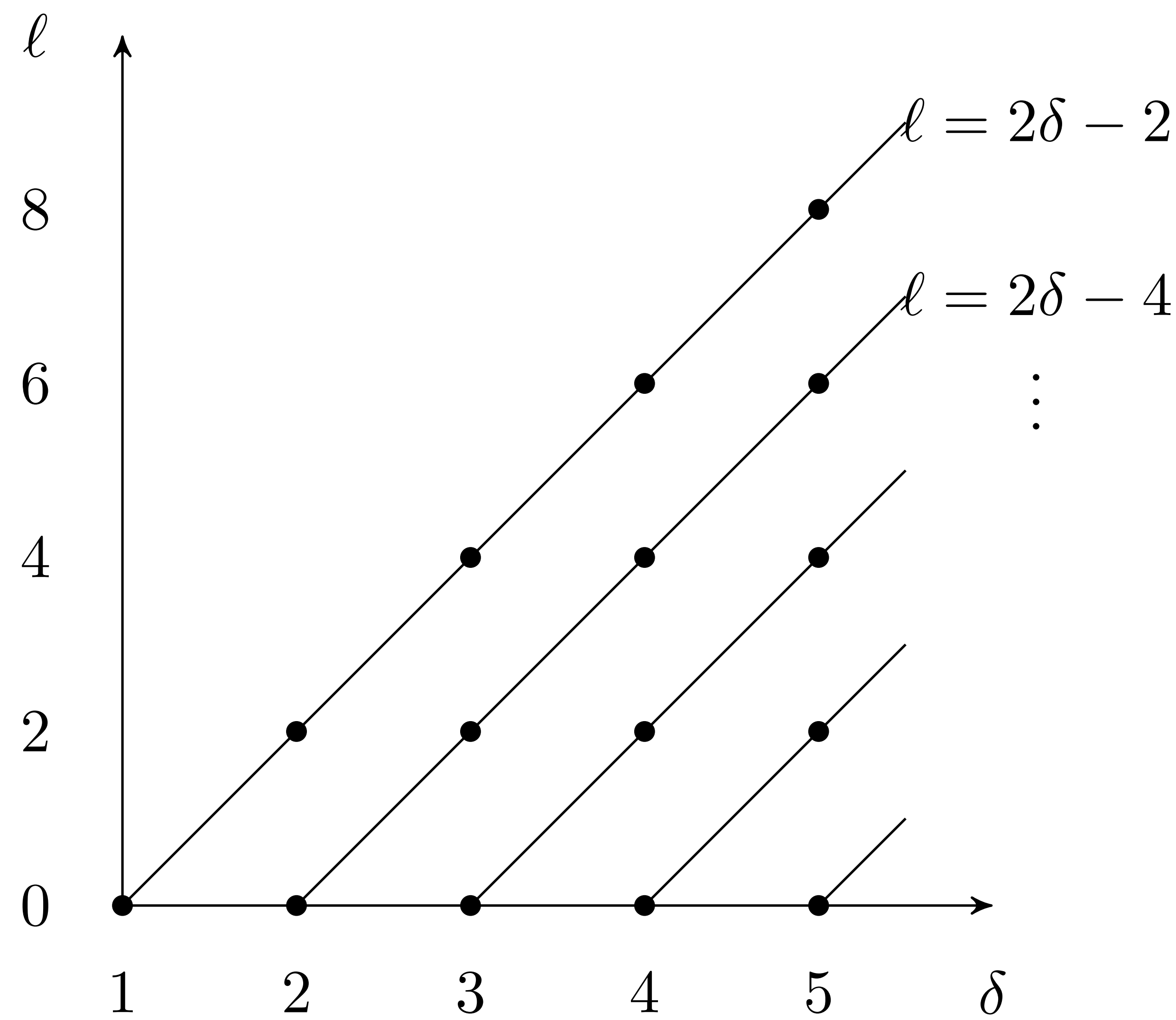


spin : $\ell \in \mathbb{Z}_{>0}$

mass level : $m^2 = \dfrac{4\delta}{\alpha'}$

History

- **Veneziano amplitude:** Chew-Frautschi plot



- **Virasoro-Shapiro amplitude:** fully crossing symmetric

Virasoro-Shapiro Amplitude

- **Virasoro-Shapiro amplitude:** 4pt tree-level string amplitude
- **String amplitude depends on**
  - String coupling $g_s \ll 1$: genus expansion, e.g. tree-level $\leftrightarrow$ genus 0
  - String tension $T \sim 1/\alpha'$: $\sqrt{\alpha'} \sim \ell_s$ = string length
- **Particles being scattered:** massless
  - Graviton/dilaton (closed string) or gluon (open string)
  - Momenta p_i & polarization: ε_i
  - Mandelstam variable: $S + T + U = 0$, where
  - $S = \alpha'(p_1 + p_2)^2, \quad T = \alpha'(p_1 + p_3)^2, \quad U = \alpha'(p_1 + p_4)^2,$

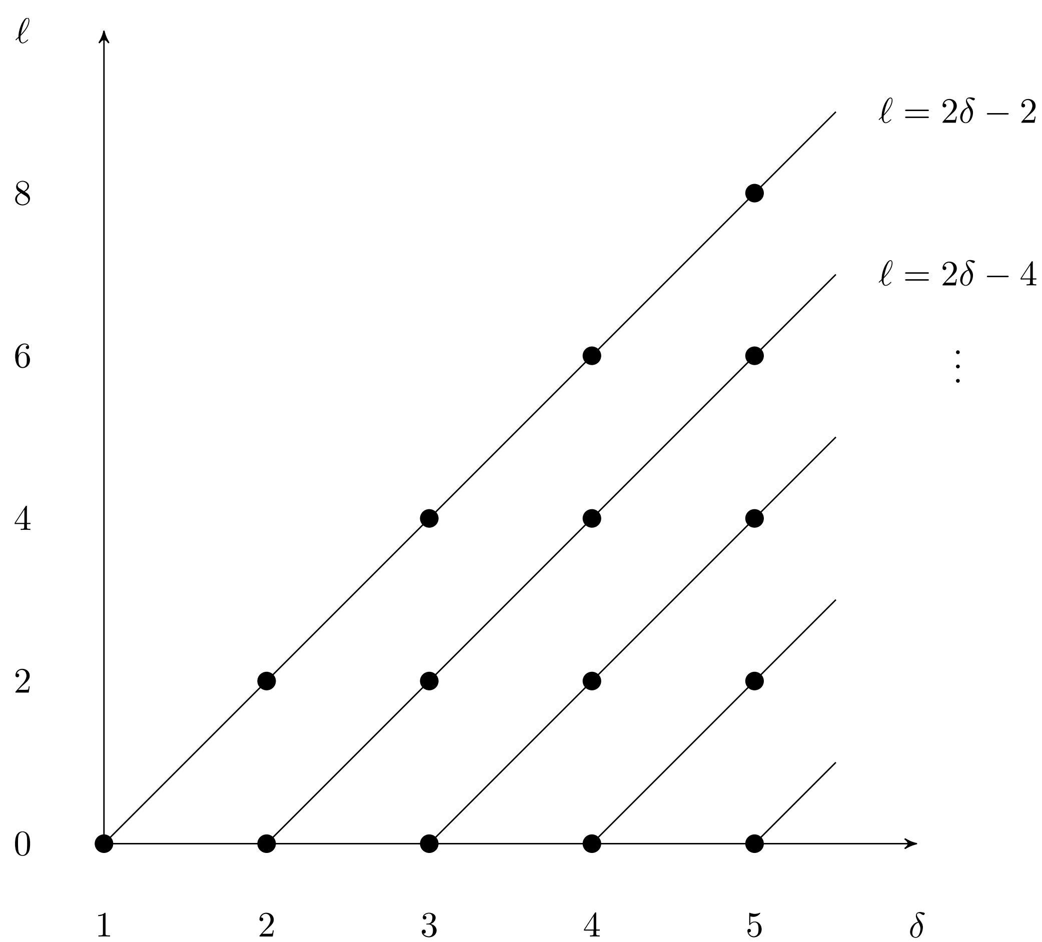
Virasoro-Shapiro Amplitude

- **Virasoro-Shapiro amplitude:** 4pt tree-level string amplitude

$$A^{(0)}(S,T) = \frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)} = -\frac{1}{U^2}\int d^2z|z|^{-2S-2}|1-z|^{-2T-2}$$

Worldsheet integral

- **Properties of VS amplitude:**
  - Poles S, T, U: massless/massive state exchange



- Crossing symmetric
- S-channel poles: $S = \delta \in \mathbb{Z}_{\geq 0} \Rightarrow m^2 = 4\delta/\alpha'$
- Residues receives contributions from even spin
- States organizes as Regge trajectories

Virasoro-Shapiro Amplitude

❖ **Virasoro-Shapiro amplitude:** 4pt tree-level string amplitude

$$\mathcal{A} = K\left(\epsilon_i, p_i\right) A^{(0)}(S,T)$$ SUSY

$$A^{(0)}(S,T) = \frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)} = -\frac{1}{U^2}\int d^2z |z|^{-2S-2}|1-z|^{-2T-2}$$

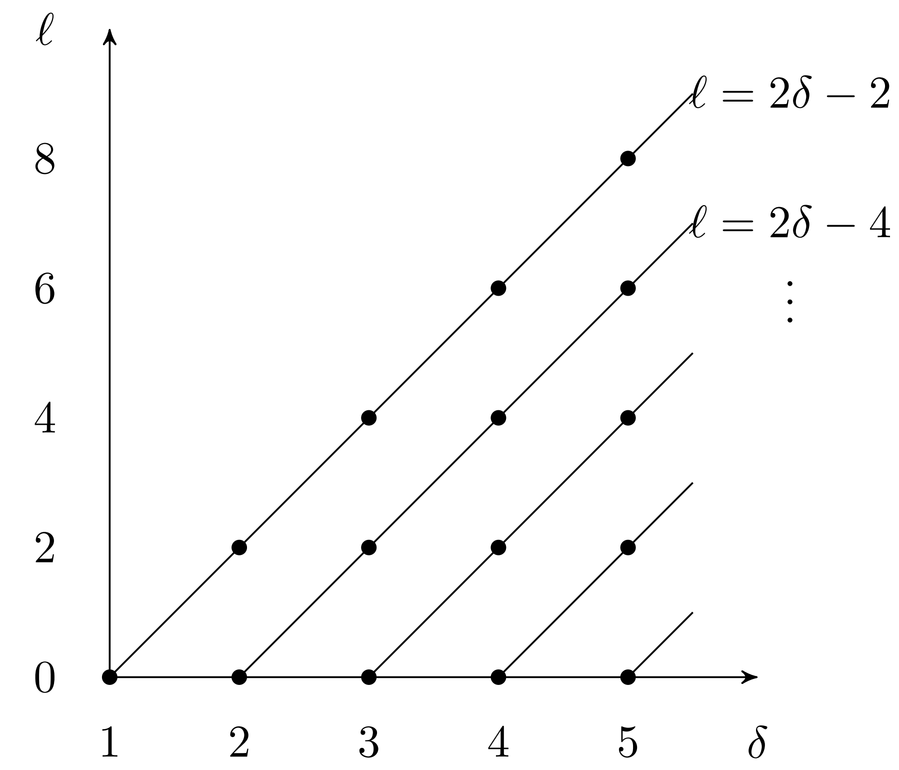
Worldsheet integral

❖ **Properties of VS amplitude:**

- Poles S, T, U: massless/massive graviton exchange
- High energy limit: S/T fixed, $S \to \infty$

$$A^{(0)}(S,T) \sim e^{-2S\log|S| - 2\log|T| - 2U\log|U|}$$

- Exponentially softer than particle scattering



Virasoro-Shapiro Amplitude

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Worldsheet integral

❖ **Properties of VS amplitude:**

- Low energy limit: $S, T, U \to 0$: point particles with *derivative interactions*

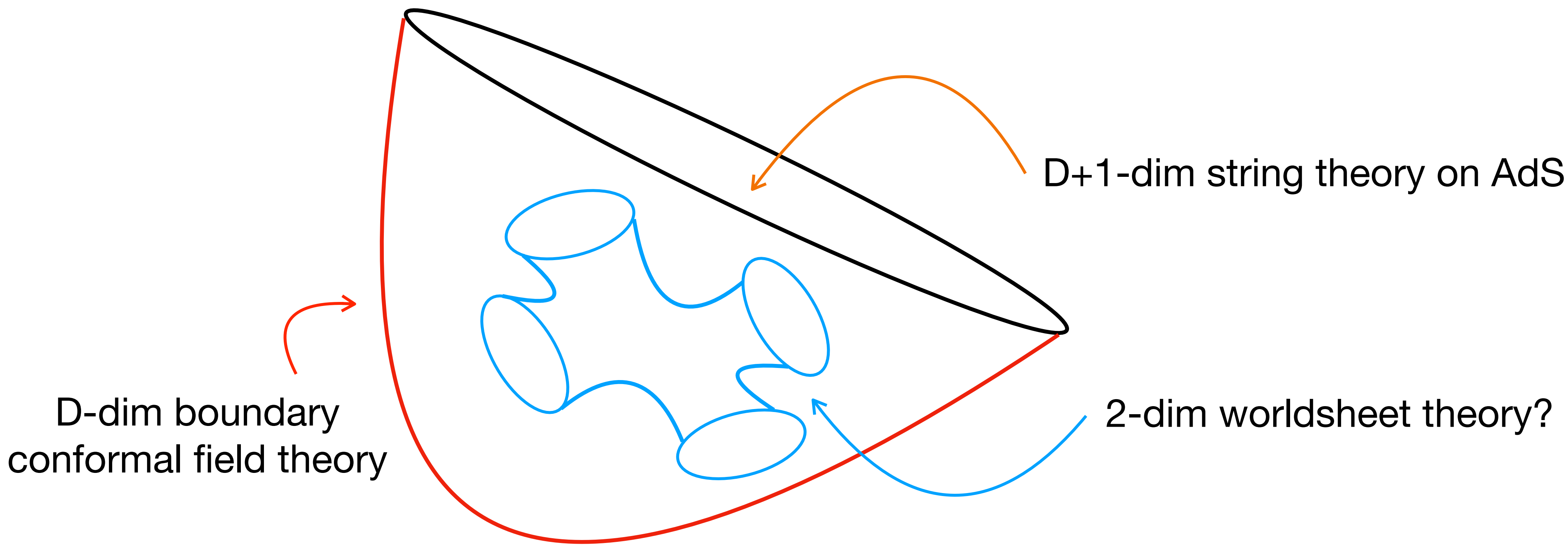
$$A^{(0)}(S,T) = -1/(STU) - 2\zeta(3) + (S^2+T^2+U^2)\zeta(5) + \ldots$$

SUGRA R^4 D^4R^4

- Only contains **odd** zeta-values ⇔ single-valued zeta values

String Amplitude

- How to formulate string theory in curved background?
- **Step one:** string amplitude
- **Still to hard!** Focus on AdS/CFT cases
- **Three equivalent ways to describe the same process**



String Theory in AdS

- **AdS/CFT:** the only non-perturbative definition of string theory
  - $AdS_5 \times S^5 \Leftrightarrow 4d\ \mathcal{N} = 4\ SU(N)$ super Yang-Mills theory
  - $AdS_4 \times \mathbb{CP}^3 \Leftrightarrow 3d\ U(N)_k \times U(N)_{-k}$ ABJM
  - $AdS_3 \times S^3 \times M_4 \Leftrightarrow 2d\ (M_4)^N / S_N$
- Closed string scattering ⇔ correlator of stress-tensor multiplet
  - Tree level in string theory ⇔ leading 1/N in CFT
- Extremely hard problem:
  - Tree AdS backgrounds involve RR flux, worldsheet not understood well
  - Dual CFT **strongly coupled** at large N

Bootstrap AdS_5 String Amplitude

❖ **Near flat space expansion:** leading order in $1/N$, all order in α'

[Alday, Silva, Hansen 22-23']

- CFT correlator ⇔ AdS amplitude via integral transformation

- Flat space limit (Borel transform) of CFT correlator

$$A(S,T) = A^{(0)}(S,T) + \frac{\alpha'}{R^2} A^{(1)}(S,T) + \left(\frac{\alpha'}{R^2}\right)^2 A^{(2)}(S,T) + \dots$$

- AdS amplitude by worldsheet integral of certain single-valued multiple polylogarithms *(SVMPLs)*. For instance:

$$A^{(1)}(S,T) = -\frac{1}{U^2} \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} \mathcal{L}_{\text{weight 3}} + \text{crossing}$$

- Comparing two ways almost fixes the answer!

AdS$_3$ String: an Introduction

❖ **AdS/CFT dictionary:**

- $AdS_3 \times S^3 \times M_4$ with $M_4 = K3$ or $T^4 \Leftrightarrow$ 2d CFT, Lagrangian not known

- Small $\mathcal{N} = 4$ SUSY, half maximal, $SU(2) \times SU(2)$ R-symmetry

- Coupling constant & dictionary: k NS-NS flux, $g_s N$ RR flux

$$\frac{R^2}{\ell_s^2} = \sqrt{g_s^2 N^2 + k^2} \equiv \sqrt{\lambda} \quad \text{'t Hooft coupling}$$

- T^4 has 20 moduli, K3 has 84 moduli, we consider the four moduli that do **not** depend on M_4 (dual to dilaton multiplet).

- Each $\Delta = 1$ 1/2-BPS tensor multiplet contains 4 moduli

- One unique multiplet $\mathcal{S}_a$ insensitive to M_4, dual to VS

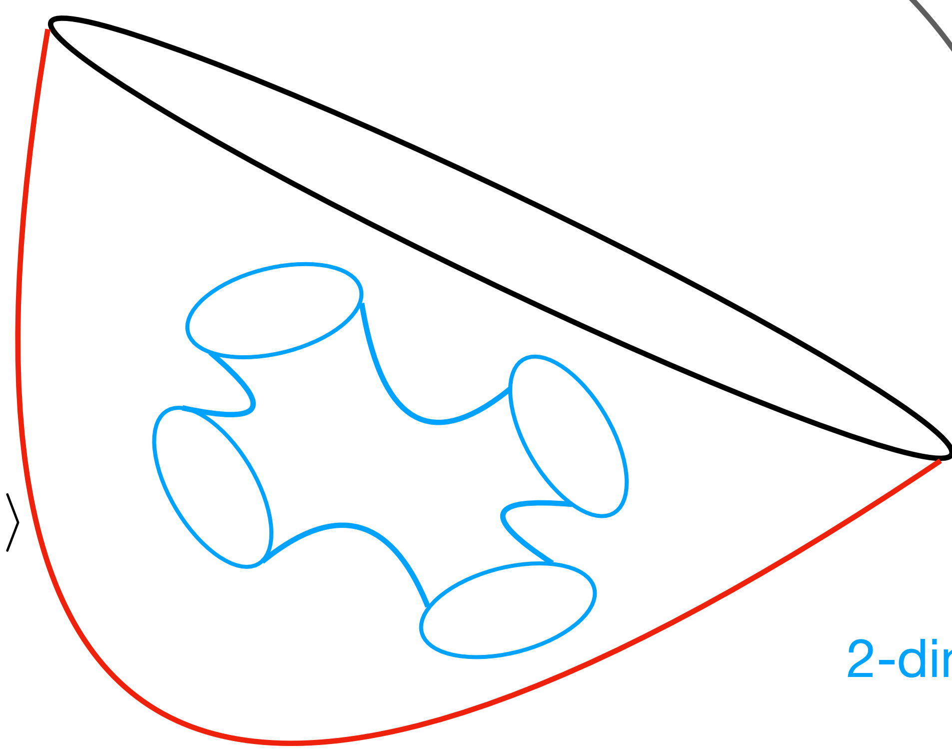
AdS VS Routine

❖ AdS VS amplitude

$\mathcal{M}(s,t)$

Mellin Tranform

Flat space limit: Borel transform



$\langle S_a(x_1) S_b(x_2) S_c(x_3) S_d(x_4) \rangle$

2-dim boundary conformal field theory

String theory on AdS_3

$$A(S,T) = A^{(0)} + \frac{A^{(1)}}{\sqrt{\lambda}} + \ldots$$

WS Ansatz

2-dim worldsheet integral

$$A^{(1)}(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G(S,T,z) + \text{crossing}$$

AdS$_3$ Correlation Function

❖ **Four-point function structure:** [Rastelli, Roumpedakis, Zhou '19]

$$\langle S_a(x_1)S_b(x_2)S_c(x_3)S_d(x_4)\rangle = \frac{1}{x_{12}^2 x_{34}^2}[\mathcal{F}^{\text{free}}_{abcd}(U,V) + \Theta_{abcd}(U,V)\mathcal{H}(U,V)]$$

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- $\mathscr{F}^{\text{free}}$ fixed from free theory, exact in $1/N$
- Θ is a kinematical prefactor, $\mathscr{H}$ contains all dynamical information, R singlet
- $\mathscr{H}$ can be expanded in terms of long superblocks

$$\mathcal{H}(U,V) = U^{-1} \sum_{l=0,2,\ldots \Delta \geq l+2} C^2_{\Delta,l}\; g_{\Delta+2,l}(U,V)$$

- Mellin space: CFT correlator ⇔ Mellin amplitude

$$\mathcal{H}(U,V) = \int \frac{dsdt}{(4\pi i)^2} U^{\frac{s}{2}+\frac{1}{3}} V^{\frac{t}{2}-\frac{2}{3}} \Gamma[\frac{2}{3}-\frac{s}{2}]^2 \Gamma[\frac{2}{3}-\frac{t}{2}]^2 \Gamma[\frac{2}{3}-\frac{u}{2}]^2 \mathcal{M}(s,t)$$

AdS_3 Correlation Function

❖ **Tree-level Mellin amplitude:**

- Mellin space: CFT correlator ⇔ Mellin amplitude, $s + t + u = 0$

$$\mathcal{H}(U,V) = \int \frac{dsdt}{(4\pi i)^2} U^{\frac{s}{2}+\frac{1}{3}} V^{\frac{t}{2}-\frac{2}{3}} \Gamma[\frac{2}{3}-\frac{s}{2}]^2 \Gamma[\frac{2}{3}-\frac{t}{2}]^2 \Gamma[\frac{2}{3}-\frac{u}{2}]^2 \mathcal{M}(s,t)$$

- Pole structure [Mack '09][Penedones, Silva, Zhiboedov '19]

$$M(s,t) \sim \frac{\lambda^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s-(\Delta-\ell+2m)}$$

- Low energy expansion:

$$M = \frac{1}{s+\frac{2}{3}} + \frac{1}{t+\frac{2}{3}} + \frac{1}{u+\frac{2}{3}} + \sum_{a,b=0}^{\infty} \frac{\sigma_2^a \sigma_3^b}{\lambda^{\frac{1}{2}+a+\frac{3}{2}b}} [\alpha^{(0)}_{a,b} + \frac{\alpha^{(1)}_{a,b}}{\sqrt{\lambda}} + \ldots]$$

- Meromorphic terms: supergravity exchange

- Polynomials in $\sigma_2 = s^2 + t^2 + u^2$, $\sigma_3 = stu$: higher derivative corrections

AdS3 Near Flat Space Expansion

❖ **Borel transform:**

- Mellin space: $\mathcal{H}(U,V) = \int \frac{dsdt}{(4\pi i)^2} U^{\frac{s}{2}+\frac{1}{3}} V^{\frac{t}{2}-\frac{2}{3}} \Gamma[\frac{2}{3}-\frac{s}{2}]^2 \Gamma[\frac{2}{3}-\frac{t}{2}]^2 \Gamma[\frac{2}{3}-\frac{u}{2}]^2 \mathcal{M}(s,t)$

- Flat space expansion: Borel resumption [Penedones '10]

$$A(S,T) = \lambda^{1/2} \int \frac{d\alpha}{2\pi i} \frac{e^{\alpha}}{\alpha^3} \mathcal{M}(\frac{2\sqrt{\lambda}S}{\alpha}, \frac{2\sqrt{\lambda}T}{\alpha})$$

- Strong coupling expansion: $A(S,T) = A^{(0)} + \frac{A^{(1)}}{\sqrt{\lambda}} + \ldots$

	$A^{(0)}$ data		$A^{(1)}$ data		$A^{(2)}$ data	
$\Delta_{\delta,\ell} =$	$2\sqrt{\delta}\lambda^{\frac{1}{4}}$	$+$	$\lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell}$	$+$	$\lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell}$	$\cdots$
$f^2_{\delta,\ell} =$	$f^{2(0)}_{\delta,\ell}$	$+$	$\lambda^{-\frac{1}{2}} f^{2(1)}_{\delta,\ell}$	$+$	$\lambda^{-1} f^{2(2)}_{\delta,\ell}$	$\cdots$

$$C^2_{\tau,\ell} = \frac{\pi^3(\delta_{\ell,0}+1)\lambda^{5/4}\tau^{-2}(\delta,\ell)}{4^{\ell+\tau(\delta,\ell)}\sin^2(\frac{\pi}{2}\tau(\delta,\ell))} f(\delta,\ell)$$

AdS$_3$ Near Flat Space Expansion

❖ **Leading Order residue comparison at** $S = \delta \in \mathbb{Z}_{\geq 0}$**:**

- On one hand, residue given by Borel transformation of block

$$M(s,t) \sim \frac{\lambda^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s - (\Delta - \ell + 2m)} \;\Rightarrow A(S,T)|_{poles} = \sum_{\ell=0,2,\ldots \delta=1,2\ldots} f^2_{\tau(\delta),\ell} \sum_{i=0}^{\infty} \frac{R^{(i)}_{\tau(\delta),\ell}(S,T)}{\lambda^{i/4}}$$

- On the other hand, we know $A^{(0)}$ is the VS amplitude

$$A^{(0)}(S,T) = -\frac{S^2+T^2+U^2}{4} \frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

- Comparing residues we find $\tau_0(\delta, l) = 2\sqrt{\delta}$ and f_0, f_0 vanishes if $\ell > 2\delta$

- Leading trajectory: $\delta = \ell/2$, higher trajectories are degenerate so we can only compute the **average** of degenerate operators $\langle f \rangle$

AdS$_3$ Near Flat Space Expansion

❖ **Subleading order $\mathcal{O}(\lambda^{-1/4})$ residue comparison**

- On one hand, residue given by Borel transformation of block

$$M(s,t) \sim \frac{\lambda^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s-(\Delta-\ell+2m)} \Rightarrow A(S,T)|_{poles} = \sum_{\ell=0,2,\ldots \delta=1,2\ldots} f^2_{\tau(\delta),\ell} \sum_{i=0}^{\infty} \frac{R^{(i)}_{\tau(\delta),\ell}(S,T)}{\lambda^{i/4}}$$

- On the other hand, no contribution from A

- Amplitude expanded in $\mathcal{O}(1/\sqrt{\lambda})$, so no contribution from A

- We find $\tau_1 = -\ell - 1$, and $\langle f_1 \rangle_{\delta,\ell} = -\dfrac{(5+4\ell)\langle f_0 \rangle_{\delta,\ell}}{4\sqrt{\delta}}$

Averaged OPE

AdS_3 Near Flat Space Expansion

❖ **Subsubleading order $\mathcal{O}(\lambda^{-1/2})$ residue comparison**

- On one hand, residue given by Borel transformation of block

$$M(s,t) \sim \frac{\lambda^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s-(\Delta-\ell+2m)} \Rightarrow A(S,T)|_{poles} = \sum_{\ell=0,2,\dots \delta=1,2\dots} f^2_{\tau(\delta),\ell} \sum_{i=0}^{\infty} \frac{R^{(i)}_{\tau(\delta),\ell}(S,T)}{\lambda^{i/4}}$$

- We do not know $A^{(1)}$, but we make ansatz with 11 unknowns (3 ambiguities)

$$A^{(1)}(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G(S,T,z) + \text{crossing}$$

- G: deg 2 polynomials in S, T times SVMPL for RR case

- Comparing residues fixes BOTH!

MPL & SVMPL

❖ Multiple polylogarithms

- Iterated integral: $w_i \in \{0,1\}$

$$L_{w_1\ldots w_r}(z) = \int_{0\le t_r\le\ldots\le t_1\le z} \frac{dt_1}{t_1 - w_1} \cdots \frac{dt_r}{t_r - w_r}$$

- Multivalued, holomorphic function
- For instance, $L_1(z) = \log(1 - z)$
- Cannot be integrated on the worldsheet!

❖ Single-valued multiple polylogarithms

- Single-valued, non-holomorphic function constructed from MPL
- For instance, $\mathscr{L}_1(z) = \log|1 - z|^2$

Why MPL?

- **Strings in weakly curved background** [Alday, Silva, Hansen 22-23']

  - Small curvature (large radius R) expansion ⇔ strong coupling expansion

$$A(S,T) = A^{(0)}(S,T) + \frac{\alpha'}{R^2} A^{(1)}(S,T) + \left(\frac{\alpha'}{R^2}\right)^2 A^{(2)}(S,T) + \ldots$$

  - Intuition from non-linear sigma model:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) \qquad G_{\mu\nu}(X) = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{R^2} + \ldots$$

$$= S_{\text{flat}} + \frac{1}{R^2} \lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} V^{\mu\nu}_{\text{graviton}}(q) + \ldots \qquad h_{\mu\nu} \sim X_\mu X_\nu \sim \lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} e^{iq\cdot X}$$

  - Curvature expansion ⇒ extra soft graviton insertion

$$A^{(1)}(S,T) \sim \lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} \langle V_1 V_2 V_3 V_4 V^{\mu\nu}_{\text{graviton}}(q) \rangle_{\text{flat}}$$

AdS_3 Near Flat Space Expansion

- **Subsubleading order $\mathscr{O}(\lambda^{-1/2})$ residue comparison**

  - For instance, the leading trajectory operator

$$\tau(\ell/2, \ell) = \lambda^{1/4}\sqrt{2l} - \ell - 1 + \frac{3\ell^2 - 2\ell + 2}{4\lambda^{1/4}\sqrt{2\ell}} + \mathcal{O}(\lambda^{-3/4})$$

  - No integrability results, only semiclassical result

  - **Idea**: Assume ℓ and R-charge J are large, compute folded string solution to Green-Schwarz action, then extrapolate to finite ℓ, J:

$$E = \lambda^{\frac{1}{4}}\sqrt{2\ell} + \lambda^{-\frac{1}{4}}[\frac{3\ell^{\frac{3}{2}}}{4\sqrt{2}} + \frac{J^2}{2\sqrt{2\ell}} + \mathcal{C}\sqrt{2\ell}] + ...$$

  - Match $J = 1$ superdesendent, predict 1-loop constant $\mathscr{C} = -1/4$

  - Also find OPE coefficients, contains ζ_3

AdS$_3$ KK Mode

[Aprile, Vieira 20']
[Wang, Wu, Yuan 25']
[Jiang, DLZ 25']

- **We can generalized to KK modes on S^3**
  - Labeled by $\Delta = p$, R-charge $\left(\frac{p}{2}, \frac{p}{2}\right)$ with $p \in \mathbb{Z}_{>=0}$
  - **Idea**: use the *generating function* of KK modes, the so-called AdS $\times$ S formalism
  - Mellin amplitude now contains a (discrete) sum on the sphere
  - Make a WS ansatz that is symmetric under AdS $\times$ S, fix parameters on $\langle 11pp \rangle$
  - Checked against semiclassical computation

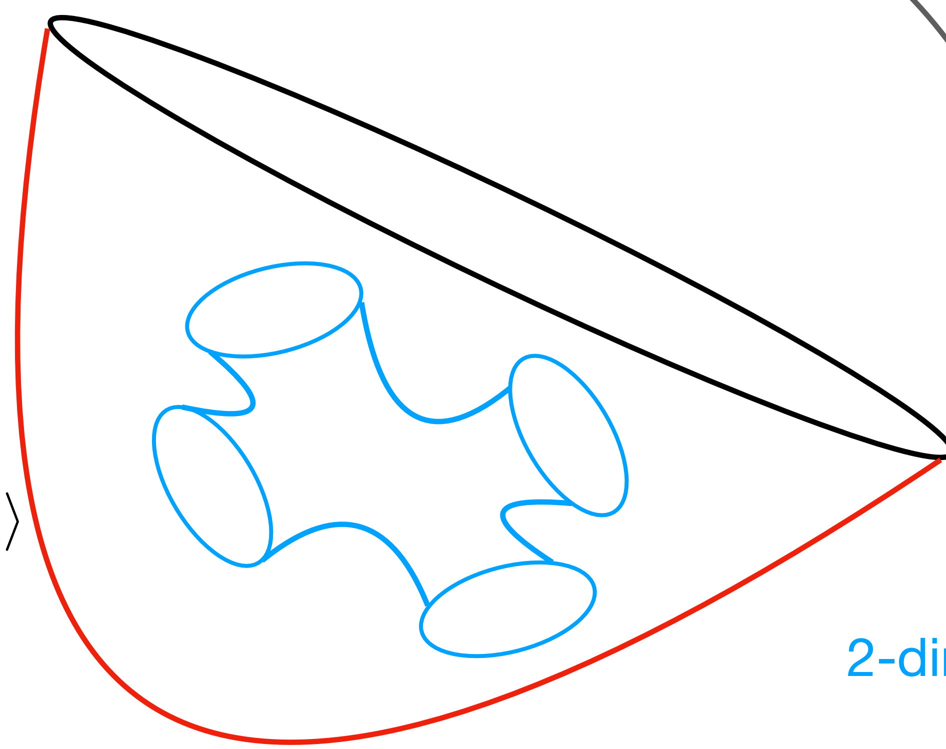
Summary

❖ **AdS VS amplitude**

$\mathcal{M}(s,t)$

Mellin Tranform

Flat space limit: Borel transform



String theory on AdS_3

$$A(S,T) = A^{(0)} + \frac{A^{(1)}}{\sqrt{\lambda}} + \ldots$$

WS Ansatz

$\langle S_a(x_1) S_b(x_2) S_c(x_3) S_d(x_4) \rangle$

2-dim boundary
conformal field theory

2-dim worldsheet integral

$$A^{(1)}(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G(S,T,z) + \text{crossing}$$

❖ **Fixes all OPE data up to $\mathcal{O}(1/\sqrt{\lambda})$, all new predictions**

Outlooks

- **Compute $A^{(2)}$ correction/ other observables (EE)**

  [Ren, Wang, Wen 26']

- **Mixed RR and NS-NS flux for AdS:** new WS basis

  [Ekhammar, Gromov,Stefanski,Thull 26']

- **Correlator/anomalous dim from string field theory/integrability**

  [Basso, Georgoudis,Klemenchuk 22']

  [Basso, Georgoudis 25']

- $AdS_3 \times S^3 \times S^3 \times S^1$**:** large $\mathcal{N} = 4$ sym, need Ward identity

- Correlator of other 1/2 BPS multiplet: T^4 case nontrivial under $SO(4)$ T-duality

- Hint for the worldsheet theory?

谢谢！