Gauge Theory Bootstrap:
predicting pion dynamics from QCD

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with Martin Kruczenski

+ WIP

Low energy physics of asymptotically free gauge theory

asymptotically free gauge theory $SU(N_c)$ with N_f massive quarks $m_q \ll \Lambda_{\text{QCD}}$

confinement & chiral symmetry breaking

$$\mathcal{L} = i \sum_{j}^{N_f} \bar{q}_j \slashed{D} q_j - \sum_{j}^{N_f} m_q \bar{q}_j q_j - \frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a + \text{gauge fixing} + \text{ghost}$$

gauge theory parameters: N_c N_f m_q Λ_{QCD}

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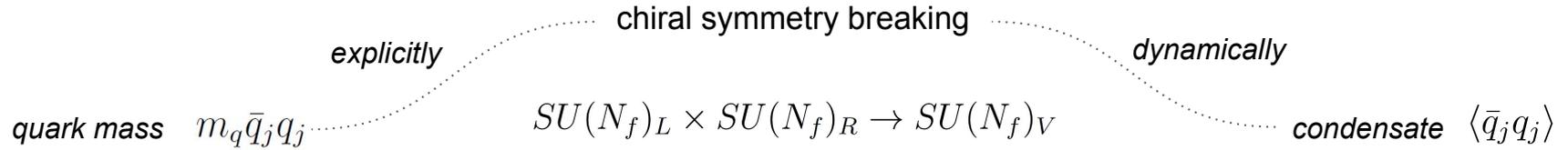
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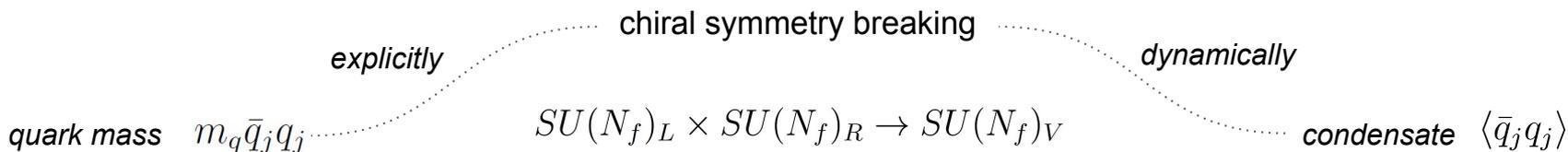
gauge theory parameters: N_c N_f m_q Λ_{QCD}

What is the low energy physics?

Physics of Goldstone bosons

chiral symmetry breaking

explicitly

dynamically

quark mass $m_q \bar{q}_j q_j$

$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$

condensate $\langle \bar{q}_j q_j \rangle$

pseudo-Goldstone bosons dominate the low energy physics

Physics of Goldstone bosons

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quark mass $m_q \bar{q}_j q_j$ $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$ *condensate* $\langle \bar{q}_j q_j \rangle$

pseudo-Goldstone bosons dominate the low energy physics

e.g. $N_f = 2$ pions $\pi_0 = \pi^3$ $\pi_\pm = \frac{1}{\sqrt{2}} \left(\pi^1 \pm i\pi^2 \right)$

NLSM $\mathcal{L} = \frac{f_\pi^2}{4} \left\{ \mathrm{Tr} \left(\partial_\mu U \partial^\mu U^\dagger \right) + m_\pi^2 \mathrm{Tr} \left(U + U^\dagger \right) \right\}$ $U = e^{i \frac{\vec{\tau} \cdot \vec{\pi}}{f\pi}}$

$\mathcal{L}_2^{2\pi} = \frac{1}{2} \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - \frac{1}{2} m_\pi^2 \vec{\pi}^2$ $\mathcal{L}_2^{4\pi} = \frac{1}{6f_\pi^2} \left((\vec{\pi} \cdot \partial_\mu \vec{\pi})^2 - \vec{\pi}^2 (\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi}) \right) + \frac{m_\pi^2}{24 f_\pi^2} (\vec{\pi}^2)^2$ ...

The EFT approach

non-renormalizable, add new terms with unknown coefficients:

e.g.
$$
\begin{aligned}
\mathcal{L}_4 \;=\; & \frac{l_1}{4}\left\{\mathrm{Tr}[D_\mu U (D^\mu U)^\dagger]\right\}^2 + \frac{l_2}{4}\mathrm{Tr}[D_\mu U (D_\nu U)^\dagger]\mathrm{Tr}[D^\mu U (D^\nu U)^\dagger] \\
& + \frac{l_3}{16}[\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)]^2 + \frac{l_4}{4}\mathrm{Tr}[D_\mu U (D^\mu \chi)^\dagger + D_\mu \chi (D^\mu U)^\dagger] \\
& + l_5\left[\mathrm{Tr}(f^R_{\mu\nu} U f^{\mu\nu}_L U^\dagger) - \frac{1}{2}\mathrm{Tr}(f^L_{\mu\nu} f^{\mu\nu}_L + f^R_{\mu\nu} f^{\mu\nu}_R)\right] \\
& + i\frac{l_6}{2}\mathrm{Tr}[f^R_{\mu\nu} D^\mu U (D^\nu U)^\dagger + f^L_{\mu\nu}(D^\mu U)^\dagger D^\nu U] \\
& - \frac{l_7}{16}[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)]^2
\end{aligned}
$$

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&- \frac{l_7}{16}[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)]^2
\end{aligned}$$

χ**PT**: unknown coefficients determined from fitting amplitude with experimental data

in principle should be computed from UV gauge theory

The strong coupling problem

The strong coupling problem

chiSB *breaks down* **?** *breaks down* asymptotic freedom

weakly coupled pions (EFT) **strongly coupled regime** **weakly coupled quarks/gluons (pQCD)**

energy

The strong coupling problem → Gauge Theory Bootstrap

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compute the strongly coupled hadron dynamics:
amplitudes, form factors, correlation functions, spectrum/couplings

Gauge Theory Bootstrap

rules:

assume — *chiral symmetry breaking* & *confinement*

input — N_c N_f m_q α_s

defining gauge theory

m_π set the unit

f_π size of pion

theoretical/numerical computation:
- not using experimental scattering data as input
- no assumption on spectrum

The strong coupling problem → Gauge Theory Bootstrap

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Gauge Theory Bootstrap

this talk:
mostly test with
$N_f = 2$ $N_c = 3$

rules:

assume — *chiral symmetry breaking* & *confinement*

input — N_c N_f m_q α_s — *defining gauge theory* m_π set the unit f_π size of pion
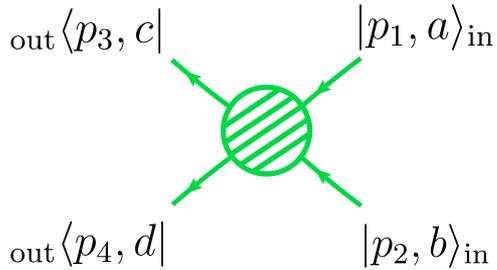
theoretical/numerical computation:
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2-to-2 amplitude

$$|\psi_1\rangle = |p_1, a; p_2, b\rangle_{\text{in}} \qquad |\psi_2\rangle = |p_1, a; p_2, b\rangle_{\text{out}} \qquad |\psi_3\rangle = \mathcal{O}_\ell^a(p)|0\rangle$$

[Karateev, Kuhn, Penedones, 2019]

$$_{out}\langle p_3, c; p_4, d|p_1, a; p_2, b\rangle_{in} = A(s,t,u)\delta_{ab}\delta_{cd} + A(t,s,u)\delta_{ac}\delta_{bd} + A(u,t,s)\delta_{ad}\delta_{bc}$$
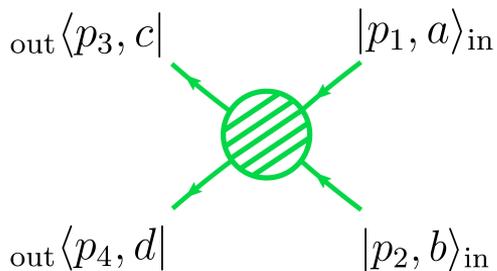
Lorentz invariants:
$$s = (p_1 + p_2)^2 \qquad t = (p_1 - p_3)^2 \qquad u = (p_1 - p_4)^2$$

CoM energy: $\sqrt{s}$ $\qquad\qquad p_i^2 = m_\pi^2 \qquad s + t + u = 4m_\pi^2$

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[Karateev, Kuhn, Penedones, 2019]

$_{\text{out}}\langle p_3, c|$ $|p_1, a\rangle_{\text{in}}$

$_{\text{out}}\langle p_4, d|$ $|p_2, b\rangle_{\text{in}}$

$_{out}\langle p_3, c; p_4, d|p_1, a; p_2, b\rangle_{in} = A(s,t,u)\delta_{ab}\delta_{cd} + A(t,s,u)\delta_{ac}\delta_{bd} + A(u,t,s)\delta_{ad}\delta_{bc}$

Lorentz invariants: $\quad s = (p_1 + p_2)^2 \qquad t = (p_1 - p_3)^2 \qquad u = (p_1 - p_4)^2$

CoM energy: $\quad \sqrt{s} \qquad\qquad p_i^2 = m_\pi^2 \qquad s + t + u = 4m_\pi^2$

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alternatively, consider 2-particle irreps states

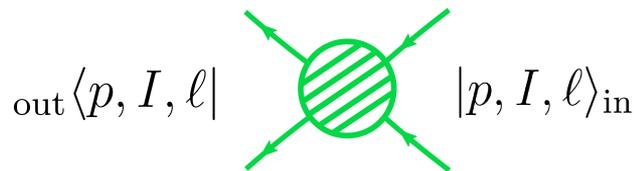$$|p_1, a; p_2, b\rangle \to |p, I, \ell\rangle$$

isospin   angular momentum

$$_{\text{out}}\langle p', I', \ell'|p, I, \ell\rangle_{\text{in}} = S_\ell^I(s)(2\pi)^4\delta^{(4)}(p - p')\delta^{II'}\delta_{\ell\ell'}$$

phase shift

2-pion partial amplitude $\quad S_\ell^I(s) = \eta_\ell^I(s)\, e^{2i\delta_\ell^I(s)}, \quad \forall \ell, I, s > 4m_\pi^2$

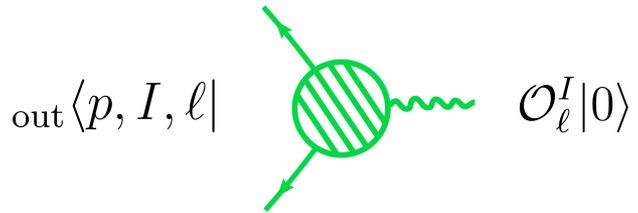$_{\text{out}}\langle p, I, \ell|$ $|p, I, \ell\rangle_{\text{in}}$

2-pion form factor and 2-point correlation functions

$$|\psi_1\rangle = |p_1, a; p_2, b\rangle_{\text{in}} \qquad |\psi_2\rangle = |p_1, a; p_2, b\rangle_{\text{out}} \qquad |\psi_3\rangle = \mathcal{O}_\ell^a(p)|0\rangle$$

2-pion form factor

$$_{\text{out}}\langle p, I, \ell|\mathcal{O}_\ell^I(0)|0\rangle = F_\ell^I(s) \qquad s = p^2$$

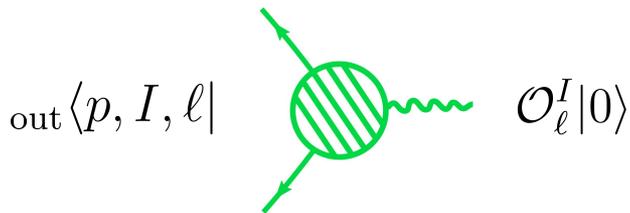$$_{\text{out}}\langle p, I, \ell| \qquad \mathcal{O}_\ell^I|0\rangle$$

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[Karateev, Kuhn, Penedones, 2019]

2-pion form factor

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$$_{\text{out}}\langle p, I, \ell| \qquad \mathcal{O}_\ell^I|0\rangle$$

2-point correlation function

$$\langle 0|\mathcal{O}_\ell^{I\dagger}(p)\mathcal{O}_\ell^I(p)|0\rangle = \rho_\ell^I(s) \qquad s = p^2$$

$$\langle 0|\mathcal{O}_\ell^{I\,\dagger} \qquad \mathcal{O}_\ell^I|0\rangle$$

time-ordered:

$$\Pi_\ell^I(s) = i \int \frac{d^4x}{(2\pi)^4} e^{ip\cdot x} \langle 0|\hat{T}\left\{\mathcal{O}_\ell^{I\dagger}(x)\mathcal{O}_\ell^I(0)\right\}|0\rangle$$

Generic constraints: analyticity
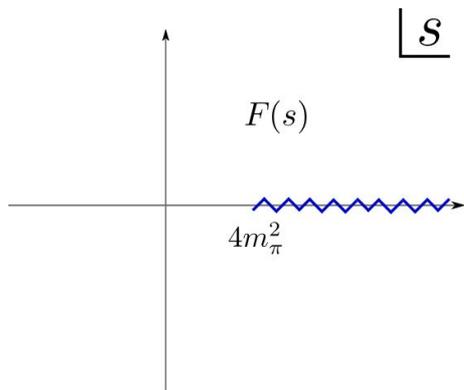
Analyticity

$$A(s,t,u) \quad s+t+u = 4m_\pi^2 \qquad \text{analytic function of two variables}$$

(causality)

cuts $\quad s,t,u > 4m_\pi^2$
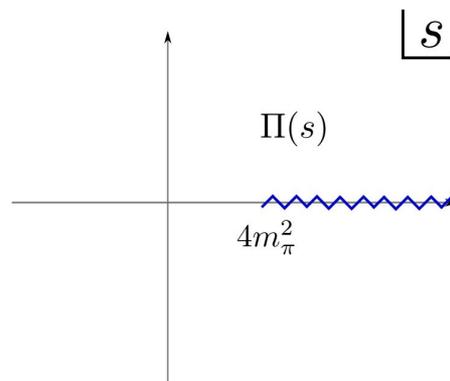
cuts: multi-particle states

$F(s)$

$\Pi(s)$

analytic function

cuts $\quad s > 4m_\pi^2$

$$\rho(s) = 2\text{Im}\Pi(s + i\epsilon)$$

$$s > 4m_\pi^2$$

Generic constraints: crossing and unitarity

Crossing (exchange symmetry)

$$A(s, t, u) = A(s, u, t)$$

Generic constraints: crossing and unitarity

Crossing   (exchange symmetry)

$$A(s,t,u) = A(s,u,t)$$

Unitarity   (probability conservation)

$$\langle \psi_a | \psi_b \rangle = \begin{array}{c} {}^{\pi\pi}_{\text{out}}\langle p,I,\ell| \\ {}^{\pi\pi}_{\text{in}}\langle p,I,\ell| \\ \langle 0|\mathcal{O}_\ell^{I\,\dagger} \end{array} \begin{array}{ccc} |p,I,\ell\rangle^{\pi\pi}_{\text{out}} & |p,I,\ell\rangle^{\pi\pi}_{\text{in}} & \mathcal{O}_\ell^I|0\rangle \end{array}$$

$$\langle \psi_a | \psi_b \rangle = \begin{bmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I(s) \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*}(s) \\ \mathcal{F}_\ell^{I*}(s) & \mathcal{F}_\ell^I(s) & \rho_\ell^I(s) \end{bmatrix} \succeq 0 \qquad \forall \ell, I, s > 4m_\pi^2$$

positive semidefinite matrix

[Paulos, Penedones, Toledo, van Rees, Vieira 2016 & 2017]
[Karateev, Kuhn, Penedones, 2019]

Non-perturbative parametrizations

$$\rho_2(x,y) = \rho_2(y,x)$$

$$A(s,t,u) = \frac{1}{\pi^2} \int_{4m_\pi^2}^{\infty} dx \int_{4m_\pi^2}^{\infty} dy \left[\frac{\rho_1(x,y)}{(x-s)(y-t)} + \frac{\rho_1(x,y)}{(x-s)(y-u)} + \frac{\rho_2(x,y)}{(x-t)(y-u)} \right] \text{+subtractions}$$

Analyticity
&
Crossing

$$F_\ell^I(s) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} dx \frac{\mathrm{Im}F_\ell^I(x)}{x-s} \text{ +subtractions}$$

$$\rho_\ell^I(s) \quad \textit{supported at} \quad s > 4m_\pi^2$$

parameters: $\left\{ \rho_1(x,y), \rho_2(x,y) = \rho_2(y,x), \mathrm{Im}F_\ell^I(x), \rho_\ell^I(x), \dots \right\}, \quad \forall I, \ell, x, y \in (4m_\pi^2, \infty)$

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Analyticity & Crossing

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$$\begin{pmatrix} 1 & S & \mathcal{F} \\ S^* & 1 & \mathcal{F}^* \\ \mathcal{F}^* & \mathcal{F} & \rho \end{pmatrix} \succeq 0$$

subject to

Unitarity

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numerics: discretize $\longrightarrow$ bootstrap variables

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$\rho_\ell^I(s)$ *supported at* $s > 4m_\pi^2$

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Unitarity

parameters: $\left\{ \rho_1(x,y), \rho_2(x,y) = \rho_2(y,x), \text{Im}F_\ell^I(x), \rho_\ell^I(x), \dots \right\}, \quad \forall I, \ell, x, y \in (4m_\pi^2, \infty)$

numerics: discretize $\longrightarrow$ bootstrap variables

If we can solve these variables, we can use them to construct the full analytic functions from them, we know these observables at all energies, i.e. along the whole RG flow

S-matrix bootstrap: old

Bootstrap (pre-QCD): solve the theory of **strong interaction** *from these constraints*   [Chew... , 1960s]

amplitudes

Symmetry+Analyticity+Crossing+Unitarity

constrained bootstrap variables $\{\rho_{1,2}(x,y),\ldots\}$

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convex space of generic amplitudes

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i.e. space of constrained bootstrap variables $\{\rho_{1,2}(x,y),\dots\}$

generic, infinitely many solutions

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maximize linear functional

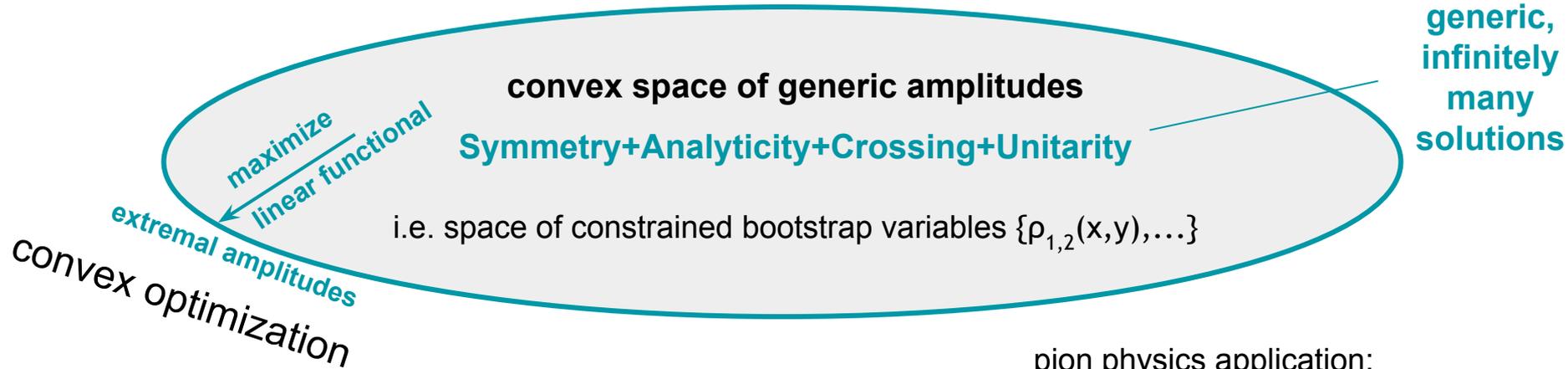extremal amplitudes

convex optimization
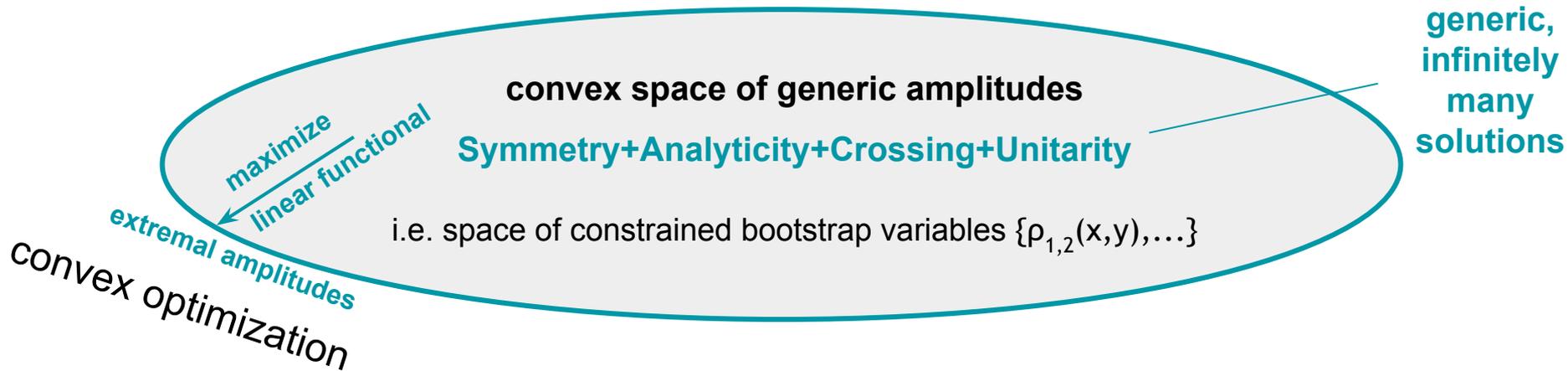
Modern S-matrix bootstrap: bound physical quantities

[Paulos, Penedones, Toledo, van Rees, Vieira 2016 & 2017]

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Modern S-matrix bootstrap: bound physical quantities

[Paulos, Penedones, Toledo, van Rees, Vieira 2016 & 2017]

recent pheno application:

fit experimental scattering data [Guerrieri, Haring, Su, 2025]

pion physics application:

bound low energy parameters

(Paulos, Penedones, Toledo, van Rees, Vieira, Guerrieri,

Chen, Fitzpatrick, Karateev, Sever, Miro, Gumus, Albert,

Rastelli, Henriksson, Vichi, Riva, Fernandez, Pomarl,

Sciotti, Ma, De Rham, Tolley, Wang, Zhou…)

Gauge Theory Bootstrap: the philosophy

instead of putting generic bounds

back to the original motivation of bootstrap:

***solve* the theory of strong interaction**

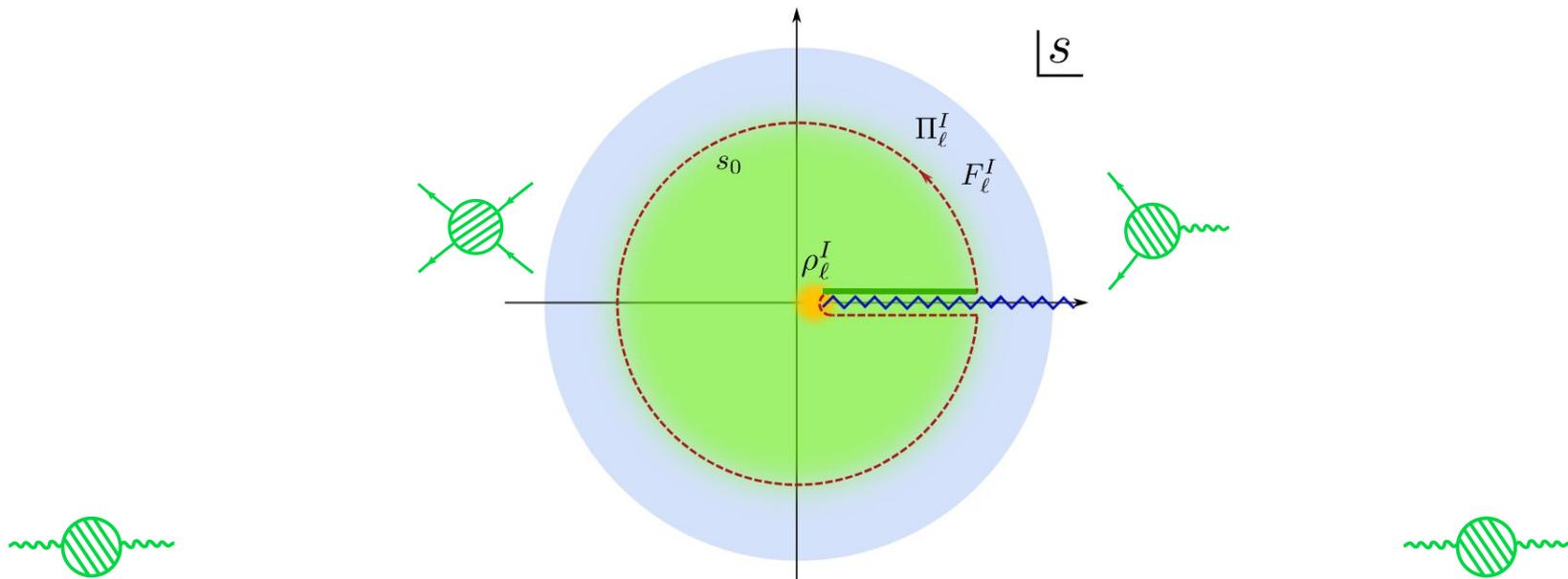**do computations of strong dynamics**

with the dynamical UV information of gauge theory (QCD)

naive expectation:

given enough theoretical UV input, should find unique solution (within errors)

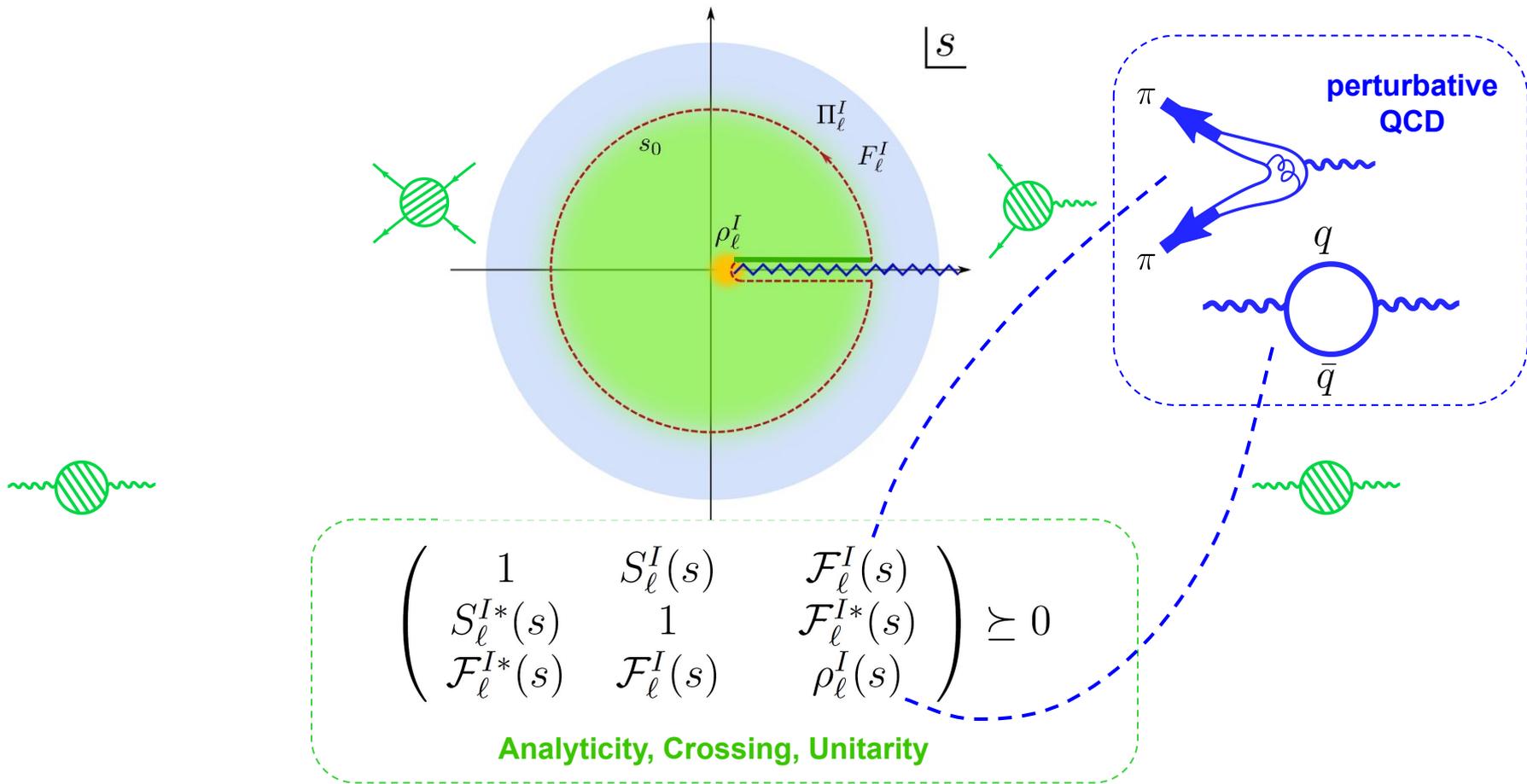***closer to first-principle computation of strongly coupled dynamics of QCD***
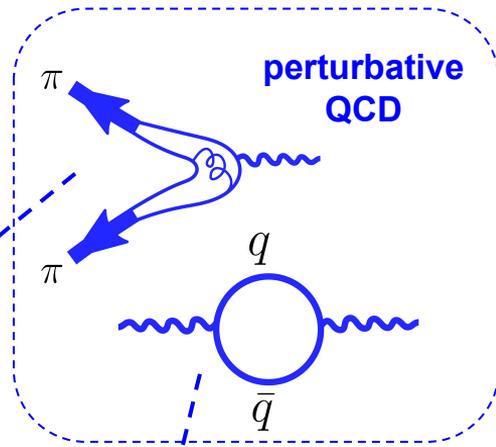
Gauge Theory Bootstrap: the recipe

$$\begin{pmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I(s) \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*}(s) \\ \mathcal{F}_\ell^{I*}(s) & \mathcal{F}_\ell^I(s) & \rho_\ell^I(s) \end{pmatrix} \succeq 0$$

Analyticity, Crossing, Unitarity

Gauge Theory Bootstrap: the recipe

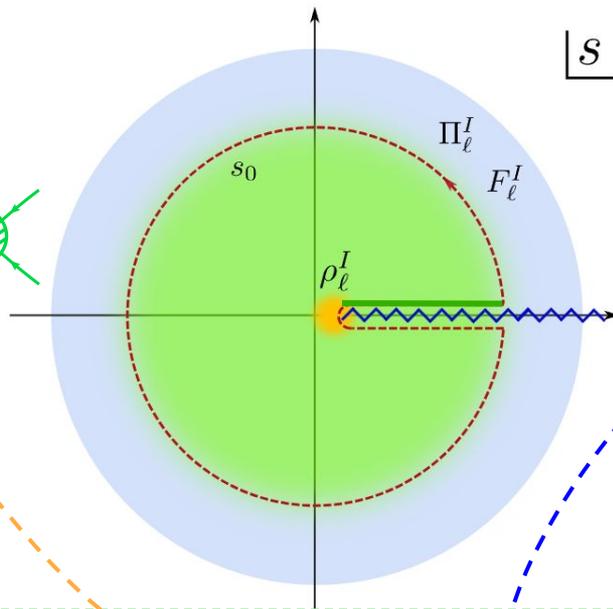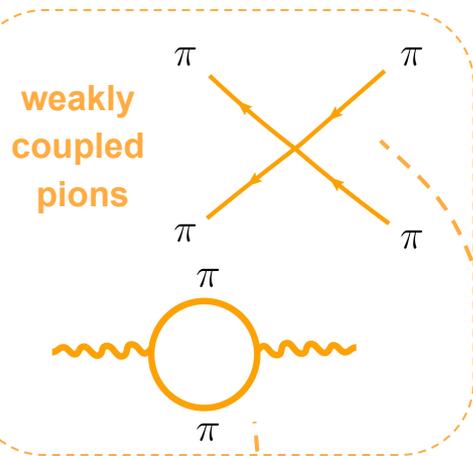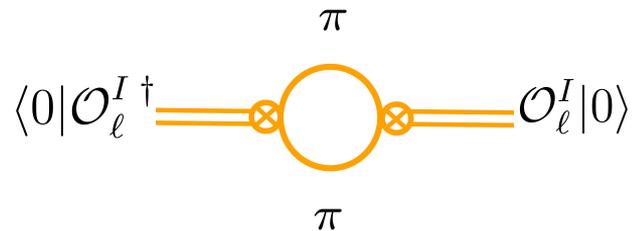# Gauge Theory Bootstrap: the recipe

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weakly coupled pions

perturbative QCD

Analyticity, Crossing, Unitarity

IR limit: free pion current correlator

Free pion Lagrangian

$$\mathcal{L}_2^{2\pi} = \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{1}{2}m_\pi^2\vec{\pi}^2$$

$$\langle 0|\mathcal{O}_\ell^{I\,\dagger}\quad\quad\quad\mathcal{O}_\ell^I|0\rangle$$

e.g.

SU(2)$_V$ vector current $\quad I = 1, \ell = 1 \quad\quad \mathcal{O}_{\ell=1}^{I=1} \simeq \epsilon^{abc}\pi^b\partial_\mu\pi^c + \mathcal{O}(\pi^4)$

energy-momentum tensor $\quad I = 0, \ell = 2 \quad\quad \mathcal{O}_{\ell=2}^{I=0} \simeq \partial_\mu\pi^a\partial_\nu\pi^a - \frac{1}{2}(\partial_\alpha\pi^a\partial^\alpha\pi^a - m_\pi^2\pi^a\pi^a)\eta_{\mu\nu} + \mathcal{O}(\pi^4)$

$$\vdots$$

IR limit: free pion current correlator

Free pion Lagrangian

$$\mathcal{L}_2^{2\pi} = \frac{1}{2}\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - \frac{1}{2}m_\pi^2 \vec{\pi}^2$$

$$\langle 0|\mathcal{O}_\ell^{I\,\dagger} \qquad \mathcal{O}_\ell^I |0\rangle$$

π

π

e.g.

SU(2)$_V$ vector current $\quad I = 1, \ell = 1 \qquad \mathcal{O}_{\ell=1}^{I=1} \simeq \epsilon^{abc}\pi^b \partial_\mu \pi^c + \mathcal{O}(\pi^4)$
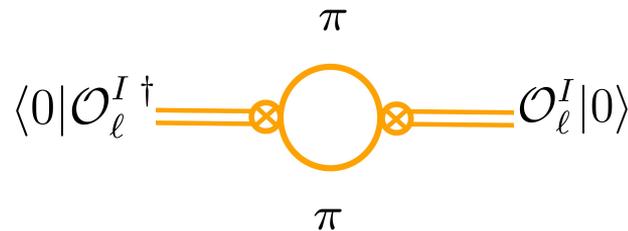
energy-momentum tensor $\quad I = 0, \ell = 2 \qquad \mathcal{O}_{\ell=2}^{I=0} \simeq \partial_\mu \pi^a \partial_\nu \pi^a - \frac{1}{2}(\partial_\alpha \pi^a \partial^\alpha \pi^a - m_\pi^2 \pi^a \pi^a)\eta_{\mu\nu} + \mathcal{O}(\pi^4)$

$\vdots$

leading low energy behavior
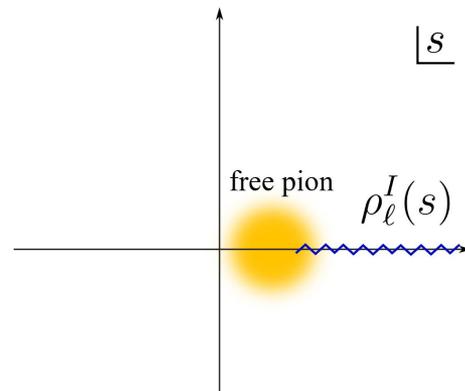of spectral density

[Gasser, Leutwyler, 1983]

$$\rho_1^1(s) \simeq \frac{1}{(2\pi)^4}\frac{s}{24\pi}\left(1 - \frac{4m_\pi^2}{s}\right)^{\frac{3}{2}}$$

$$\rho_2^0(s) \simeq \frac{1}{(2\pi)^4}\frac{s^2}{160\pi}\left(1 - \frac{4m_\pi^2}{s}\right)^{\frac{5}{2}}$$

$\lfloor s$

free pion $\quad \rho_\ell^I(s)$

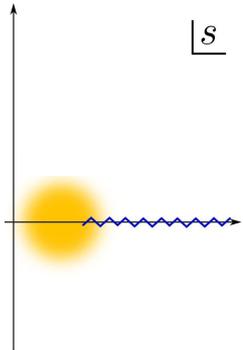Numerics: parameterize the spectral density with this low energy threshold behavior

IR limit: tree-level amplitude

interaction: $\mathcal{L}_2^{4\pi} = \dfrac{1}{6f_\pi^2}\Big((\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})\Big) + \dfrac{m_\pi^2}{24f_\pi^2}(\vec{\pi}^2)^2$

${}_{\text{out}}\langle p, I, \ell|$ $|p, I, \ell\rangle_{\text{in}}$

tree-level amplitude: $A_{\text{tree}}(s,t,u) = \dfrac{4}{\pi}\dfrac{s - m_\pi^2}{32\pi f_\pi^2}$ [Weinberg, 1966]

S0: $f_{0,\text{tree}}^0(s) = \dfrac{2}{\pi}\dfrac{2s - m_\pi^2}{32\pi f_\pi^2}$ P1: $f_{1,\text{tree}}^1(s) = \dfrac{2}{\pi}\dfrac{s - 4m_\pi^2}{96\pi f_\pi^2}$ S2: $f_{0,\text{tree}}^2(s) = \dfrac{2}{\pi}\dfrac{2m_\pi^2 - s}{32\pi f_\pi^2}$

Numerics: require partial waves to match this at very low energy

UV limit: current correlator at high energy

QCD Lagrangian
$$\mathcal{L} = i \sum_{j}^{N_f} \bar{q}_j \slashed{D} q_j - \sum_{j}^{N_f} m_q \bar{q}_j q_j - \frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a$$

e.g.

SU(2)$_V$ vector current $\quad I = 1, \ell = 1 \quad \mathcal{O}_{\ell=1}^{I=1} = \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)$

energy-momentum tensor $\quad I = 0, \ell = 2 \quad \mathcal{O}_{\ell=2}^{I=0} = \bar{u}\,i\gamma^{(\mu}D^{\nu)}u + \bar{d}\,i\gamma^{(\mu}D^{\nu)}d + G^{a\,\mu\lambda}G^{a\,\nu}{}_\lambda + g^{\mu\nu}\mathcal{L}_{\mathrm{QCD}}$
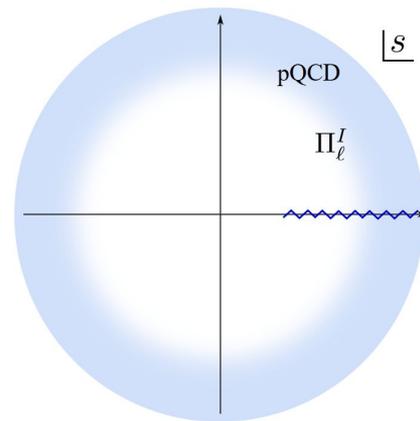
$$\vdots$$

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$\vdots$

time-ordered 2-point correlation function

$$\Pi_\ell^I(s) = i \int \frac{d^4x}{(2\pi)^4} e^{ip\cdot x} \langle 0|\hat{T}\left\{\mathcal{O}_\ell^{I\dagger}(x)\mathcal{O}_\ell^I(0)\right\}|0\rangle$$

asymptotic freedom: compute in perturbative QCD

good at short distance — large momenta region

Short distance expansion

[Shifman, Vainshtein, Zakharov, 1979]

at the short distance
operator product expansion:
$$T\{\mathcal{O}^{\dagger}(x)\mathcal{O}(0)\} \overset{x\to 0}{=} C_{\mathbb{1}}(x)\,\mathbb{1} + \sum_{\mathcal{O}} C_{\mathcal{O}}(x)\,\mathcal{O}(0)$$

$$\langle 0|T\{\mathcal{O}^{\dagger}(x)\mathcal{O}(0)\}|0\rangle \overset{x\to 0}{=} C_{\mathbb{1}}(x) + \dots$$

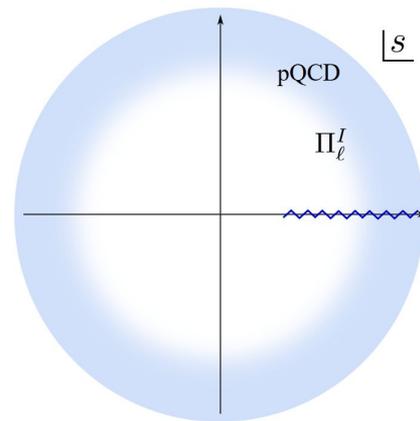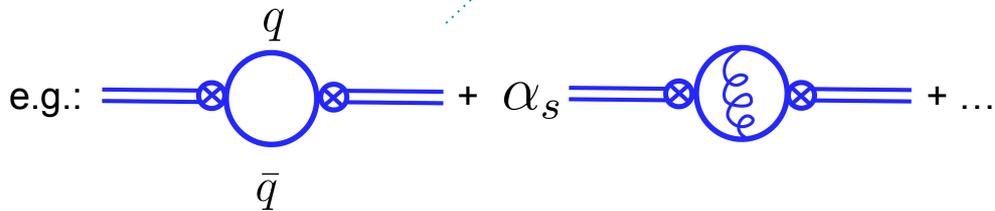**OPE coefficients: perturbative QCD**

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OPE coefficients: perturbative QCD

$$\Pi(s) = i \int \frac{d^4 x}{(2\pi)^4} e^{ip\cdot x} \langle 0|\hat{T}\{\mathcal{O}^\dagger(x)\mathcal{O}(0)\}|0\rangle$$

e.g.:

large energy expansion of time-ordered two-point correlator: e.g. spin 1 vector current

$$\Pi_1^1(s) = \frac{1}{2}\frac{1}{(2\pi)^4}\left\{-\frac{N_c}{12\pi^2}\left(1+\frac{\alpha_s}{\pi}\right) s \ln(-\frac{s}{\mu^2}) + \dots\right\}$$
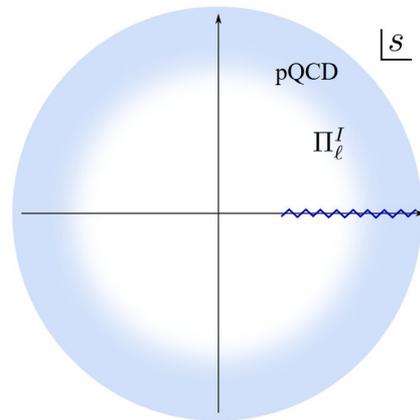
Finite energy sum rules
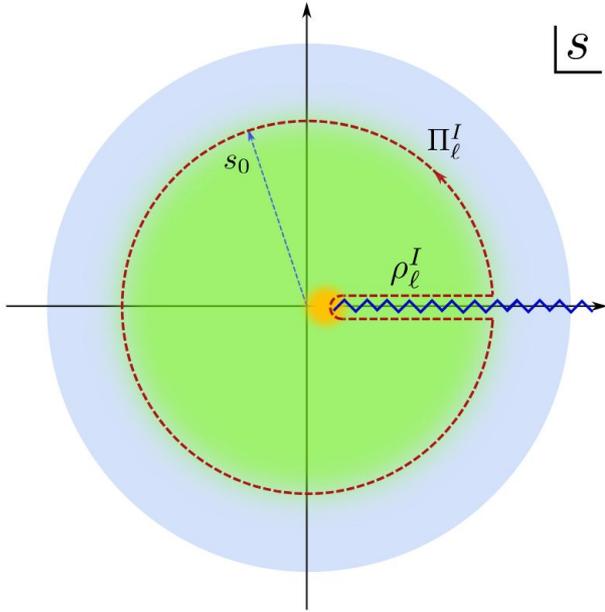
connect strongly coupled with short distance region at large s_0

contour integral $\quad s^n \Pi(s) \quad$ vanishes

Finite energy sum rules
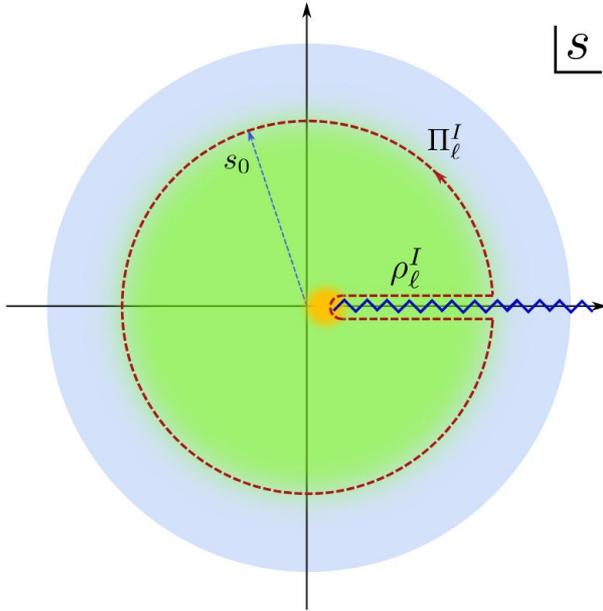
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moments of correlator

large momenta expansion

$$\int_{4m_\pi^2}^{s_0} \rho(x) x^n dx = -s_0^{n+1} \int_0^{2\pi} e^{i(n+1)\varphi} \Pi(s_0 e^{i\varphi}) d\varphi$$

Finite energy sum rules
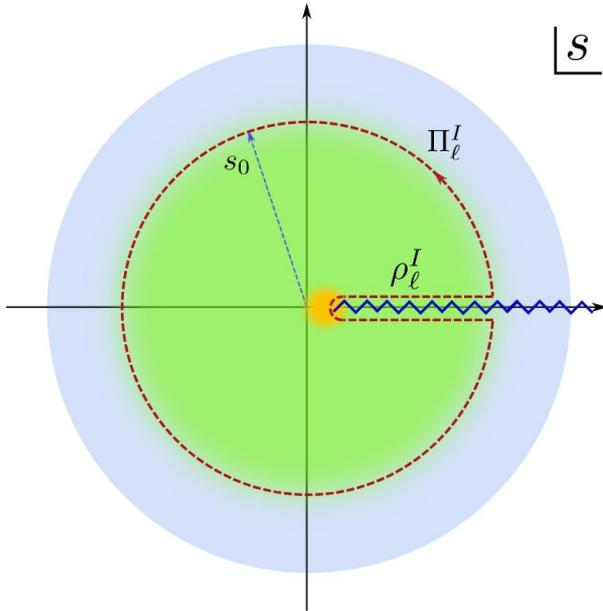
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microscopic theory parameters enter

Numerics: constrain the moments using pQCD results

e.g.: spin 1 vector current

$$P1 \; : \; \frac{1}{s_0^{n+2}} \int_{4m_\pi^2}^{s_0} \rho_1^1(x)x^n dx = \frac{1}{2(2\pi)^4} \left\{ \frac{N_c}{6\pi(n+2)} \left(1 + \frac{\alpha_s}{\pi}\right) + \dots \right\}, \; n \geq -1$$

UV limit: form factor at high momentum transfer

[Lepage, Brodsky, 1979]

pQCD predict the the form factors at asymptotically high momentum transfer

Leading contribution:

$$s = p^2$$

$$\alpha_s$$

$$q$$

$$\bar{q}$$

$$\langle \pi | \qquad | \pi \rangle$$

$$\frac{1}{R_\pi^2} \sim f_\pi^2$$

$$pQCD \qquad \lfloor s$$

$$F_\ell^I$$

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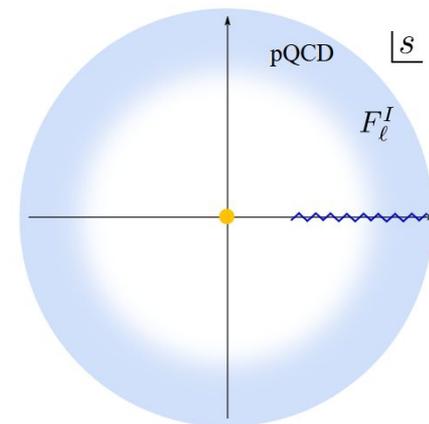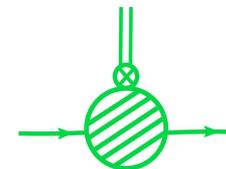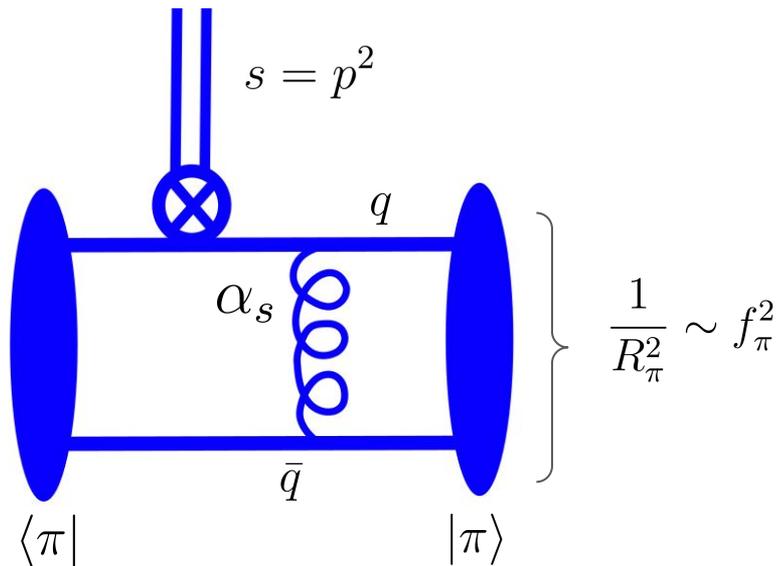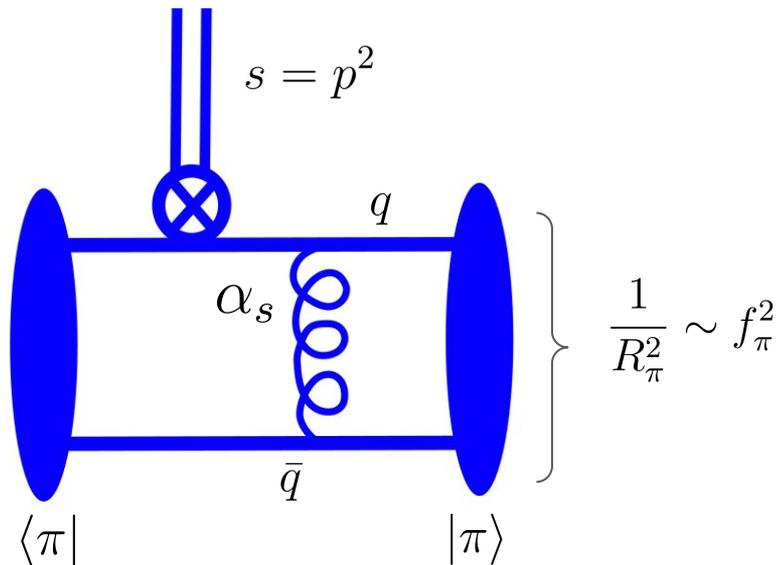
the only f_π input:

the size of pion

e.g. spin 1 vector current

$$F_\pi(s) \simeq - \frac{16\pi \alpha_s(s) f_\pi^2}{s}$$

Numerics: impose this behavior above s_0

pQCD $\lfloor s$

F_ℓ^I

Summary of the GTB constraints

e.g. $I = 1, \ell = 1$ $\qquad$ $\epsilon^{abc}\pi^b\partial_\mu\pi^c \simeq$ $\quad$ j_V^μ $= \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)$ $\qquad$ *identify local operator*

IR $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ **UV**

free pion kinematics $\qquad\qquad$ **two-point function** $\qquad\qquad$ *pQCD sum rules*

$$\rho_1^1(s) \simeq \frac{1}{(2\pi)^4}\frac{s}{24\pi}\left(1 - \frac{4m_\pi^2}{s}\right)^{\frac{3}{2}} \qquad \langle 0|\mathcal{O}_\ell^{I\,\dagger} \text{〜〜} \mathcal{O}_\ell^I|0\rangle \qquad \frac{1}{s_0^{n+2}}\int_{4m_\pi^2}^{s_0}\rho_1^1(x)x^n dx \simeq \frac{1}{(2\pi)^4}\frac{1}{4\pi(n+2)}\left(1 + \frac{\alpha_s}{\pi}\right)$$

normalization $\qquad\qquad\qquad$ **form factor** $\qquad\qquad\qquad$ *asymptotics*

$$F_1^1(0) = 1 \qquad\qquad {}_{\text{out}}\langle p, I, \ell| \text{〜} \mathcal{O}_\ell^I|0\rangle \qquad\qquad F_\pi(s) \simeq -\frac{16\pi\alpha_s(s)f_\pi^2}{s}$$

analyticity
crossing
unitarity
$$\begin{pmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I(s) \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*}(s) \\ \mathcal{F}_\ell^{I*}(s) & \mathcal{F}_\ell^I(s) & \rho_\ell^I(s) \end{pmatrix} \succeq 0 \quad s > 4m_\pi^2, \;\; \forall I, \ell$$

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how to find solution?

Intuition: saturating the matrix of state overlaps

Ignoring other interactions, pion (and nucleon) is the only stable particle in QCD

Hilbert space spanned by
n-pion in/out states:

$|\pi\rangle, \ |\pi\pi\rangle, \ |\pi\pi\pi\rangle, \ |\pi\pi\pi\pi\rangle, \ \ldots$

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States created by local operators
encoding QCD information:

$$|\pi\rangle, \ |\pi\pi\rangle, \ |\pi\pi\pi\rangle, \ |\pi\pi\pi\pi\rangle, \ \ldots \qquad \mathcal{O}_1|0\rangle, \ \ \mathcal{O}_2|0\rangle, \ \ \mathcal{O}_3|0\rangle, \ \ \ldots$$

$\langle\pi|$
$\langle\pi\pi|$
$\langle\pi\pi\pi|$
$\langle\pi\pi\pi\pi|$
$\vdots$
$\langle 0|\mathcal{O}_1^\dagger$
$\langle 0|\mathcal{O}_2^\dagger$
$\langle 0|\mathcal{O}_3^\dagger$
$\vdots$

$$\succeq 0$$

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$\langle\pi\pi|$
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$\langle\pi\pi\pi\pi|$
$\vdots$
$\langle 0|\mathcal{O}_1^\dagger$
$\langle 0|\mathcal{O}_2^\dagger$
$\langle 0|\mathcal{O}_3^\dagger$
$\vdots$

n-pion in/out-states are each complete

⬇

- *in/out-states can be expanded in terms of each other*
- *states created by local operators can be expanded in terms of n-pion states*

⬇

matrix has zero modes

$\succeq 0$

saturate

Physics of the zero modes

Currently, consider only two-pion (in & out) states and one operator

$$\begin{bmatrix} _{\text{out}}\langle\pi\pi|\pi\pi\rangle_{\text{out}} & _{\text{out}}\langle\pi\pi|\pi\pi\rangle_{\text{in}} & _{\text{out}}\langle\pi\pi|\mathcal{O}|0\rangle \\ _{\text{in}}\langle\pi\pi|\pi\pi\rangle_{\text{out}} & _{\text{in}}\langle\pi\pi|\pi\pi\rangle_{\text{in}} & _{\text{in}}\langle\pi\pi|\mathcal{O}|0\rangle \\ \langle 0|\mathcal{O}^\dagger|\pi\pi\rangle_{\text{out}} & \langle 0|\mathcal{O}^\dagger|\pi\pi\rangle_{\text{in}} & \langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle \end{bmatrix} \succeq 0 \qquad \forall \ell, I, s > 4m_\pi^2$$

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$$\mathcal{F} \sim {}_{\text{out}}\langle\pi\pi|\mathcal{O}|0\rangle$$

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zero modes of the matrix

$$\rho \sim \langle0|\mathcal{O}^\dagger\mathcal{O}|0\rangle = \langle0|\mathcal{O}^\dagger|\pi\pi\rangle_{\text{in}} \, {}_{\text{in}}\langle\pi\pi|\mathcal{O}|0\rangle + \ldots \simeq |\mathcal{F}|^2$$

insert a complete
set of in-states:

$$\mathbb{1} = |X\rangle\langle X| = |\pi\rangle_{\text{in}} \, {}_{\text{in}}\langle\pi| + |\pi\pi\rangle_{\text{in}} \, {}_{\text{in}}\langle\pi\pi| + |\pi\pi\pi\rangle_{\text{in}} \, {}_{\text{in}}\langle\pi\pi\pi| + \ldots$$

$$\mathcal{F} \sim {}_{\text{out}}\langle\pi\pi|\mathcal{O}|0\rangle = {}_{\text{out}}\langle\pi\pi|\pi\pi\rangle_{\text{in}} \, {}_{\text{in}}\langle\pi\pi|\mathcal{O}|0\rangle + \ldots \simeq S\mathcal{F}^*$$

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equality exact up to certain energy

insert a complete set of in-states:

$$\mathbb{1} = |X\rangle\langle X| = |\pi\rangle_{\text{in}}\ {}_{\text{in}}\langle\pi| + |\pi\pi\rangle_{\text{in}}\ {}_{\text{in}}\langle\pi\pi| + |\pi\pi\pi\rangle_{\text{in}}\ {}_{\text{in}}\langle\pi\pi\pi| + \ldots$$

approximate above — enlarge the matrix with more states

$$\mathcal{F} \sim {}_{\text{out}}\langle\pi\pi|\mathcal{O}|0\rangle = {}_{\text{out}}\langle\pi\pi|\pi\pi\rangle_{\text{in}}\ {}_{\text{in}}\langle\pi\pi|\mathcal{O}|0\rangle + \ldots \simeq S\mathcal{F}^*$$

write QCD operator states in terms of on-shell pion states and vice versa

The zero modes

Zero modes: nonlinear in bootstrap variables, hard to solve. Some observations:

$$G_{\mathrm{SDP}} = \begin{pmatrix} 1 & S_\ell(s) & \mathcal{F}_\ell(s) \\ S_\ell^*(s) & 1 & \mathcal{F}_\ell^*(s) \\ \mathcal{F}_\ell^*(s) & \mathcal{F}_\ell(s) & \rho(s) \end{pmatrix} \longrightarrow G_{\mathrm{z.m.}} = \begin{pmatrix} 1 & \frac{\mathcal{F}_\ell(s)}{\mathcal{F}_\ell^*(s)} & \mathcal{F}_\ell(s) \\ \frac{\mathcal{F}_\ell^*(s)}{\mathcal{F}_\ell(s)} & 1 & \mathcal{F}_\ell^*(s) \\ \mathcal{F}_\ell^*(s) & \mathcal{F}_\ell(s) & \mathcal{F}_\ell(s)\mathcal{F}_\ell^*(s) \end{pmatrix}$$

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two zero modes:

$$v_1 = \begin{pmatrix} -\mathcal{F}_\ell(s) \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -\frac{\mathcal{F}_\ell(s)}{\mathcal{F}_\ell^*(s)} \\ 1 \\ 0 \end{pmatrix}$$

generically SDP satisfy:

$$\mathrm{Tr}(M \cdot G_{\mathrm{SDP}}) \geq 0, \quad \forall \, M = vv^\dagger$$

satisfying

minimum

$$\mathrm{Tr}(M_i \cdot G_{\mathrm{z.m.}}) = 0, \quad M_i = v_i v_i^\dagger, \quad i = 1, 2$$

Iterative procedure for finding GTB solution

- start with an arbitrary initial solution satisfying GTB constraints

e.g. maximizing some linear functional, or an arbitrary feasible point by min 0

-

-

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⬇

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$$\mathcal{F}_\ell^{(\text{old})}(s) \longrightarrow v \longrightarrow M^{(\text{old})} = vv^\dagger$$

linear functional in new variables

$$\min \sum \text{Tr}\big(M^{(\text{old})} \cdot G^{(\text{new})}\big) \longrightarrow G^{(\text{new})}$$

convex optimization, efficient to solve

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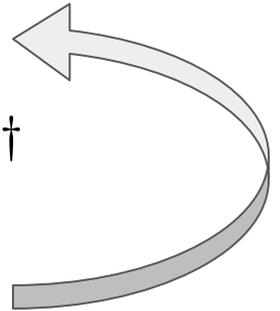
- Iterate until converge   **GTB solution is a fix point of the procedure**

Test: Computing pion quartic coupling

$$\lambda = \frac{1}{32\pi} \, \mathcal{M}(\pi^0 \pi^0 \to \pi^0 \pi^0)\big|_{s=t=u=\frac{4}{3}m_\pi^2} = \frac{3\pi}{4} A\left(\frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2\right)$$

Shuang-Yong's talk

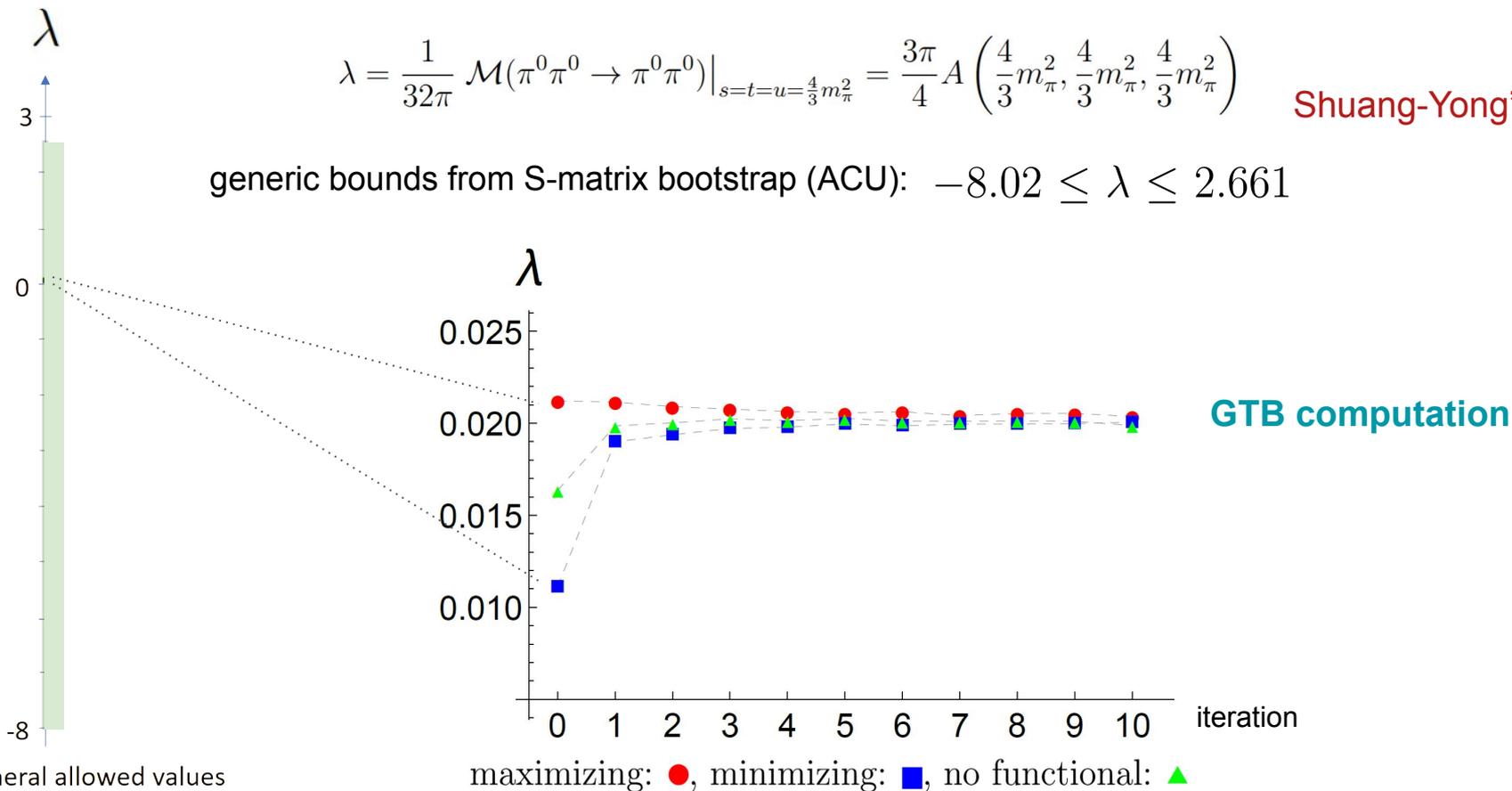generic bounds from S-matrix bootstrap (ACU): $-8.02 \leq \lambda \leq 2.661$

λ

3

0

-8

General allowed values

Test: Computing pion quartic coupling

$$\lambda = \frac{1}{32\pi} \mathcal{M}(\pi^0\pi^0 \to \pi^0\pi^0)\big|_{s=t=u=\frac{4}{3}m_\pi^2} = \frac{3\pi}{4} A\left(\frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2\right)$$

Shuang-Yong's talk

generic bounds from S-matrix bootstrap (ACU): $-8.02 \le \lambda \le 2.661$

GTB computation

General allowed values

maximizing: ●, minimizing: ■, no functional: ▲

Partial waves from 1-10 iterations

starting from
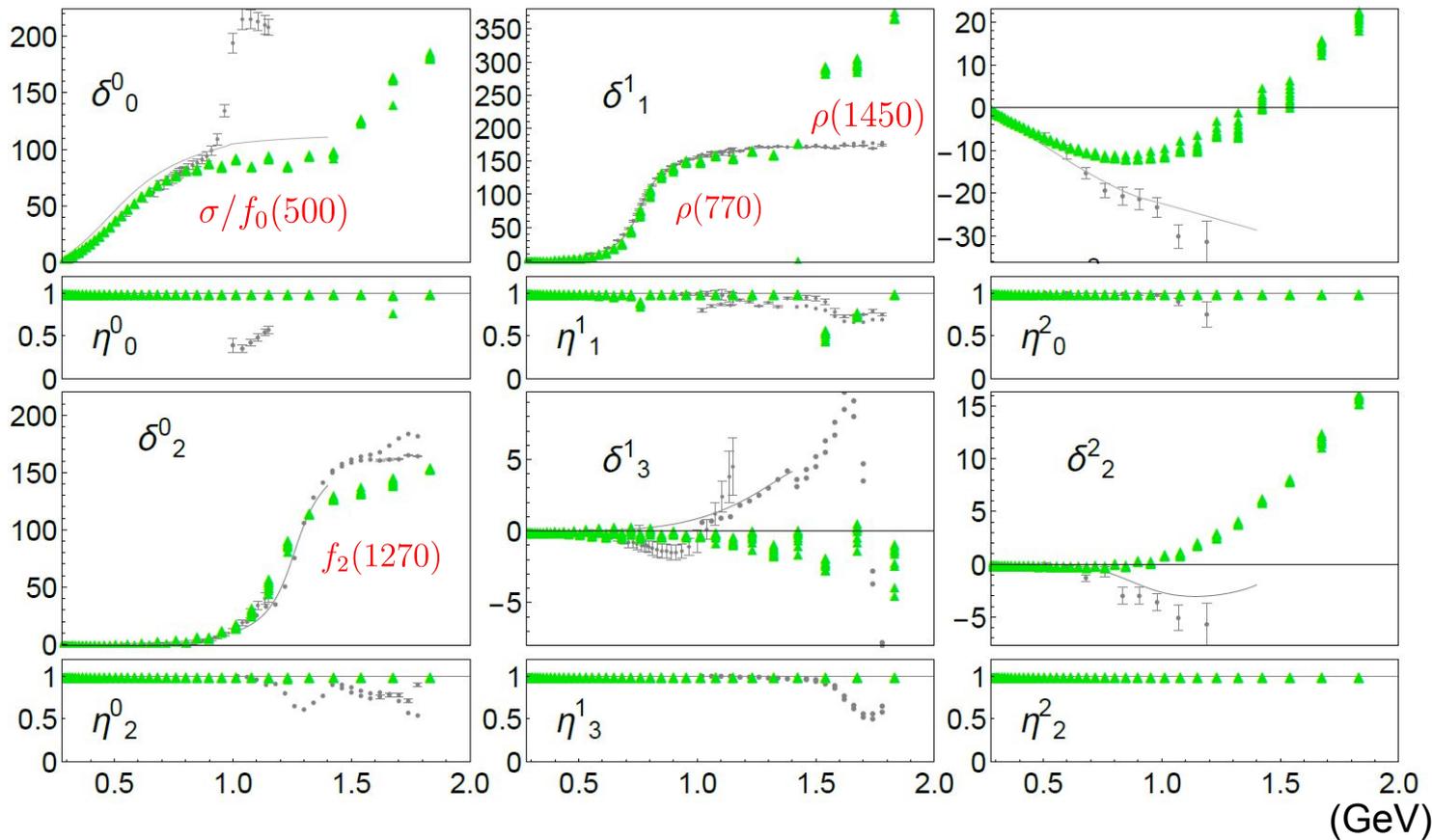a generic
feasible point

**GTB
solution**

$N_f = 2 \quad N_c = 3$

experimental data
(gray dots)
[Protopopescu et al, 1973]
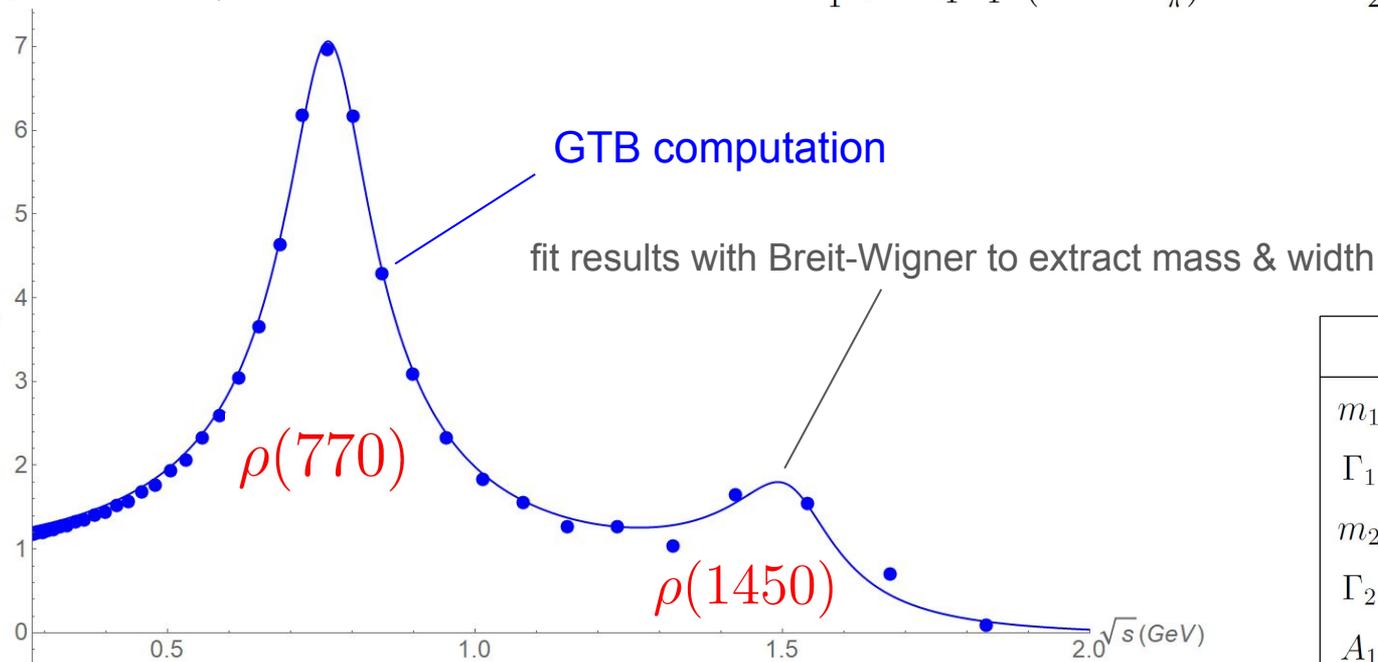[Losty et al, 1974]
pheno fit
(gray line)
[Pelaez, Yndurain, 2005]

δ^0_0
$\sigma / f_0(500)$

δ^1_1
$\rho(1450)$
$\rho(770)$

η^0_0
η^1_1
η^2_0

δ^0_2
$f_2(1270)$

δ^1_3

δ^2_2

η^0_2
η^1_3
η^2_2

(GeV)

1-loop amplitude & form factors: computing LEC

1-loop renormalized coefficients at $\mu = m_\pi$

	GTB	GL	Bij	CGL
		theory	fitting experiment	
$\bar{\ell}_1$	1.6	-2.3 ± 3.7	-1.7 ± 1.0	-0.4 ± 0.6
$\bar{\ell}_2$	5.5	6.0 ± 1.3	6.1 ± 0.5	4.3 ± 0.1
$\bar{\ell}_3$	7.8	2.9 ± 2.4		
$\bar{\ell}_4$	4.7	4.3 ± 0.9	4.4 ± 0.3	4.4 ± 0.2
$\bar{\ell}_6$	14.3	18.7 ± 1.1	16.0 ± 0.5 ± 0.7	

	GTB	Exp.	W
λ	0.02		0.023
f_π (MeV)	101	92	

pion coupling

Vector form factor $F_1^1(s)$

$$F_1^1(s) = \frac{-A_1 m_1^2}{s - m_1^2 + i m_1 \Gamma_1 \theta(s - 4m_\pi^2)} + \frac{A_2 m_2^2}{s - m_2^2 + i m_2 \Gamma_2 \theta(s - 4m_\pi^2)}$$
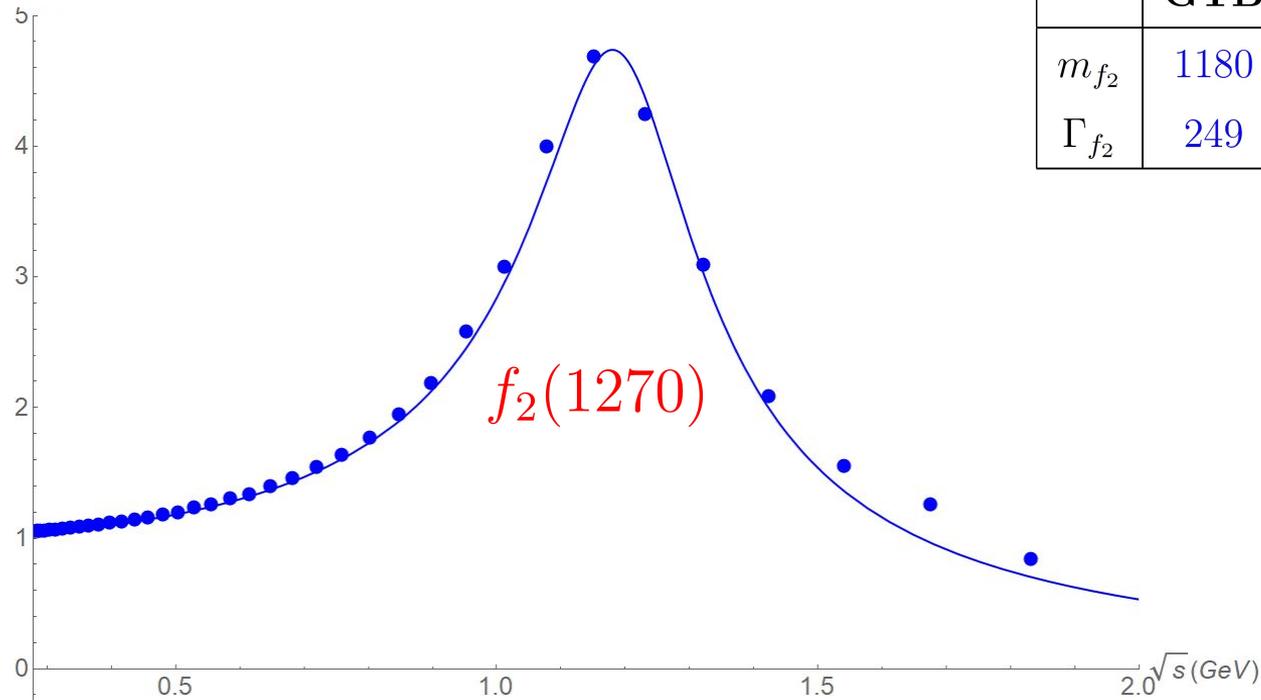
$F_1^1(s) \sim \langle 0 | j_V^\mu | \pi^+ \pi^- \rangle$

GTB computation

fit results with Breit-Wigner to extract mass & width

$\rho(770)$

$\rho(1450)$

	GTB	PDG
m_1	758	775 ± 0.23 MeV
Γ_1	137	149.1 ± 0.8 MeV
m_2	1514	1465 ± 25 MeV
Γ_2	162	400 ± 60 MeV
A_1	1.28	
A_2	0.18	

$e^+e^- \rightarrow$ hadrons

GTB computation

$|F_1^1(s)|^2$

hadronic R ratio

ρ

ω

ϕ

Gravitational form factor $F_2^0(s)$

$F_2^0(s) \sim \langle 0|T^{\mu\nu}|\pi^+\pi^-\rangle$

	GTB	PDG
m_{f_2}	1180	1275.4 ± 0.6 MeV
Γ_{f_2}	249	186.6 ± 2.3 MeV

$f_2(1270)$

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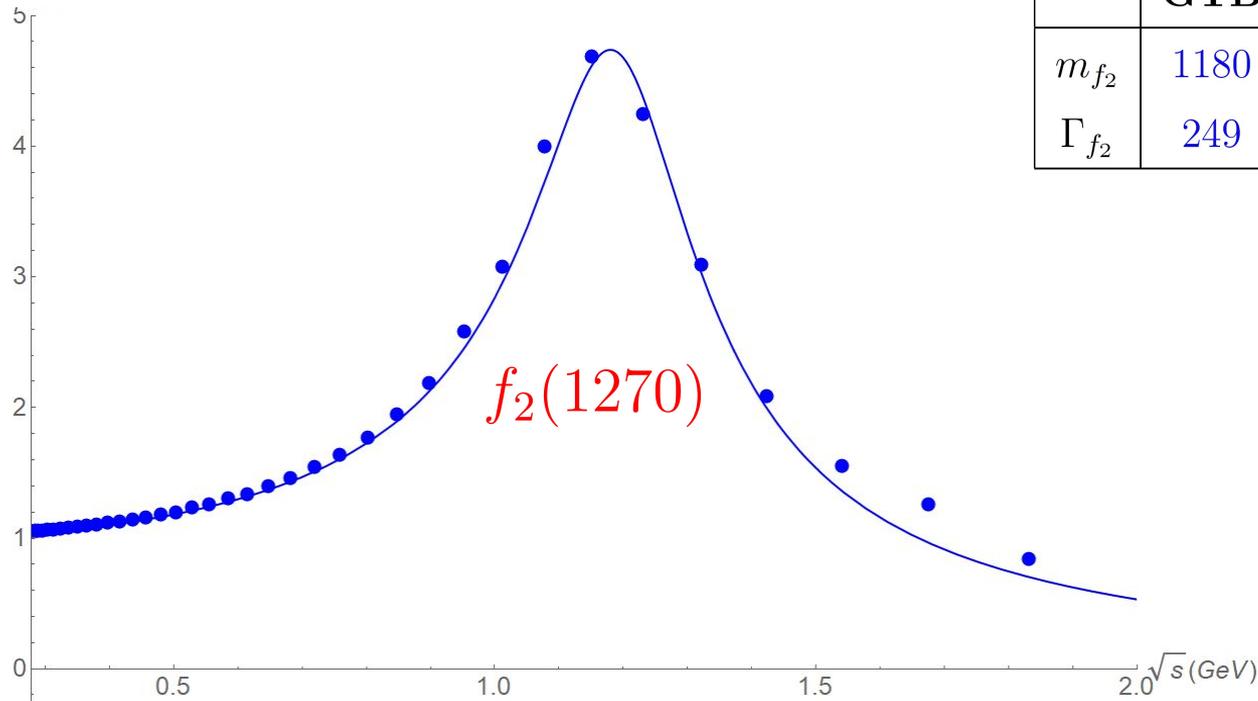
Thermodynamics of dilute interacting pion gas

deviation from ideal gas dominated by binary collisions

consider cluster expansion

$$\Xi = e^{\beta pV} = 1 + z \sum_{\nu, N_\nu = 1} e^{-\beta E_\nu} + z^2 \sum_{\nu, N_\nu = 2} e^{-\beta E_\nu} + \mathcal{O}(z^3)$$

$$\beta P = \lim_{V \to \infty} \frac{1}{V} \ln \Xi = \sum_{n=1}^{\infty} b_n(T) z^n \qquad \text{fugacity} \quad z = e^{\beta \mu}$$

second virial coefficient encodes corrections from interaction

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second virial coefficient encodes corrections from interaction

can be related to phase shift: [Beth, Uhlenbeck, 1937]
[Dashen, Ma, Bernstein, 1969]

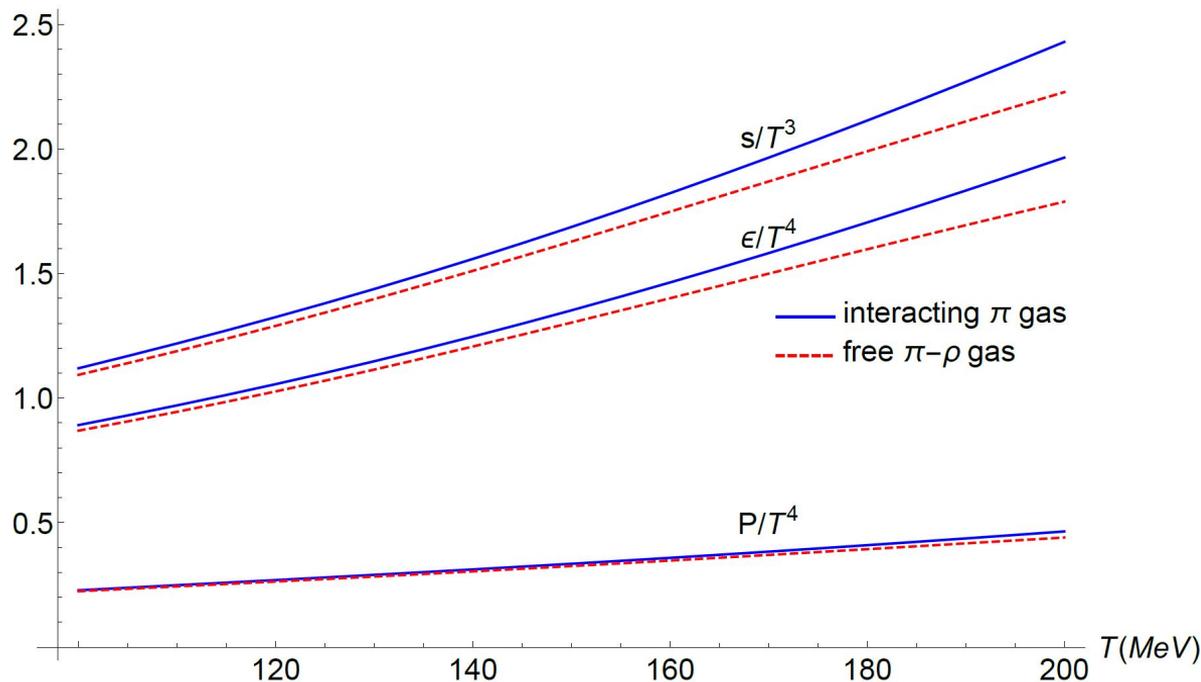can now do this with a
theoretical computation

recent application to QCD string: [Baratella, Miro, Gendy, 2024]

$$b_2 = \frac{1}{2\pi^3 \beta} \int_{2m_\pi}^{\infty} dM \, M^2 K_2(\beta M) \sum_{I\ell}{}' (2I+1)(2\ell+1) \frac{\partial \delta_\ell^I}{\partial M}$$
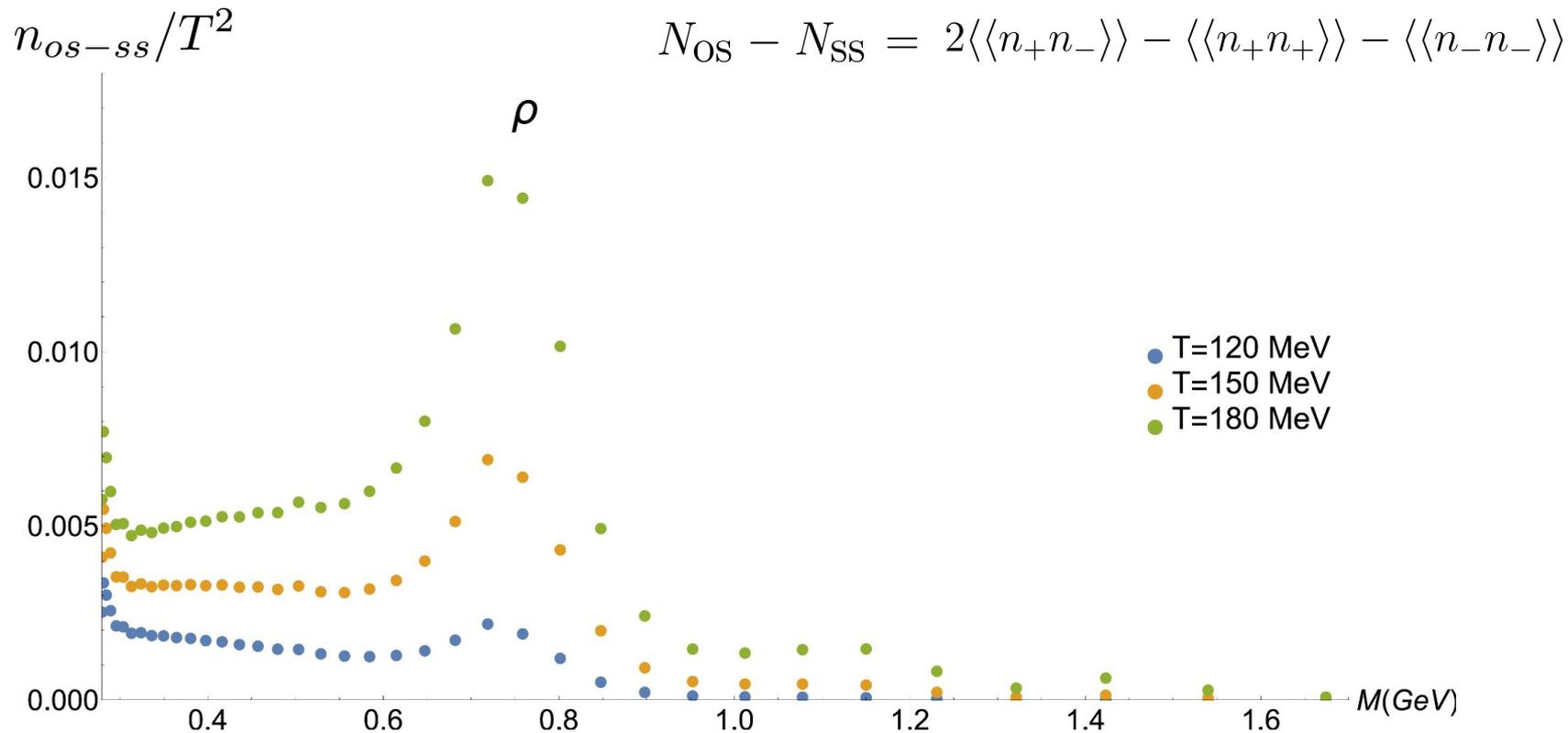
[Venugopalan,
Parakash, 1992]

Thermodynamic quantities at temperature near pion mass

$$P_{\text{int}} = Tb_2 \qquad \epsilon_{\text{int}} = -\frac{\partial b_2}{\partial \beta} \qquad s_{\text{int}} = \left[b_2 \left(1 - 2\mu\beta \right) - \beta \frac{\partial b_2}{\partial \beta} \right]$$

Invariant mass distribution of pion pairs with opposite and same charges

$$n_{os-ss}/T^2$$

$$N_{\mathrm{OS}} - N_{\mathrm{SS}} = 2\langle\langle n_+ n_- \rangle\rangle - \langle\langle n_+ n_+ \rangle\rangle - \langle\langle n_- n_- \rangle\rangle$$

ρ

- T=120 MeV
- T=150 MeV
- T=180 MeV

$M(GeV)$

Conclusions

- Gauge Theory Bootstrap:

  using only $\quad N_c \; N_f \; m_q \; \Lambda_{\mathrm{QCD}}$ $\qquad\qquad m_\pi \qquad\qquad f_\pi$

  $\underbrace{\phantom{N_c \; N_f \; m_q \; \Lambda_{\mathrm{QCD}}}}_{\textit{gauge theory parameters}}$ $\qquad\quad$ set the unit $\qquad$ size of pion

  strongly coupled low energy physics of asymptotically free gauge theories

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  f_π

  size of pion

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Conclusions

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- Not precision age yet: need more robust computations, set error bars, etc….
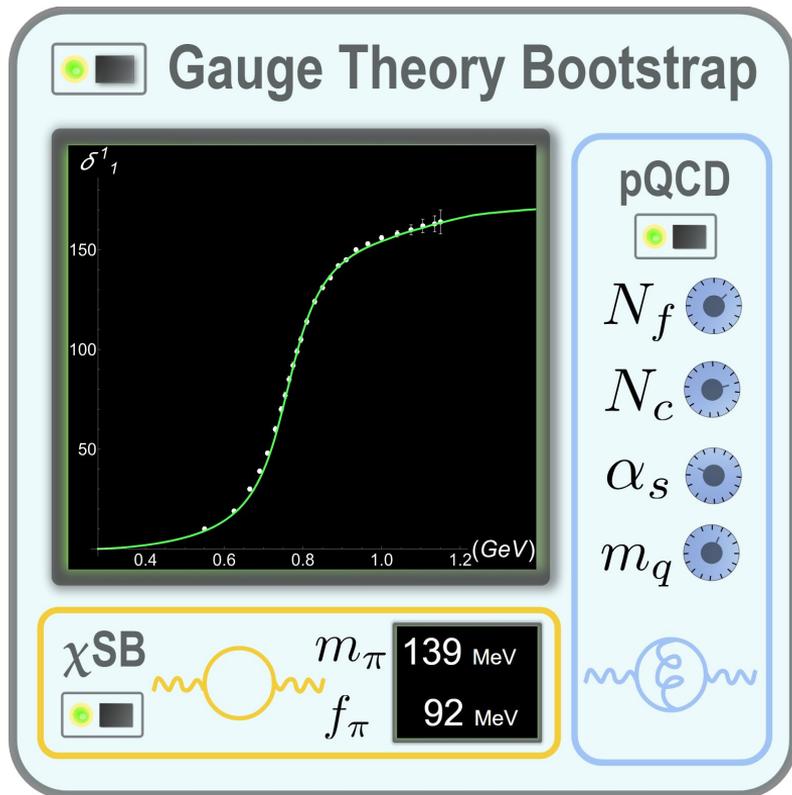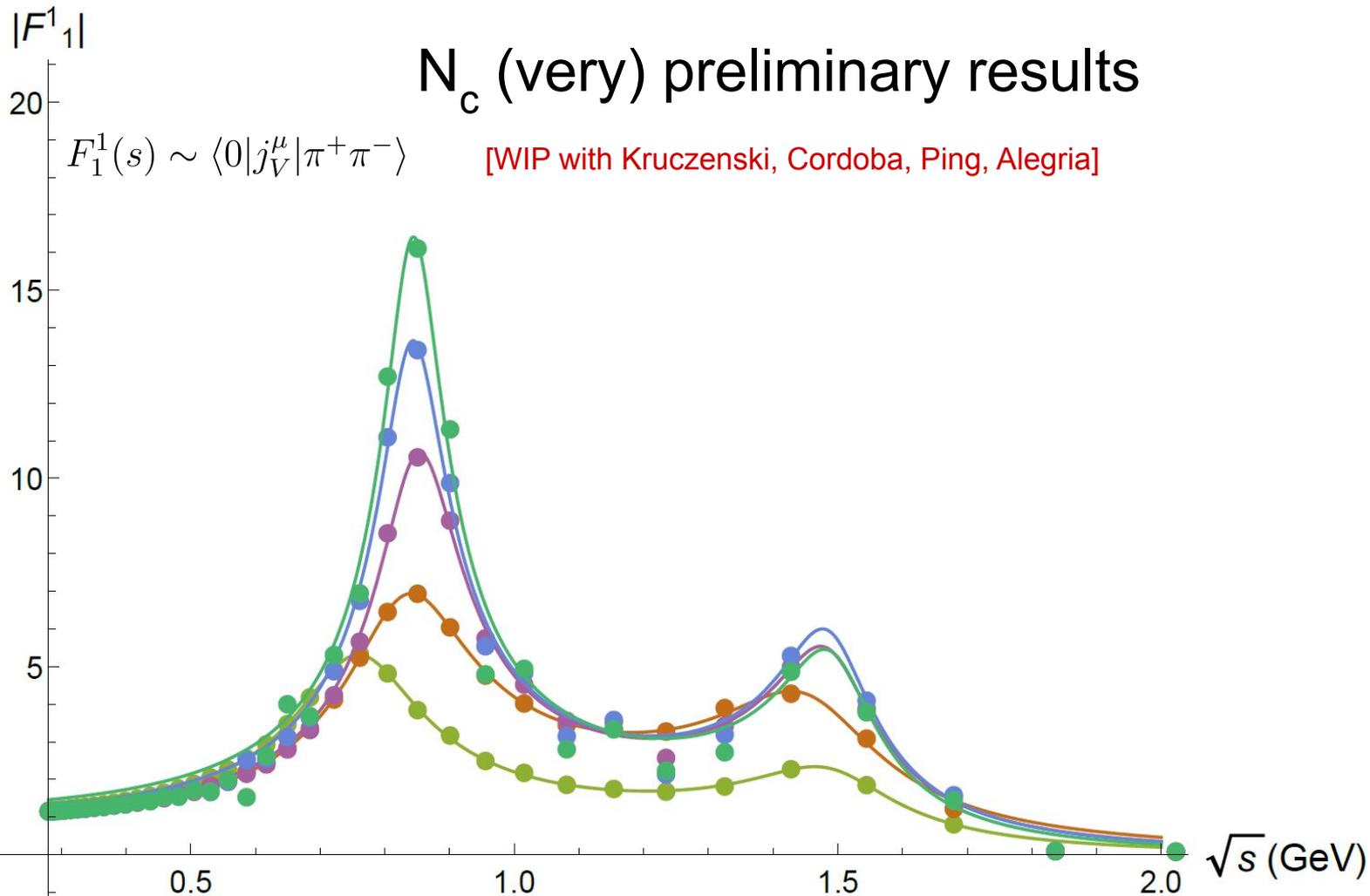
Github repository:

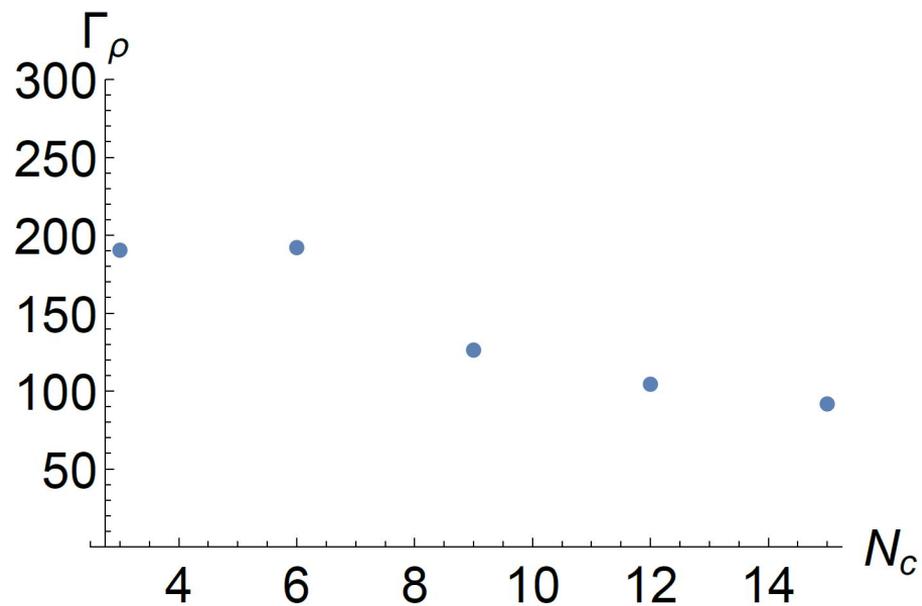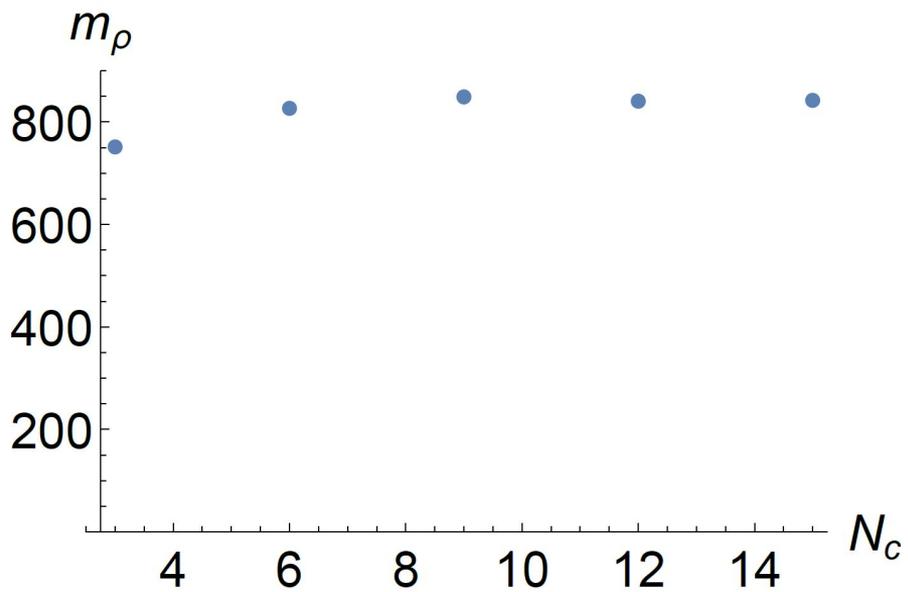use machine precision convex optimization solver Mosek (cvx)

N_c (very) preliminary results

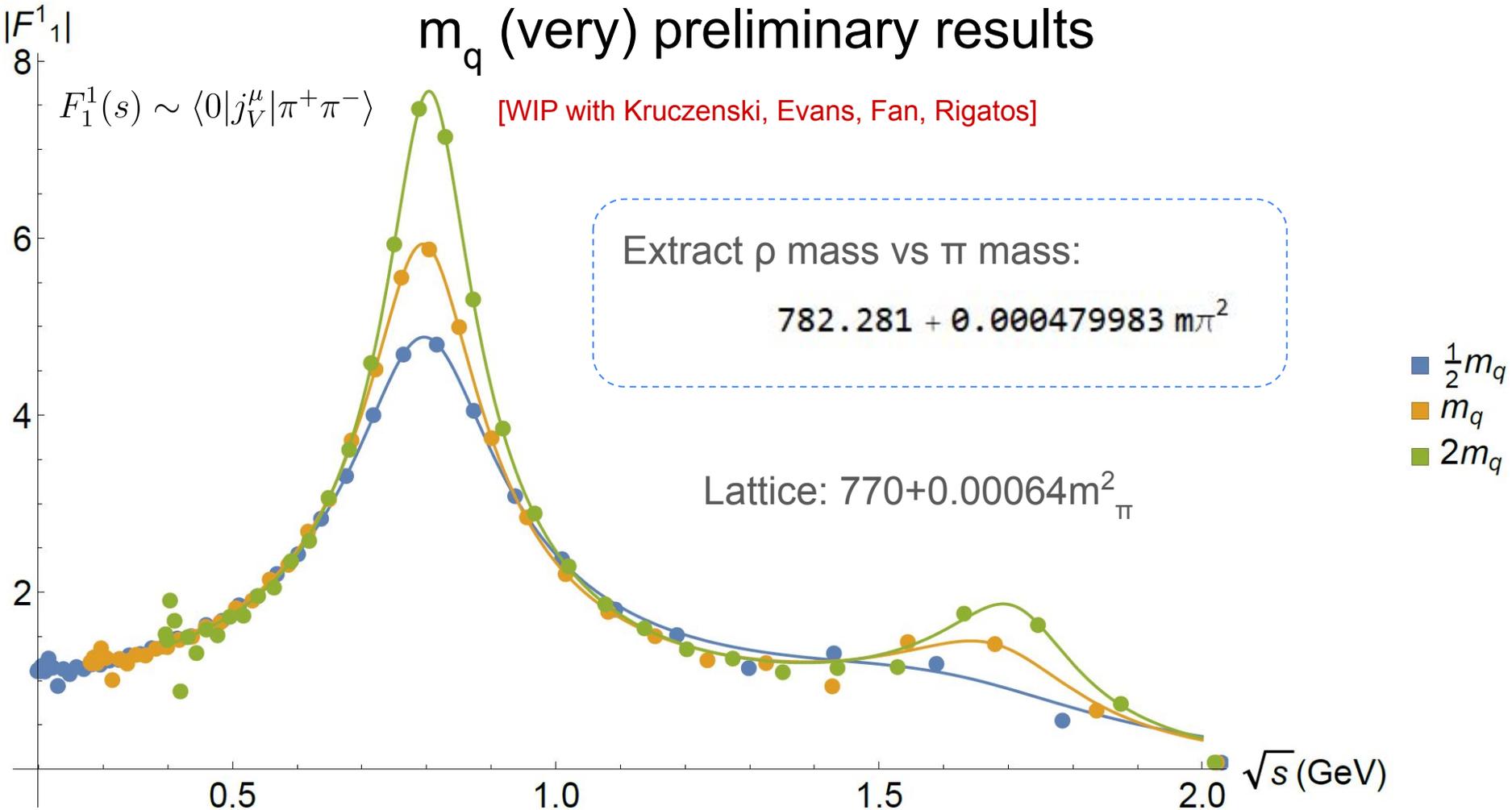$F_1^1(s) \sim \langle 0 | j_V^\mu | \pi^+ \pi^- \rangle$

[WIP with Kruczenski, Cordoba, Ping, Alegria]

Legend:
- $N_c = 3$
- $N_c = 6$
- $N_c = 9$
- $N_c = 12$
- $N_c = 15$

Axes: $|F^1_1|$ vs $\sqrt{s}$ (GeV)

N_c (very) preliminary results

m_q (very) preliminary results

$F_1^1(s) \sim \langle 0|j_V^\mu|\pi^+\pi^-\rangle$

[WIP with Kruczenski, Evans, Fan, Rigatos]

Extract ρ mass vs π mass:

$$782.281 + 0.000479983\, m\pi^2$$

Lattice: $770 + 0.00064 m_\pi^2$

$\frac{1}{2}m_q$

m_q

$2m_q$

Some outlook

GTB idea: bootstrap bridge between the IR and UV

In the context of QCD, would be interesting to extend GTB computations to include non-local operators (e.g. detector operators)

Thank you!