Symmetries in kernels of B-meson LCDA and soft functions

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Leading-twist distribution amplitude in HQET

Definition [A. Grozin, M. Neubert (1997)]

$$\langle 0|\big[\bar q(zn)\, \not n [zn,0]\gamma_5 h_v(0)\big]_R|\bar B(v)\rangle = iF_B(\mu)\ \Phi_+(z,\mu)$$

- v_μ is a time-like vector, $h_v(0) = [0,+\infty v]_F$

- **RGE** $$\Big(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} + \mathcal{H}_{\rm LN}(a)\Big)\Phi_+(z,\mu) = 0\,,$$ [B. Lange, M. Neubert (2003)]

- **One-loop evolution kernel**

$$[\mathcal{H}^{(1)}_{\rm LN}\Phi_+](z,\mu) = 4C_F\left\{[\ln(i\widetilde{\mu}z)+1/2]\Phi_+(z,\mu) + \int_0^1 du\frac{\bar u}{u}\,[\Phi_+(z,\mu)-\Phi_+(\bar u z,\mu)]\right\}$$

where $\widetilde{\mu} = \mathrm{e}^{\gamma_E}\mu_{\overline{\rm MS}}$ and $\bar u = 1-u$. [A. Grozin, M. Neubert, (1997); V. Braun, D. Ivanov, G. Korchemsky, (2004)]

- **Solution to one-loop RGE** [G. Bell, T. Feldmann, Y.-M. Wang, M. W. Y. Yip, (2013); V. Braun, A. Manashov (2014)]

$$\Phi_+(z,\mu) = -\frac{1}{z^2}\int_0^\infty ds\, s\, e^{is/z}\,\eta_+(s,\mu)\,,$$

$$\eta_+(s,\mu) = R(s,\mu,\mu_0)\eta_+(s,\mu_0)\,, \qquad R(s,\mu,\mu_0) \propto s^{\frac{2C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}}$$

Conformal symmetry in Lange-Neubert kernel

- $\mathcal{H}^{(1)}_{\rm LN}$ commute with special conformal generator of light field $S_+ \sim v^\mu \mathbf{K}_\mu$ but not S_0

$$S_+ = z^2\partial_z + 2jz\,,\ S_0 = z\partial_z + j\,,\ S_- = -\partial_z\,,\ [S_+, S_-] = 2S_0$$

$$[S_+, \mathcal{H}^{(1)}_{\rm LN}] = 0\,,\quad [S_0, \mathcal{H}^{(1)}_{\rm LN}] = 4C_F = \Gamma^{(1)}_{\text{cusp}}\,,\quad [S_0, S_\pm] = \pm S_\pm\,,$$

[M. Knödlseder, N. Offen (2011)]

solution: $$\mathcal{H}^{(1)}_{\rm LN} = \Gamma^{(1)}_{\text{cusp}} \ln(i\mu S_+) + \text{const}$$ (for $\bar q(nz)\not{n} h(0)$, $j = 1$)

[V. Braun, A. Manashov (2014)]

- light $\mapsto$ heavy reduction

[V. Braun, YJ, A. Manashov, (2018)]

$$S_+^{(h)} \mapsto \lambda^{-1} S_+^{(h)}\,,\qquad S_-^{(h)} \mapsto \lambda S_-^{(h)} \mapsto \mu\,,\qquad S_0^{(h)} \mapsto S_0^{(h)}\,,\qquad \lambda \sim m_b \to \infty$$

$$\Longrightarrow\quad \mathcal{H}^{(1)}_{\bar q q}(\mathbb{C}^2_{\bar q q}) \mapsto \mathcal{H}_{\rm LN} \equiv \mathcal{H}_{\bar q h} = \Gamma^{(1)}_{\text{cusp}} \ln(i\mu S_+) + \text{const}$$

- eigenfunction of $\mathcal{H}^{(1)}_{\rm LN}$ coincides with that of S_+:

$$Q_s(z) = -\frac{e^{is/z}}{z^2}$$

[V. Braun, A. Manashov (2014)]

Conformal symmetry in Lange-Neubert kernel

- **To all orders** $\boxed{\mathcal{H}_{\rm LN} = \Gamma_{\rm cusp}(a)\ln(i\bar\mu S_+) + \Gamma_+(a)}$ [V. Braun, YJ, A. Manashov (2019)]

Evolution kernels in the MS-like schemes are ϵ-independent

Exact conformal symmetry in $d = 4 - 2\epsilon$ at the critical point $\beta(a_*) = 0$

$$(1) \quad \left[S_+^{\rm full}, \mathcal{H}_{\rm LN}(a_*)\right] = 0$$

Conformal generators receive quantum corrections:

$$S_+^{(0)} = z^2\partial_z + 2z \mapsto S_+^{\rm full}(a_*) = S_+^{(0)} + z\left[-\epsilon + \Delta(a_*)\right],$$
$$S_0^{(0)} = z\partial_z + 1 \mapsto S_0^{\rm full}(a_*) = S_0^{(0)} - \epsilon + \mathcal{H}_{\rm LN}(a_*)$$

$\Delta(a_*) = a_*\Delta^{(1)} + a_*^2\Delta^{(2)} + \ldots$ is called conformal anomaly satisfying

from (1) and $SL(2)$ algebra $\Longrightarrow$ $\boxed{(2) \quad [z\partial_z, S_+^{\rm full}(a_*)] = S_+^{\rm full}(a_*)}$

$\ln\mu z$ enters $\mathcal{H}_{\rm LN}$ only linearly with coefficient $\Gamma_{\rm cusp}$ [G. Korchemsky, A. Radyushkin (1992)]

$$(3) \quad [z\partial_z, \mathcal{H}_{\rm LN}(a_*)] = \Gamma_{\rm cusp}(a_*)$$

$(1) \Longrightarrow \mathcal{H}_{\rm LN}(a_*) = f(S_+^{\rm full}(a_*)) \overset{(2),(3)}{\Longrightarrow} zf'(z) = \Gamma_{\rm cusp}(a_*) \Longrightarrow$ **Proposition**

Conformal generators at one-loop in HQET

- **Two-loop evolution of twist-2 DA** [V. Braun, YJ, A. Manashov (2019)]

$$\mathcal{H}^{(2)}_{\rm LN}(a_*) = \Gamma^{(2)}_{\rm cusp}(a_*)\ln(i\bar\mu S^{(1)}_+(a_*)) + \Gamma^{(2)}_+(a_*)\,,$$
$$S^{(1)}_+(a_*) = S^{(0)}_+ + z\big(-\epsilon(a_*) + a_*\Delta^{(1)}\big)$$

$$\bar\mu = \widetilde\mu e^{\gamma_E} = \mu_{\overline{\rm MS}}\, e^{2\gamma_E}$$

$$\epsilon(a_*) = -\beta_0 a_* + O(a_*^2)$$

One-loop conformal anomaly   *four one-loop diagrams*

$$\Delta^{(1)}\mathcal{O}(z) = C_F\left\{3\mathcal{O}(z) + 2\int_0^1 d\alpha\left(\frac{2\bar\alpha}{\alpha} + \ln\alpha\right)[\mathcal{O}(z) - \mathcal{O}(\bar\alpha z)]\right\}$$

- **The scheme-dependent constant $\Gamma^{(2)}_+(a)$ is found from Feynman diagrams**

Two-loop kernel in integral representation

- **Integral representation for $\mathcal{H}_h^{t=2}$ is usually preferred**

Ansatz

$$\mathcal{H}(a)\mathcal{O}(z) = \Gamma_{\text{cusp}}(a)\left[\ln(i\widetilde{\mu}z)\mathcal{O}(z) + \int_0^1 d\alpha\, \frac{\bar{\alpha}}{\alpha}\big(1+h(a,\alpha)\big)\big(\mathcal{O}(z)-\mathcal{O}(\bar{\alpha}z)\big)\right] + \gamma_+(a)\mathcal{O}(z)$$

- $\Delta^{(1)}$ **and** $\epsilon(a_*) = -\beta_0 a_* + O(a_*^2)$ **dictate** $h(a,\alpha)$ *going to Mellin space*

$$h(a,\alpha) = a\ln\bar{\alpha}\left\{\beta_0 - 2C_F\left(\frac{3}{2} + \ln\frac{\alpha}{\bar{\alpha}} + \frac{\ln\alpha}{\bar{\alpha}}\right)\right\} + O(a^2)$$

- γ_+ **requires additional calculation** *scheme-dependent*, $\gamma_{\phi_+} = \gamma_+ - \gamma_F$

$$\gamma_+^{\overline{\text{MS}}}(a) = -aC_F + a^2 C_F\left\{4C_F\left[\frac{21}{8} + \frac{\pi^2}{3} - 6\zeta_3\right] + C_A\left[\frac{83}{9} - \frac{2\pi^2}{3} - 6\zeta_3\right] + \beta_0\left[\frac{35}{18} - \frac{\pi^2}{6}\right]\right\} + \dots$$

also obtainable from known anomalous dimensions!

- **LN kernel at three-loops: to appear**

Solution to one-loop evolution of higher-twist DAs

- **One-loop evolution kernels of three-particle DAs are pairwise.**
  Example

$$2F_B\Psi_3(z_1,z_2) = \langle 0|\,\bar{q}(z_1n)\not{n}[z_1n,z_2n]G_{\mu\nu}(z_2n)\gamma_\perp^\mu n^\nu[z_2n,0]h_v(0)\,|B\rangle$$

where the one-loop kernel takes the form $\mathcal{H}^{(1)}_{\Phi_3} = \mathcal{H}^{(1)}_{qg} + \mathcal{H}^{(1)}_{gh} + \mathcal{H}^{(1)}_{qh}$ with

$$[\mathcal{H}^{(1)}_{qh}f](z_1) = \frac{-1}{N_c}\left\{\int_0^1 \frac{d\alpha}{\alpha}\,[f(z_1)-\bar{\alpha}f(\bar{\alpha}z_1)] + \left[\ln(i\mu z_1)-\frac{5}{4}\right]f(z_1)\right\},$$

$$[\mathcal{H}^{(1)}_{gh}f](z_2) = N_c\left\{\int_0^1 \frac{d\alpha}{\alpha}\,\left[f(z_2)-\bar{\alpha}^2f(\bar{\alpha}z_2)\right] + \left[\ln(i\mu z_2)-\frac{1}{2}\right]f(z_2)\right\},$$

$$[\mathcal{H}^{(1)}_{qg}\varphi](z_1,z_2) = N_c\left\{\int_0^1 \frac{d\alpha}{\alpha}\,\left[2\varphi(z_1,z_2)-\bar{\alpha}\varphi(z_{12}^\alpha,z_2)-\bar{\alpha}^2\varphi(z_1,z_{21}^\alpha)\right] - \frac{3}{4}\varphi(z_1,z_2)\right\} - \frac{2}{N_c}\int_0^1 d\alpha\int_{\bar{\alpha}}^1 d\beta\,\bar{\beta}\,\varphi(z_{12}^\alpha,z_{21}^\beta)\,,$$

where $z_{12}^\alpha = \bar{\alpha}z_1 + \alpha z_2$. [M. Knödlseder, N. Offen (2011); V. Braun, A. Manashov, J Rohrwild, (2009); YJ, A. Belitsky (2014)]

- **Eigenfunction?**

One-loop evolution of higher-twist DAs

- **RGE for $\Phi_3(z_1, z_2, \mu)$ is integrable at large N_c limit.**

  Two conserved charges (hidden symmetries)

$$[\mathbb{Q}_1, \mathbb{Q}_2] = \left[\mathbb{Q}_1, \mathcal{H}^{(1)}_{\Phi_3}\right] = \left[\mathbb{Q}_2, \mathcal{H}^{(1)}_{\Phi_3}\right] = 0$$

  explicitly [V. Braun, A. Manashov, N. Offen (2015)]

$$\mathbb{Q}_1 = i(S_q^+ + S_g^+)\,,$$
$$\mathbb{Q}_2 = \frac{9}{4} i S_g^+ - i S_g^+\left(S_g^+ S_q^- + S_g^0 S_q^0\right) - i S_g^0\left(S_q^0 S_g^+ - S_g^0 S_q^+\right)$$

  $S^+ = z^2\partial_z + 2jz\,, S^0 = z\partial_z + j\,, S^- = -\partial_z$, j conformal spin.

- **Two DOF in $\Phi^{(1)}_{\Phi_3} \implies \mathcal{H}^{(1)}_{\Phi_3}$ and $\{\mathbb{Q}_1, \mathbb{Q}_2\}$ share the same eigenfunction.**

- **Integrability of RGE $\Leftrightarrow$ Integrable spin chains** [V. Braun, YJ, A. Manashov (2018)]

One-loop evolution of higher-twist DAs

- **Solving for eigenfunction of $\{\mathbb{Q}_1, \mathbb{Q}_2\}$ leads to (complete orthonormal basis)**

$$\phi_-(\omega,\mu) = \int_\omega^\infty \frac{d\omega'}{\omega'}\phi_+(\omega',\mu) + \int_0^\infty ds\, J_0(2\sqrt{\omega s})\, \eta_3^{(0)}(s,\mu)$$

$$\phi_3(\underline{\omega},\mu) = \int_0^\infty ds\Big[\eta_3^{(0)}(s,\mu)\, Y_3^{(0)}(s\,|\,\underline{\omega}) + \frac{1}{2}\int_{-\infty}^\infty dx\, \eta_3(s,x,\mu)\, Y_3(s,x\,|\,\underline{\omega})\Big],$$

where $Y_3^{(0)}(s\,|\,\underline{\omega}) = Y_3(s, x = i/2\,|\,\underline{\omega})$ and

$$Y_3(s,x\,|\,\underline{\omega}) = -\int_0^1 du\, \sqrt{us\omega_1}\, J_1(2\sqrt{us\omega_1})\,\omega_2\, J_2(2\sqrt{\bar{u}s\omega_2})\, {}_2F_1\left(\begin{matrix}-\frac{1}{2}-ix, -\frac{1}{2}+ix \\ 2\end{matrix}\,\Big|\, -\frac{u}{\bar{u}}\right)$$

Solving RGE for $\phi_3(\underline{\omega},\mu)$ up to $1/N_c^2$ gives

$$\eta_3(s,x,\mu) = L^{\gamma_3(x)/\beta_0} R(s;\mu,\mu_0)\, \eta_3(s,x,\mu_0)$$

$$\eta_3^{(0)}(s,\mu) = L^{N_c/\beta_0} R(s;\mu,\mu_0)\eta_3^{(0)}(s,\mu_0)$$

$L = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$ and $\gamma_3(x) = N_c[\psi(3/2+ix) + \psi(3/2-ix) + 2\gamma_E]$.

- $1/N_c^2 \sim \mathcal{O}(10^{-1})$ **taken perturbatively**
- **also possible at twist-4 level: three conserved charges of 2×2 matrices**

[V. Braun, YJ, A. Manashov (2017)]

Evolution kernels of soft functions

- $h \to \triangle_b \to \gamma\gamma$ **at subleading power induces the soft function** [ZL. Liu, M. Neubert, (2020)]

$$\mathcal{S}_1 \equiv \langle 0|\, \mathcal{O}_1 \,|\gamma\gamma\rangle \;, \quad \text{with} \quad \mathcal{O}_1(z_1, z_2) = \bar{q}(z_1 n_1)[z_1 n_1, 0]\not{n}_1 \not{n}_2[0, z_2 n_2]q(z_2 n)$$

necessary to solve the RGE of $\mathcal{S}_1$ to resum large logs $\sim \ln(\Lambda_{\rm QCD}/m_b)$

one-loop: [ZL. Liu, M. Neubert, (2020)] (**from RG consistency**), [G. Bodwin, J. Ee, J. Lee and X-P. Wang (2021)] (**from diagrammatic calculations**)

- **Exploiting translation + conformal symmetry +** γ_Q**:** [M. Beneke, YJ, X. Wang (2024)]
  - $\mathbb{H}_{\mathcal{O}_1}$ **satisfies the same commutation ("double copy") relation as** $\mathcal{H}_{\rm LN}$

$$[\mathcal{K}, \mathbb{H}_{\mathcal{O}_1}] = 0, \qquad [\mathcal{D}, \mathbb{H}_{\mathcal{O}_1}] = \Gamma^{\rm F}_{\rm cusp}$$

  - **Analysis of the local limit** $s, t \to 0$ **yields the relation**

$$\mathbb{H}_{\mathcal{O}_1}(s,t) = \mathcal{H}_{\rm LN}(s) + \mathcal{H}_{\rm LN}(t) - 2\gamma_Q$$

  - **Two loop** $\mathbb{H}_{\mathcal{O}_1}$ **obtained directly** [M. Beneke, YJ, X. Wang (2024)]

Evolution kernels of soft functions

- **Similarly $h \to \triangle_b \to gg$ requires** [ZL. Liu, M. Neubert, M. Schnubel and X. Wang (2022)]

$$\mathcal{S}_2 \equiv \langle 0| \mathcal{O}_2 |gg\rangle \ ,$$
$$\mathcal{O}_2(z_1, z_2) = \bar{q}(z_1 n_1)[z_1 n_1, \infty n_1] T^a [\infty n_1, 0][0, \infty n_2] T^b [\infty n_2, z_2 n_2] q(z_2 n_2)$$

- **lightlike infinite Wilson lines spoils conformal properties**
  - **only one (possible) commutation relation:** [M. Beneke, YJ, X. Wang (2024)]

$$[\mathcal{K}, \mathbb{H}_{\mathcal{O}_2}] \neq 0, \qquad [\mathcal{D}, \mathbb{H}_{\mathcal{O}_2}] \stackrel{?}{=} \Gamma^{\mathrm{F}}_{\mathrm{cusp}} - \Gamma^{\mathrm{A}}_{\mathrm{cusp}}$$

  - **One-loop via diagrammatic calculation using background field method** [M. Beneke, YJ, X. Wang (2024)]
- **new "emergence symmetry" to fully constrain $\mathbb{H}_{\mathcal{O}_2}$?**

Evolution kernels of soft functions

- **Nonperturbative effects in $B \to \gamma\gamma$ with soft gluon emission is captured by**

[Q. Qin, Y-L. Shen, C. Wang, Y-M. Wang (2022)]; [Y-K. Huang, YJ, Y-L. Shen, C. Wang, Y-M. Wang, X-C. Zhao (2023)]

$$\mathcal{S}_3 = \langle 0 | \mathcal{O}_3 | B \rangle$$
$$\mathcal{O}_3(z_1, z_2) = \bar{q}(z_1 n_1) \not{n}_1 [z_1 n_1, 0][0, z_2 n_1] G_{\mu\nu}(z_2 n_2) \gamma_\perp^\mu n_2^\nu \gamma^5 [z_2 n_2, 0] h_v(0)$$

- **Evolution kernel at one-loop constructible directly from conformal symmetry**

[to appear: YJ, Y-M. Wang, H-X. Yu]

- **complication of constants at higher-loops (Γ, C)**

$$\left[\mathbb{H}_{\mathcal{O}_3} \mathcal{O}_3\right](z_1, z_2) = \Gamma^{(1)} \left[\int_0^1 \frac{d\alpha}{\alpha} f_1(\alpha) \left[\mathcal{O}(z_1, z_2) - \mathcal{O}(\bar{\alpha} z_1, z_2)\right] + \ln(\mu z_1) \right] + \Gamma^{(2)} \left[\int_0^1 \frac{d\alpha}{\alpha} f_2(\alpha) \left[\mathcal{O}(z_1, z_2) - \mathcal{O}(z_1, \bar{\alpha} z_2)\right] + \ln(\mu z_2) \right] + C$$

[to appear: YJ, Y-M. Wang, H-X. Yu]

- **How to get $\Gamma^{(1)}$ and $\Gamma^{(2)}$ directly beyond two-loops?**

Evolution kernels of soft functions

- **Similarly $B \to X_s\gamma$ at subleading power requires:**

[M. Benzke, S-J. Lee, M. Neubert, and G. Paz (2010)]; [R. Bartocci, P. Böer, and T. Hurth (2024)]

$$\mathcal{S}_4 = \langle B|\, \mathcal{O}_4 \,|B\rangle$$
$$\mathcal{O}_4(z_1, z_2) = \bar{h}_v(z_1 n_-)\not{n}\_[z_1 n_-, 0][0, z_2 n_+]G_{\mu\nu}(z_2 n_+)\gamma_\perp^\mu n_+^\nu \gamma^5 [z_2 n_+, 0]h_v(0)$$

- **At one-loop, $\mathbb{H}_{\mathcal{O}_4}$ obtainable form symmetry consideration** [to appear: YJ, Y-M. Wang, H-X. Yu]
- **determination of constants again becomes nontrivial at higher-loops**

$$[\mathbb{H}_{\mathcal{O}_3}\mathcal{O}_3]\,(z_1, z_2) = \widetilde{\Gamma}\left[\int_0^1 \frac{d\alpha}{\alpha}\, f_2(\alpha)\,[\mathcal{O}(z_1, z_2) - \mathcal{O}(z_1, \bar{\alpha} z_2)] + \ln(\mu z_2)\right] + \widetilde{C}$$

[to appear: YJ, Y-M. Wang, H-X. Yu]

- **Simplification beyond two-loops? $\widetilde{\Gamma}$ from Wilson loop anomalous dimension?**

Evolution kernels of soft functions

- **Subleading power corrections to Drell-Yan process near threshold involving**

[M. Beneke, A. Broggio, M. Garny, S. Jaskiewicz, R. Szafron, L. Vernazza, J. Wang (2018)]
[M. Beneke, A. Broggio, S. Jaskiewicz, L. Vernazza (2019)]; [M. Beneke, Y. Ji, E. Sünderhauf, and X. Wang (2025)]

$$\mathcal{S}_5 = \langle 0|\, \mathcal{O}_5 \,|0\rangle$$

$$\mathcal{O}_5(x_0; z_1, z_2) = \frac{g^2}{N_c C_F}\mathrm{tr}\bar{\mathbf{T}}\left[\left(\bar{q}Y_{n_-}\right)(x_0 + z_2 n_-)T^a Y^\dagger_{n_-}(x_0) Y_{n_+}(x_0)\right]\frac{\not{n}_-}{4}$$
$$\times\, \mathbf{T}\left[Y^\dagger_{n_+}(0) Y_{n_-}(0) T^a \left[Y_{n_-} q_s\right](s_1 n_-)\right]$$

- $\mathbb{H}_{\mathcal{O}_5}$ **computed at one-loop** [M. Beneke, Y. Ji, E. Sünderhauf, and X. Wang (2025)]
- **Symmetry analysis shows both $\mathbb{H}_{\mathcal{O}_2}$ and $\mathbb{H}_{\mathcal{O}_5}$ related to $\mathbb{H}_{\phi_3}$:**

$$2F_B \Psi_3(z_1, z_2) = \langle 0|\, \bar{q}(z_1 n)\not{n}[z_1 n, z_2 n] G_{\mu\nu}(z_2 n)\gamma_\perp^\mu n^\nu [z_2 n, 0] h_v(0)\,|B\rangle$$

Conclusion and Outlook

Conclusion

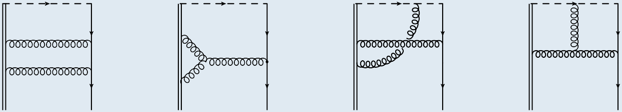
- Symmetries reduces the computation workload of various evolution kernels needed for large log resummations in many exclusive processes
- also allows establishing relations between different kernels

Outlook for future work

- Extending the symmetry analysis to NLP jet functions and other perturbative functions
- Computing evolution kernel of higher-twist B-LCDAs at higher orders via symmetry considerations

There are ~ 30 diagrams in Feynman gauge:

- **Exchange diagrams**



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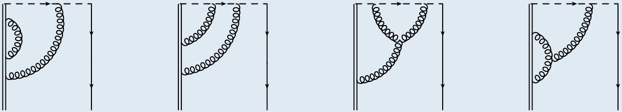
- **Cusp diagrams**



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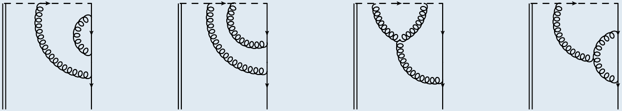
- **Light vertices**



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Two-loop kernels from Feynman diagrams

$$I \sim \int_0^1 du \int \frac{d^D l_1}{(2\pi)^D} \int \frac{d^D l_2}{(2\pi)^D} \frac{\mathrm{e}^{-ixn\cdot(k_2-\bar{u}l_2+ul_1)}}{v\cdot(k_1-l_1)l_2^2(l_1+l_2)^2 l_1^2(k_2-l_2)^2} \bar{q}(k_1)\not{n}\gamma_5 q(k_2)\Big|_{\mathrm{FT}}$$

- **Exchange diagrams contribute to both $h(a,\alpha)$ and γ_+** *(many are UV-finite)*
- **Cusp diagrams generate $\sim \ln z$ and contribute to γ_+**
- **Light vertices contribute to $h(a,\alpha)$ only, known**

[V. Braun, A. Manashov, S. Moch, and M. Strohmaier (2016)]

- **RGEs for twist-4 DAs are also integrable** [V. Braun, YJ, A. Manashov (2017)]

  Light fields mixing: kernels of 2×2 matrices. [V. Braun, A. Manashov, J. Rohrwild (2009); YJ, A. Belitsky (2014)]

  Three conserved charges $\{\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3\}$

$$\Phi_4(\underline{\omega}) = \frac{1}{2}\int_0^\infty ds \int_{-\infty}^{\infty} dx\, \eta_4^{(+)}(s,x,\mu)\, Y_{4;1}^{(+)}(s,x\,|\underline{\omega})\,,$$

$$(\Psi_4 + \widetilde{\Psi}_4)(\underline{\omega}) = -\int_0^\infty ds \int_{-\infty}^{\infty} dx\, \eta_4^{(+)}(s,x,\mu)\, Y_{4;2}^{(+)}(s,x\,|\underline{\omega})\,,$$

$$(\Psi_4 - \widetilde{\Psi}_4)(\underline{\omega}) = 2\int_0^\infty \frac{ds}{s}\left(-\frac{\partial}{\partial\omega_2}\right)\left\{\eta_3^{(0)}(s,\mu)\, Y_3^{(0)}(s\,|\underline{\omega}) + \frac{1}{2}\int_{-\infty}^{\infty} dx\, \eta_3(s,x,\mu)\, Y_3(s,x\,|\underline{\omega})\right\}$$
$$- \int_0^\infty ds \int_{-\infty}^{\infty} dx\, \varkappa_4^{(-)}(s,x,\mu)\, Z_{4;2}^{(-)}(s,x\,|\underline{\omega})\,,$$

$$\eta_4^{(+)}(s,x,\mu) \overset{\mathcal{O}(1/N_c^2)}{=} L^{\gamma_4(x)/\beta_0} R(s;\mu,\mu_0)\, \eta_4^{(+)}(s,x,\mu_0)$$
$$\varkappa_4^{(-)}(s,x,\mu) \overset{\mathcal{O}(1/N_c^2)}{=} L^{\gamma_4(x)/\beta_0} R(s;\mu,\mu_0)\, \varkappa_4^{(-)}(s,x,\mu_0)$$

- **Redundant operators are traded for others using EOMs and Lorentz symmetry.**

- **Evolution kernel of $\mathcal{O}_\ell = \bar{q}(nz_1)\not{n}q(nz_2)$ in integral form**

$$[\mathcal{H}_\ell\varphi](z_1,z_2) = \int_0^1 du\, h(u)\Big[2\varphi(z_1,z_2) - \varphi(z_{12}^u,z_2) - \boxed{\varphi(z_1,z_{21}^u)}\Big] + \boxed{\int_0^1 du \int_0^{\bar{u}} dv\, \chi(u,v)[\varphi(z_{12}^u,z_{21}^v) + \varphi(z_{12}^v,z_{21}^u)]} + c\varphi(z_1,z_2)$$

- **drop terms in boxes and $z_2 \to 0$ to obtain $\mathcal{H}_h^{\text{ex}} + \mathcal{H}_h^{\text{lv}}$.**
  ↰ **Location of the heavy quark is fixed!**

  explicit expressions for $\mathcal{H}_l^{(2)}$ available [V. Braun, A. Manashov, S. Moch, M. Strohmaier (2016)]
- **adding contribution of cusp diagrams again gives us $\mathcal{H}_{\mathrm{LN}}^{(2)}$.**
- **can be proved by noting $h_v(0)$ is fixed in space**

- **Operator $\mathcal{O}(z)$ in Mellin space**

$$\mathcal{O}(z) = \int_{-i\infty}^{+i\infty} dj \, (i\mu_{\overline{\text{MS}}} e^{\gamma_E} z)^j \mathcal{O}(j)$$

gives rise to the Mellin-space RGE:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(a)\frac{\partial}{\partial a} - \Gamma_{\text{cusp}}(a)\frac{\partial}{\partial j} + V(j,a)\right)\mathcal{O}(j,a,\mu) = 0$$

explicit expression for $V(j,a)$ at two-loop available in [V. Braun, YJ, A. Manashov, 1912.03210]

- **Mellin moment j as the second coupling, with Γ_{cusp} as the β-function**