Kerr Black Hole Dynamics from an Extended Polyakov Action

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Based on 2503.20538 (N. Emil. J. Bjerrum-Bohr, Gang Chen, Chenliang Su, **TW**) and work in progress

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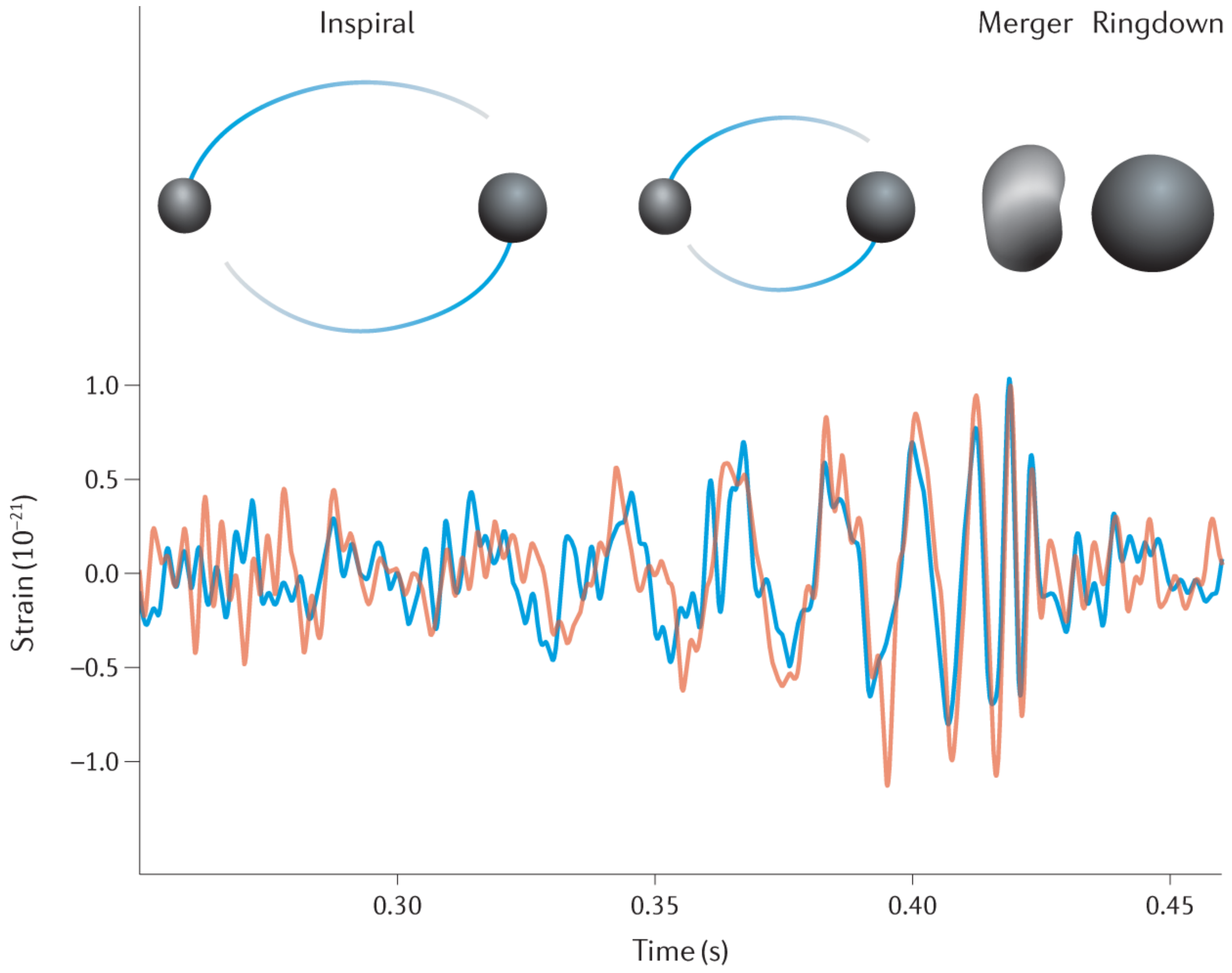
GW Physics Meets QFT/Amplitudes

- Gravitational waves from binary systems of black holes/neutron stars detected by LIGO/Virgo, upcoming third-generation detectors (LiSA/KAGRA/ET/TianQin/…)



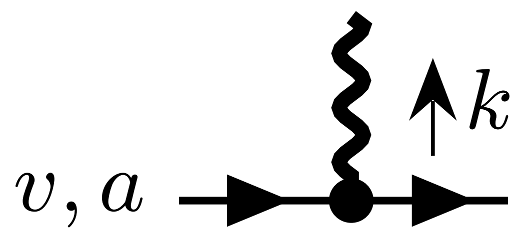
[Image credit: Nature Review Physics]

- Post-Newtonian-based methods for computing the dynamics in the inspiral phase: numerical relativity, EOB, NRGR, worldline EFT
  [Abbott et al, Buonanno, Damour, Goldberger, Rothstein, Kol, Smolkin, Gilmore, Ross, Foffa, Sturani, Mastrolia, Sturm, Porto, Blümlein, Maier, Marquard, Schäfer, …]

- Post-Minkowskian-based/“Amplitude-flavoured”:
  - Scattering amplitudes in the classical (soft) limit (EFT matching, heavy-mass EFT, Kosower-Maybee-O’Connell formalism, etc…)
    [Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove, Bern, Cheung, Rothstein, Solon, Roiban, Shen, Zeng, Luna, Parra-Martinez, Brandhuber, Chen, Travaglini, Wen, Aoude, Haddad, Heissenberg, Helset, Chiodaroli, Johansson, Pichini, ….]
  - Worldline models (effective action, WQFT)
    [Goldberger, Rothstein, Porto, Levi, Steinhoff, Jakobssen, Mogull, Plefka, Steinhoff, Ben-Maoer, Sauer, Shi, Caomberati, Gonzo, Driesse, Klemm, Nega, Usovitsch]

Classical Gravitational "Compton" Amplitudes

- Three-point (one graviton) amplitude $\kappa = \sqrt{32\pi G_N}$



$$= -i\kappa(mv\cdot\varepsilon)\left(mv\cdot\varepsilon\cosh(k\cdot a) + ik\cdot S\cdot\varepsilon\frac{\sinh(k\cdot a)}{k\cdot a}\right)$$

  - Originally obtained from symmetry requirements

    [Arkani-Hamed, Huang, Huang, Vines, Guevara, Ochirov, Chung, Kim, Lee, Chen]

- Four-point (two graviton) amplitude

  - AHH amplitudes have spurious poles at $s \geq 2$ in the opposite-helicity configurations, related to ambiguities in the contact terms in the classical limit
  - Various proposals based on fundamental symmetry requirements and their respective assumptions of choice (spin-shift symmetry, double copy, higher-spin, etc)

    [Aoude, Haddad, Helset] [Bern, Kosmopoulos, Luna, Roiban, Scheopner, Teng] [Menezes, Sergola] [Febres Cordero, Kraus, Ruf, Zeng] [De Angelis, Gonzo, Novichkov] [Bjerrum-Bohr, Chen, Skowronek] [Cangemi, Chiodaroli, Johansson, Ochirov, Pichini, Skvortsov]

  - All agree up to the quartic power in spin and differ at higher orders in contact contributions

    [Bautista, Guevara, Kavanagh, Vines]

  - Matching with the Teukovsky equation approach involves subtleties in the treatment of the bookkeeping parameters, requiring further clarification

Worldline Models for Kerr

- Black hole as a point particle coupled to background gravity
  - Worldline effective field theory
    [Goldberger, Rothstein, Porto, Levi, Steinhoff]

"Minimal coupling" (up to linear order in spin)

$$S = \int d\sigma \left\{ -m\sqrt{u^2} - \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + L_{\rm SI}\left[u^\mu, S_{\mu\nu}, g_{\mu\nu}(y^\mu)\right] \right\}$$

Spin-induced multipole expansion accounting for finite-size effects

$$L_{\rm SI} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{ES}^{2n}}{m^{2n-1}} D_{\mu_{2n}} \cdots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} \cdots S^{\mu_{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{BS}^{2n+1}}{m^{2n}} D_{\mu_{2n+1}} \cdots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} \cdots S^{\mu_{2n}}$$

Electric and magnetic components of the Weyl curvature tensor

Wilson coefficients (for Kerr, C=1 for all Cs)

  - Worldline QFT (N=2 SUSY model accounts for up to quadratic orders in spin, an ansatz-based Lagrangian for higher orders)
    [Jakobssen, Mogull, Plefka, Steinhoff, Ben-Maoer, Sauer, Shi, Caomberati, Gonzo, Haddad, Xu]
  - Worldline action with a quadratic-in-Riemann operator matching with higher spin constructions
    [Maor, Cangemi, Johansson]

Geometry of Kerr

- Key features of the Kerr geometry:

  [Israel]

  - An equatorial disk centred around the axis of symmetry
  - Intrinsically flat and of a radius |a|
  - The ringlike boundary of the disk comprises the geometrical singularity of the metric

- 3-metric of Kerr at (r=0) in terms of intrinsic coordinates:

$$g_{ab}\xi^a\xi^b = a^2\cos^2\zeta d\zeta^2 + a^2\sin^2\zeta d\phi^2 - dt^2$$

What about modelling Kerr, which is an extended body, as…

Well… an extended body?

Outline

Hypersurface model in flat background

"Minimal" generalisations to curved background

Matching with the three-point amplitude for Kerr results in a simple hypersurface action

Singularity structures due to the hypersurface action (for our own curiosity mostly)

From hypersurface to gravitational Compton amplitude (WIP)

Outlook (all wishes great and small)

Hypersurface Model in Flat Space

Hypersurface Action

- Starting point: (2+1)-dimensional worldvolume extension of the Polyakov action
  - Diffeomorphism on the worldvolume
  - Diffeomorphism on the target space
- Constraints on the worldvolume metric:
  - τ-independence
  - Respecting rotational symmetries expected for a spinning object that resembles a disc
  - Crucial difference from "stringy models": the necessity of a nontrivial worldvolume metric

Spacetime coordinates

$$C \int_{\mathbb{M}^2 \times \mathbb{R}} d^3\sigma \sqrt{\gamma} \left(\partial_a Z^\mu \partial_b Z^\nu \gamma^{ab} \eta_{\mu\nu} \right)$$

Compact 2d manifold

$\sigma^a = (\tau, \vec{\sigma})$

Worldvolume metric

$$ds^2 = d\tau^2 - a_w^2 \left(d\theta^2 + f(\theta) d\phi^2 \right)$$

With length dimension

$$f(\theta) = \sin^2 \theta$$

Fixed by physical constraints

Free Action & Dynamics in Flat Background

- Physical Constraint: the free equation of motion must admit solutions that match with the weak-field expectations

- Free action

$$S_0 = -\frac{m}{8\pi a_w^2}\int_{\mathbb{S}^2\times\mathbb{R}} d^3\sigma\sqrt{\gamma}\Big(\partial_a Z^\mu \partial_b Z^\nu \gamma^{ab}\eta_{\mu\nu}+1\Big)$$

  - Mass term included for classical physics
  - Consistent with reparameterization invariance of τ
  - Normalisation to match with the point-mass (worldline) action in the limit $a_w \to 0$

- Free EOM: $\Box Z^\mu = \gamma^{ab}\Big(\partial_a\partial_b Z^\mu - \Gamma^c_{ab}(\partial_c Z^\mu)\Big) = 0$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot(\theta) \qquad \Gamma^\theta_{\phi\phi} = -\sin(\theta)\cos(\theta)$$

- Solutions: $Z^\mu = X^\mu + Y^\mu$

$$X^\mu = x^\mu + v^\mu\tau$$

Leading-order (straight line) trajectory of the COM

$$Y^\mu = \sum_{l>j\geq 0} Y^\mu_{l,j}$$

Relative coordinates

$$Y^\mu_{l,j} = a_w\Big(c_{l,j}\,\beta^\mu_{\rm x}\cos\big(\frac{\sqrt{l(l+1)}}{a_w}\tau + j\phi\big) + c'_{l,j}\,\beta^\mu_{\rm y}\sin\big(\frac{\sqrt{l(l+1)}}{a_w}\tau + j\phi\big)\Big)\mathcal{P}^j_l\big(\cos(\theta)\big)$$

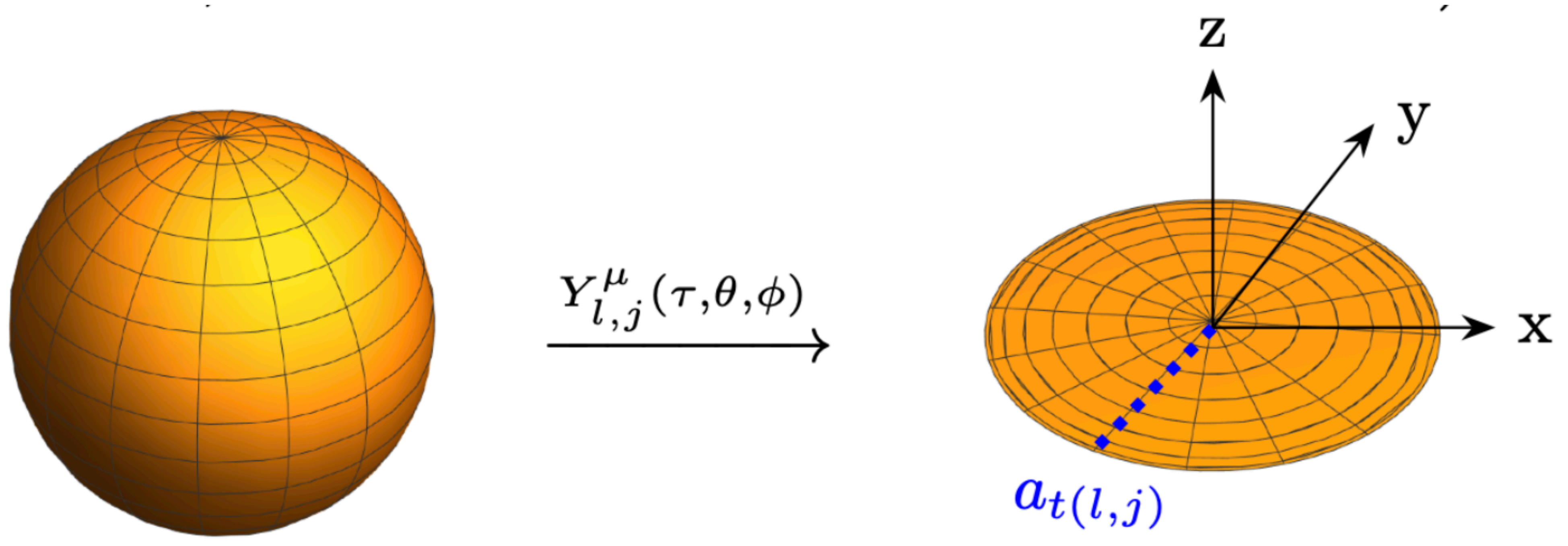
Set to be equal to keep the rotational symmetry

Unit vectors

Harmonic Legendre polynomials

From Sphere to Disc

- The modes $Y^{\mu}_{l,j}$ can be interpreted as maps from the worldvolume of $\mathbb{S}^2 \times \mathbb{R}$ to solutions on a spinning disc in the target space of the radius $a_{t(l,j)} \equiv a_w c_{l,j}$



~~$Y^{\mu}_{l,0}$: only depend (τ, θ), hence dimension-reducing~~

$Y^{\mu}_{1,1}$: single-covering map

$Y^{\mu}_{l,j}$ $(l > 1 \text{ or } j > 1)$: multi-covering map

  - Consistent with the starting point that the source of a Kerr black hole is a spinning disc

- For our purpose of modelling the Kerr black hole classically coupled to weak background gravity, our main focus from now on will be the (1,1)-mode

  - Geometrically, it is the closest to the disc

  - We believe the existence of multi-modes with different frequencies or wrapping numbers is unnatural in classical physics

Classical Spin

- Angular momentum due to the rotation around the COM

$$S^{\mu\nu} = \frac{m a_w^2}{4\pi a_w^2} \int_{\mathbb{S}^2} d\theta\, d\phi\, \sin(\theta)\, (Y^\mu_{l,j} \partial_\tau Y^\nu_{l,j} - Y^\nu_{l,j} \partial_\tau Y^\mu_{l,j}) = \frac{\sqrt{l(l+1)}(l+j)!(m a_w)}{(2l+1)(l-j)!} c^2_{l,j} (\beta^\mu_{\mathrm{x}} \beta^\nu_{\mathrm{y}} - \beta^\mu_{\mathrm{y}} \beta^\nu_{\mathrm{x}})$$

  - Identifying with the standard spin tensor $S^{\mu\nu} = m\epsilon^{\mu\nu\rho\lambda} v_\rho a_\lambda$ and imposing the spin supplementary condition $v_\mu S^{\mu\nu} = 0$, we are required to set $v \cdot \beta_{\mathrm{x}} = v \cdot \beta_{\mathrm{y}} = 0$

  - Without loss of generality, we also set $\beta_{\mathrm{x}} \cdot \beta_{\mathrm{y}} = 0$

  - Hence we have $S^{\mu\nu} = m|a|(\beta^\mu_{\mathrm{x}} \beta^\nu_{\mathrm{y}} - \beta^\mu_{\mathrm{y}} \beta^\nu_{\mathrm{x}})$ $\qquad v^\mu v^\mu - \eta^{\mu\nu} - \frac{a^\mu a^\nu}{|a|^2} = \beta^\mu_{\mathrm{x}} \beta^\nu_{\mathrm{x}} + \beta^\mu_{\mathrm{y}} \beta^\nu_{\mathrm{y}}$

  - Spin length related to the radius of the sphere

$$|a| = \frac{\sqrt{l(l+1)}(l+j)!}{(2l+1)(l-j)!} a_w c^2_{l,j}$$

Hypersurface Model in Curved Space

Generalisation to Curved Background

- Immediate generalisation of the Polyakov action: $\eta_{\mu\nu} \rightarrow \mathcal{G}_{\mu\nu}(Z)$ ← Background metric
- In principle, one can add any covariant couplings that respect the fundamental symmetries of the system. But we choose to stick to the "minimal" action. Hence, we make further restrictions.
  - Exclude potential terms that spoil the free EOM $\left[\partial_a Z^\mu \partial_b Z^\nu \gamma^{ab} \mathcal{G}_{\mu\nu}(Z)\right]^n, \quad n \geqslant 2$
  - Exclude explicit polynomials of Z to avoid couplings to the field X and only keep first-order derivatives of Z.
  - Each point Z couples to the background metric in a simple way, either directly or to the local Riemann tensor $\mathcal{R}_{\mu\nu\rho\sigma}(Z)$. Higher-dimensional interactions are also excluded.
  - Worldvolume geometry can come into play through the worldvolume metric or its respective Riemann tensor ϱ^{abcd} whose non-vanishing components are

$$\varrho^{\theta\phi\theta\phi} = -\varrho^{\phi\theta\theta\phi} = -\varrho^{\theta\phi\phi\theta} = \varrho^{\phi\theta\phi\theta} = \frac{-1}{a_w^6 \sin^2(\theta)}$$

Action in Curved Background

- Ansatz for the action

$$(\mathcal{D}Z)^2 \equiv \partial_a Z^\mu \partial_b Z^\nu \gamma^{ab} \mathcal{G}_{\mu\nu}(Z)$$

$$S = -\frac{m}{8\pi a_w^2} \int_{\mathbb{S}^2\times\mathbb{R}} d^3\sigma \sqrt{\gamma} \Big[(\mathcal{D}Z)^2 + 1 + a_w^2 \mathcal{R}_{\mu\nu\rho\lambda}(Z) \partial_a Z^\mu \partial_b Z^\rho \partial_c Z^\nu \partial_d Z^\lambda$$
$$\times \Big(\gamma^{ab}\gamma^{cd} (\sum_{j=0}^{\infty} \xi_{2j+1} [(\mathcal{D}Z)^2]^j) + a_w^2 \varrho^{acbd} (\sum_{j=0}^{\infty} \xi_{2j+2} [(\mathcal{D}Z)^2]^j) \Big) \Big]$$

Free parameters to be determined

  - Respect both the worldvolume and the target space diffeomorphism

- Perturbation around the flat metric (post-Minkowskian expansion):

$$\mathcal{G}_{\mu\nu}(Z) = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}(Z) \qquad \kappa = \sqrt{32\pi G_N}$$

From Action to Three-Point Amplitude

- Classical three-point one graviton amplitude to the leading PM order can be directly read off from the action

$$\frac{ie^{ik\cdot x}}{(2\pi)^4} 2\pi\delta(mv\cdot k) A_3(v,a,k) = -2h_{\mu\nu}(k)\frac{\delta S}{\delta h_{\mu\nu}(k)}\Big|_{h_{\mu\nu}\to 0}$$

Integrating out the worldvolume coordinates

  - Evaluating the RHS in momentum space using the Fourier transform

$$h_{\mu\nu}(Z) = \int \frac{d^4k}{(2\pi)^4} h_{\mu\nu}(k) e^{ik\cdot X(\tau)} e^{ik\cdot Y(\sigma)}$$

  - Outgoing graviton is taken to be a plane wave and without loss of generality, we assume the polarisation tensor of the graviton is a product $\varepsilon_\mu(k)\,\varepsilon_\nu(k)$

  - For a given single mode, the three-point amplitude is obtained by substituting the (classical) solution of the free EOM

$$A_3^{(l,j)}(v,a,k) \equiv A_3(v,a,k)|_{Y^\mu \to Y^\mu_{l,j}}$$

For Kerr, we consider the amplitude evaluated on the (1,1) mode.

Worldvolume Integration

- Spin counting:
  - Each factor of Y from the exponential contributes one spin power, while the first-derivative of Z does not contribute. The parameter a_w is proportional to the spin length.
  - Interactions involving the curvature tensor come in at quadratic and higher orders in spin.
  - The zeroth and linear powers in spin are determined by the "Polyakov" part of the action and hence universal, as expected from the minimal-coupling worldline action.
- ϕ-integrals vanish for odd powers of Y (including its derivatives) and are easily evaluated for even powers, independent of τ, which trivialises the. τ-integral.
- Tensor prefactors of the form $\partial_{a_1} Y^{\mu_1} \cdots \partial_{a_n} Y^{\mu_n} Y^{\nu_1} \cdots Y^{\nu_m}$ can be divided into a sequence of pairs and each pair is identified using

$$[\partial_\tau Y^\mu Y^\nu] \to [S^{\mu\nu}] \qquad [Y^\mu Y^\nu], \quad [\partial_b Y^\mu \partial_b Y^\nu] \to \left[v^\mu v^\mu - \eta^{\mu\nu} - \frac{a^\mu a^\nu}{|a|^2} \right]$$

Worldvolume Integration

- Remaining θ-integrals evaluated on the (1,1)-mode (setting $c_{1,1} = 3/(2\sqrt{2})$ such that the radius of the target space disc is the same as the Kerr disc/the spin length)

	even	odd
$\mathcal{I}_P^{(1,1)e/o}$	$\left(\frac{17}{8} - \frac{27}{8k\cdot a}\partial_{k\cdot a}\right)\frac{\sinh(k\cdot a)}{k\cdot a}$	$\frac{3}{k\cdot a}\partial_{k\cdot a}\frac{\sinh(k\cdot a)}{k\cdot a}$
$\mathcal{I}_{\xi_1}^{(1,1)e/o}$	$\frac{25}{8}\cosh(k\cdot a) - \left(\frac{79}{8} + \frac{81}{4k\cdot a}\partial_{k\cdot a}\right)\frac{\sinh(k\cdot a)}{k\cdot a}$	$\left(3 - \frac{9}{k\cdot a}\partial_{k\cdot a}\right)\frac{\sinh(k\cdot a)}{k\cdot a}$
$\mathcal{I}_{\xi_2}^{(1,1)e/o}$	$\frac{9}{4}\cosh(k\cdot a) - \frac{9}{4}\frac{\sinh(k\cdot a)}{k\cdot a}$	0
$\mathcal{I}_{\xi_4}^{(1,1)e/o}$	$-\frac{9}{32}\cosh(k\cdot a) + \left(\frac{369}{16} - \frac{2187}{32k\cdot a}\partial_{k\cdot a}\right)\frac{\sinh(k\cdot a)}{k\cdot a}$	0

- Rewriting the exponential factor as a differential operator, all the remaining interactions can be evaluated as the action of the differential operator on the first two terms.

Hypersurface Action for Kerr

- Matching with the three-point amplitude of the Kerr black hole, we find

$$\xi_1 = \frac{1}{3}\,, \quad \xi_2 = -\frac{1}{3^4}\,, \quad \xi_4 = \frac{4}{3^4}\,, \quad \xi_i = 0\,, \;\; \text{for } i = 3 \text{ or } i \geqslant 4$$

- Action for Kerr

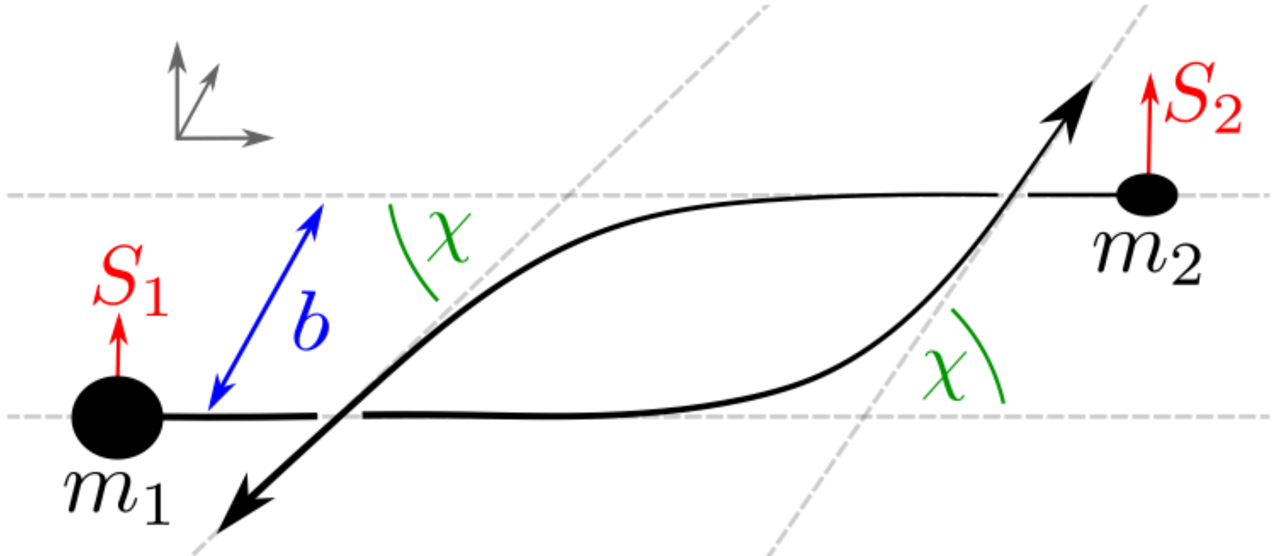
$$S = -\frac{m}{8\pi a_w^2}\int_{\mathbb{S}_2\times\mathbb{R}} d^3\sigma\sqrt{\gamma}\Bigg[(\mathcal{D}Z)^2 + 1 + a_w^2 \mathcal{R}_{\mu\nu\rho\lambda}(Z)\partial_a Z^\mu \partial_b Z^\rho \partial_c Z^\nu \partial_d Z^\lambda \\ \times\left(\frac{1}{3}\gamma^{ab}\gamma^{cd} - \frac{1}{3^4}a_w^2\varrho^{acbd} + \frac{4}{3^4}a_w^2\varrho^{acbd}(\mathcal{D}Z)^2\right)\Bigg]$$

- Completeness of the full action: we have tested up to $\mathcal{O}(|a|^{99})$ and confirmed that the spin expansions of $\mathcal{I}_{\xi_i}$'s form a complete basis of one-variable entire functions. Hence the action can describe any three-point amplitude, provided that it admits a regular spin expansion of $(k\cdot a)$ only.

Singularity Structures (for fun)

Singularity Structure of Scattering Angle

- Scattering angle of a binary system with a spinless black hole



[Image credit: Antonelli, Kavanagh, Khalil, Steinhoff, Vines]

$$A_3^{(1,1)}(v,a,k) = (-i\kappa)\sum_{i=1}^{5} c_i B_i$$

$$\chi_{1\mathrm{PM}}^{(1,1)}(\xi_1,\xi_2,\xi_4) = \sum_{i=1}^{5} c_i \chi_{1\mathrm{PM}}^{(1,1)}(B_i)$$

reorganised by a basis of entire functions

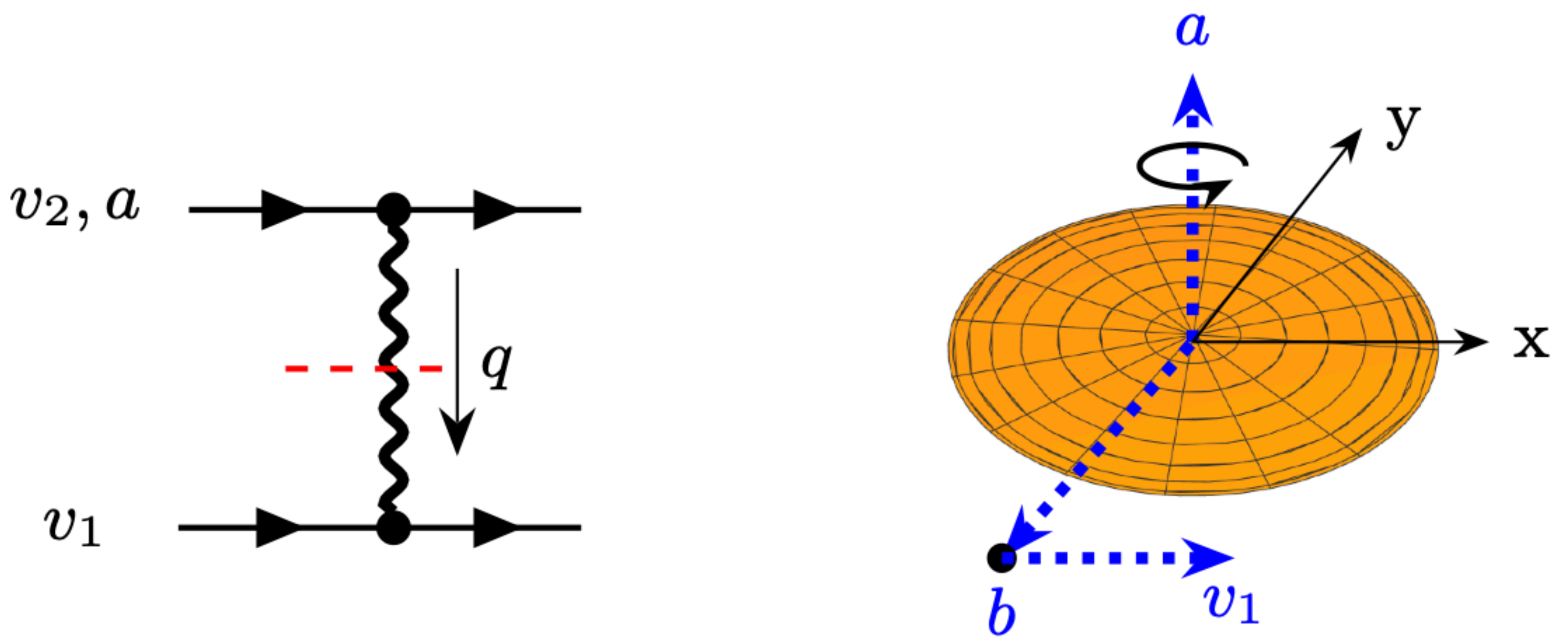


FIG. 2: The related vectors are $v_2^\mu = (1,0,0,0)$, $v_1^\mu = (y, \sqrt{y^2-1}, 0, 0)$, $b^\mu = (0,0,-|b|,0)$, $a^\mu = (0,0,0,|a|)$.

$$B_1 = m^2 (v\cdot\varepsilon)^2 \cosh(k\cdot a)$$

$$B_2 = m^2 (v\cdot\varepsilon)^2 \frac{\sinh(k\cdot a)}{k\cdot a}$$

$$B_3 = m^2 (v\cdot\varepsilon)^2 \frac{\partial_{k\cdot a}}{k\cdot a}\frac{\sinh(k\cdot a)}{k\cdot a}$$

$$B_4 = i(mv\cdot\varepsilon)(k\cdot S\cdot\varepsilon)\frac{\sinh(k\cdot a)}{k\cdot a}$$

$$B_5 = i(mv\cdot\varepsilon)(k\cdot S\cdot\varepsilon)\frac{\partial_{k\cdot a}}{k\cdot a}\frac{\sinh(k\cdot a)}{k\cdot a}$$

$$c_1 = \frac{25}{8}\xi_1 + \frac{9}{4}\xi_2 - \frac{9}{32}\xi_4$$

$$c_2 = \frac{17}{8} - \frac{79}{8}\xi_1 - \frac{9}{4}\xi_2 + \frac{369}{16}\xi_4$$

$$c_3 = \frac{27}{32}(24\xi_1 - 81\xi_4 - 4)$$

$$c_4 = 3\xi_1$$

$$c_5 = 3 - 9\xi_1$$

Singularity Structure of Scattering Angle

$$\chi^{(1,1)}_{1\mathrm{PM}}(B_1) = C_\chi \frac{(2y^2-1)|b|}{|b|^2-|a|^2}$$

$$\chi^{(1,1)}_{1\mathrm{PM}}(B_2) = C_\chi \frac{(2y^2-1)\,\mathrm{arctanh}(\frac{|a|}{|b|})}{|a|}$$

$$\chi^{(1,1)}_{1\mathrm{PM}}(B_3) = C_\chi \frac{(2y^2-1)(|a||b|+(|a|^2-|b|^2)\,\mathrm{arctanh}(\frac{|a|}{|b|}))}{2|a|^3}$$

$$\chi^{(1,1)}_{1\mathrm{PM}}(B_4) = C_\chi \frac{-2y\sqrt{y^2-1}|a|}{|a|^2-|b|^2}$$

$$\chi^{(1,1)}_{1\mathrm{PM}}(B_5) = C_\chi \frac{2y\sqrt{y^2-1}(|b|\,\mathrm{arctanh}(\frac{|a|}{|b|})-|a|)}{|a|^2}$$

$$C_\chi = \frac{\kappa^2\sqrt{s}}{16\pi\,(y^2-1)}$$

- For Kerr $A^{(1,1)}_{3,\mathrm{Kerr}}(v,a,k) = -i\kappa(B_1+B_4)$

  The scattering angle has only simple poles and agrees with known results in the literature.

- For generic parameters (ξ_1, ξ_2, ξ_4)

  The scattering angle has curious logarithmic divergences.

- At higher orders, the singularities always lie at the same location. Only Kerr is free of logarithmic divergences.

Higher Modes

- The three-point amplitude can be evaluated analytically on the (l,l) and (l,l-1)-modes for all the examples we have considered
  - They are expressed in terms of hypergeometric functions which are also entire functions.
  - The analytical property of the resulting scattering angle for a given higher mode suggests that its respective singular ring lies outside the radius of the disc, if we choose the mode coefficient such that the radius equals the spin length.
- For other higher modes, their three-point amplitudes are unknown analytically, while numerical evaluations suggest that they still converge.

Singularity Structure of Metric

- Metric generated by the presence of the hypersurface and its interaction with gravity
  [Mougiakakos, Vanhove, Damgaard, Lee]
  - Einstein equation

$$R^{\mu\nu} - \frac{R}{2} g^{\mu\nu} = 8\pi G_N T^{\mu\nu} = 8\pi G_N \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}(x)}$$

  - At the first post-Minkowskian order

Directly related to the three-point amplitude

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa^2 \mathfrak{h}_{\mathbf{1}\,\mu\nu}(x) + \mathcal{O}(\kappa^4)\,, \qquad T^{\mu\nu}(x) = T_{\mathbf{0}}^{\mu\nu}(x) + \mathcal{O}(\kappa)$$

Only the leading-order stationary contribution

  - Fourier transform to momentum space

$$\widetilde{\mathfrak{h}}_{\mathbf{1}}^{\mu\nu}(q) = \frac{\widetilde{T}_{\mathbf{0}}^{\mu\nu}(q) - \frac{1}{2}\widetilde{T}_{\mathbf{0}\rho}^{\rho}(q)\eta^{\mu\nu}}{2q^2} = \frac{2\pi\delta(mv\cdot q)}{2q^2} \sum_{\text{spin sum}} \frac{A_3^{(1,1)}(v,a,q)}{-i\kappa} \varepsilon^{*\mu}\varepsilon^{*\nu}$$

Singularity Structure of Metric

- 1PM metric in the spheroidal coordinate (also organised by entire functions)

$$ds^2_{(1)}(B_1) = G_N m \frac{2r}{r^2 + |a|^2 \cos^2(\theta)} [ds^2_{\text{flat}} - 2dt^2] \,,$$

$$ds^2_{(1)}(B_2) = G_N m \frac{2}{|a|} \arctan(\frac{|a|}{r}) [ds^2_{\text{flat}} - 2dt^2] \,,$$

$$ds^2_{(1)}(B_3) = G_N m \left[\frac{(1 - 3\cos^2(\theta)) W(r)}{2} - \frac{\sin^2(\theta) \arctan(\frac{|a|}{r})}{|a|} \right] [ds^2_{\text{flat}} - 2dt^2] \,,$$

$$ds^2_{(1)}(B_4) = G_N m \frac{4|a| r \sin^2(\theta)}{r^2 + |a|^2 \cos^2(\theta)} [d\phi dt] \,,$$

$$ds^2_{(1)}(B_5) = G_N m \; 2 \sin^2(\theta) |a| W(r) [d\phi dt] \,,$$

Where

$$W(r) = \frac{\left(|a|^2 + r^2\right) \arctan\left(\frac{|a|}{r}\right) - |a| r}{|a|^3} \,,$$

$$ds^2_{\text{flat}} = \left[dt^2 - \frac{\left(|a|^2 \cos^2(\theta) + r^2\right)}{\left(|a|^2 + r^2\right)} dr^2 - \left(|a|^2 \cos^2(\theta) + r^2\right) d\theta^2 - \left(|a|^2 + r^2\right) \sin^2(\theta) d\phi^2 \right] .$$

Singularity Structure of Metric

- Entire functions B1 and B4 give rise to the ring singularity in the leading-order stationary metric.
- Entire functions B2, B3 and B5 only contribute to the regular part of the leading-order stationary metric.
- For Kerr, given by B1+B4, the leading-order stationary metric vanishes as r ->0, suggesting that the flat-space solution remains stable and unaffected by its own generated metric.
- For more generic cases, the leading order stationary metric receives contributions from B2, B3 and B5, which do not vanish as r ->0, suggesting that the flat-space solution can be modified.
- This observation may corroborate the stability of the Kerr black hole.

Toward Gravitational Compton

From Stress-Energy Tensor to Compton

- Gravitational Compton amplitude can be extracted from the the solutions to the Einstein equation with the stress-energy tensor

[Scheopner, Vines]

$$T^{\mu\nu}(x) = \frac{m}{4\pi a_w^2 \sqrt{-g}} \int d^3\sigma \sqrt{\gamma}\,\delta^4(x-Z)(\partial_a Z^\mu \gamma^{ab} \partial_b Z^\nu) + \dots$$

- The Polyakov term in the hypersurface action suffices to compute the Compton amplitude up to linear power in spin.

- The stress-energy tensor in the post-Minkowskian expansion $T^{\mu\nu} = T_{\mathbf{0}}^{\mu\nu} + \kappa\, T_{\mathbf{1}}^{\mu\nu} + \cdots$

- The metric receives contributions from the background and the stationary contributions, and is expanded around flat space $g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa \mathrm{h}_{\mu\nu}$

$$\mathrm{h}_{\mu\nu} = h_{\mathbf{0}\,\mu\nu} + \kappa\, h_{\mathbf{1}\,\mu\nu} + \kappa^2\, h_{\mathbf{2}\,\mu\nu} + \cdots$$

Incoming plane wave

$$h_{\mathbf{0}\,\mu\nu} = \lambda\, \varepsilon_{1\mu}\varepsilon_{1\nu} e^{-ik_1\cdot x}$$

Stationary contributions

Outgoing wave contribution

$$h_{\mathbf{1}\,\mu\nu} = \mathfrak{h}_{\mathbf{1}\,\mu\nu} + \mathcal{O}(\lambda^2) \qquad h_{\mathbf{2}\,\mu\nu} = \mathfrak{h}_{\mathbf{2}\,\mu\nu} + \lambda\delta h_{\mathbf{2}\,\mu\nu} + \mathcal{O}(\lambda^2)$$

From Stress-Energy Tensor to Compton

- Compton amplitude as the linear response to the interaction with the black hole [Scheopner, Vines]
- Einstein equation at $\mathcal{O}(\kappa^3\lambda)$ in the de Donder gauge $\partial_\mu h_{\mathbf{i}}^{\mu\nu} - \frac{1}{2}\partial^\nu h^{\mu}_{\mathbf{i}\,\mu} = 0$

$$-\partial^2 \delta h_{\mathbf{2}}^{\mu\nu} + \eta^{\mu\nu}\partial_\rho\partial_\sigma \delta h_{\mathbf{2}}^{\rho\sigma} = \frac{T_{\mathbf{1}}^{\mu\nu}}{2} - M_{\mathbf{1}}^{\mu\nu}$$

The ONLY unknown

Graviton scattering from the stationary metric (all contributions involved at this order are known)

$$\begin{aligned} M_{\mathbf{1}}^{\mu\nu} = &\frac{1}{2} h_{\mathbf{0}}^{\rho\sigma}\partial_\rho\partial_\sigma \mathfrak{h}_{\mathbf{1}}^{\mu\nu} + \frac{1}{2}(\partial^{(\mu} h_{\mathbf{0}\,\rho\sigma})\partial^{\nu)}\mathfrak{h}_{\mathbf{1}}^{\rho\sigma} \\ &- (\partial_\rho h_{\mathbf{0}}^{\sigma(\mu})\partial_\sigma \mathfrak{h}_{\mathbf{1}}^{\nu)\rho} + h_{\mathbf{0}}^{\rho(\mu}\partial^2 \mathfrak{h}_{\mathbf{1}\rho}^{\nu)} + h_{\mathbf{0}}^{\rho(\mu}\partial^{\nu)}\partial_\sigma \mathfrak{h}_{\mathbf{1}\rho}^{\sigma} \\ &- \frac{1}{2} h_{\mathbf{0}}^{\mu\nu}\partial_\rho\partial_\sigma \mathfrak{h}_{\mathbf{1}}^{\rho\sigma} + \frac{1}{2} h_{\mathbf{0}}^{\rho\sigma}\partial^\mu\partial^\nu \mathfrak{h}_{\mathbf{1}\rho\sigma} + (\partial^\rho h_{\mathbf{0}}^{\sigma(\mu})\partial_\rho \mathfrak{h}_{\mathbf{1}\sigma}^{\nu)} \\ &- h_{\mathbf{0}}^{\rho(\mu}\partial_\rho\partial_\sigma \mathfrak{h}_{\mathbf{1}}^{\nu)\sigma} - h_{\mathbf{0}}^{\rho\sigma}\partial_\rho\partial^{(\mu}\mathfrak{h}_{\mathbf{1}\sigma}^{\nu)} + \frac{1}{2}(\partial^\mu\partial^\nu h_{\mathbf{0}}^{\rho\sigma})\mathfrak{h}_{\mathbf{1}\rho\sigma} \\ &- h_{\mathbf{0}}^{\rho(\mu}\partial^{\nu)}\partial_\sigma \mathfrak{h}_{\mathbf{1}\rho}^{\sigma} + \frac{1}{2}(\partial_\rho\partial_\sigma h_{\mathbf{0}}^{\mu\nu})\mathfrak{h}_{\mathbf{1}}^{\rho\sigma} + (\cdots) \end{aligned}$$

- Fourier transform to momentum space

Containing the massive channel

$$A_4(v, a, k_1, k_2) = \frac{\kappa^2}{\lambda}\left(-\frac{\widetilde{T}_{\mathbf{1}}^{\mu\nu}}{2}\varepsilon_{2\mu}\varepsilon_{2\nu} + \widetilde{M}_{\mathbf{1}}^{\mu\nu}\varepsilon_{2\mu}\varepsilon_{2\nu}\right)$$

Containing the massless channel

Stress-Energy Tensor & EOM at NLO

- First correction to the Fourier-transformed stress-energy tensor

$$\widetilde{T}_{\mathbf{1}}^{\mu\nu}(k_2) = \frac{\lambda}{4\pi a_w^2} \int d^3\sigma \sqrt{\gamma} e^{ik_2\cdot Z_{\mathbf{0}}} \Big[ik_2\cdot Z_{\mathbf{1}} \partial_a Z_{\mathbf{0}}^{\mu} \gamma^{ab} \partial_b Z_{\mathbf{0}}^{\nu} + 2\partial_a Z_{\mathbf{0}}^{\mu} \gamma^{ab} \partial_b Z_{\mathbf{1}}^{\nu} \Big]$$

  - Relevant EOM $\partial_a\left(\sqrt{\gamma}\gamma^{ab}\partial_b Z^{\rho}\right) + \sqrt{\gamma}\partial_a Z^{\mu}\gamma^{ab}\partial_b Z^{\nu}\Gamma^{\rho}_{\mu\nu}(Z) = 0$

    Equivalent to the conservation of the stress-energy tensor up to this order

  - PM expansion of the coordinate $Z^{\mu}(\sigma) = Z_{\mathbf{0}}^{\mu}(\sigma) + \kappa\lambda Z_{\mathbf{1}}^{\mu}(\sigma) + \cdots$, $\qquad Z_{\mathbf{0}}^{\mu} \equiv X_{\mathbf{0}}^{\mu}(\tau) + Y_{1,1}^{\mu}(\tau,\theta,\phi)$

  - PM expansion of the background metric $\mathcal{G}_{\rho\nu}(Z) = \eta_{\rho\nu} + \kappa\lambda\varepsilon_{1\rho}\varepsilon_{1\nu}e^{-ik_1\cdot Z} + \cdots$

- EOM with the ansatz $Z_{\mathbf{1}}^{\rho} = e^{-ik_1\cdot X_{\mathbf{0}}}\bar{Z}_{\mathbf{1}}^{\rho}$

$$\partial_a\left(\sqrt{\gamma}\gamma^{ab}\partial_b\bar{Z}_{\mathbf{1}}^{\rho}\right) - a_w^2\sin(\theta)\left((k_1\cdot v)^2\bar{Z}_{\mathbf{1}}^{\rho} + 2ik_1\cdot v\partial_\tau\bar{Z}_1^{\rho}\right) = ie^{-ik_1\cdot Y_{\mathbf{0}}}\sqrt{\gamma}\gamma^{ab}\partial_b(\varepsilon_1\cdot Z_{\mathbf{0}})\left(\partial_a(k_1\cdot Z_{\mathbf{0}})\varepsilon_1^{\rho} - \frac{1}{2}\partial_a(\varepsilon_1\cdot Z_{\mathbf{0}})k_1^{\rho}\right)$$

Solve EOM at NLO

- Further decompose the ansatz into a complete orthonormal basis

$$\bar{Z}_{\mathbf{1}}^{\rho}(\sigma) = \sum_{l \geq |m| = 0}^{\infty} \Psi_{\mathbf{1},l,m}^{\rho}(\tau) \mathcal{Y}_{l,m}(\theta, \phi)$$

Spherical harmonic functions

  - The coefficients can be easily solved
  - The boundary conditions are chosen to be consistent with physical expectations
  - Explicit solutions relevant for the Compton amplitude up to linear power in spin

$$\Psi_{\mathbf{1},0,0}^{\rho} = \frac{i\sqrt{\pi} v_1 \cdot \varepsilon_1 \left(k_1^{\rho} v_1 \cdot \varepsilon_1 - 2k_1 \cdot v_1 \varepsilon_1^{\rho}\right)}{\left(k_1 \cdot v_1\right)^2} - \frac{2\sqrt{2\pi} c_{1,1}^2 a_w}{3\left(k_1 \cdot v_1\right)^2} \left(k_1^{\rho} v_1 \cdot \varepsilon_1 - k_1 \cdot v_1 \varepsilon_1^{\rho}\right) \left(k_1 \cdot \beta_y \beta_x \cdot \varepsilon_1 - k_1 \cdot \beta_x \beta_y \cdot \varepsilon_1\right) + \mathcal{O}(a_w^2)$$

$$\Psi_{\mathbf{1},1,\pm 1}^{\rho} = i\sqrt{\frac{\pi}{6}} c_{1,1} a_w e^{\pm \frac{i\sqrt{2}\tau}{a_w}} \left(\varepsilon_1^{\rho} \left(\beta_y \cdot \varepsilon_1 \pm i\beta_x \cdot \varepsilon_1\right) + \frac{v_1 \cdot \varepsilon_1 \left(\varepsilon_1^{\rho} \left(k_1 \cdot \beta_y \pm i k_1 \cdot \beta_x\right) - k_1^{\rho} \left(\beta_y \cdot \varepsilon_1 \pm i \beta_x \cdot \varepsilon_1\right)\right)}{k_1 \cdot v_1}\right) + \mathcal{O}(a_w^2)$$

Gravitional Compton Amplitude at O(S)

- Evaluating the next-to-leading order stress-energy tensor with

$$Z_{\mathbf{1}}^{\rho} = e^{-ik_1\cdot X_{\mathbf{0}}}\left(\Psi^{\rho}_{\mathbf{1},0,0}(\tau)\mathcal{Y}_{0,0}(\theta,\phi) + \Psi^{\rho}_{\mathbf{1},1,1}(\tau)\mathcal{Y}_{1,1}(\theta,\phi) + \Psi^{\rho}_{\mathbf{1},1,-1}(\tau)\mathcal{Y}_{1,-1}(\theta,\phi)\right) + \mathcal{O}(a_w^2)$$

- We arrive at

Zeroth order in spin

$$\begin{aligned}\widetilde{T}_{\mathbf{1}}^{\mu\nu}\varepsilon_{2\mu}\varepsilon_{2\nu} =& \lambda\,\delta(v\cdot k_2 - v\cdot k_1)\Bigg[\Bigg[\frac{v_1\cdot\varepsilon_1 v_1\cdot\varepsilon_2}{2(k_1\cdot v_1)^2}\Big(2v_1\cdot F_1\cdot F_2\cdot v_1 - 2\varepsilon_1\cdot\varepsilon_2\,(k_1\cdot v_1)^{\,2} + k_1\cdot k_2 v_1\cdot\varepsilon_1 v_1\cdot\varepsilon_2\Big)\\ &- \frac{i}{2m(k_1\cdot v_1)^2}\Big(v_1\cdot\varepsilon_2 k_1\cdot S\cdot\varepsilon_1(-2v_1\cdot F_1\cdot F_2\cdot v_1 + k_1\cdot v_1 k_2\cdot\varepsilon_1 v_1\cdot\varepsilon_2 - k_1\cdot k_2 v_1\cdot\varepsilon_1 v_1\cdot\varepsilon_2) + (1\leftrightarrow 2)\Big)\Bigg]\\ &+ \Bigg[\frac{i}{2m}\Big(-\frac{v_1\cdot\varepsilon_1 v_1\cdot\varepsilon_2 \mathrm{tr}\,(F_1\cdot F_2\cdot S)}{k_1\cdot v_1} - v_1\cdot\varepsilon_2\varepsilon_1\cdot F_2\cdot S\cdot\varepsilon_1 + v_1\cdot\varepsilon_1\varepsilon_2\cdot F_1\cdot S\cdot\varepsilon_2 - \varepsilon_1\cdot\varepsilon_2 k_1\cdot v_1\varepsilon_1\cdot S\cdot\varepsilon_2\Big)\Bigg]\Bigg]\\ &+ \mathcal{O}(a_w^2),\end{aligned}$$

First order in spin

where $F_i^{\mu\nu} = k_i^{\mu}\varepsilon_i^{\nu} - \varepsilon_i^{\mu}k_i^{\nu}$

Gravitional Compton Amplitude at O(S)

- Together with the contribution from the scattering with the stationary metric, we obtain the gravitational Compton amplitude from our model

$$A_4(v, a, k_1, k_2) = \kappa^2 \delta(v\cdot k_2 - v\cdot k_1)\Big[\Big(\frac{v\cdot F_1\cdot F_2\cdot v}{v\cdot k_1}\Big)^2 - \frac{i v\cdot F_1\cdot F_2\cdot v}{2k_1\cdot k_2\,(v\cdot k_1)^{\,2}} \times \Big(v\cdot k_1 \mathrm{tr}\,(S\cdot F_2\cdot F_1) + v\cdot F_1\cdot F_2\cdot S\cdot k_2 + k_1\cdot S\cdot F_1\cdot F_2\cdot v\Big)\Big]$$

  - Which agrees with the literature up to the linear power in spin

- At the zeroth order in spin, our formalism reduces to the worldline models straightforwardly.

- At the first order in spin, while the final amplitudes are identical, our treatment of the spin degree of freedom is different from that in the worldline models.

Higher Powers in Spin

- Beyond the linear order in spin, the extra couplings to the Riemann tensor also contribute
  - Corrections to the solutions of the NLO EOM at higher powers in spin
  - More modes
- Preliminary results:
  - The NLO EOM can be solved straightforwardly for all modes at any power in the spin expansion
  - The part of the Compton amplitude on the massive cut obtained from the stress-energy tensor agrees with known results in the literature at the quadratic order in spin

Outlook

When Newtonmas comes to town…

- Gravitational Compton amplitude
    - Complete agreement with known results in the literature up to the quartic order in spin
    - Consistency with factorisation and symmetry requirements at the fifth powers and beyond
- Contact contributions, spin resummations
- Extraction of the multipole expansion from the stress-energy tensor, MPD equation
- Tidal effects, higher multiplicities, etc…

“Futuresque” directions

- Extension to a broad class of spinning black holes
- Deformation, quantisation, black hole stability, physical interpretation of higher modes …

Thank you!