Mellin Amplitudes for Energy Correlators

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New Frontiers of Quantum Field and Gravity

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Based on work in progress with Ian Moult, Di Wu, Anastasia Volovich

Asymptotic observables for Quantum Field Theory:

## correlation function of detector operators

$$\langle\Psi|E(n_1)E(n_2)\dots E(n_N)|\Psi\rangle \qquad \mathcal{E}(n) = \int_{-\infty}^{+\infty} du \lim_{r\to\infty} r^2 T_{0i}(t = u + r, r\,\vec{n})n^i$$

integrated correlation functions || infinite sum over squared amplitudes

[Belisky,Hohenegger,Korchemsky,Sokatchev, Zhiboedov; Henn,Sokatchev,Yan,Zhiboedov; Dixon, Luo, Shtabovenko,Yang,Zhu;He,Jiang,Yang, Zhang,…]

-- insights into higher point data very limited

Mellin space as a new language for the multi-point asymptotic observables
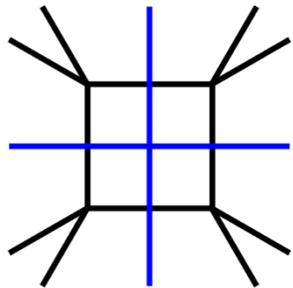
$$\langle\phi_{\Delta_1}(x_1)\cdots\phi_{\Delta_n}(x_n)\rangle = \int [\mathrm{d}\delta_{ij}]\, M(\delta_{ij}) \prod_{i<j}^{n} \Gamma(\delta_{ij})\, x_{ij}^{-\delta_{ij}}$$

Mellin formalism developed in the context of AdS/CFT correlation functions[Mack, Fitzpatrick, Kaplan, Penedones, Raju, van Rees; Nandan, Volovich ,Wen]

Powerful in conformally invariant setting : conformal correlator/dual-conformal feynman integrals [Nandan,Paulos,Spradlina,Volovich]
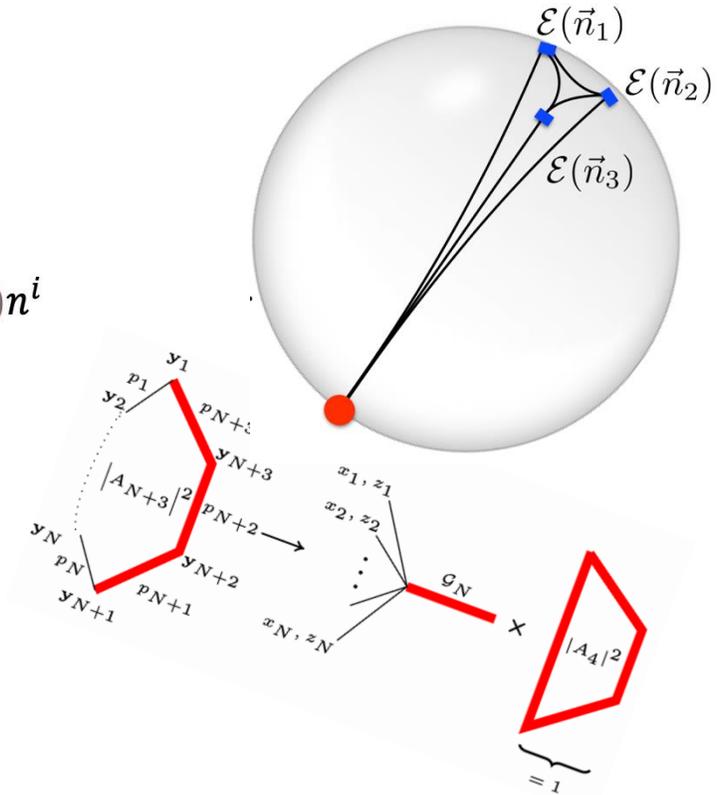
Why Mellin space?

Physics of correlators transparent, allowing conformal block decomposition.

simple integro-differential relations between various types of multi-loop integrals[Drummond, Henn, Trnka;Dixon,Drummond,Henn; Ferro], complementary to standard IBPs
mapping onto basic building blocks: n-point star integral, as volumes of simplices in hyperbolic space[Schnetz,Goncharov, Bourjaily,Gardi,McLeod,Vergu,Ren,Spradlina,Vergu,Volovich…]

Profit from extensive studies on star integrals, extending knowledge on the unknowns
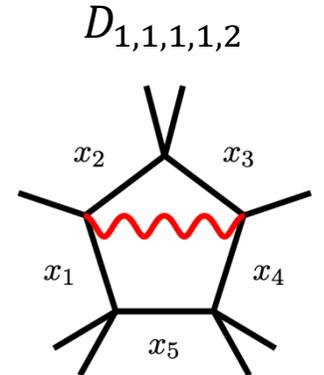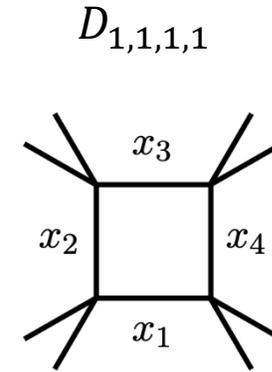
Symbology for physical observables:
What can we learn from physical constraints on the observables?
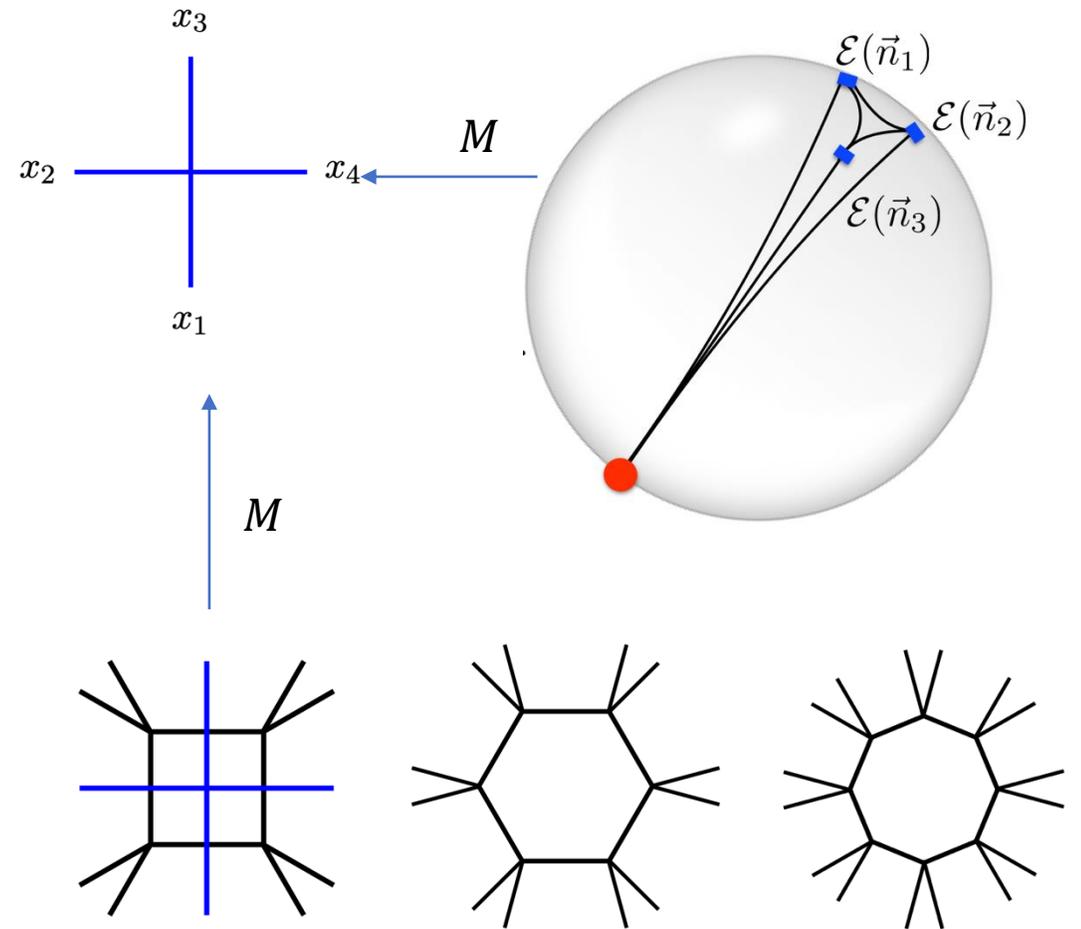
Preliminary: star formula

$$I_n \equiv \pi^{-h} \int \mathrm{d}^d Q \prod_{i=1}^{n} \frac{\Gamma(\Delta_i)}{(-P_i \cdot Q)^{\Delta_i}}$$

$$:= D_{\Delta_1, \cdots, \Delta_n}$$
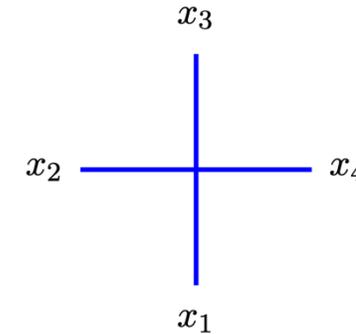
$D_{1,1,1,1}$

$D_{1,1,1,1,2}$

$$I_n = 2 \int_0^{+\infty} \prod_i \frac{\mathrm{d}t_i}{t_i} t_i^{\Delta_i} \exp\left(\sum_{i<j} t_i t_j P_{ij}\right)$$

Conformal constraints

$$\delta_{ii} = -\Delta_i, \qquad \sum_j \delta_{ij} = 0.$$

$$= \frac{1}{2} \oint \mathrm{d}\delta_{ij} \prod_{i<j} \Gamma(\delta_{ij}) P_{ij}^{-\delta_{ij}}$$

$$\delta_{ij} = k_i \cdot k_j$$

$$\sum_i k_i = 0.$$

$$k_i^2 = -\Delta_i$$

$$\# \ of \ \Gamma(\delta_{ij}): \quad \frac{n(n-1)}{2} \qquad \# \ of \ independent \ \delta_{ij}' s \ : \frac{n(n-3)}{2}$$

For N-point energy correlators, in collinear limit :

\# of (multiplicatively) independent kinematic variables: $\frac{N(N-1)}{2} - 1 = \frac{(N+1)(N-2)}{2}$ same as $(N+1)$-gon

Preliminary: N-point EC in the multi-collinear limit

N-fold iterated integral over $1 \to N$ collinear splitting function

Toy example: ϕ^4-theory
$$|\text{Split}|^2_{1\to 3} = \frac{1}{s^2_{123}} \qquad s_{I...J} := \sum_{(i,j)\in[I,J]} x_i x_j |z_{ij}|^2$$
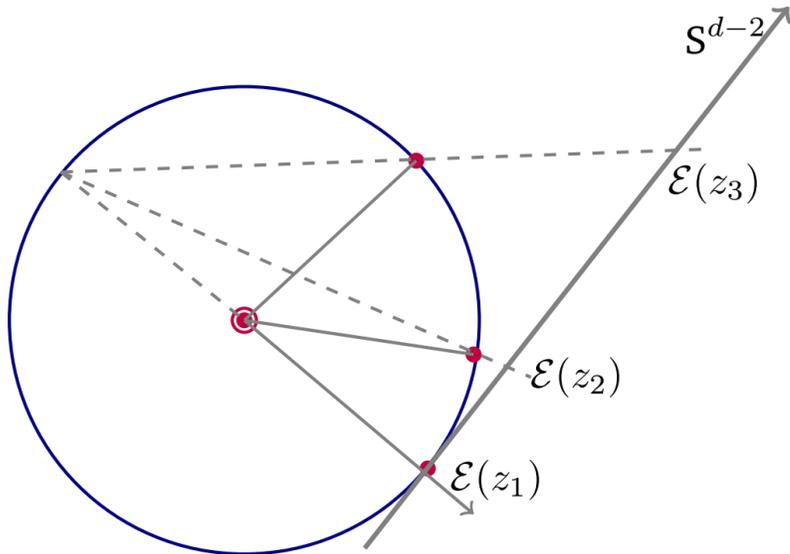
$$G_3 := |z_{12}z_{23}|^2 \int_0^1 dx_1 dx_2 dx_3\, \delta(1 - x_1 - x_2 - x_3)\, \frac{x_1^{-J_1-1} x_2^{-J_2-1} x_3^{-J_3-1}}{s^2_{123}}$$

For energy correlators: $\quad J_1 = J_2 = J_3 = 1 - d = -3$

Introduce an auxiliary point P_{N+1} in embedding space

$$P_i = (z_i^\mu, |z_i|^2, 1), \qquad P_{N+1} = (0, 1, 0)\,.$$

$$|z_{ij}|^2 = \frac{P_i \cdot P_j}{(P_i \cdot P_4)(P_4 \cdot P_j)} \qquad i,j = 1,2,3$$

Writes into projective integral $\quad x_i \rightarrow x_i P_{i4}$

$$\frac{P_{12}P_{23}}{P_{14}P_{24}P_{24}P_{34}} \times \prod_{i=1}^{3} P_{i4}^{-J_i} \int \frac{d^3 x}{\mathrm{GL}(1)} \frac{\prod_i x_i^{-J_i-1}}{\left(\sum_{i,j=1}^{3} x_i x_j P_{ij}\right)^2 \left(\sum_i x_i P_{i4}\right)^{-J-4}} \qquad J := J_1 + J_2 + J_3$$

Introduce x_{N+1} , promote linear singularity into quadratic

$$\int d^4 x \prod_{i=1}^{3} x_i^{-J_i-1} x_4^{-J-4-1} \exp\left[-\sum_{i,j=1}^{4} x_i x_j P_{ij}\right] := D_{-J_1,-J_2,-J_3,-J-4}$$

$$\boxed{D_{3,3,3,5} \rightarrow? \int \circ \hat{\partial}^k [D_{1,1,1,1}]}$$

Kinematics mapping: $\quad u = \dfrac{P_{12}P_{24}}{P_{13}P_{24}} \rightarrow \dfrac{|z_{12}|^2}{|z_{13}|^2}$

$$v = \frac{P_{23}P_{14}}{P_{13}P_{24}} \rightarrow \frac{|z_{23}|^2}{|z_{13}|^2}$$

Solving conformal constraints

$$\delta_{14} + \delta_{24} + \delta_{34} = 3$$
$$\delta_{12} + \delta_{23} + \delta_{24} = 3$$
$$\delta_{13} + \delta_{23} + \delta_{34} = 3$$
$$\delta_{14} + \delta_{24} + \delta_{34} = 5$$

Shift mellin variables and map onto the scalar box

$$\delta_{14} + \delta_{24} + \delta_{34} = 1$$
$$\delta_{12} + \delta_{23} + \delta_{24} = 1$$
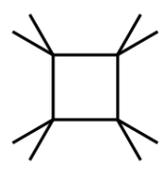$$\delta_{13} + \delta_{23} + \delta_{34} = 1$$
$$\delta_{14} + \delta_{24} + \delta_{34} = 1$$

$\delta_{14} \rightarrow \delta_{14} + 1, \delta_{24} \rightarrow \delta_{24} + 1,$
$\delta_{24} \rightarrow \delta_{24} + 2, \delta_{13} \rightarrow \delta_{13} + 1$

$$M[D_{1,1,1,1}] = \prod \Gamma(\delta_{ij}) = \Gamma^2(\delta_u)\,\Gamma^2(\delta_u)\Gamma^2(1 - \delta_u - \delta_v)$$
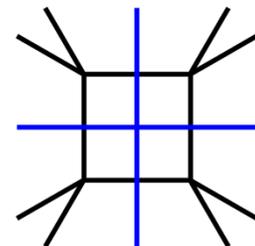
$\delta_{12} = \delta_u, \delta_{23} = \delta_v, \delta_{13}$
$= 1 - \delta_u - \delta_v$

$$M[D_{3,3,3,5}] \rightarrow \delta_u\,\delta_v\,(1 - \delta_v - \delta_u)^2(2 - \delta_u - \delta_v)M[D_{1,1,1,1}]$$

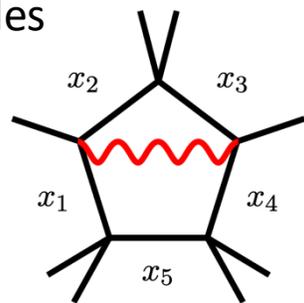$\delta_u \rightarrow u\partial_u, \delta_v \rightarrow v\partial_v$

$$\square = P_{13}P_{24} \oint \mathrm{d}\delta_{ij} \prod_{i<j}^{4} \Gamma(\delta_{ij}) \, P_{ij}^{-\delta_{ij}} \qquad \implies \qquad M = 1$$
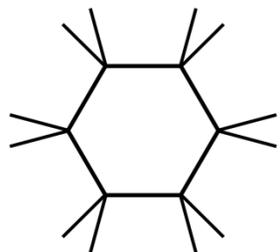
2 variables

$$G_3^{\phi^4} \implies M = \left(\frac{s}{2}+1\right)\left(\frac{u}{2}+1\right)\left(\frac{t}{2}+1\right)^2\left(\frac{t}{2}+\frac{3}{2}\right)$$

5 variables

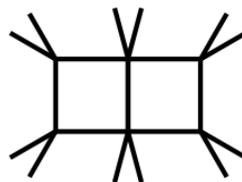$$I_n^1 = \frac{1}{\pi^2} \int \mathrm{d}^4 Q \, (-Q \cdot Y) \prod_{i=1}^{n} \frac{1}{(-P_i \cdot Q)} \qquad \implies \qquad M = 1$$

9 variables

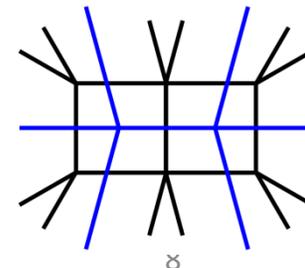$$\implies \qquad M = 1 \qquad\qquad \implies \qquad M = \frac{1}{1 - s_{123}}$$

EC in N=4 super Yang Mills Theory

$$E^N C \overset{\text{coll.}}{=} \frac{1}{|z_{12} \cdots z_{N-1N}|^2} \int \frac{d^N x}{\text{GL}(1)} (x_1 + \cdots + x_N)^{-N} \mathcal{G}_N$$
$$+ \text{perm}\,(z_1, \cdots, z_N)$$

$$s_{I\ldots J} := \sum_{(i,j)\in[I,J]} x_i x_j |z_{ij}|^2$$

$N = 3$:

$$\mathcal{G}_3 = \frac{s_{12}\,x_{123}}{s_{123}\,x_{12}} + \frac{s_{23}\,x_{123}}{s_{123}\,x_{23}} + \frac{x_1\,x_3}{x_{12}\,x_{23}}$$

$$x_{I,\ldots,J} := x_I + \cdots + x_J$$

Introduce a deforming parameter t

$$\frac{s_{12}}{s_{123}\,x_{12}\,x_{123}^2} \longrightarrow \int d^4x\,dt \prod_{i=1}^{3} x_1 x_2 x_4^2 \exp\left[-\sum_{i,j=3}^{4} x_i x_j P_{ij} + x_4 t (P_{14} + P_{24}) + x_4 (P_{14} + P_{24} + P_{34}) \right]$$
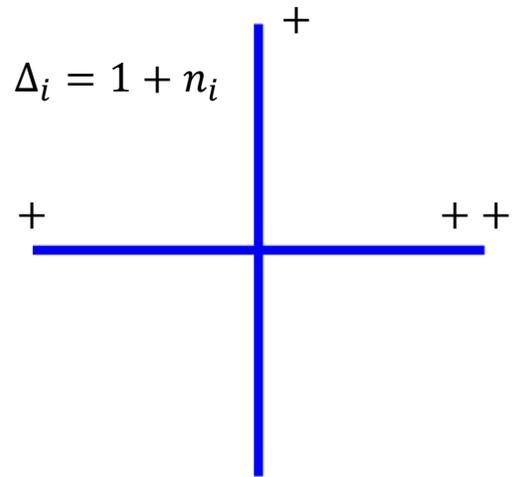
Euler type integral over t: $\rightarrow \dfrac{1}{\delta_{14}+\delta_{24}-1}$

$$\frac{s_{12}}{s_{123}\, x_{12}\, x_{123}^2} \xrightarrow{M} P_{12} \times \frac{1}{\delta_{14} + \delta_{24} - 1} \times M\left[D_{2,2,1,3}\right]$$

$$\delta_{14} \to \delta_{14} + 1, \delta_{24} \to \delta_{24} + 1$$

$$\prod \Gamma(\delta_{ij})$$
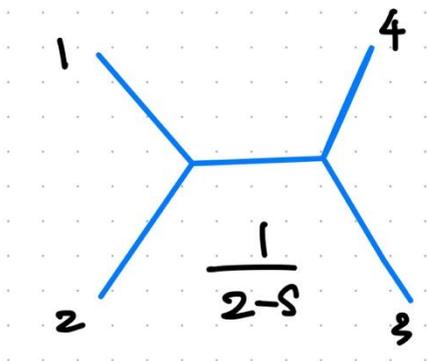
$$\sum_{j \neq i} k_i \cdot k_j = -k_i^2 = \Delta_i = 1 + n_i$$

$$M[D_{2,2,1,3}] \to \frac{1}{P_{14}P_{24}} \times \delta_{14}\delta_{24} \times M[D_{1,1,1,1}] \qquad M[D_{1,1,1,1}] = \frac{1}{P_{13}P_{24}}$$

$$\frac{s_{12}}{s_{123}\, x_{12}\, x_{123}^2} \xrightarrow{M} \frac{1}{P_{14}P_{24}P_{34}}\left[\frac{P_{12}P_{34}}{P_{13}P_{24}}\right] \times \frac{\delta_{14}\delta_{24}}{\delta_{14} + \delta_{24} + 1}$$
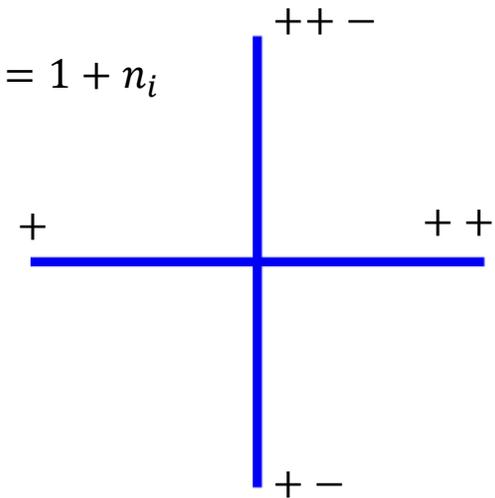
Solving conformal constraints

$$[u] \times \frac{\delta_v(1 - \delta_u - \delta_v)}{2 - \delta_u}$$

$$\delta_{12} = \delta_u, \delta_{23} = \delta_v, \delta_{13} = 1 - \delta_u - \delta_v$$

$$\delta_{14} \to \delta_{14} + 1, \delta_{34} \to \delta_{34} + 1, \delta_{12} \to \delta_{12} + 1,$$
$$\delta_{13} \to \delta_{13} - 1$$

$$\prod \Gamma(\delta_{ij})$$

$$\sum_{j \neq i} k_i \cdot k_j = -k_i^2 = \Delta_i = 1 + n_i$$

$$M[D_{2,2,1,3}] \to \frac{P_{13}}{P_{12}P_{14}P_{34}} \times \frac{\delta_{12}\delta_{14}\delta_{34}}{\delta_{13} - 1} \times M[D_{1,1,1,1}]$$

$$\frac{s_{12}}{s_{123} \, x_{12} \, x_{123}^2} \xrightarrow{M} \frac{1}{P_{14}P_{24}P_{34}} \times \frac{\delta_{12}\delta_{14}\delta_{34}}{(\delta_{14} + \delta_{24})(\delta_{13} - 1)}$$

Solving conformal constraints

$$-\frac{\delta_u^2 \, \delta_v}{(1 - \delta_u)(\delta_u + \delta_v)}$$
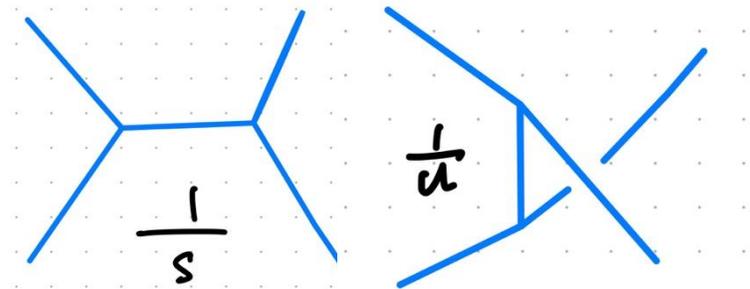
Contour shift: constructing UT integrals

$$[u] \times \frac{\delta_v(1 - \delta_u - \delta_v)}{2 - \delta_u}$$
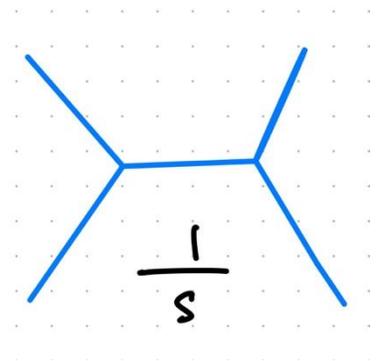
$$\xrightarrow{\delta_u \to \delta_u + 1}$$

$$-\frac{\delta_u^2 \, \delta_v}{(1 - \delta_u)(\delta_u + \delta_v)}$$

$$\Gamma(\delta_u)^2 \Gamma(\delta_v)^2 \Gamma(1 - \delta_u - \delta_v)^2 \; v^{-\delta_v} u^{-\delta_u}$$

$$-\frac{\delta_u^2 \, \delta_v}{(1 - \delta_u)(\delta_u + \delta_v)} = -\frac{\delta_u \delta_v}{1 - \delta_u} + \boxed{\frac{\delta_v^2 \, \delta_u}{(1 - \delta_u)(\delta_u + \delta_v)}}$$

$$\delta_v \to \delta_v - 1$$

$$-[1] \times \frac{\delta_u \delta_v}{1 - \delta_u} - [v] \times \frac{\delta_u(1 - \delta_u - \delta_v)}{1 - \delta_u}$$
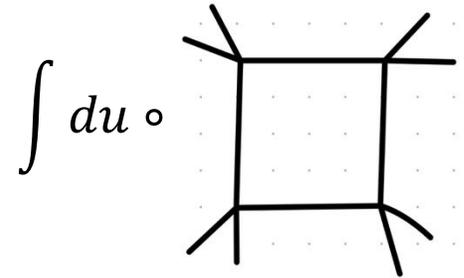
$$\frac{1}{s}$$

$$E^3 C \xrightarrow{M}$$

$$-[1] \times \left(\frac{\delta_u \delta_v}{1 - \delta_u} + \frac{\delta_u \delta_v}{1 - \delta_v} \right) - [v] \times \frac{\delta_u (1 - \delta_u - \delta_v)}{1 - \delta_u} - [u] \times \frac{\delta_v (1 - \delta_u - \delta_v)}{1 - \delta_v}$$

+ boundary constants

$$\hat{\partial}_u := u \, \partial_u \, , \hat{\partial}_v := v \, \partial_v$$

$$\mathcal{I}_{box'} := \int_0^1 dt \, I_4(ut, v) \qquad \xrightarrow{M} \qquad \frac{1}{1 - \delta_u}$$

$$[\widehat{\partial}_u + 1] \, \mathcal{I}_{box'} = I_4 \qquad \xrightarrow{M} \qquad 1$$
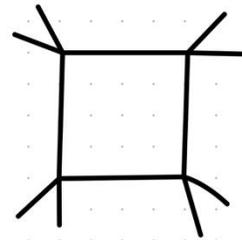
$$\widehat{\partial}_v \, \mathcal{I}_{box'} = g_5^1 \qquad \xrightarrow{M} \qquad \frac{\delta_v}{\delta_u - 1} \qquad \xrightarrow{M^{-1}} \qquad \frac{x_1}{s_{123} \, x_{12} \, x_{123}}$$

$$\int du \circ$$
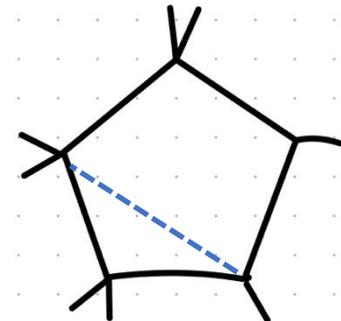
E3C given by second derivatives acting on a transcendental weight 3 integral

$$\int du \circ$$

$\hat{\partial}_v$

$\hat{\partial}_u + 1$

$\hat{\partial}_u$

$\hat{\partial}_u$

$\mathcal{E}(\vec{n}_1)$

$\mathcal{E}(\vec{n}_2)$

$\mathcal{E}(\vec{n}_3)$
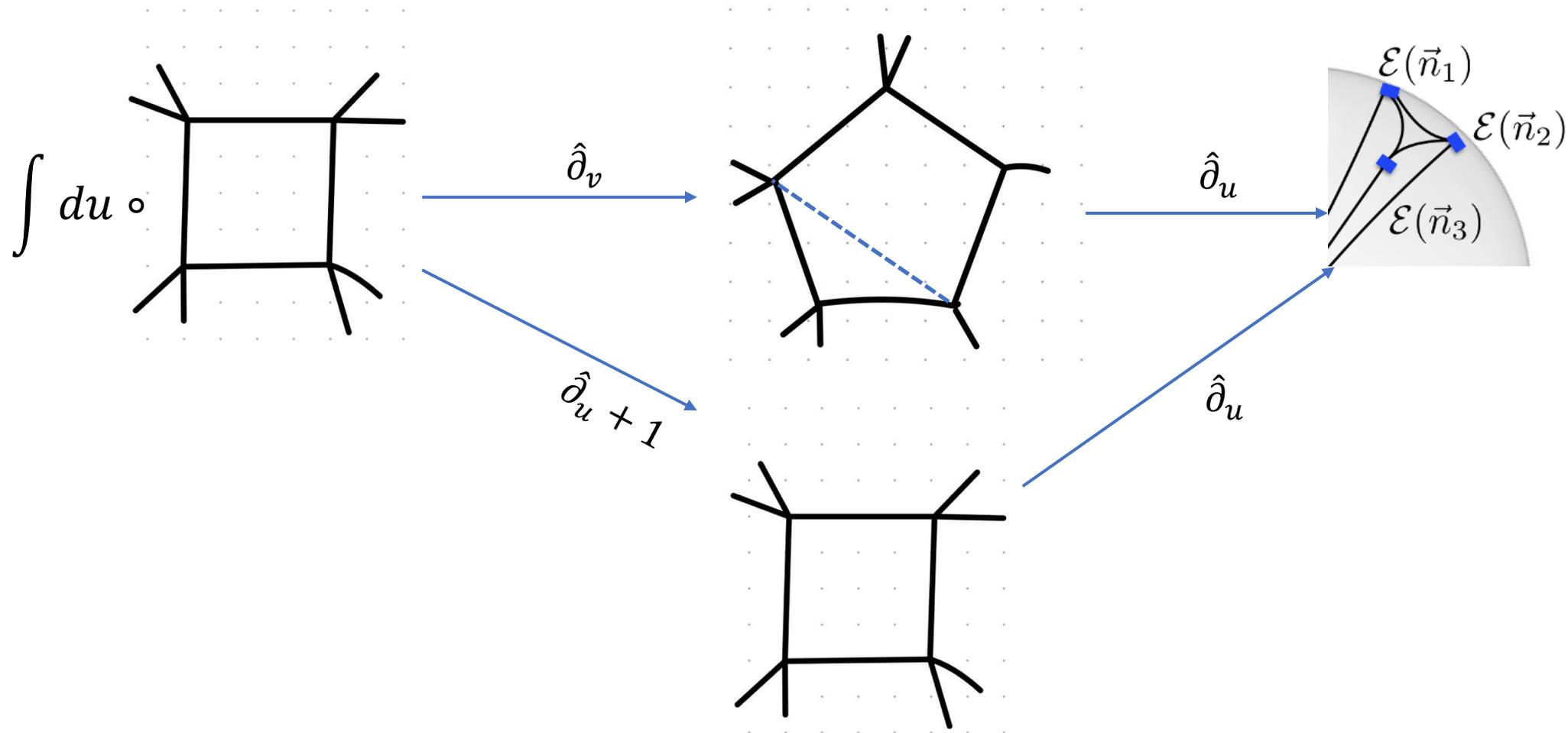
$$\mathcal{S}[\mathcal{I}_{box'}] =$$

$$|1-z|^2 \otimes (1-|1-z|^2) \otimes |1-z|^2 - \frac{1}{2}|1-z|^2 \otimes |z|^2 \otimes |1-z|^2 - \frac{1}{2}|z|^2 \otimes |1-z|^2 \otimes |1-z|^2$$

$$+ \frac{1}{2}|z|^2 \otimes \frac{1-z}{1-\bar{z}} \otimes \frac{1-z}{1-\bar{z}} - \frac{1}{2}|1-z|^2 \otimes \frac{z}{\bar{z}} \otimes \frac{1-z}{1-\bar{z}}$$

First entry $\otimes u, \otimes v$

Last entry $\otimes 1-z, \otimes 1-\bar{z}$

Determines the LS of weight-2 functions

Key input for bootstrapping the observable in QCD

[Gong, Pokraka,Yan,Zhang]

$$\begin{pmatrix} \frac{1}{2}d\log v & \frac{1}{2}d\log\frac{1-u+v+\sqrt{\Delta}}{1-u+v-\sqrt{\Delta}} & 0 & 0 & 0 \\ & & d\log\frac{(1-v)^2}{u} & -d\log v & 0 \\ & & d\log\frac{1+u-v+\sqrt{\Delta}}{1+u-v-\sqrt{\Delta}} & d\log\frac{1-u+v-\sqrt{\Delta}}{1-u+v+\sqrt{\Delta}} & 0 \\ & & & & d\log v \\ & & & & d\log u \\ & & & & 0 \end{pmatrix} \begin{matrix} \\ \\ B_1 \\ B_2 \\ \\ \\ \end{matrix}$$
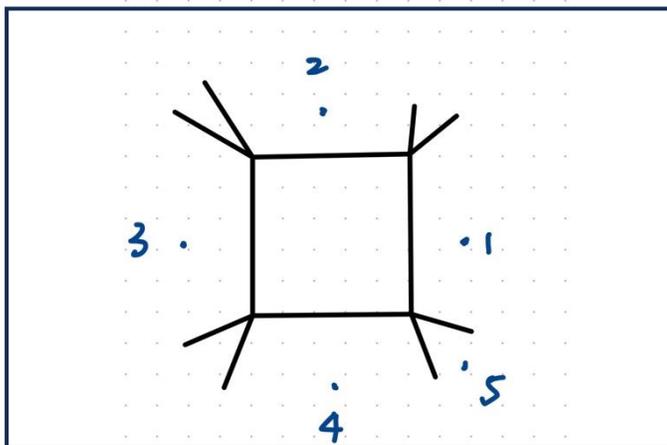
B_i: UT master integrals

$Li_2(1-v) - \frac{1}{2}\ln v \ln u \ (D_+ -$
function [1912.11050])

B_1
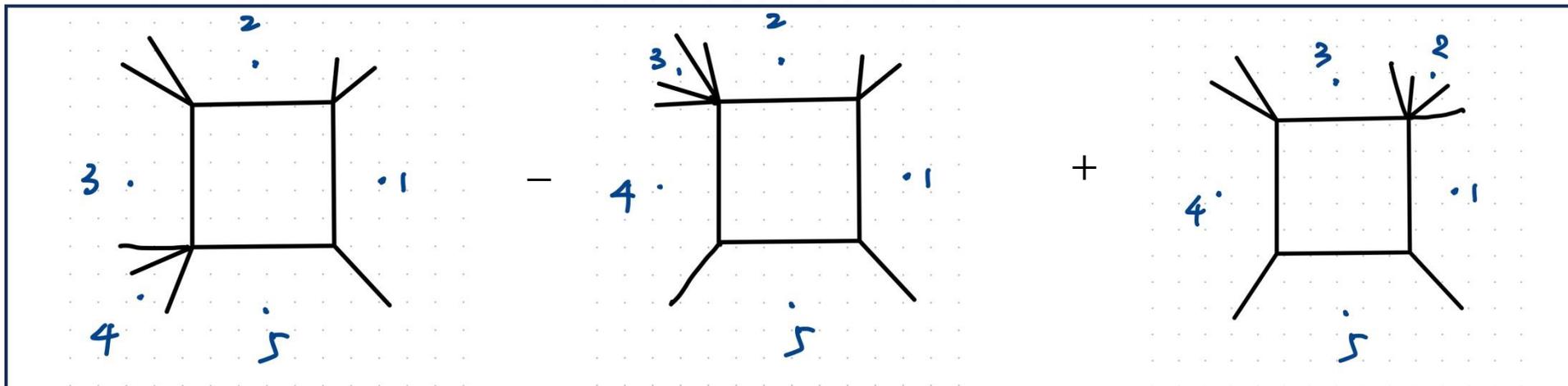
B_2

$\Phi_2(z, \bar{z}) \ (D_- -$
function
[1912.11050])

$$\to \frac{1}{1-v}B_1 + \frac{1}{\sqrt{\Delta}}B_2$$
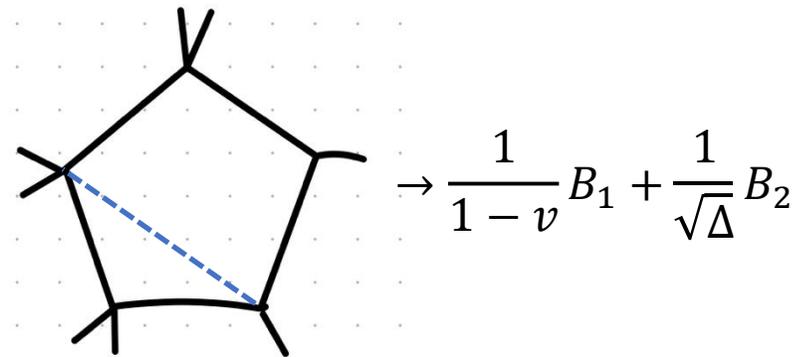
$$G(z) = \frac{(1 + |z|^2 + |1 - z|^2)}{2|z|^2|1 - z|^2}(1 + \zeta_2) + \frac{(-1 + |z|^2 + |z|^4 - |z|^6 - |1 - z|^4 - |z|^2|1 - z|^4 + 2|1 - z|^6)}{2|z|^2|1 - z|^2(z - \bar{z})^2}\log|1 - z|^2$$

$$+ \frac{(-1 - |z|^4 + 2|z|^6 + |1 - z|^2 - |z|^4|1 - z|^2 + |1 - z|^4 - |1 - z|^6)}{2|z|^2|1 - z|^2(z - \bar{z})^2}\log|z|^2$$

$$+ \frac{|z|^4 - 1}{2|z|^2|1 - z|^4}D_2^+(z) + \frac{|1 - z|^4 - 1}{2|z|^4|1 - z|^2}D_2^+(1 - z) + \frac{(|z|^2 - |1 - z|^2)(|z|^2 + |1 - z|^2)}{2|z|^2|1 - z|^2}D_2^+\left(\frac{z}{z - 1}\right)$$

$$+ \frac{2iD_2^-(z)}{2|1 - z|^4|z|^4(z - \bar{z})^3}p_3(|z|^2, |1 - z|^2)$$

$$p_3(|z|^2, |1 - z|^2) = (-1 + |z|^2 - |1 - z|^2)(1 + |z|^2 - |1 - z|^2)(-1 + |z|^2 + |1 - z|^2)$$

$$\times \Big[(-1 + |1 - z|^2)^2|1 - z|^2 + |z|^6(1 + |1 - z|^2)$$

$$- 2|z|^4(1 + |1 - z|^2)^2 + |z|^2(1 + |1 - z|^2)(1 + (-5 + |1 - z|^2)|1 - z|^2)\Big].$$

$$\hat{\partial}_u[-\hat{\partial}_v + v(1 + \hat{\partial}_u + \hat{\partial}_v)] \circ \int du \circ$$
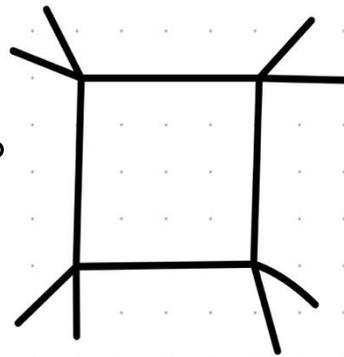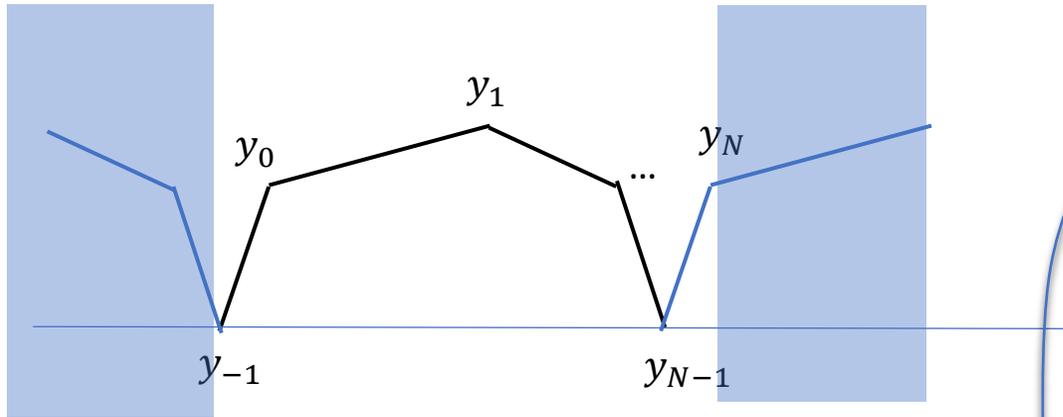

+ perms.

Moving onto Higher Point EC

The multi-collinear limit of $|F_{N+1}|^2$ can be written in terms of dual conformal invariants including $N + 2$ points: $[y_{-1}, \cdots, y_N]$

$$p_i = y_i - y_{i-1}, \ p_{i+N+1} := p_i, \ s_{i+1,\ldots,k} = y_{ik}^2$$

In the $N-$particle collinear limit,

$$y_{iN}^2 \to x_{i+1} + \cdots + x_{N-1}$$

$$y_{-1i}^2 \to x_1 + \cdots + x_i$$

$1 \to N$ splitting function with $N = 3,4 \ $ (NMHV/MHV): $\qquad (a,b,c,d) := \dfrac{y_{ab}^2 y_{cd}^2}{y_{ac}^2 y_{bd}^2}$

$$N = 3: \quad (-1,1,2,4) + (-1,3,2,0) + (3,1,0,4)$$

$$N = 4:$$

$$-1 + (-1,1,2,5) + (-1,2,3,5) + (-1,4,3,0) + (4,1,0,5) + (-1,3,2,0) + (4,2,1,5)$$
$$+ (0,4,3,1) + (0,4,3,1)(-1,1,3,5) + (-1,4,3,1)(3,1,0,5)$$
$$+ (-1,4,2,0)(0,2,3,5) + (-1,1,2,4)(4,2,0,5) + (-1,3,2,0)(4,2,0,5) + (-1,4,2,0)(4,2,1,5)$$
$$+ (-1,4,3,0)(-1,1,2,4) + (4,1,0,5)(0,2,3,5) + (-1,4,3,1)(-1,4,2,0) + (3,1,0,5)(4,2,0,5)$$
$$+ (-1,4,3,1)(-1,1,2,5) + (3,1,0,5)(-1,2,3,5) + (-1,1,2,4)(-1,1,3,5) + (0,2,3,5)(-1,1,3,5)$$

[Chicherin, Moult, Sokatchev, Yan, Zhu 24']

Four − point correlator

$$G_4 := \frac{|F_5|^2}{|F_{5,MHV}|^2}$$

$$s_{I\dots J} := \sum_{(i,j)\in[I,J]} x_i x_j |z_{ij}|^2$$
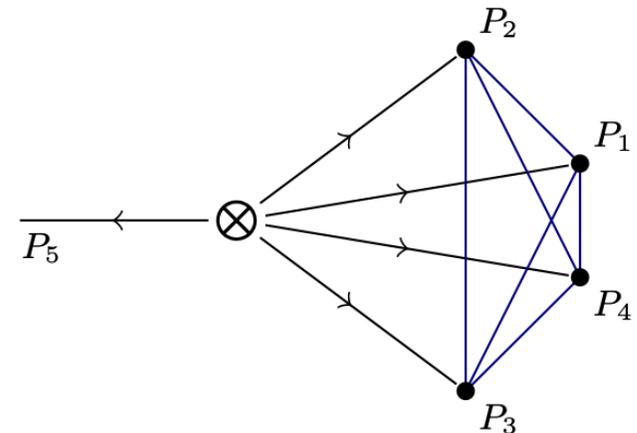
$$(-1,4,3,0) = \frac{s_{123}\, x_{1234}}{s_{1234}\, x_{123}}$$

$$(-1,4,2,0)(0,2,3,5) = \frac{s_{\overline{12}}\, x_{1234}x_4}{s_{1234}s_{123}\, x_{12}x_{34}}$$

$$(-1,4,3,0)(-1,1,2,4) = \frac{s_{123}s_{34}\, x_{1234}x_1}{s_{1234}s_{234}\, x_{12}x_{123}}$$

$$(0,4,3,1)(-1,1,3,5) = \frac{s_{1234}s_{23}}{s_{123}s_{234}}\frac{x_1 x_4}{x_{123}x_{234}}$$
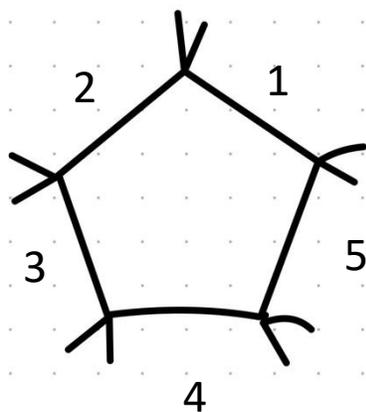
Singularities depend on 5 angles : five mass chiral pentagon as matching kernel
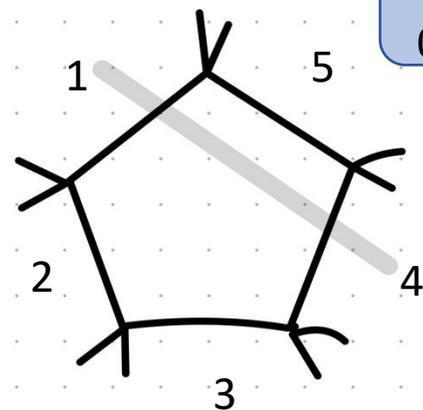
Singularities depend on 4 angles: impose $P_{14} = 0$ in the matching kernel

Rules for Mellin transform

$$\int [d^N x] \frac{x_1^{a_1} \cdots x_N^{a_N}}{D_1 D_2 \cdots D_k \, L_1^{a_{N+1}-m+2} L_2 \cdots L_m}$$

$$L_1 := x_{123\cdots N}$$

$$\int [d^{N+1}x][dt][ds] \, x_1^{a_1} \cdots x_{N+1}^{a_{N+1}} \exp[-t_1 D_1 - \cdots t_k D_k - x_{N+1} L_1 - \cdots s_m x_{N+1} L_m]$$

exponentiate the (extended) quadratic form

$$\oint [d\delta] \prod \Gamma(\delta_{ij}) P_{ij}^{-\delta_{ij}} \int [dx] \, x_1^{a_1} \cdots x_{N+1}^{a_{N+1}} \prod (x_i x_j)^{-\delta_{ij}} \left| \int [dt][ds][f(s,t,\delta_{ij})] \right.$$

Star formula for $D_{a_1+1,\cdots,a_{N+1}+1}$

Euler-type integrals
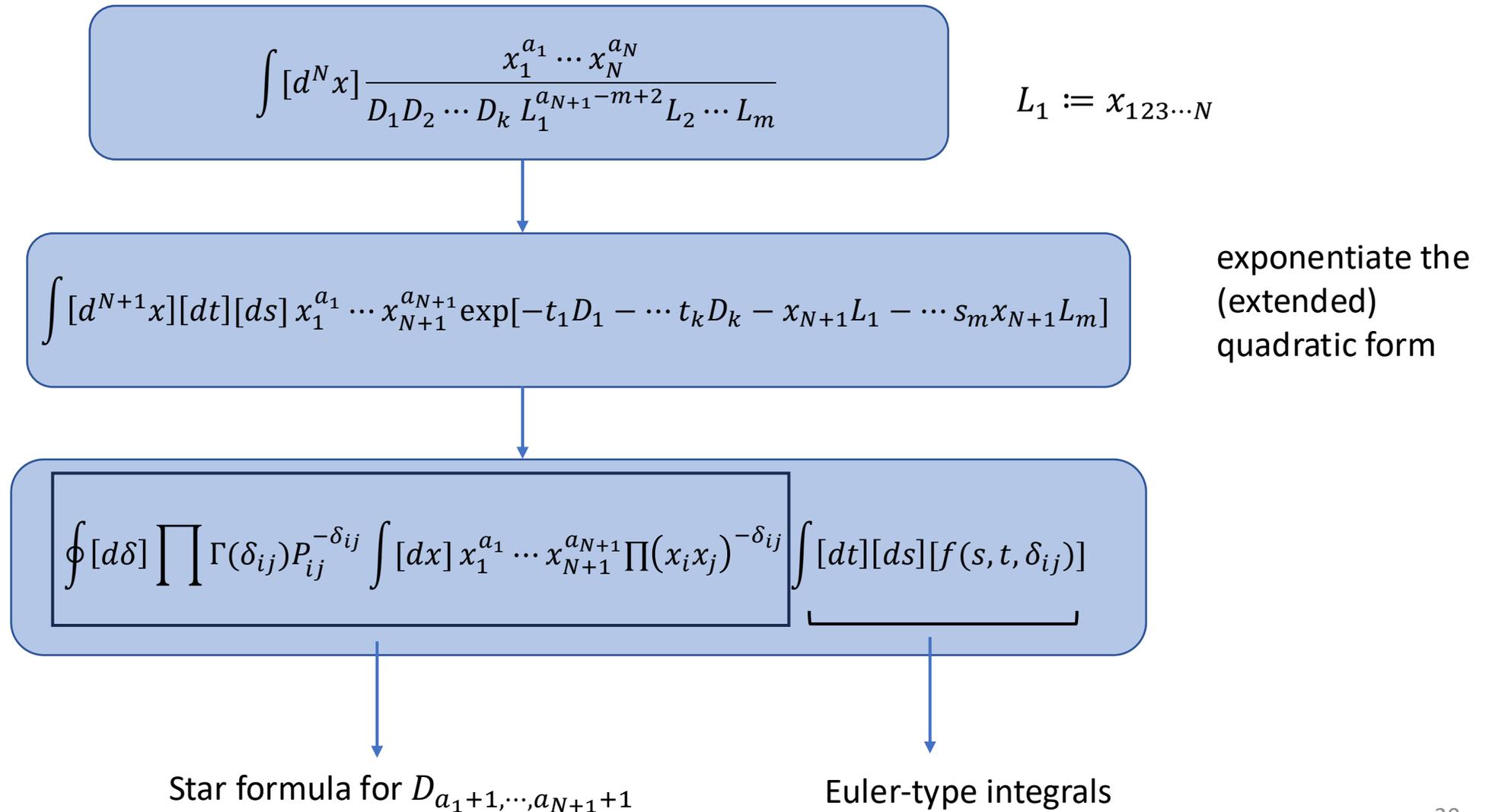
Resolving overlapping singularities

$$\delta_{\sigma_I} := \sum_{(i,j)\in\sigma_I} \delta_{ij},$$

$$\sigma_I := \{(i,j)|\, P_{ij}\ appears\ in\ D_I, but\ not\ in\ others\}$$

Euler-type integral for a k-propagator topology:

$$\int_0^\infty [d^k t]\, t_1^{-\delta_{\sigma_1}} t_2^{-\delta_{\sigma_2}} \cdots (t_1 + t_2)^{-\delta_{\sigma_{12}}} \cdots (t_1 + t_2 + t_3)^{-\delta_{\sigma_{123}}} \cdots$$

compatible poles: $\dfrac{1}{s_{123\cdots N}\, s_{23\cdots N}\cdots}, \dfrac{1}{x_{123\cdots N}\, x_{23\cdots N}\cdots}$

$$t_1 < t_1 + t_2 < t_1 + t_2 + t_3 \cdots$$

$$\delta\left(\sum_\sigma \delta_\sigma - k\right) \times \frac{1}{\delta_{\sigma_{12}} - n_{12}} \times \frac{1}{\delta_{\sigma_{123}} - n_{123}} \times \cdots$$

non-compatible poles, e.g. $\dfrac{1}{s_{123}\, s_{234}}$:

$$\int [dt_1 dt_2]\, t_1^{-\delta_{12}-\delta_{13}} t_2^{-\delta_{24}-\delta_{34}} (t_1 + t_2)^{-\delta_{23}} = \delta\left(\sum_{(ij)\neq(14)} \delta_{ij} - 2\right) \frac{\Gamma(1 - \delta_{12} - \delta_{13})\Gamma(1 - \delta_{24} - \delta_{34})}{\Gamma(\delta_{23})}$$

Mapping onto Feynman Integrals

Matching Kernel: N+1 -point Star Integral (in even dimension)

Search for simpler integrals (to map onto) (e.g. higher-point star integrals with restricted kinematics)

Generalized Euler Operator:
Differential Operator $\hat{\partial}_i := u_i \partial_{u_i}$
Integral Operator $\left(\hat{\partial}_i + n_i\right)^{-1} \rightarrow \int dt\, t^{n_i-1}\, I_{N+1}(u_i t)$

$$\frac{s_{234}\, x_{1234}}{s_{1234}\, x_{234}} + \frac{s_{12}s_{234}\, x_{1234}x_4}{s_{123}s_{1234}\, x_{234}x_{34}} + \frac{s_{23}s_{34}\, x_{1234}^2}{s_{123}s_{1234}\, x_{234}x_{34}}$$
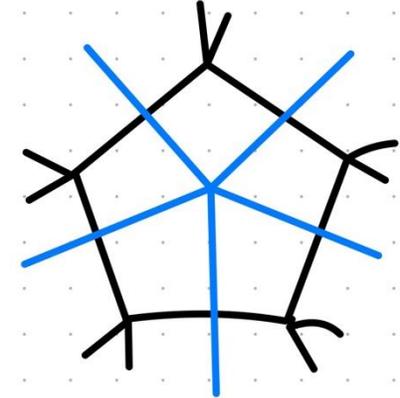
$$+ \frac{s_{12}^2\, x_{1234}x_4}{s_{123}s_{1234}\, x_{12}x_{34}} + \frac{s_{12}s_{34}\, x_{123}x_{1234}}{s_{123}s_{1234}\, x_{12}x_{34}} + (1 \leftrightarrow 4, 2 \leftrightarrow 3)$$

M

$$c_5 := \delta_{12}, \qquad c_2 := \delta_{34},$$
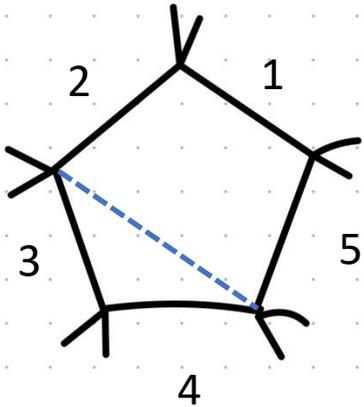$$c_3 := \delta_{123}, \qquad c_4 := \delta_{234},$$
$$c_1 := \delta_{23}$$

$$\frac{1}{-\mathsf{c[1]} + \mathsf{c[3]} + \mathsf{c[4]}}$$

$$\left(\frac{\mathsf{c[3]}\,\mathsf{c[4]}^2}{2\,(-2+\mathsf{c[4]})} - \frac{\mathsf{c[1]} \times \mathsf{c[2]} \times \mathsf{c[4]}}{(-1+\mathsf{c[4]})\,(-1+\mathsf{c[2]}-\mathsf{c[5]})} - \frac{\mathsf{c[4]}\,(-\mathsf{c[1]}+\mathsf{c[4]}+\mathsf{c[3]} \times \mathsf{c[4]})\,\mathsf{c[5]}}{2\,(-2+\mathsf{c[4]})\,(-2+\mathsf{c[2]}-\mathsf{c[5]})} + \frac{\mathsf{c[2]} \times \mathsf{c[4]} \times \mathsf{c[5]}}{2\,(-1+\mathsf{c[2]}-\mathsf{c[5]})} - \right.$$

$$\left. \frac{\mathsf{c[2]} \times \mathsf{c[4]}\,(-1+\mathsf{c[2]}+\mathsf{c[3]}-\mathsf{c[5]})\,\mathsf{c[5]}}{2\,(-2+\mathsf{c[2]}-\mathsf{c[5]})\,(1+\mathsf{c[2]}-\mathsf{c[5]})} + \frac{\mathsf{c[3]} \times \mathsf{c[4]}\,\mathsf{c[5]}^2}{2\,(-2+\mathsf{c[2]}-\mathsf{c[5]})\,(1+\mathsf{c[2]}-\mathsf{c[5]})} \right)$$

$+(\,\mathsf{c[3]} \leftrightarrow \mathsf{c[4]}\,,\,\mathsf{c[2]} \leftrightarrow \mathsf{c[5]})$

$$c_3 + c_4 - c_1 = \delta_{14} - 1, \qquad c_5 - c_2 = \delta_{35} + \delta_{45} - 1,$$

Amplitude side: matching kernels

Five-mass chiral pentagon in 4d
$$I_n^1 = \frac{1}{\pi^2} \int \mathrm{d}^4 Q \, (-Q \cdot Y) \prod_{i=1}^{n} \frac{1}{(-P_i \cdot Q)} \qquad Y \cdot P_{1,2,3,4} = 0$$

$$I_5 = \frac{(-Y \cdot P_5)}{P_{25} \, P_{14} \, P_{35}} \oint \prod_{i=1}^{5} \left(\frac{\mathrm{d}c_i}{(2\pi i} u_i^{c_i-1} \Gamma(1 - c_i) \right) \Gamma\left(c_1 + c_2 - c_4 - 1 \right)$$

$$\times \, \Gamma\left(c_3 + c_4 - c_1 \right) \Gamma\left(c_2 + c_3 - c_5 \right) \Gamma\left(c_1 + c_5 - c_3 - 1 \right) \Gamma\left(c_4 + c_5 - c_2 \right)$$

$$u_1 = \frac{P_{14} \, P_{23}}{P_{13} \, P_{24}}, \quad u_2 = \frac{P_{25} \, P_{34}}{P_{24} \, P_{35}}, \quad u_3 = \frac{P_{13} \, P_{45}}{P_{14} \, P_{35}}, \quad u_4 = \frac{P_{15} \, P_{24}}{P_{14} \, P_{25}}, \quad u_5 = \frac{P_{12} \, P_{35}}{P_{13} \, P_{25}}$$

$$u_i = u_{i,i+2} = \frac{P_{i,i+3} \, P_{i+1,i+2}}{P_{i,i+2} \, P_{i+1,i+3}}, \qquad i = 1, \ldots, 5.$$
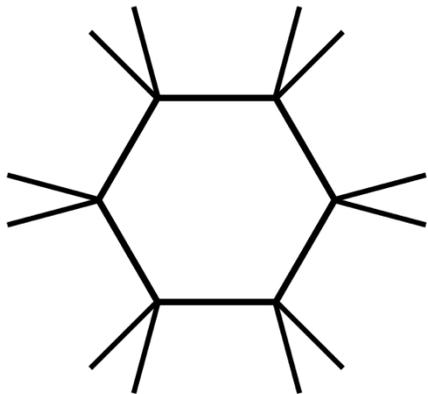
Reduction for a Hexagon with restricted kinematics:

$$P_{16} = P_{56} = 0, \frac{P_{62}}{P_{63}} = \frac{P_{52}}{P_{53}}, \frac{P_{64}}{P_{63}} = \frac{P_{54}}{P_{53}}$$

$$\hat{I}_6 \equiv (P_{14}P_{25}P_{36}) I_6 = \oint \prod_{i=1}^{9} \frac{dc_i}{2\pi i} u_i^{-c_i} \left(\prod_{i=4}^{9} \Gamma(c_i) \right) \Gamma(c_1 - c_5 - c_6) \Gamma(c_2 - c_6 - c_7)$$

$$\times \Gamma(1 - c_1 - c_3 + c_5 + c_8) \Gamma(c_3 - c_7 - c_8) \Gamma(c_1 - c_8 - c_9) \Gamma(1 - c_1 - c_2 + c_6 + c_9)$$

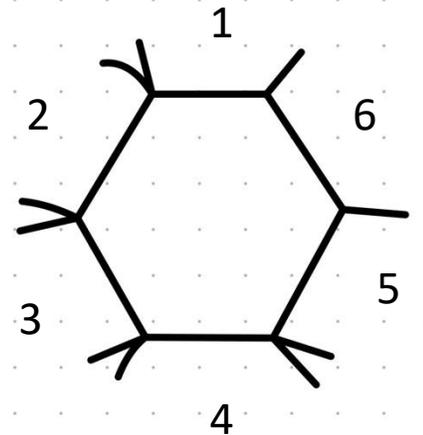$$\times \Gamma(1 - c_2 - c_3 + c_4 + c_7) \Gamma(c_3 - c_4 - c_5) \Gamma(c_2 - c_4 - c_9),$$

$$u_{ij} \equiv \frac{P_{i,j+1} \, P_{i+1,j}}{P_{ij} \, P_{i+1,j+1}}.$$

$$u_i = u_{i,i+3}, \qquad i = 1, 2, 3,$$
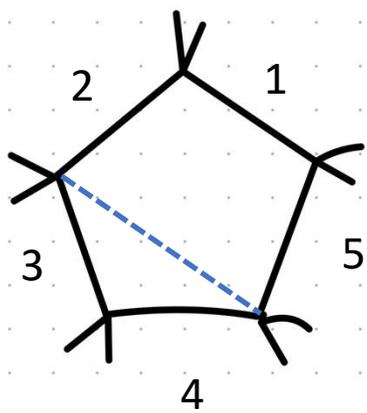
$$u_{i+3} = u_{i+1,i+5}, \qquad i = 1, \ldots, 6$$

Set $u_7 = u_8 = 0, u_2 = u_9 = 1$
Apply Bannes Lemma to
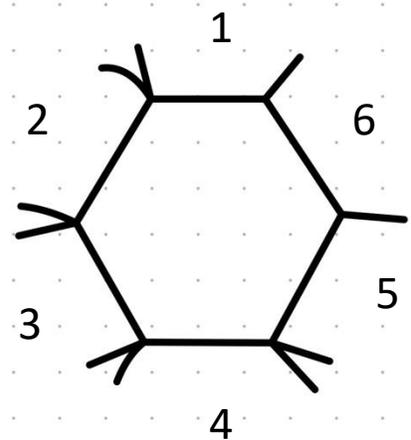integrate out a_2, a_9

$$\xrightarrow{\quad M \quad} \frac{1}{1 - c_4}$$

normalize by I_5^1

$$Hex_A$$
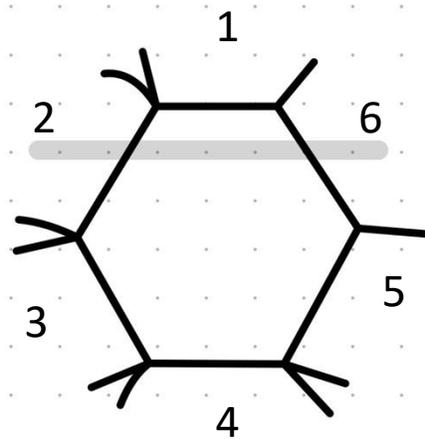
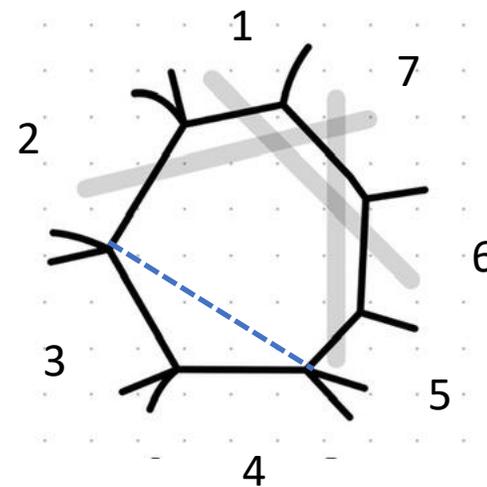$$Hex_B$$

$$M \longrightarrow \frac{1}{1-\delta_{234}}$$

$$M \longrightarrow \frac{1}{\delta_{12}-\delta_{34}}$$

$$M \longrightarrow \frac{\delta_{123}}{(1-\delta_{234})(\delta_{12}-\delta_{34}+1)}$$

$$Hep_A \quad D_{1,1,1,2,1,1,1}$$

Deformed Hexagon

$$\int \frac{dt}{(1+t)} Hex_{A,B}((1+t)u_3)$$

Each contains >3000 terms in its symbol

First 2 entries:

$\{-1+w, \ w, \ -1+wb, \ wb, \ -1+z, \ z, \ -w+z, \ -1+zb, \ zb, \ -wb+zb, \ \boxed{-1+z\,zb}\}$

3rd entry\first 2 :

$\{z-zb, \ -wb\,z+w\,zb, \ \boxed{-wb\,z+w\,zb, \ -wb+w\,zb, \ -w+wb\,z,}$

$\boxed{1-w-wb+wb\,z, \ 1-w-wb+w\,z, \ -1+w+wb-wb\,z-w\,zb+z\,zb\}}$

Last entry\first 3 :

$z \leftrightarrow w, \ \bar{z} \leftrightarrow \bar{w}$

$\{1-w\,wb, \ 1-wb\,z, \ wb\,z-zb, \ 1-w\,zb, \ z-w\,zb,$

$1-z-zb+w\,zb, \ 1-z+wb\,z-zb, \ 1-w\,wb-z+wb\,z-zb+w\,zb\}$

$z_1 = 0$

$z_4 = \frac{z-w}{1-w}$

$z_2 = z$

$z_3 = 1$
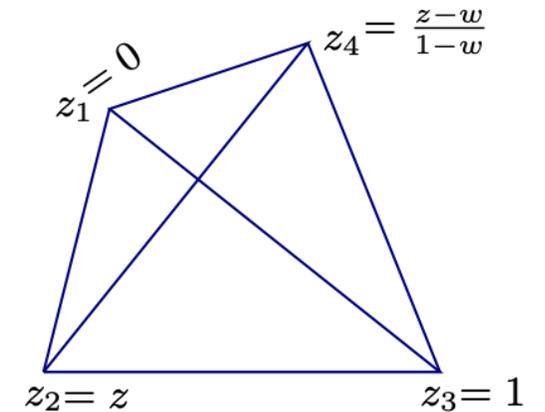
Summary

- Systematic studies on deformed star integrals

--extra structures in the symbol alphabets,

-- universal patterns independent of supersymmetry?

- Interpretation on the Mellin amplitudes

---Physical v.s. spurious poles in Mellin space; connection to symbol letters

---Last entry: physical origin of singularities in the LS

- How to build differential equations systematically (without/with minimum IBP)?

- Higher loop @ Lower point : generalize with dimensional regularization