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Primal S-matrix bootstrap with dispersion relations

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New Frontiers of Quantum Field and Gravity, 16 Jan 2026, Beijing, PKU

de Rham, Tolley, Zhuo-Hui Wang & SYZ, [2506.22546](#)

Outline

- Introduction
- Primal bootstrap with dispersion relations
 - Bounds on leading coefficients
 - Regge behaviors
- Constraining glueball couplings
- Fractionally subtracted dispersion relations
- Summary

Dual bootstrap vs Primal bootstrap for S-matrix

QFT theory space

Dual bootstrap

analyticity + unitarity

- rule out unviable space
- exclude amplitudes
- use dispersion relation
- impose full crossing

Positivity bounds for EFTs:

[de Rham, Melville, Tolley & SYZ, 2017](#)
[Arkani-Hamed, Huang & Huang, 2017](#)

.....

[de Rham, Kundu, Reece, Tolley & SYZ, Snowmass White Paper, 2203.06805](#)

[Guerrieri, Sever, 2106.10257](#)

Primal bootstrap

analyticity + unitarity

- rule in viable space
- build amplitudes
- full crossing built-in

[Paulos, Penedones, Toledo, van Rees, Vieira, 1708.06765](#)

[Kruczenski, Penedones & van Rees, Snowmass White paper, 2203.02421](#)

Duality gap: ideally, it shrinks to zero

Many models worked out in both approaches

Unitarity

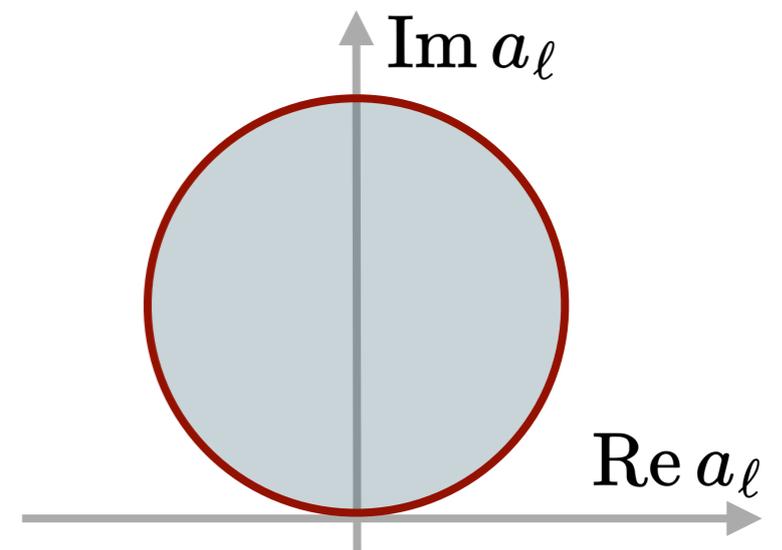
Partial wave expansion

$$\mathcal{M}(s, t) \propto \sum_{\ell \geq 0} (2\ell + 1) a_{\ell}(s) P_{\ell} \left(1 + \frac{2t}{s - 4m^2} \right)$$

Partial wave unitarity

$$2 \operatorname{Im} a_{\ell} \geq (\operatorname{Re} a_{\ell})^2 + (\operatorname{Im} a_{\ell})^2$$

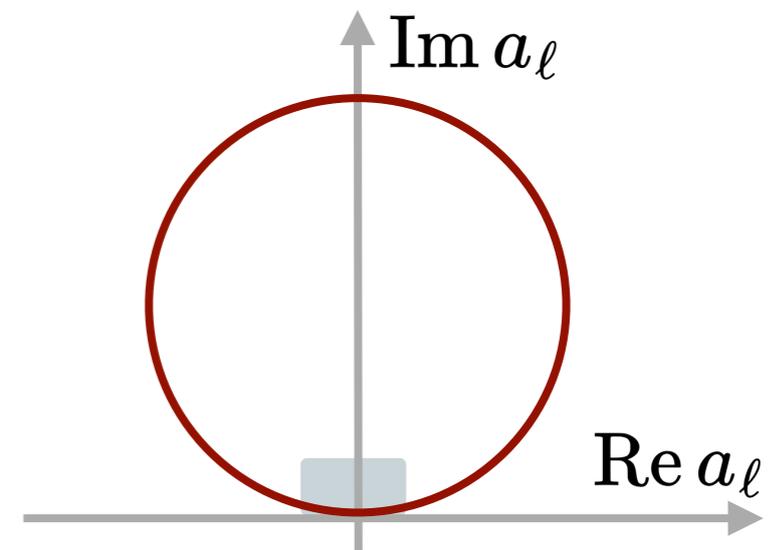
important for strongly coupled theory



For weakly coupled EFTs

$$\operatorname{Im} a_{\ell} \geq 0 \quad \Rightarrow \quad \text{positivity bounds}$$

already rich bounds on Wilson coefficients



Causality \Rightarrow Analyticity

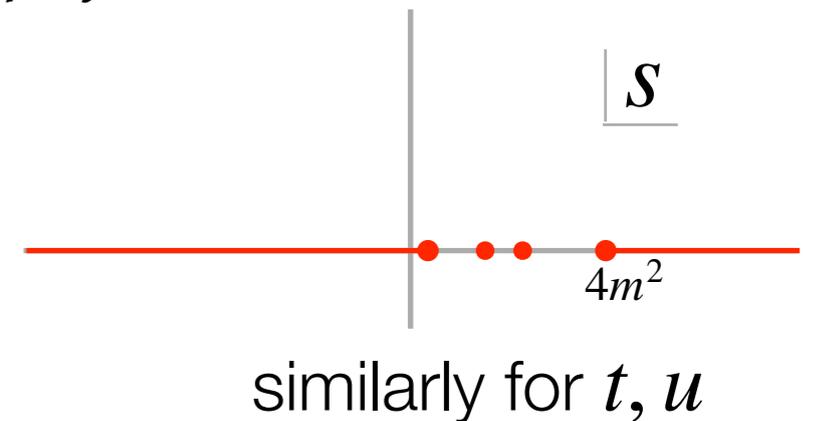
Maximal analyticity (Mandelstam conjecture)

previous primal

physical-sheet amplitude is analytic *except for*

$$s \geq 4m^2, \quad t \geq 4m^2, \quad u \geq 4m^2$$

(up to bound state poles)



Martin's proven analyticity

physical-sheet amplitude is analytic *within*

$$\text{doubly cut } s \text{ plane, } |t| \leq 4m^2$$

(up to bound state poles)

our choice

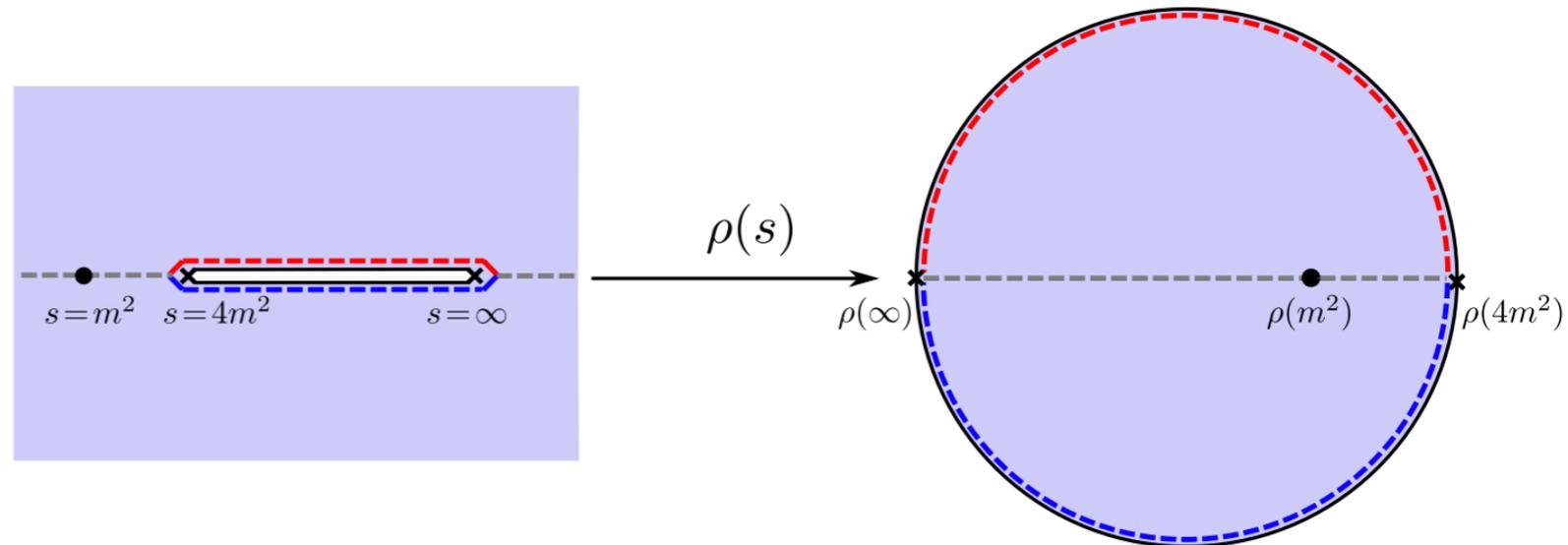
Primal bootstrap with full amplitudes (1)

Paulos, Penedones, Toledo, van Rees, Vieira, 1708.06765

Conformal mapping

$$s \rightarrow \rho(s)$$

$$\rho(x) = \frac{\sqrt{4m^2 - x_0} - \sqrt{4m^2 - x}}{\sqrt{4m^2 - x_0} + \sqrt{4m^2 - x}}$$



Build all possible amplitudes

(crossing symmetric)

$$\mathcal{M}(s, t) = \sum_{a+b+c \leq N_{\max}} d_{abc} \rho^{(a)}(s) \rho^{(b)}(t) \rho^{(c)}(u)$$

parametrize amplitudes

subject to

$$s + t + u = 4m^2 \rightarrow \text{polynomial constraints on } \rho(s), \rho(t), \rho(u)$$

Primal bootstrap with full amplitudes (2)

Paulos, Penedones, Toledo, van Rees, Vieira, 1708.06765

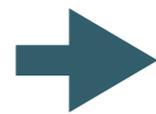
Partial wave projection

$$a_\ell(s) \sim \int_{-(s-4m^2)}^0 dt P_\ell \left(1 + \frac{2t}{s-4m^2} \right) \mathcal{M}(s, t)$$

Impose partial wave unitarity

$$\mathbf{S}_\ell \mathbf{S}_\ell^\dagger = \mathbf{1}$$

$$\mathbf{S}_\ell = \begin{pmatrix} S_\ell & \cdots \\ \vdots & \ddots \end{pmatrix}$$



$$S_\ell S_\ell^\dagger \preceq \mathbf{1} \quad \text{or}$$

$$|1 + iS_m a_\ell(s)| \leq 1$$

physical region $s \geq 4m^2$

$$S_m = \sqrt{(s-4m^2)/s}$$



$$\begin{pmatrix} 1 - \frac{1}{2}S_m \text{Im} a_\ell(s), & S_m^{1/2} \text{Re} a_\ell(s) \\ S_m^{1/2} \text{Re} a_\ell(s), & 2\text{Im} a_\ell(s) \end{pmatrix} \succeq 0 \quad s \geq 4m^2$$

$a_\ell = (\text{linear in decision variables } d_{abc}) \Rightarrow$ semi-definite program (SDP)

Primal bootstrap with full amplitudes (3)

Bounds on amplitude coefficients

$$\mathcal{M}(s, t) = \sum_{i,j=0}^{\infty} g_{i,j} (s - s_*)^i (t - t_*)^j$$

$s_* = t_* = 4m^2/3$

non-perturbative

Many other results, for example:

[Paulos, Penedones, Toledo, van Rees, Vieira, 1708.06765, 1810.12849](#)

[Doroud, Miro, 1804.04376](#)

[Guerrieri, Penedones, Vieira, 1810.12849, 2102.02847](#)

[Miro, Guerrieri, Hebbar, Penedones, Vieira, 1906.08098](#)

[Hebbar, Karateev, Penedones, 2011.1708](#)

[Miro, Guerrieri, Gumus, 2210.01502](#)

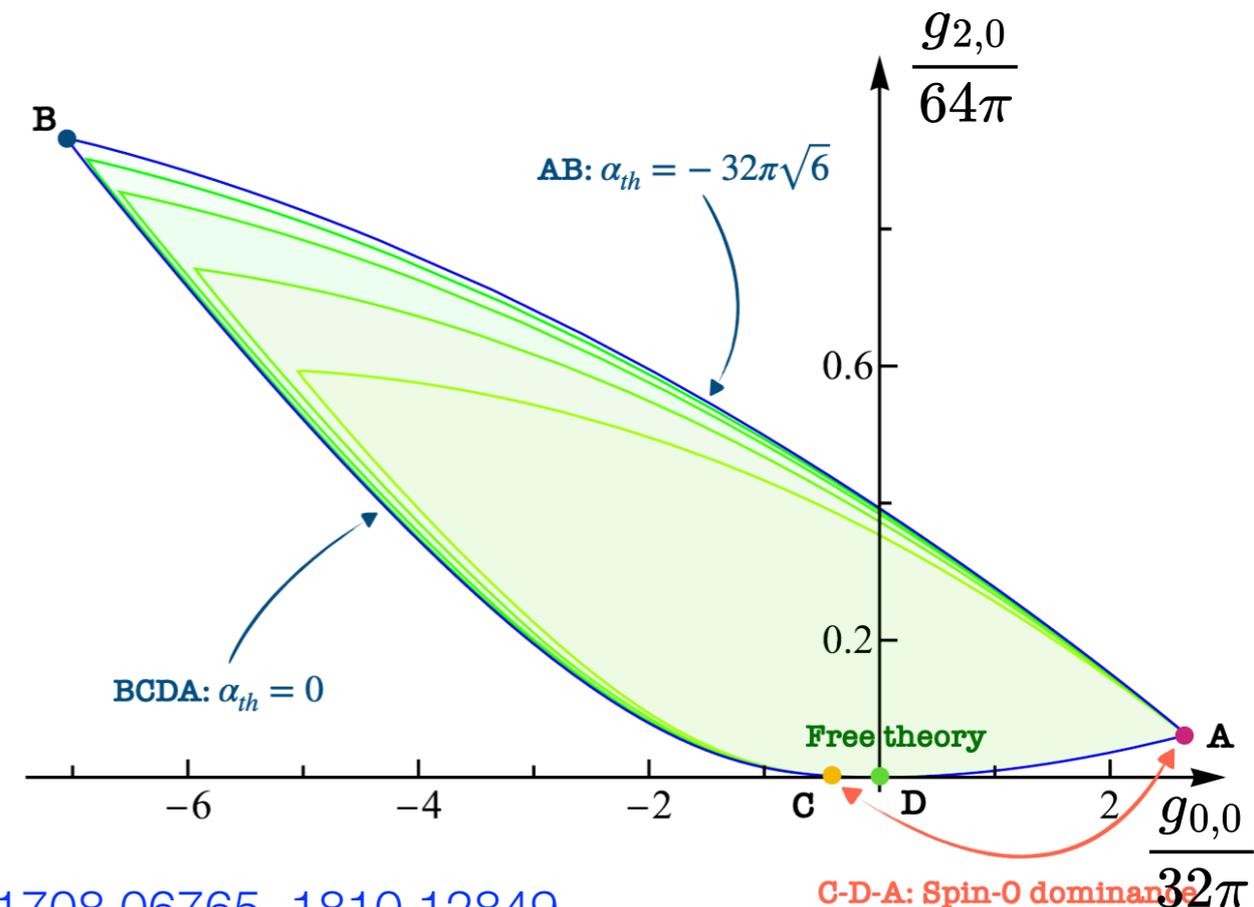
[Haring, Karateev, Meineri, Penedones, 2211.05795](#)

[He, Kruczenski, 2309.12402, 2505.19332](#)

[Guerrieri, Haring, Su, 2410.23333](#)

[Correia, Georgoudis, Guerrieri, 2506.04313](#)

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Another way to bootstrap?

- Parametrize amplitudes in physical region?
- Strictly, unitarity is outside convergence region
- How to control UV behavior of amplitude?

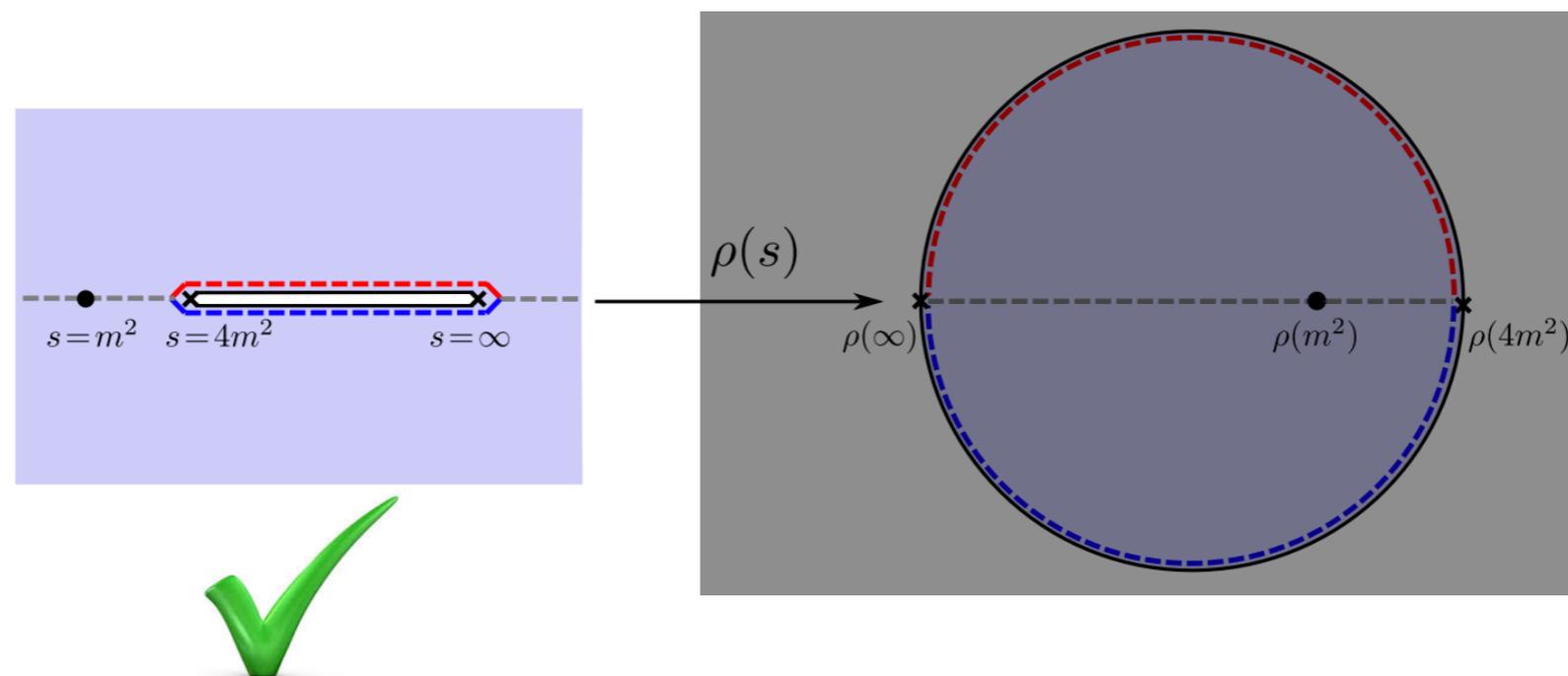
Correia, Sever, Zhiboedov
2006.08221

locality
unitarity
analyticity \Rightarrow Froissart-Martin bound

$$\mathcal{M}(s \rightarrow \infty, t) \sim s^{1+\eta(t)}$$

$$t < 4m^2, \eta(t) < 1$$

How about use Mandelstam variables?



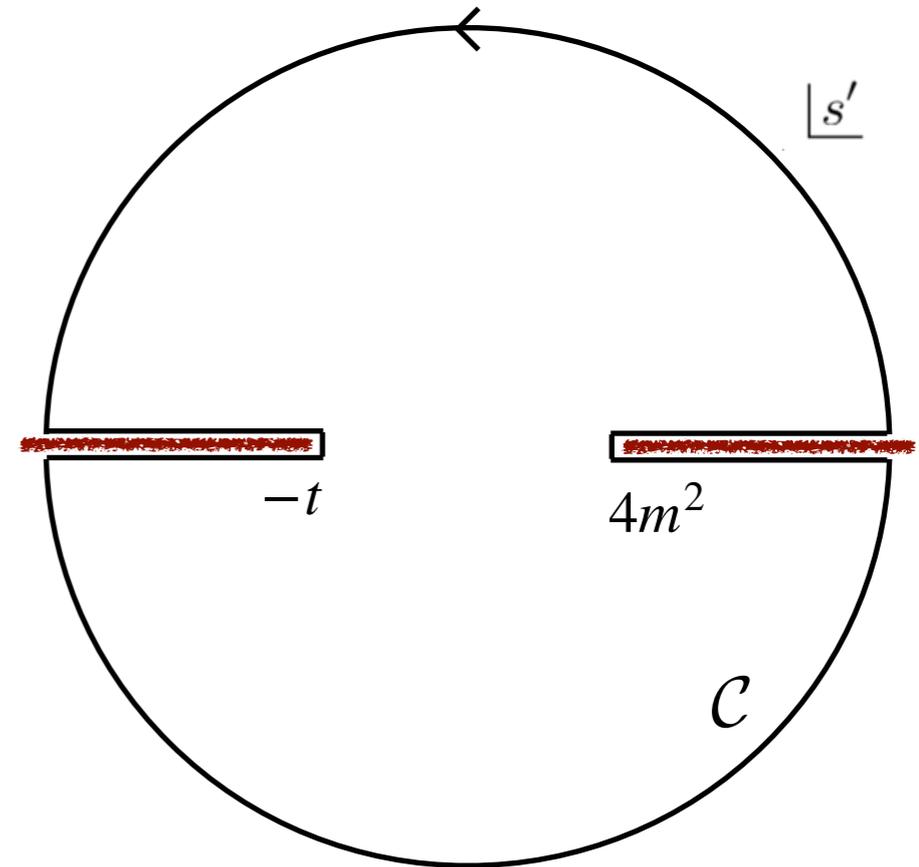
Primal bootstrap with dispersion relations

Analyticity \Rightarrow Dispersion relation

- Analyticity in complex s plane (fixed t)

$$\mathcal{M}(s, t) = \oint_C \frac{ds'}{2\pi i} \frac{\mathcal{M}(s', t)}{s' - s}$$

- Froissart-Martin bound $\mathcal{M}(s \rightarrow \infty, t) \sim s^{1+\eta}$
- su crossing symmetry $\mathcal{M}(s, t) = \mathcal{M}(s, u)$



Twice subtracted dispersion relation

$$\text{Re}\mathcal{M}(s, t) = \mathcal{M}(s_0, t_0) + \mathcal{P} \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \left(\text{Im}\mathcal{M}(\mu, t) K_{s,t}^{\mu, t_0} + \text{Im}\mathcal{M}(\mu, t_0) K_{t, t_0}^{\mu, s_0} \right)$$

$$K_{s,t}^{\mu, t_0} = \frac{1}{\mu - s} + \frac{1}{\mu - 4m^2 + s + t} - \frac{1}{\mu - t_0} - \frac{1}{\mu - 4m^2 + t + t_0}$$

analytical function is fixed by singularities and asymptotics!

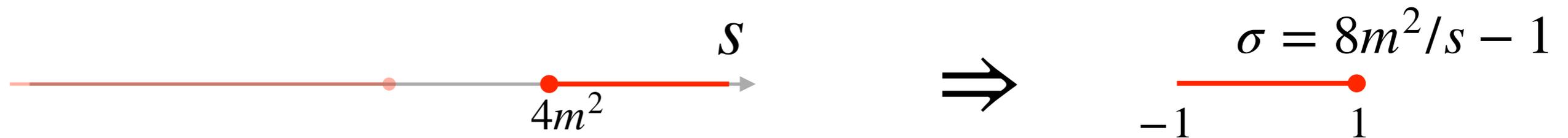
Parametrize partial waves

Partial wave expansion in physical region $s \geq 4m^2$

$$\text{Im}\mathcal{M}(s, t) = 16\pi \sum_{\text{even } \ell \geq 0} (2\ell + 1) \text{Im}a_\ell(s) P_\ell \left(1 + \frac{2t}{s - 4m^2} \right)$$

$P_\ell \sim (s - 4m^2)^{-\ell}$

Parametrize $\text{Im } a_\ell(s)$ in physical region ($s \geq 4m^2$)



$$\text{Im } a_\ell(s) = \sqrt{\frac{s}{s - 4m^2}} \left(\frac{s - 4m^2}{s} \right)^\ell \sum_{k=0}^{k_{\max}} c_{\ell, k} P_k \left(8m^2/s - 1 \right)$$

$s \geq 4m^2$

Annotations for the equation above:

- cancel P_ℓ 's (pointing to the $(s - 4m^2)^\ell$ term)
- phase space (pointing to the square root term)
- threshold divergence (pointing to the $(s - 4m^2)$ term in the fraction)
- boundedness of partial wave (pointing to the s term in the denominator of the fraction)
- decision variables (pointing to the $c_{\ell, k}$ coefficient)
- Legendre polynomials (pointing to the P_k term)

Obtain $\text{Re } a_\ell(s)$

$$\text{Re } \mathcal{M}(s, t) = \mathcal{M}(s_0, t_0) + \mathcal{P} \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \left(\text{Im } \mathcal{M}(\mu, t) K_{s,t}^{\mu, t_0} + \text{Im } \mathcal{M}(\mu, t_0) K_{t, t_0}^{\mu, s_0} \right)$$

$$\text{Re } \mathcal{M}(s, t) \sim \sum \text{Re } a_\ell(s) P_\ell \left(1 + \frac{2t}{s - 4m^2} \right) \qquad \text{Im } \mathcal{M}(\mu, t) \sim \sum \text{Im } a_\ell(\mu) P_\ell \left(1 + \frac{2t}{\mu - 4m^2} \right)$$

use dispersion relation to solve for $\text{Re } a_\ell(s)$

parametrized $\text{Im } a_\ell(\mu)$

one choice: partial wave projection \Rightarrow Roy equations

but it reduces viable s $(\dots) = \int_{4m^2-s}^0 dt \sum_\ell \int_{4m^2}^{\infty} d\mu P_\ell \left(1 + \frac{2t}{\mu - 4m^2} \right) (\dots)$

Discretize t in dispersion relation:

$$t = t_{\min} : \quad (\dots) \text{Re } a_0(s) + (\dots) \text{Re } a_2(s) + (\dots) \text{Re } a_4(s) + \dots = (\dots)$$

$$t = t_1 : \quad (\dots) \text{Re } a_0(s) + (\dots) \text{Re } a_2(s) + (\dots) \text{Re } a_4(s) + \dots = (\dots)$$

$$t = t_2 : \quad (\dots) \text{Re } a_0(s) + (\dots) \text{Re } a_2(s) + (\dots) \text{Re } a_4(s) + \dots = (\dots)$$

\vdots

\vdots

can use dispersion relation for arbitrarily large s

Unitarity and full crossing

Unitarity conditions

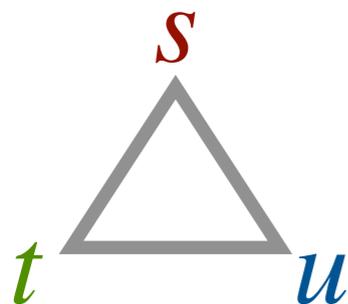
imposed on physical region

$$\begin{pmatrix} 1 - \frac{1}{2}S_m \text{Im} a_\ell(s), & S_m^{1/2} \text{Re} a_\ell(s) \\ S_m^{1/2} \text{Re} a_\ell(s), & 2\text{Im} a_\ell(s) \end{pmatrix} \succeq 0 \quad s \geq 4m^2$$

Full crossing symmetry

fixed- t dispersion relation is su crossing symmetric

$$tu \text{ null constraints: } a_{\ell=\text{odd}}(s) = 0$$

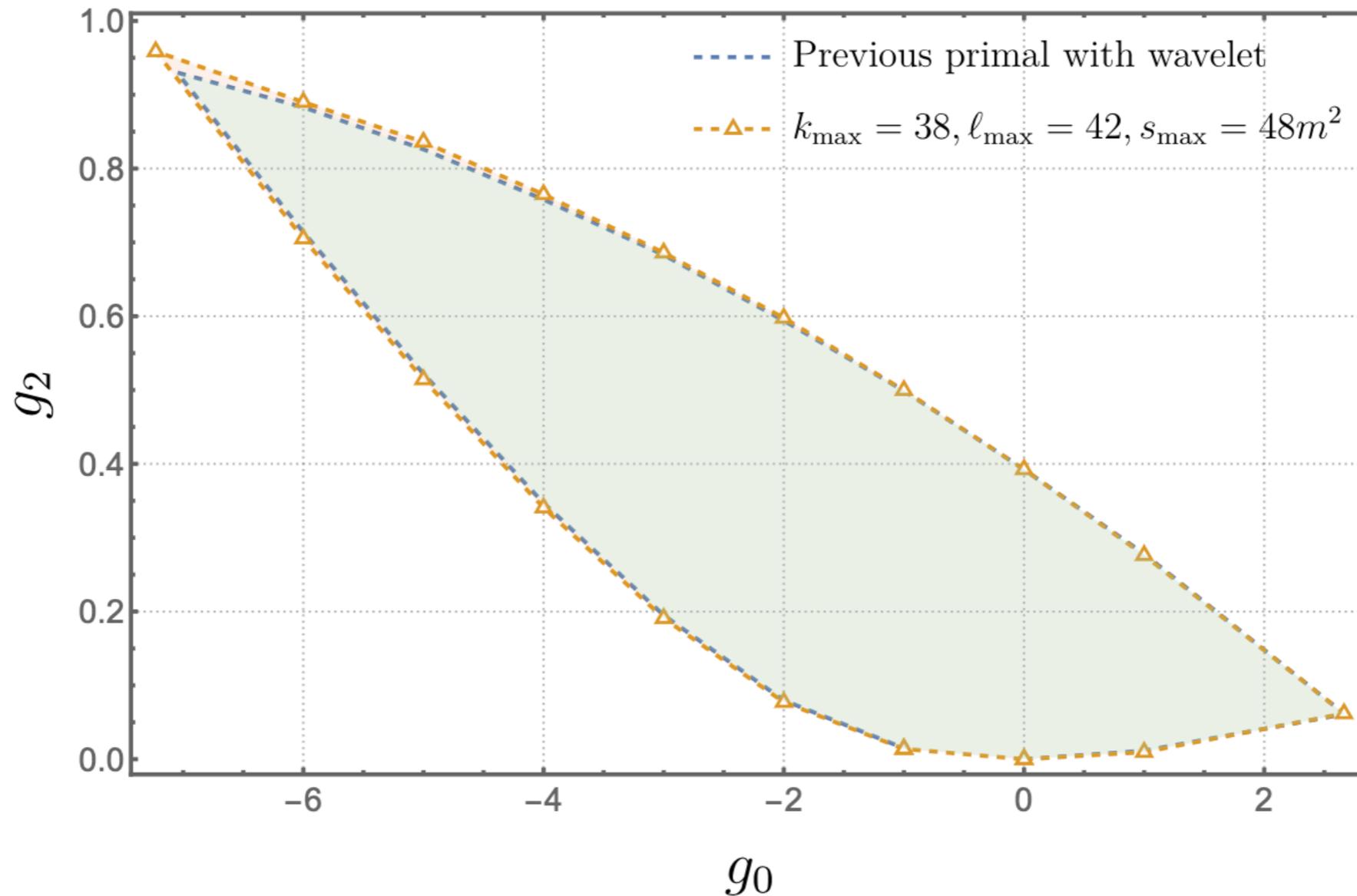


st null constraints are not necessary due to analyticity, unlike positivity bounds

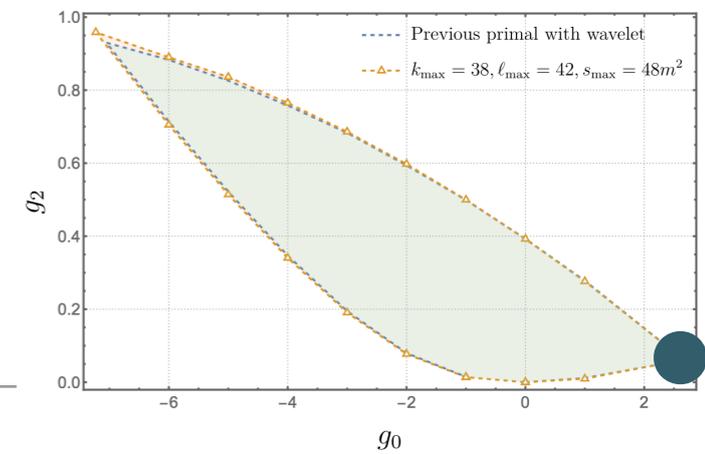
Sanity check

Bounds on leading amplitude coefficients

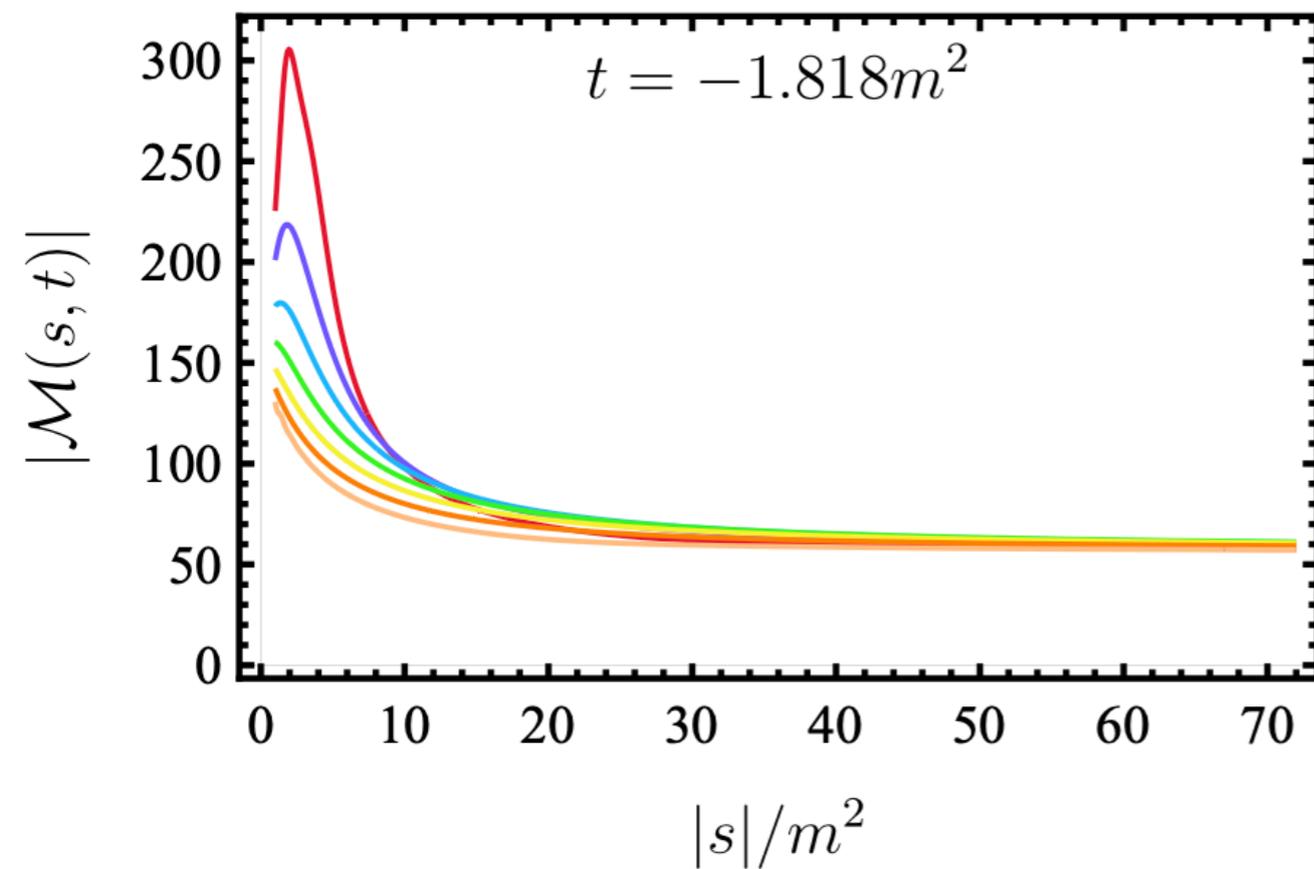
$$\mathcal{M}(s, t) \sim g_0 + g_2(s - 4m^2/3)^2$$



The right-hand kink

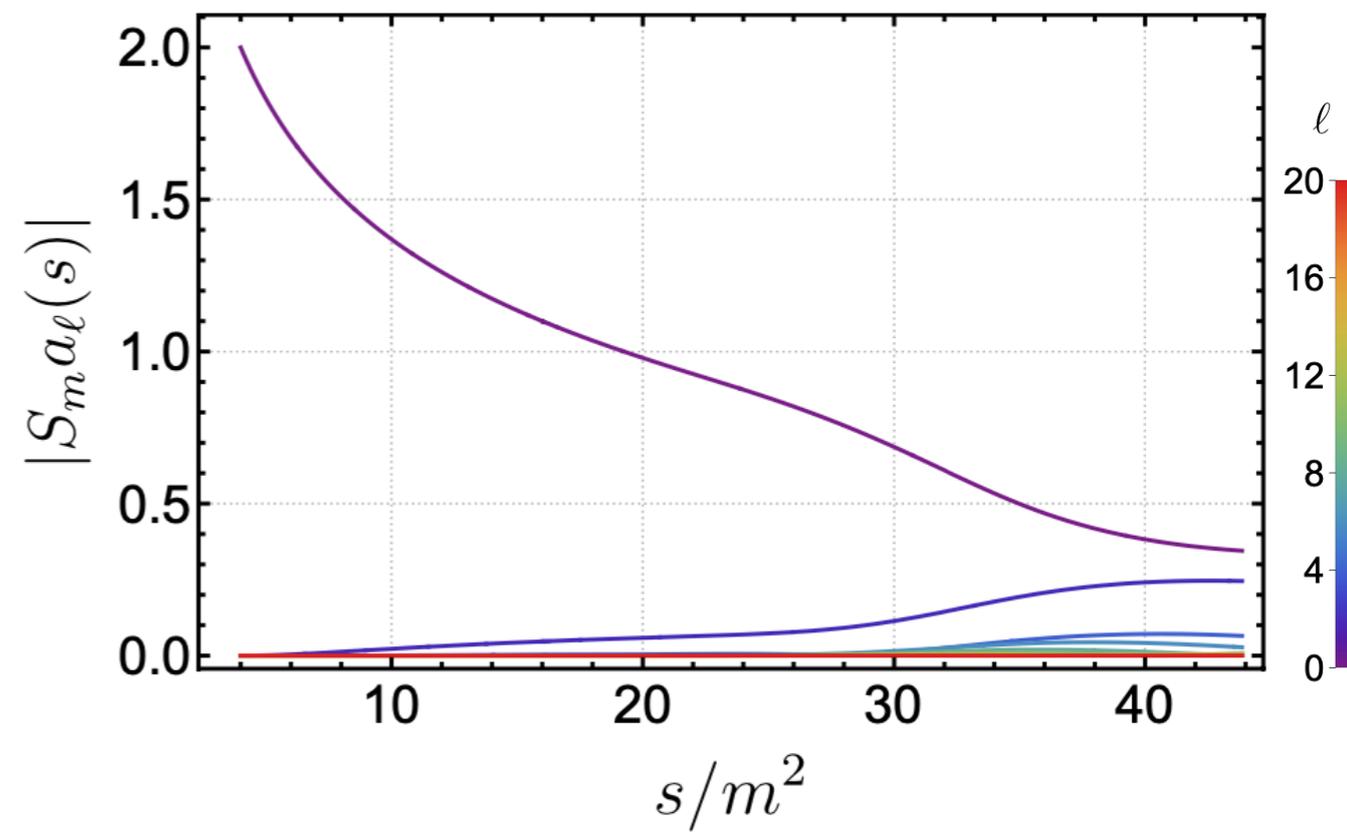


Amplitude



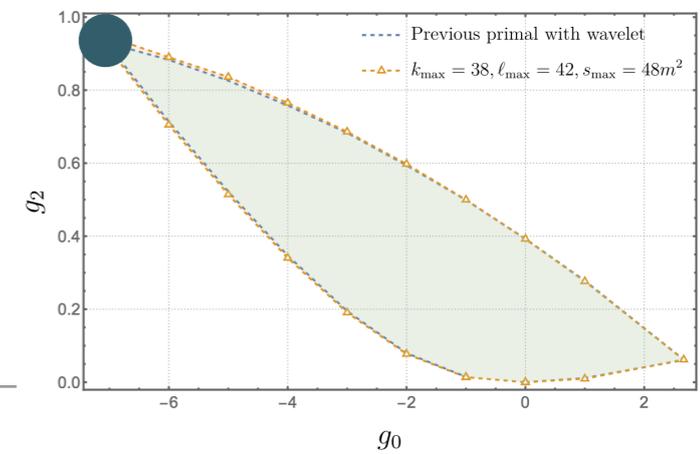
- $\pi/8$
- $2\pi/8$
- $3\pi/8$
- $4\pi/8$
- $5\pi/8$
- $6\pi/8$
- $7\pi/8$

Partial wave amplitudes



low-spin dominance

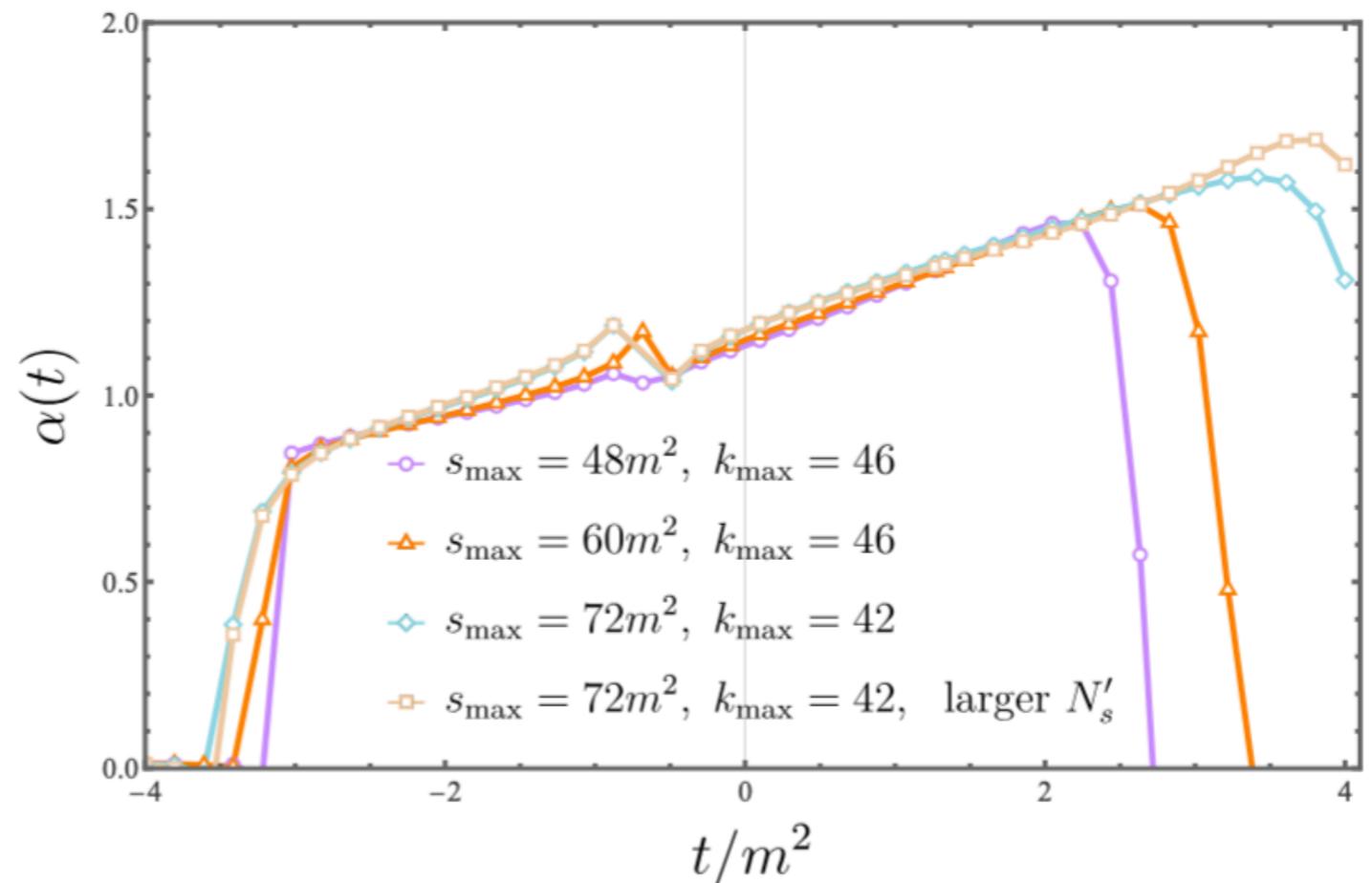
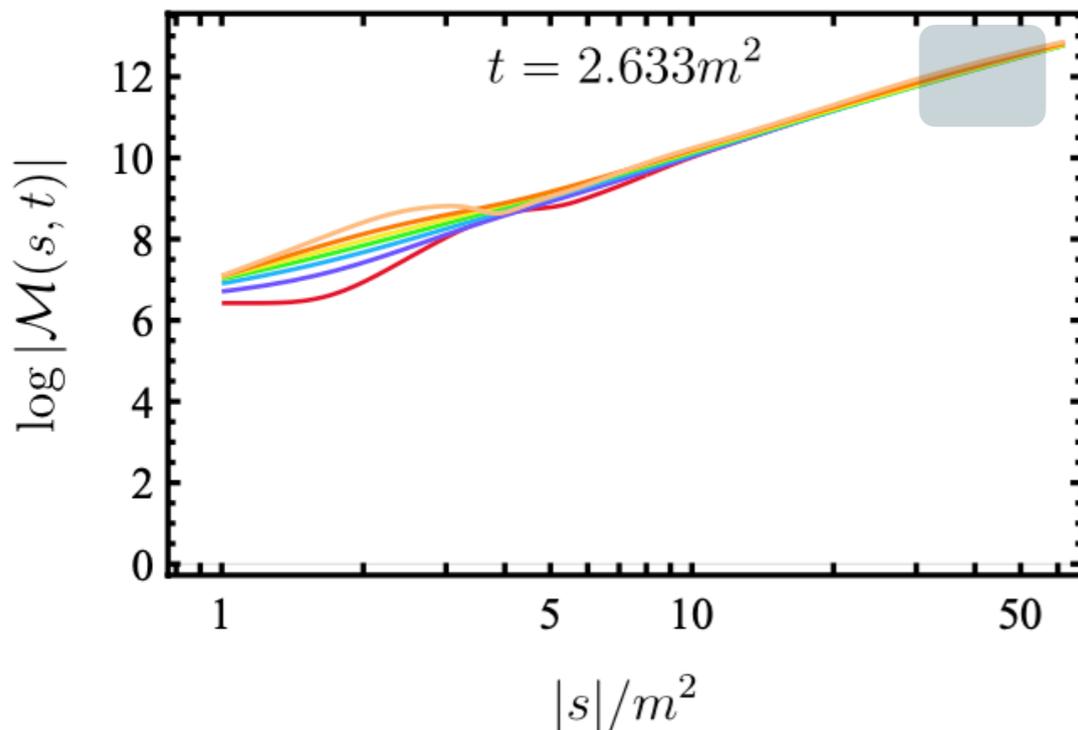
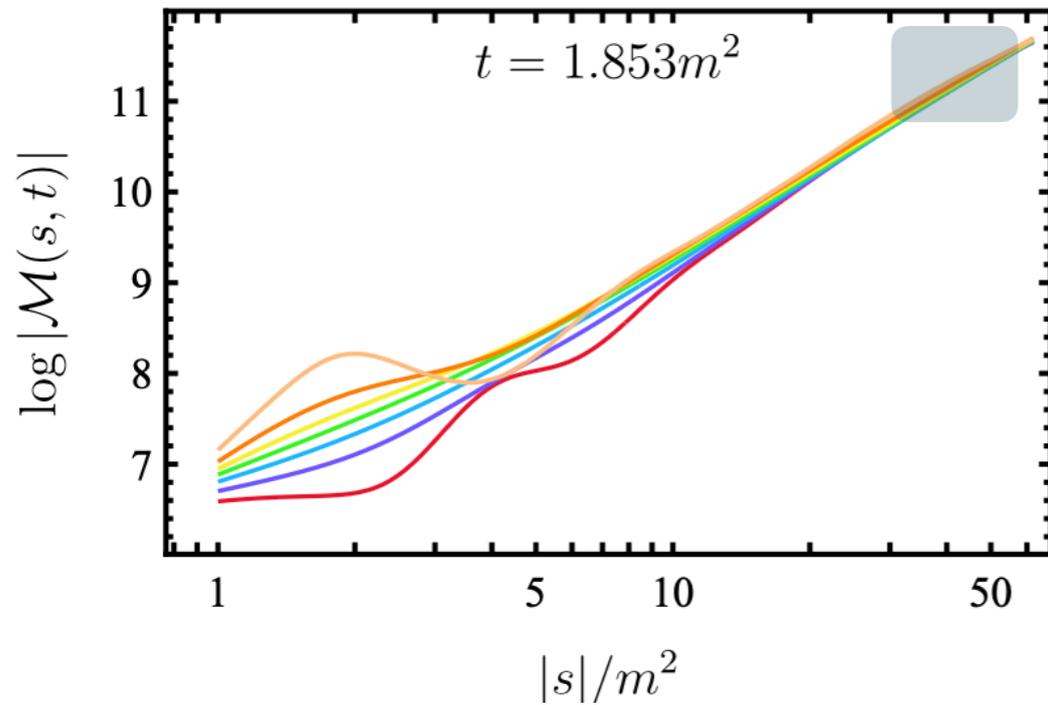
The left-hand kink (Regge behavior)



Linear Regge trajectory

$$\mathcal{M}(s, t) \sim s^{\alpha(t)}$$

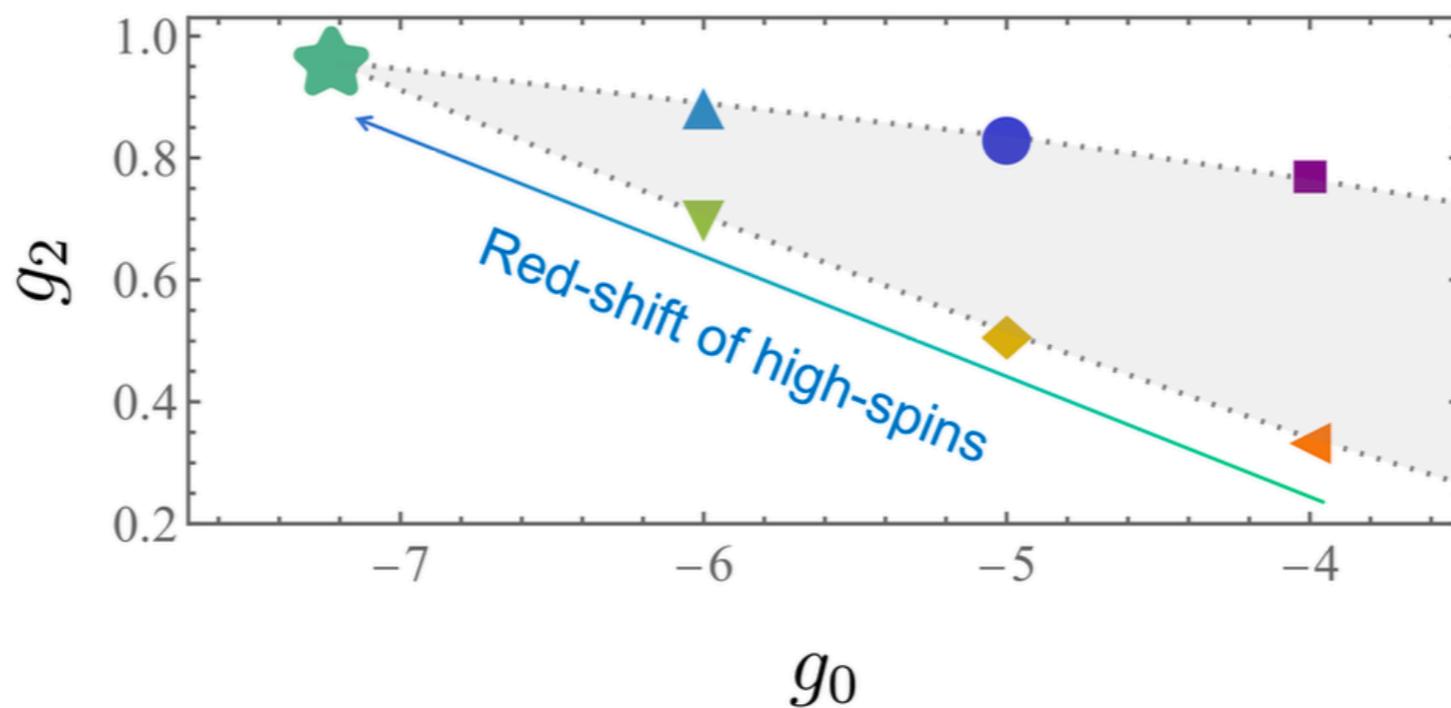
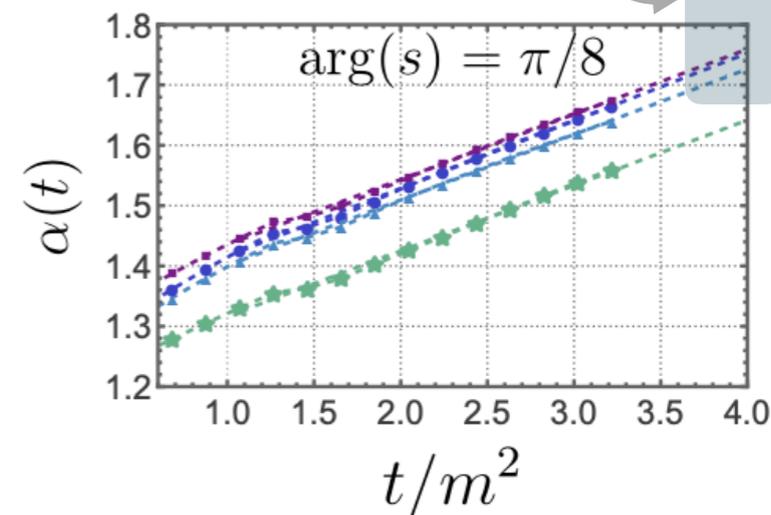
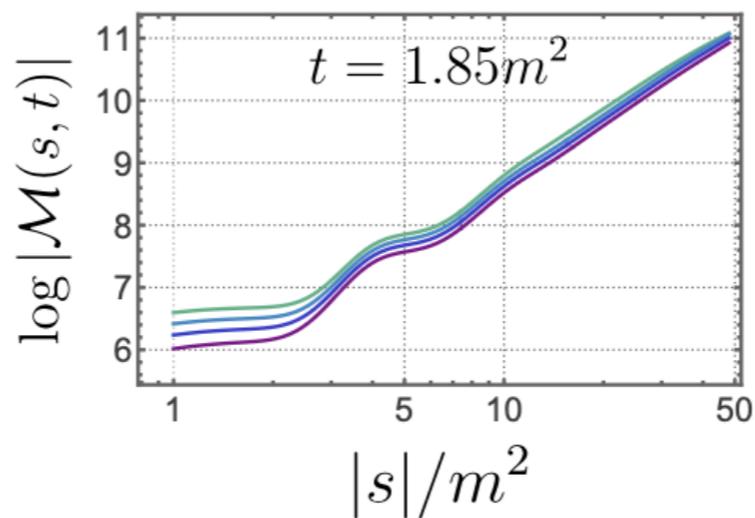
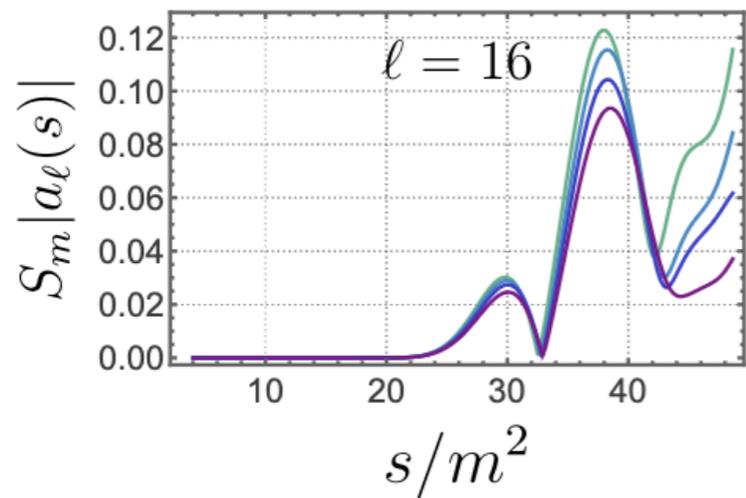
$$\alpha(t) \sim \alpha_0 + \alpha' t$$



• $\pi/8$ • $2\pi/8$ • $3\pi/8$ • $4\pi/8$ • $5\pi/8$ • $6\pi/8$ • $7\pi/8$

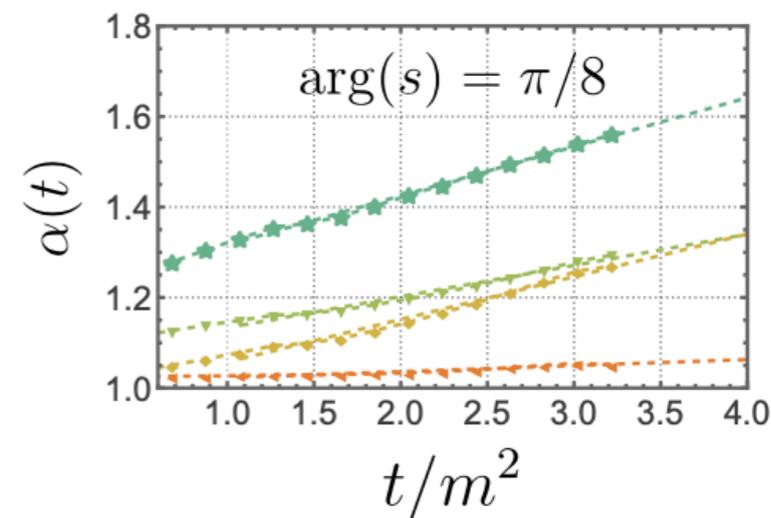
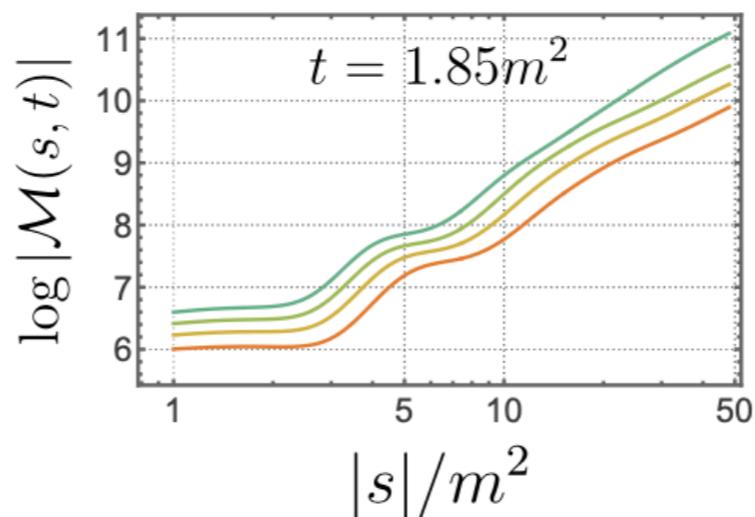
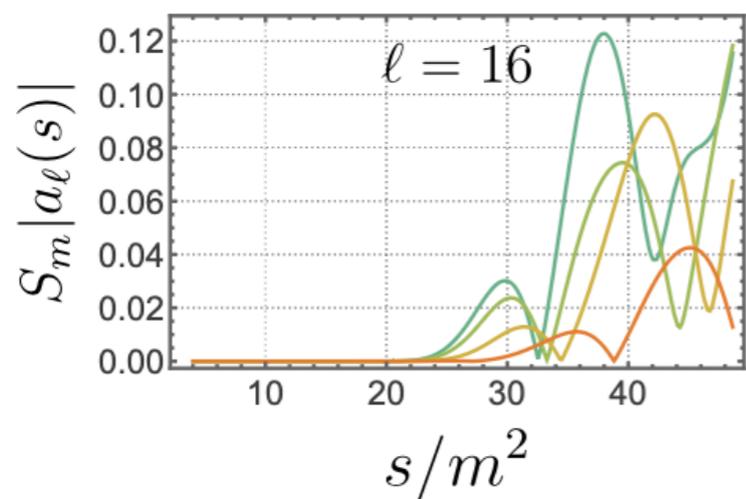
Away from right-hand kink

fastest growth $\mathcal{M} \sim s^{1.7 \sim 1.8}$



Froissart-Martin bound

$$\mathcal{M} < s^2$$



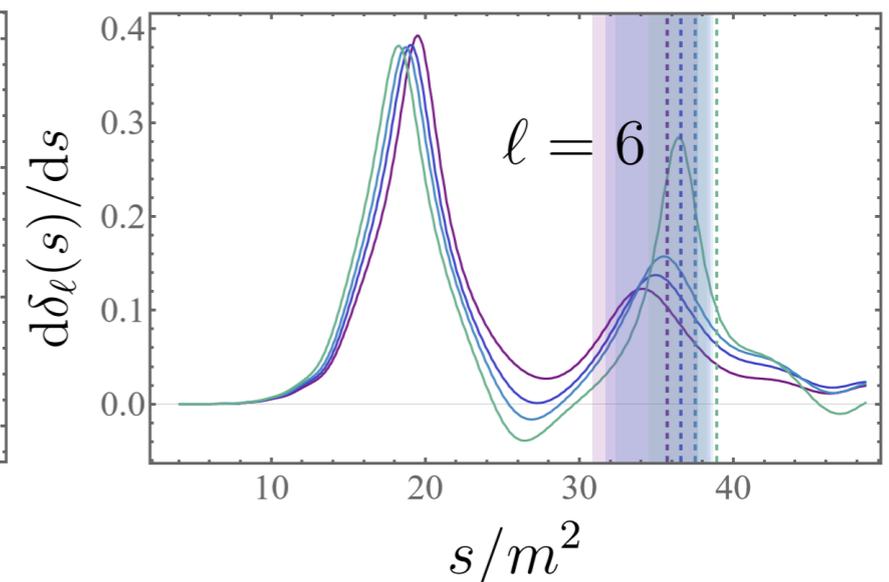
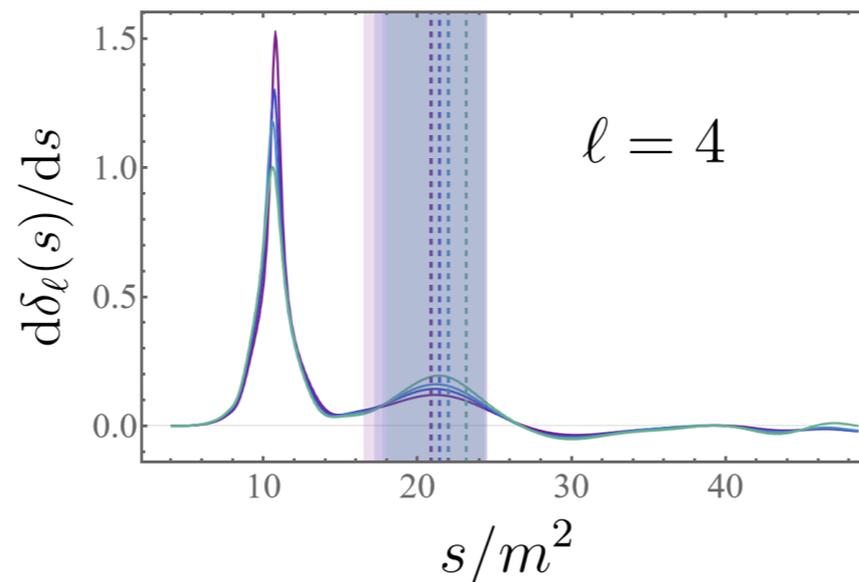
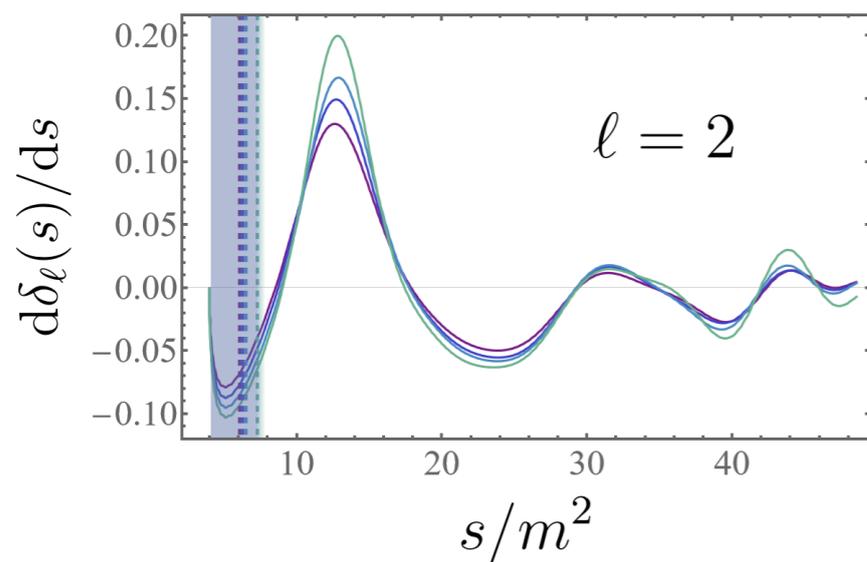
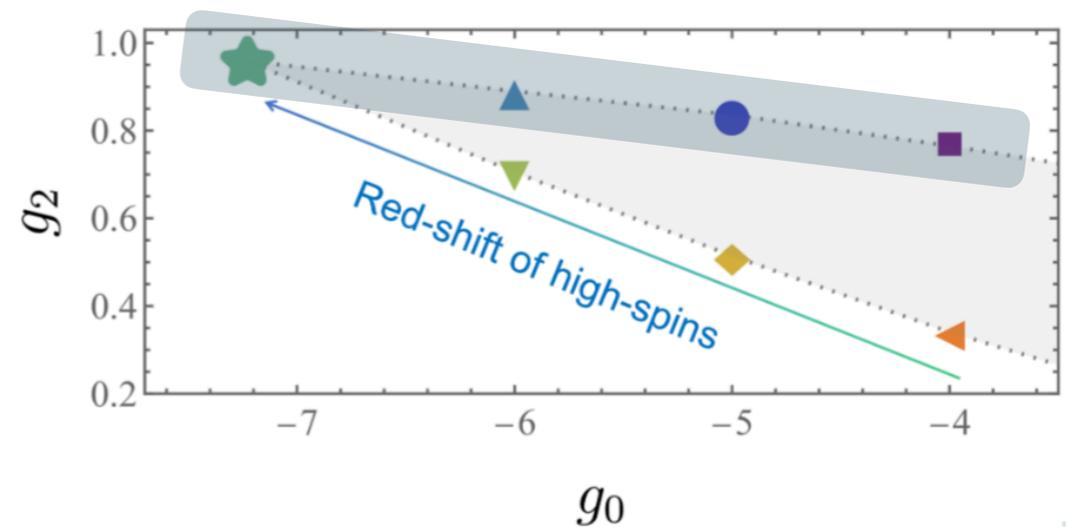
Pomeron trajectory

Extracted in two ways:

1. Regge theory with $\alpha(t) \sim \alpha_0 + \alpha' t$

$$\alpha(m_\ell^2) = \ell$$

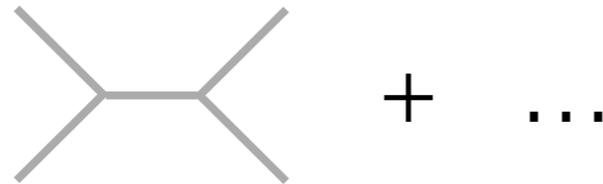
2. phase shift $\delta_\ell = \frac{1}{2} \arg(1 + iS_m a_\ell(s))$



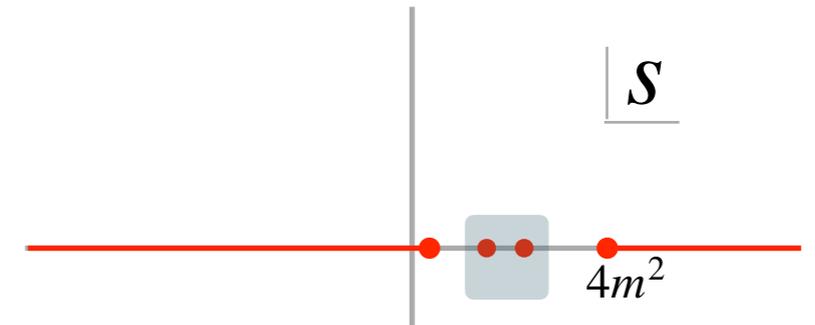
Constraining glueball couplings

Existence of scalar bound states

$$\mathcal{L} \supset \frac{g_0}{3!} \phi^3$$



+ ...



Pole subtracted amplitude

$$\mathcal{M}_{\text{cut}}(s, t) = \mathcal{M}(s, t) - g^2 \left(\frac{1}{m^2 - s} + \frac{1}{m^2 - t} + \frac{1}{m^2 - u} \right)$$

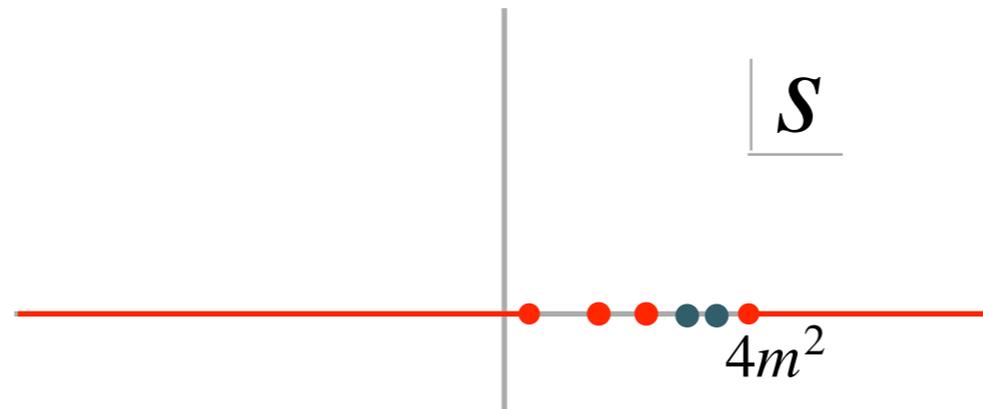
Only slightly modify unitarity conditions

$$\begin{pmatrix} 1 - \frac{1}{2} S_m \text{Im} a_\ell^{\text{cut}}(s), & S_m^{1/2} (\text{Re} a_\ell^{\text{cut}}(s) + \text{Re} a_\ell^{\text{pole}}(s)) \\ S_m^{1/2} (\text{Re} a_\ell^{\text{cut}}(s) + \text{Re} a_\ell^{\text{pole}}(s)), & 2 S_m \text{Im} a_\ell^{\text{cut}}(s) \end{pmatrix} \succeq 0$$

Bounds on g

$$0 \leq |g| < 48.8m$$

Extra spin ≥ 2 bound states



Can not directly subtract poles

$$(\text{spin-}\ell \text{ pole}) \sim \frac{s^\ell}{m_b^2 - t} \text{ violates Froissart-Martin bound}$$

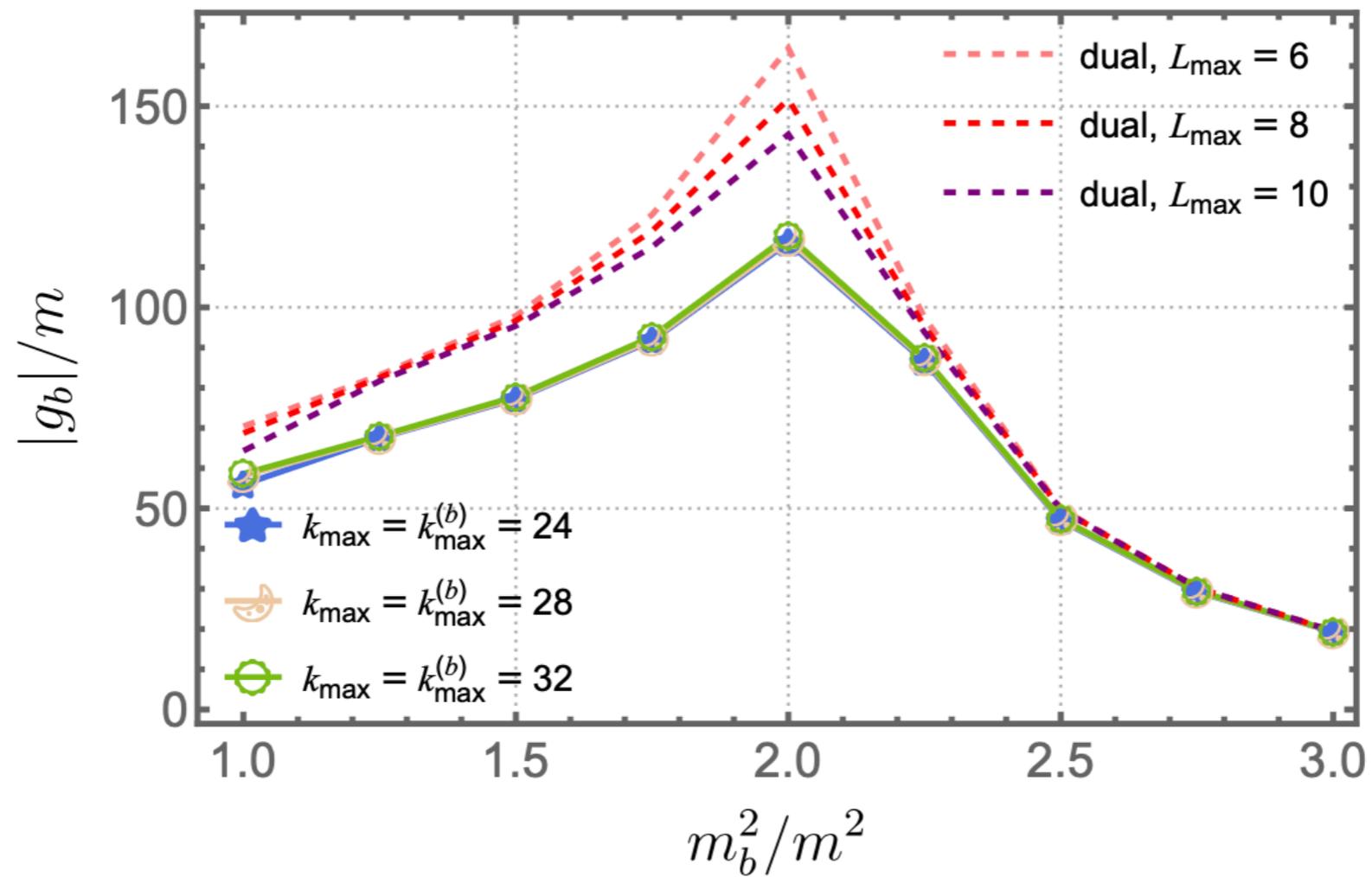
But easily fits in dispersion relation

$$\mathcal{M}(s, t) = \left(\text{cut part} \right) + \sum_b g_b^2 \left(P_\ell \left(1 + \frac{2t}{m_b^2 - 4m^2} \right) K_{s,t}^{m_b^2, t_0} + P_\ell \left(1 + \frac{2t_0}{m_b^2 - 4m^2} \right) K_{t,t_0}^{m_b^2, s_0} \right)$$

$$|t| < 4m^2 \rightarrow |t| < m_{b,\min}^2$$

Check against dual method

Toy model: Add a spin-2 pole



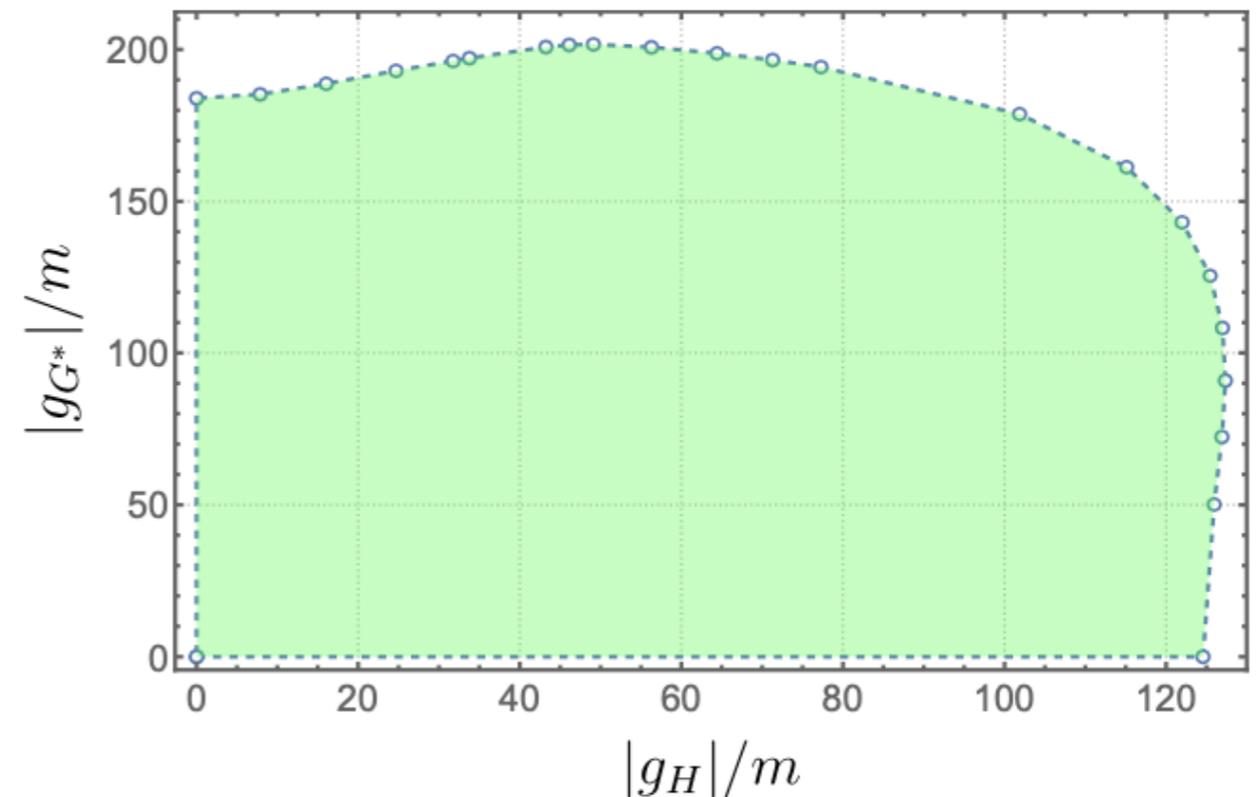
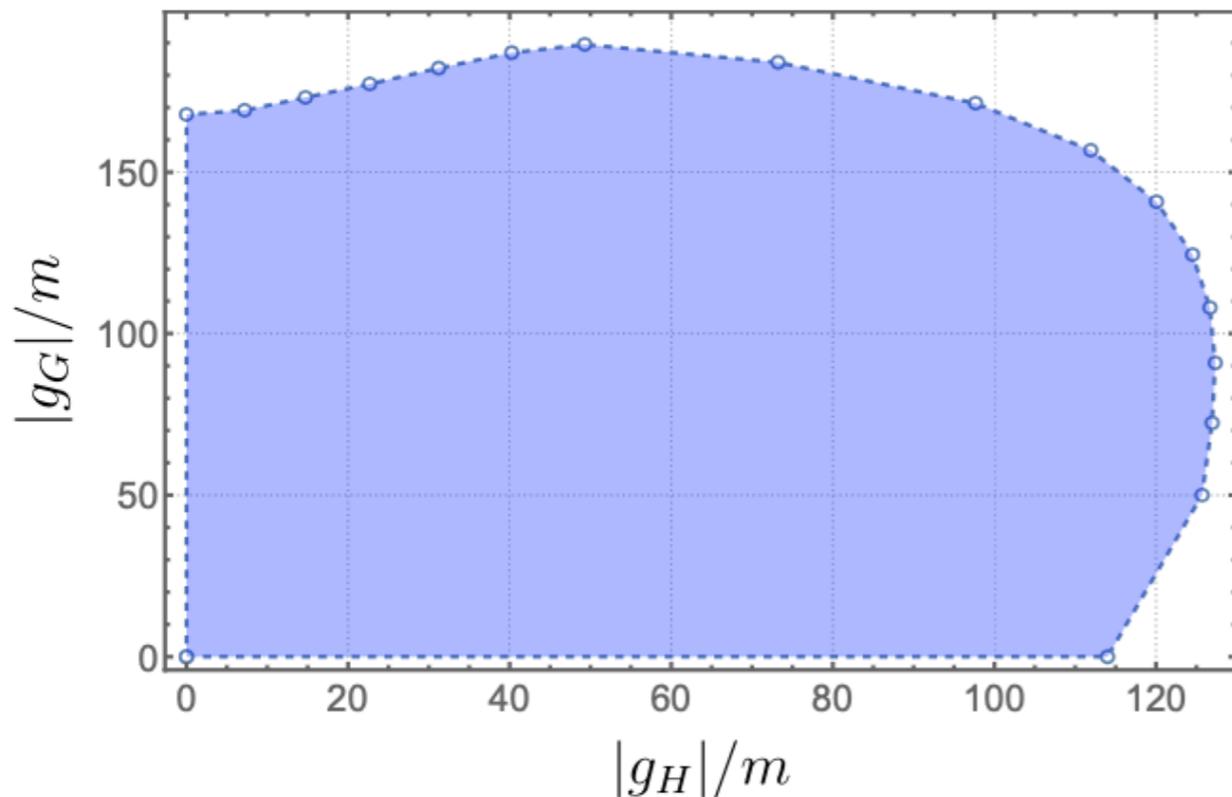
Bounds on glueball couplings

Glueballs:

massive, color-neutral, composite particles from gluons

Extra input: mass spectrum from lattice QCD

$$G(0) : 1; \quad H(2) : 1.437 \pm 0.006; \quad G^*(0) : 1.72 \pm 0.01; \quad H^*(2) : 1.99 \pm 0.01$$



Bootstrap with
fractionally subtracted dispersion relations

Slow growth and $2r$ -th subtractions

Controlled UV behavior

$$\lim_{|s| \rightarrow \infty} |\mathcal{M}(s, t)| < |s|^{2r}, \quad 0 \leq r \leq 1$$

su crossing symmetric kernel

$$G_{\{s_i\}}^{\{r_i\}}(s, t) = \prod_{i=1}^n (s_i - s)^{-r_i} (s_i - u)^{-r_i} \sim s^{-2r}$$

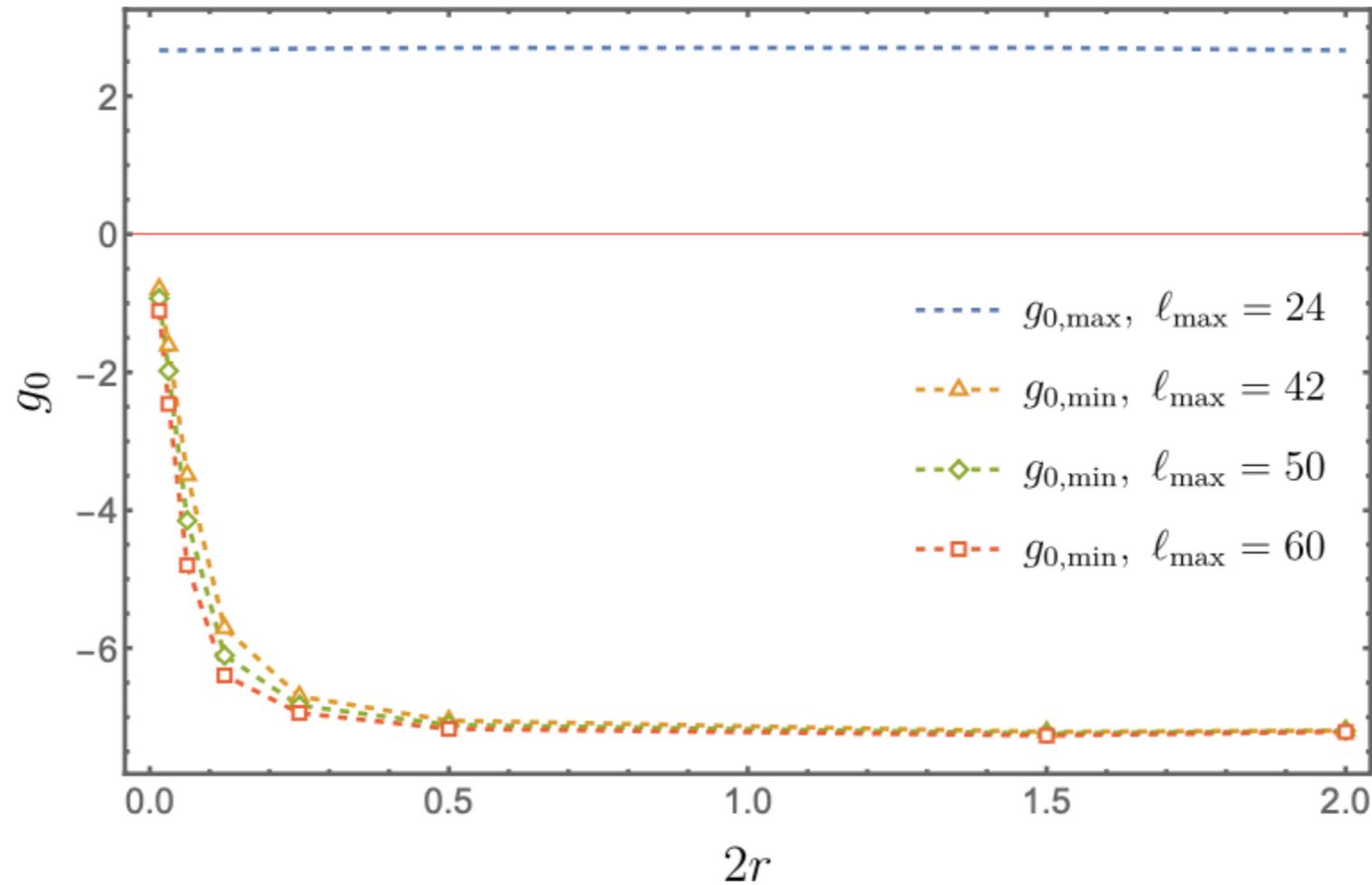
$$4m^2 = s_0 < s_1 < s_2 < \cdots < s_n < s_{n+1} = \infty, \quad 0 < r_i < 1, \quad \sum_{i=1}^n r_i = r$$

$s_j =$ multi-particle thresholds

Fractionally subtracted dispersion relations:

$$\operatorname{Re} \mathcal{M}(s, t) G_{\{s_i\}}^{\{r_i\}}(s, t) = \mathcal{P} \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \operatorname{Disc}_{\mu} \left(\mathcal{M}(\mu, t) G_{\{s_i\}}^{\{r_i\}}(\mu, t) \right) \left(\frac{1}{\mu - s} + \frac{1}{\mu - u} \right)$$

How bounds change with asymptotics



$$g_0|_{r=0} \sim \int_{4m^2}^{\infty} d\mu \frac{\text{Im } \mathcal{M}(\mu, 4m^2/3)}{\mu - 4m^2/3} \geq 0$$

most QFT space is covered by slow Regge behavior $\mathcal{M}(s, t) \lesssim s^{0.1}$?

Summary

- Found a new way to do primal S-matrix bootstrap in 4D
- Uses dispersion relations, with controlled UV behaviors
- Probed Regge/Pomeron behavior, approaching Froissart-Martin bounds
- Easy to add spin ≥ 2 bound states
- Constrained glueball couplings

Thank you!

Numerical scheme for dispersive integrals

$$\operatorname{Re}\mathcal{M}(s, t) = \mathcal{M}(s_0, t_0) + \mathcal{P} \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \left(\operatorname{Im}\mathcal{M}(\mu, t) K_{s,t}^{\mu, t_0} + \operatorname{Im}\mathcal{M}(\mu, t_0) K_{t, t_0}^{\mu, s_0} \right)$$

Two types of improper points:

- Principal-branch points: $\mu = s$, potentially $\mu = 4m^2 - t - t_0$

dealt analytically with $\mathcal{P} \int_{4m^2}^{\infty} d\mu \operatorname{Im} \mathcal{M}(y, t) \frac{y}{\mu} \frac{1}{\mu - y}$

- Boundary point: $\mu = 4m^2$

$$(4m^2, +\infty) \quad \xrightarrow{\quad} \quad (-\infty, +\infty)$$

$$\mu = 8m^2 / (x + 1)$$

$$x = \tanh \left(\frac{\pi}{2} \sinh w \right)$$

Numerical implementation

Naturally choose s_j to be multi-particle thresholds

$$s_0 = 4m^2, \quad s_1 = 9m^2, \quad s_2 = 16m^2, \quad \dots$$

Parameterize each sector independently

$$\sin \psi_j \operatorname{Re} a_\ell(\mu) + \cos \psi_j \operatorname{Im} a_\ell(\mu), \quad \psi_j = \sum_{i=1}^j r_i \pi \quad \text{for} \quad s_j < \mu < s_{j+1}$$

Solve dispersion relation for

$$\sin \psi_j \operatorname{Re} a_\ell(s) - \cos \psi_j \operatorname{Im} a_\ell(s)$$