Model-independent determination of gravitational form factors from dispersion relations

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XHC, Feng-Kun Guo, Qu-Zhi Li & De-Liang Yao, Dispersive determination of nucleon gravitational form factors, *Nat. Commun.* **16**, 6979 (2025)
XHC, Feng-Kun Guo, Qu-Zhi Li, Bo-Wen Wu & De-Liang Yao, Gravitational form factors of pions, kaons and nucleons from dispersion relations, Eur.Phys.J.ST in press

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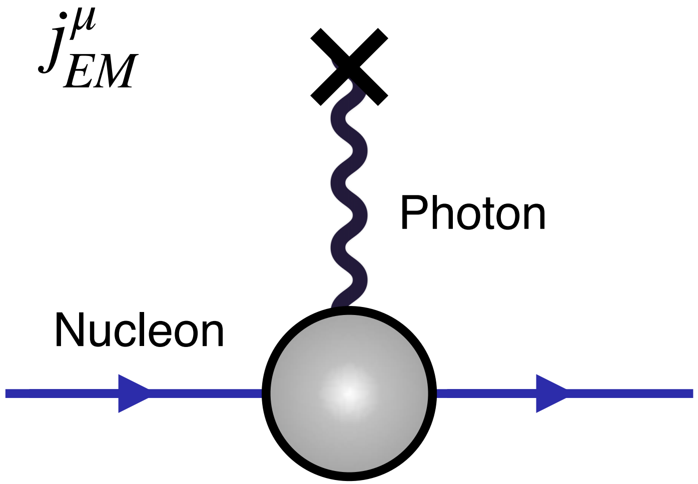
2025/9/12-9/15

EM interactions: Hadrons in external field



- Definition: $t \equiv q^2 = (p' - p)^2$

$$\langle N(p') | j^\mu_{EM} | N(p) \rangle = \bar{u}(p') \left[F_1(t)\gamma^\mu + i\frac{F_2(t)}{2m_N}\sigma^{\mu\nu}q_\nu \right] u(p)$$

- Sachs form factors: $G_E = F_1 + \frac{t}{4m^2}F_2, \quad G_M = F_1 + F_2$

- Charge radius (proton):

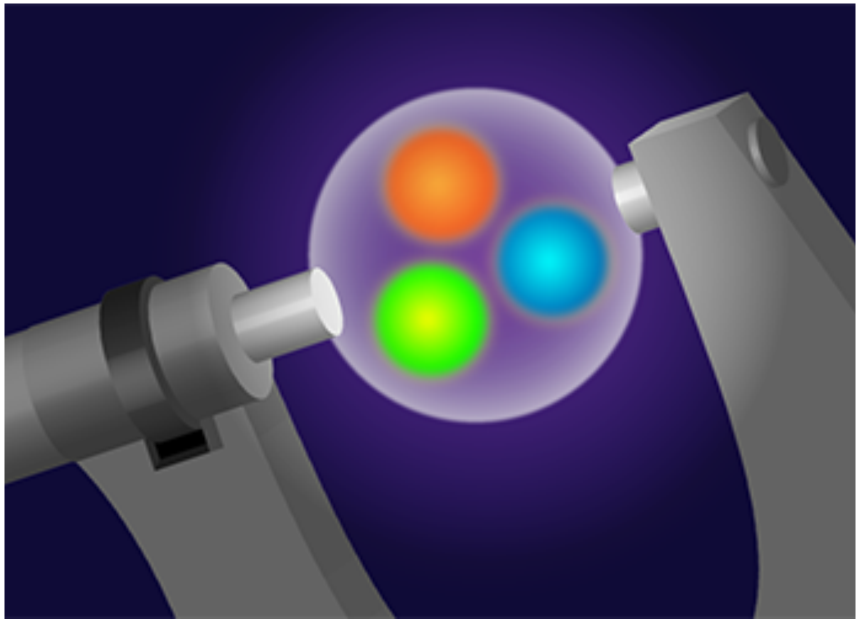
  $G_E(t) = 1 + t\langle r_C^2 \rangle/6 + \ldots$

- Extracted from the lepton-nucleon elastic scattering or the hydrogen(-like) atom spectroscopy

"proton radius puzzle"

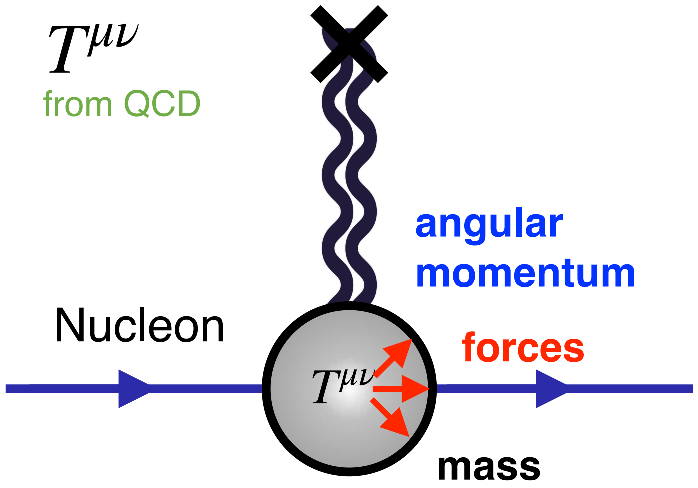
0.84 fm v.s. 0.88 fm

Gao and Vanderhaeghen, RMP (2022)



https://physics.aps.org/articles/v12/s28

Gravitational structure of nucleons



- Gravity couples to matter through energy-momentum tensor (EMT) $T^{\mu\nu}$

$$T^{\mu\nu}_{\rm QCD} = T^{\mu\nu}_q + T^{\mu\nu}_g$$

study gravitational forces within hadrons

study **mechanical properties** of hadrons

- Definition:

$$\left\langle N(p') \middle| T^{\mu\nu} \middle| N(p) \right\rangle = \frac{1}{4m_N}\bar{u}(p')\left[A(t)P^\mu P^\nu + J(t)\left(iP^{\{\mu}\sigma^{\nu\}\rho}\Delta_\rho\right) + D(t)\left(\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2\right)\right]u(p)$$

Kobzarev, Okun (1962)
Pagels (1966)

- Mass normalization:

$$m_N = \int d^3r T_{00}(r)$$

$$A(0) = 1$$

- Spin normalization:

$$J^i = \epsilon_{ijk}\int d^3r r_j T_{0k}(r)$$

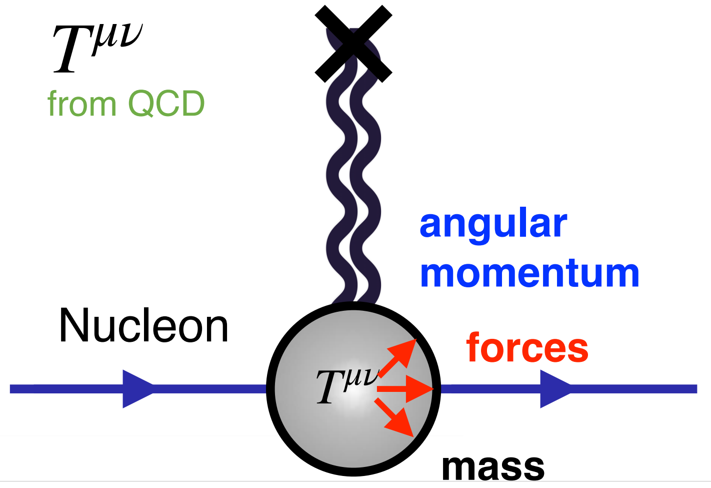
$$J(0) = 1/2$$

- D-term: $D \equiv D(0)$

$$D = -\frac{m_N}{2}\int d^3r \left(r_i r_j - \frac{1}{3}\delta_{ij}\right) T_{ij}(r)$$

$$D = ?$$

Gravitational structure of nucleons



D-term as the "last unknown global property"

Polyakov, and Schweitzer, (2018)

em: $\partial_\mu J^\mu_{\text{em}} = 0$	$\langle N'\vert J^\mu_{\textbf{em}}\vert N\rangle$	$\longrightarrow$	Q	$= 1.602176487(40) \times 10^{-19}\text{C}$
			μ	$= 2.792847356(23)\mu_N$
weak: PCAC	$\langle N'\vert J^\mu_{\textbf{weak}}\vert N\rangle$	$\longrightarrow$	g_A	$= 1.2694(28)$
			g_p	$= 8.06(55)$
gravity: $\partial_\mu T^{\mu\nu}_{\textbf{grav}} = 0$	$\langle N'\vert T^{\mu\nu}_{\textbf{grav}}\vert N\rangle$	$\longrightarrow$	m	$= 938.272013(23)\,\text{MeV}/c^2$
			J	$= \frac{1}{2}$
			D	$= ?$

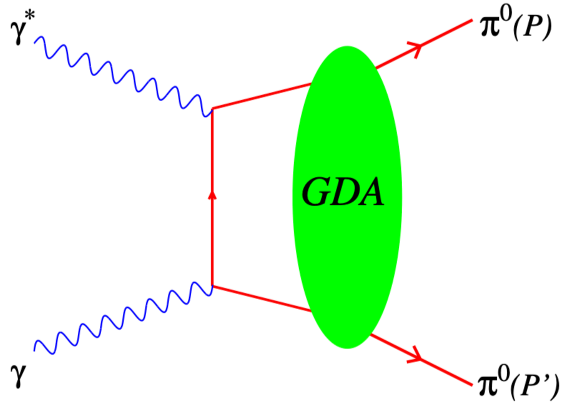
First insights from experiment

- π^0: $\gamma^*\gamma \rightarrow \pi^0\pi^0$ in e^+e^- Belle data: PRD 93, 032003 (2016)

  $D^Q_{\pi^0} \simeq -0.7$ at $\langle Q^2 \rangle = 16.6$ GeV^2

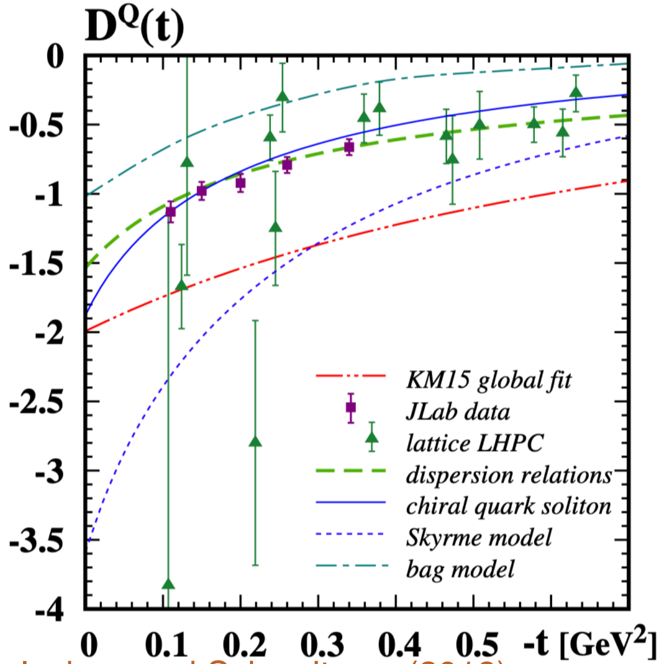
  Kumano, Song, Teryaev, PRD (2018)

  Chiral symmetry: total $D_{\pi^0} \simeq -1$ (gluons contribute the rest)

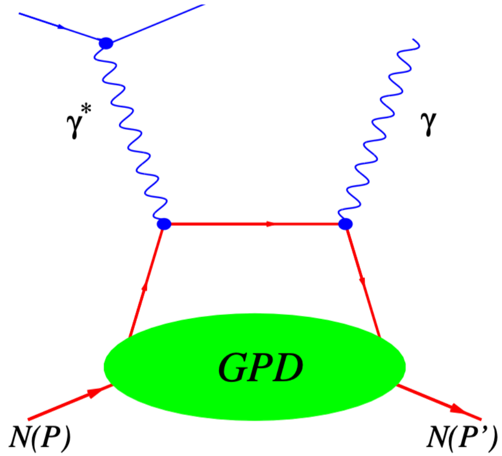


- Proton: Burkert, Elouadrhiri, Girod, Nature 557, 396 (2018)

  JLab data: PRL 100, 162002 (2008) & PRL 115, 212003 (2015)



Polyakov, and Schweitzer, (2018)



subtraction function in dispersion relations for DVCS amplitudes

- $\Delta(t,\mu) \rightarrow D^Q(t,\mu)$ scale-dependent

  model-dependent

- Explore Q^2 range at EIC & EicC

Scalar radius puzzle

- Trace FF:

$$\left\langle N(p') \left| T^{\mu}_{\ \mu} \right| N(p) \right\rangle = m_N \bar{u}(p') \left[A(t) - \frac{t}{4m_N^2}[A(t) - 2J(t) + 3D(t)] \right] u(p) \equiv \bar{u}(p') \Theta(t) u(p)$$

- Trace anomaly in QCD: $T^{\mu}_{\ \mu} = \frac{\beta(g)}{2g} F^{a,\mu\nu} F^{a,}{}_{\mu\nu} + (1 + \gamma_m) \sum_q m_q \bar{\psi}_q \psi_q$

- Scalar radius: $\Theta(t) = m_N \left(1 + t\langle r_\Theta^2 \rangle/6 + \dots\right)$

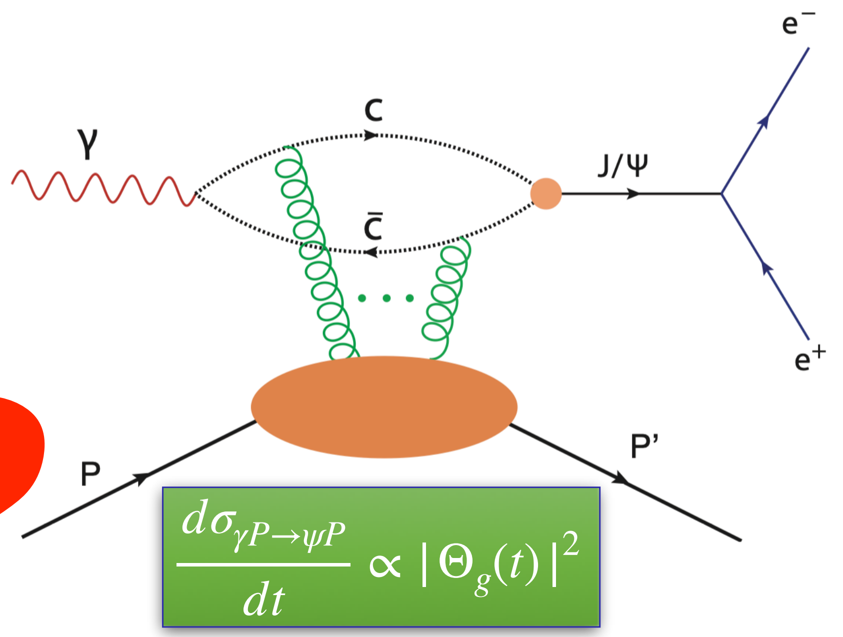
  - Kharzeev proposed it can be extracted from the threshold photoproduction of the vector-meson, e.g., J/ψ, and the fit result is ~0.55 fm

    Kharzeev, PRD (2021)……

  - Recently, two LQCD calculations at near physical quark mass result in a large scalar radius, ~1 fm

    Hackett et al., PRL (2024); Wang et al. [χQCD], PRD (2024)

?



Kharzeev, PRD (2021)

fits to low-t data ⇒ dispersive analyses of "all" data

Why data-driven dispersion theory?

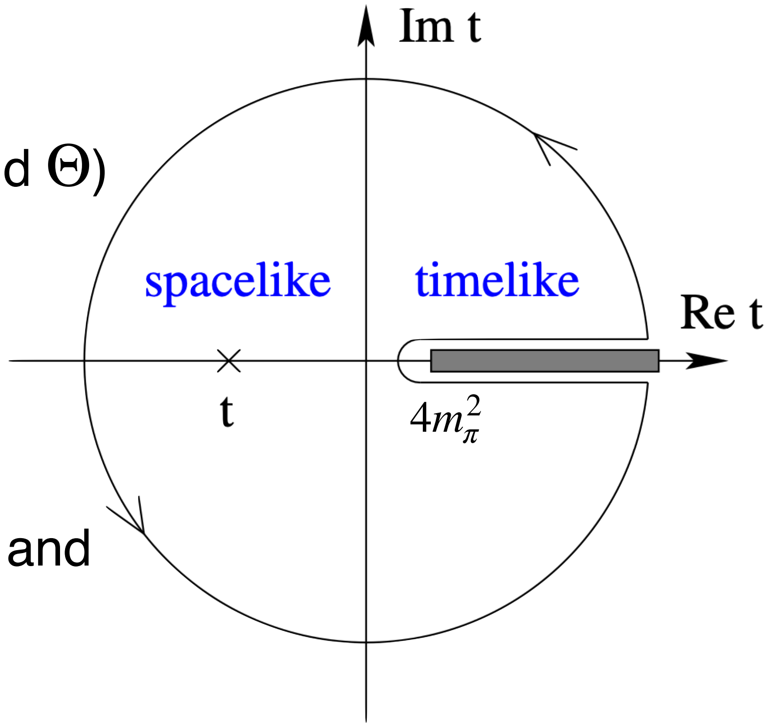
- Based on fundamental principles: unitarity, analyticity and crossing symmetry

- Simultaneous analysis of all four FFs (A, J, D and Θ)

- Connects FFs over full range of momentum transfers: timelike and spacelike data

- Connects to data from other processes ($\pi\pi, K\bar{K}$ and $\pi N, KN$ scatterings…)

- Constraints from ChPT, pQCD



Unitarity relation for pion GFFs

- Gravitational form factors (GFFs) for spin-0 particles, e.g., for pion:

$$\left\langle \pi^a(p') \left| \hat{T}^{\mu\nu}(0) \right| \pi^b(p) \right\rangle = \frac{\delta^{ab}}{2}\left[A^\pi(t)P^\mu P^\nu + D^\pi(t)\left(\Delta^\mu\Delta^\nu - \mathrm{t}g^{\mu\nu}\right)\right]$$

$$P^\mu = p'^\mu + p^\mu, \quad \Delta^\mu = p'^\mu - p^\mu$$

Crossing symmetry
for constructing dispersion relations

$$\left\langle \pi^a(p')\, \pi^b(p) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle = \frac{\delta^{ab}}{2}\left[A^\pi(t)\Delta^\mu\Delta^\nu + D^\pi(t)\left(P^\mu P^\nu - \mathrm{t}g^{\mu\nu}\right)\right]$$

- Unitarity for full amplitude ⇒discontinuity (imaginary part) of the pion GFFs

$$\mathrm{Disc}\left\langle \pi^a(p')\, \pi^b(p) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle$$

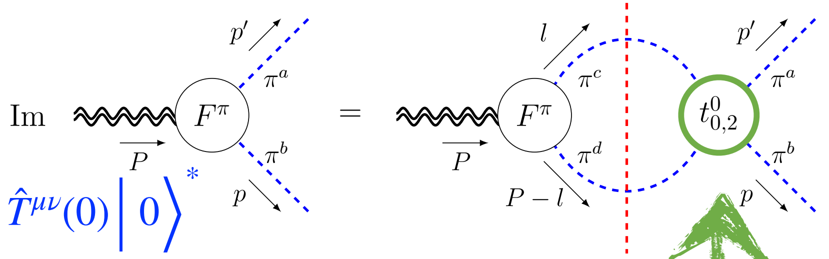
$$= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int \mathrm{d}\Omega_l \left\langle \pi^a(p')\,\pi^b(p) \mid \pi^c(l)\pi^d(P-l) \right\rangle \left\langle \pi^c(l)\pi^d(P-l) \left| \hat{T}^{\mu\nu}(0) \right| 0 \right\rangle^*$$

$$= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int \mathrm{d}\Omega_l \left[A(t,s,u)\delta^{ab}\delta^{cd} + A(s,t,u)\delta^{ac}\delta^{bd} + A(u,s,t)\delta^{ad}\delta^{bc}\right]$$

$$\times \frac{\delta^{cd}}{2}\left[\left(A^\pi(t)\right)^*(2l-P)^\mu(2l-P)^\nu + \left(D^\pi(t)\right)^*\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]$$

$$= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt{t}}\int \mathrm{d}\Omega_l \frac{\delta^{ab}}{2}A^{I=0}(t,s,u)\left[\left(A^\pi(t)\right)^*(2l-P)^\mu(2l-P)^\nu + \left(D^\pi(t)\right)^*\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]$$

S-,*D*-waves   *S*-wave



$\pi\pi$ scattering

Unitarity relation for pion GFFs

- Discontinuity of A^π and D^π: associated only with t_0^0, t_2^0 partial waves.

$$\text{Disc}\left\langle \pi^a(p')\,\pi^b(p)\right|\hat{T}^{\mu\nu}(0)\,|\,0\rangle = \frac{\delta^{ab}}{2}\left[\text{Disc}\,A^\pi(t)\Delta^\mu\Delta^\nu + \text{Disc}\,D^\pi(t)\left(P^\mu P^\nu - t g^{\mu\nu}\right)\right]$$

$$= 2i\frac{2p_\pi}{\sqrt{t}}\frac{\delta^{ab}}{2}\left[\left(A^\pi(t)\right)^*\left(\frac{4}{3t}p_\pi^2\boxed{\left(t_0^0(t)-t_2^0(t)\right)}\left(P^\mu P^\nu - t g^{\mu\nu}\right) + \boxed{t_2^0(t)}\Delta^\mu\Delta^\nu\right) + \left(D^\pi(t)\right)^*\boxed{t_0^0(t)}\left(P^\mu P^\nu - t g^{\mu\nu}\right)\right]$$

Partial-wave amplitudes of $\pi\pi$ scattering

- Partial-wave unitarity

$$\text{Im}\,A^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_2^0(t)\right)^* A^\pi(t)$$

$$\text{Im}\,D^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left[\frac{4}{3}\frac{p_\pi^2}{t}\left(t_0^0(t)-t_2^0(t)\right)^* A^\pi(t) + \left(t_0^0(t)\right)^* D^\pi(t)\right]$$

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$, matrix elements (conserved separately):

Raman, PRD (1971)

$$\left\langle \pi^a(p')\,\pi^b(p)\right|\hat{T}^{\mu\nu}(0)\,|\,0\rangle = \delta^{ab}\left\{\frac{1}{3}\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right)\Theta^\pi(t) + \left[\Delta^\mu\Delta^\nu + \frac{\Delta^2}{3t}\left(P^\mu P^\nu - t g^{\mu\nu}\right)\right]A^\pi(t)\right\}$$

Trace part: $\left\langle \pi^a(p')\,\pi^b(p)\right|\hat{T}^\mu{}_\mu(0)\left|\,0\right\rangle = \delta^{ab}\Theta^\pi(t)$, $\quad \Theta^\pi(t) = -\frac{1}{2}\left(4p_\pi^2 A^\pi(t) + 3tD^\pi(t)\right)$

Unitarity relation for pion GFFs

- Discontinuity of A^π and D^π: associated only with t_0^0, t_2^0 partial waves.

$$\text{Disc}\left\langle \pi^a(p')\,\pi^b(p)\right|\hat{T}^{\mu\nu}(0)|0\rangle = \frac{\delta^{ab}}{2}\left[\text{Disc}\,A^\pi(t)\Delta^\mu\Delta^\nu + \text{Disc}\,D^\pi(t)\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]$$

$$= 2i\frac{2p_\pi}{\sqrt{t}}\frac{\delta^{ab}}{2}\left[\left(A^\pi(t)\right)^*\left(\frac{4}{3t}p_\pi^2\boxed{\left(t_0^0(t)-t_2^0(t)\right)}\left(P^\mu P^\nu - tg^{\mu\nu}\right) + \boxed{t_2^0(t)}\Delta^\mu\Delta^\nu\right) + \left(D^\pi(t)\right)^*\boxed{t_0^0(t)}\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]$$

Partial-wave amplitudes of $\pi\pi$ scattering

- Partial-wave unitarity

$$\text{Im}\,A^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_2^0(t)\right)^* A^\pi(t)$$

$$\text{Im}\,D^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left[\frac{4}{3}\frac{p_\pi^2}{t}\left(t_0^0(t)-t_2^0(t)\right)^* A^\pi(t) + \left(t_0^0(t)\right)^* D^\pi(t)\right] \;\Rightarrow\; \text{Im}\,\Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}}\left(t_0^0(t)\right)^*\Theta^\pi(t)$$

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$, matrix elements (conserved separately):

$$\left\langle \pi^a(p')\,\pi^b(p)\right|\hat{T}^{\mu\nu}(0)|0\rangle = \delta^{ab}\left\{\frac{1}{3}\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2}\right)\Theta^\pi(t) + \left[\Delta^\mu\Delta^\nu + \frac{\Delta^2}{3t}\left(P^\mu P^\nu - tg^{\mu\nu}\right)\right]A^\pi(t)\right\}$$

Trace part: $\left\langle \pi^a(p')\,\pi^b(p)\right|\hat{T}^\mu{}_\mu(0)\left|0\right\rangle = \delta^{ab}\Theta^\pi(t)$, $\quad \Theta^\pi(t) = -\frac{1}{2}\left(4p_\pi^2 A^\pi(t) + 3tD^\pi(t)\right)$

A^π: D-wave single channel

- **Single-channel**: Watson's theorem ⇒ phase of FF = scattering phase shift

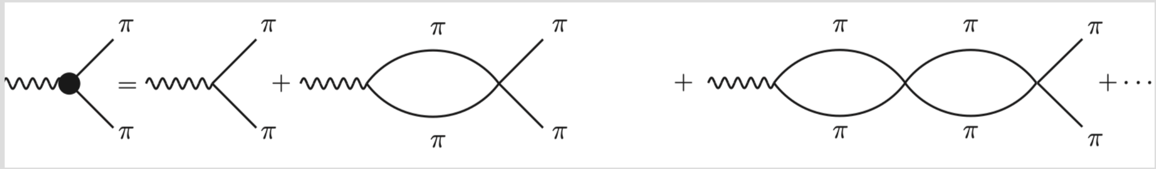
$$\text{disc}\, A^\pi(t) = 2iA^\pi(t)\sin\delta_2^0(t)e^{-i\delta_2^0(t)}\theta\left(t-t_\pi\right)$$

$$\left|t_2^0\right| e^{i\phi_2^0} = \frac{\eta_2^0 e^{2i\delta_2^0}-1}{2i\sigma_\pi}$$

δ_2^0 replaced by the phase of $\pi\pi$ partial wave ϕ_2^0 to account for inelasticity
Hoferichter et al., Phys. Rept. (2016)

Omnes solution

$$A^\pi(t) = P_2^\pi(t)\Omega_2^0(t), \quad \Omega_2^0(t) \equiv \exp\left\{\frac{t}{\pi}\int_{t_\pi}^{\infty}\frac{\mathrm{d}t'}{t'}\frac{\phi_2^0(t')}{t'-t}\right\}$$

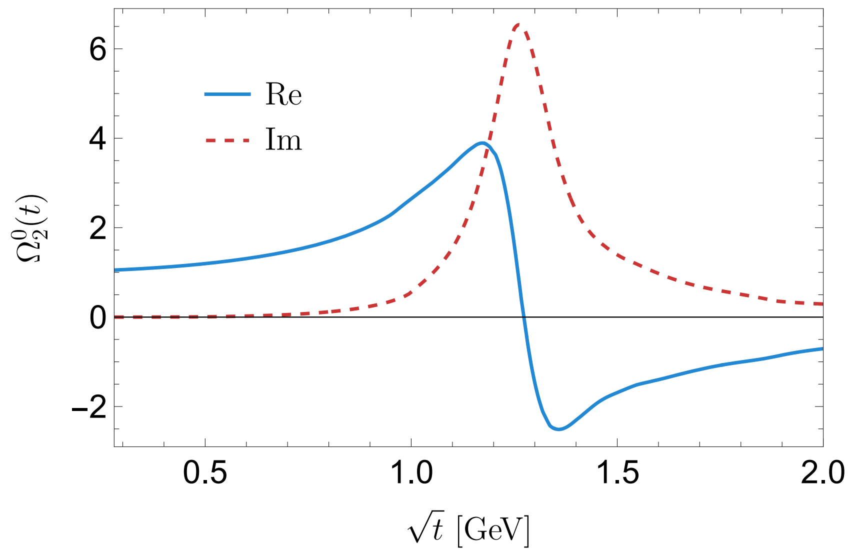


- δ_2^0, η_2^0 up to $E_0 \simeq 2$ GeV from dispersive analysis   Bydžovský et al., PRD (2016)
- Polynomial: $P_2^\pi(t) = 1+\alpha t$ matched to NLO ChPT

$$\Rightarrow A^\pi(t) = 1 - \frac{2L_{12}^r}{F_\pi^2}t$$

Tensor-meson dominance estimate: $L_{12}^r = -\frac{F_\pi^2}{2m_{f_2}^2}$

$$P_2^\pi(t) = 1 + \left[\frac{1}{m_{f_2}^2} - \dot{\Omega}_2^0(0)\right] t \simeq 1 - \left(0.01\mathrm{GeV}^{-2}\right) t$$



Θ^π: S-wave $\pi\pi - K\bar{K}$ coupled channel

- $\pi\pi$ S-wave phase shifts known precisely from Roy(-like) equation analyses

Bern group; Madrid-Krakow group

- Generalisation to coupled channels: isoscalar-scalar $f_0(500), f_0(980)$ mesons

  - Unitarity relation for Θ^π ⇒

$$\mathrm{Im}\,\Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}} \left(t_0^0(t)\right)^* \Theta^\pi(t)$$

Matrix relation for coupled channels (both pion and kaon trace GFFs):

$$\mathrm{Im}\boldsymbol{\Theta}(t) = [\mathbf{T}_0^0(t)]^* \, \boldsymbol{\Sigma}_0^0(t) \, \boldsymbol{\Theta}(t)$$

$$\boldsymbol{\Theta}(t) = \begin{pmatrix} \Theta^\pi(t) \\ \frac{2}{\sqrt{3}}\Theta^K(t) \end{pmatrix}$$

Phase-space factor

$$\boldsymbol{\Sigma}_0^0(t) \equiv \mathrm{diag}(\sigma_\pi \theta(t-t_\pi), \sigma_K \theta(t-t_K))$$

With $\sigma_i(t) = \sqrt{1-4m_i^2/t}\;(i=\pi,K)$

$\pi\pi$-$K\bar{K}$ T-matrix

$$\mathbf{T}_0^0(t) = \begin{pmatrix} \dfrac{\eta_0^0(t)e^{2i\delta_0^0(t)}-1}{2i\sigma_\pi} & |g_0^0(t)|\,e^{i\Psi_0^0(t)} \\ |g_0^0(t)|\,e^{i\Psi_0^0(t)} & \dfrac{\eta_0^0(t)e^{2i(\Psi_0^0(t)-\delta_0^0(t))}-1}{2i\sigma_K} \end{pmatrix}.$$

$$\eta_0^0(t) = \sqrt{1-4\sigma_\pi\sigma_K|g_0^0(t)|^2\theta\left(t-t_K\right)}$$

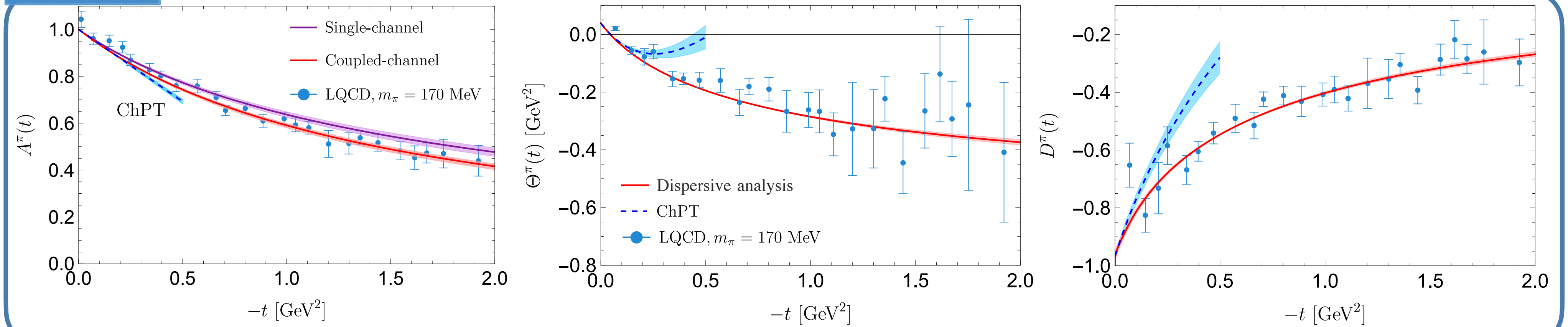
Muskhelishvili-Omnes problem

Donoghue, Gasser, Leutwyler, NPB (1991)

Pion and Kaon GFFs

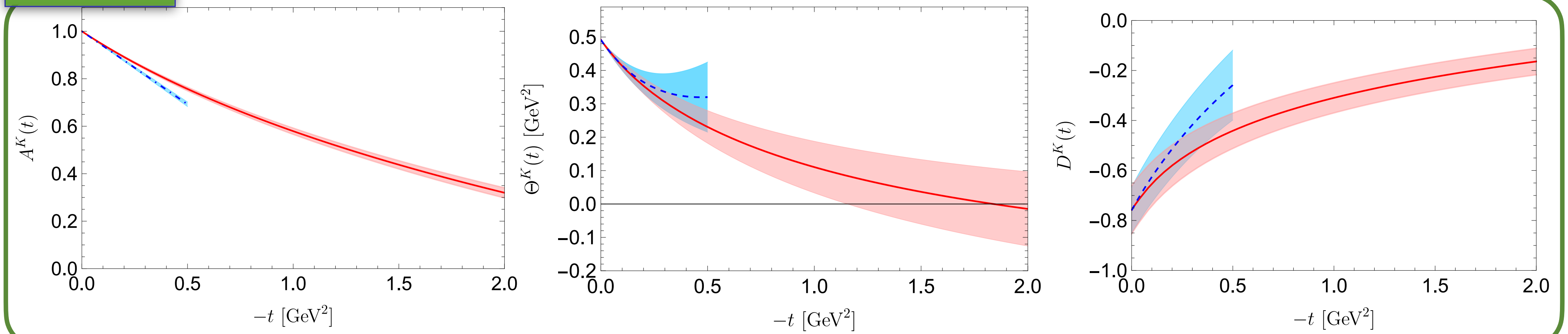
Prediction, Not fit

Pion



LQCD ($m_\pi \simeq 170$ MeV): Hackett et al., PRD (2023)

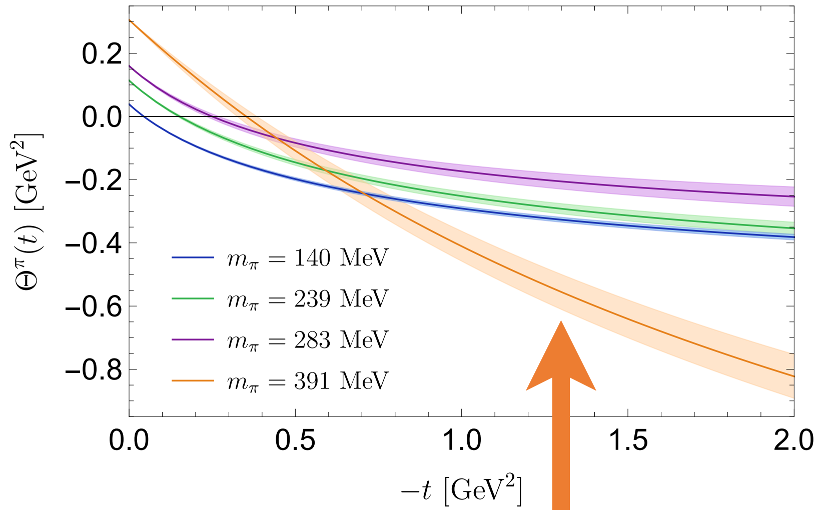
Kaon

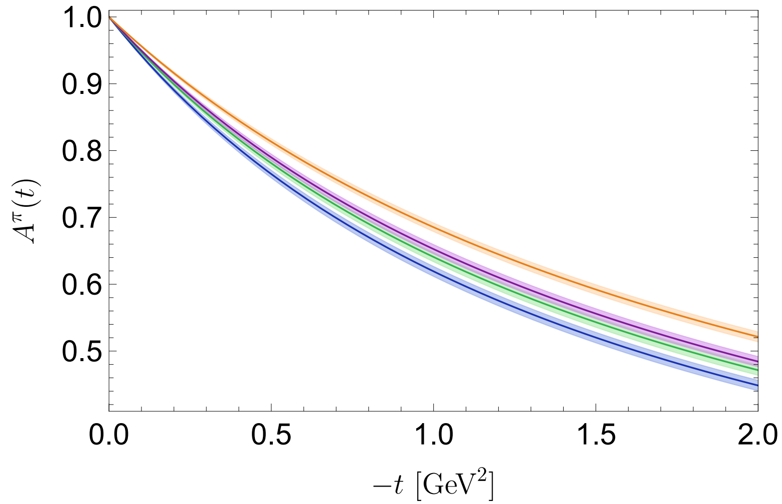


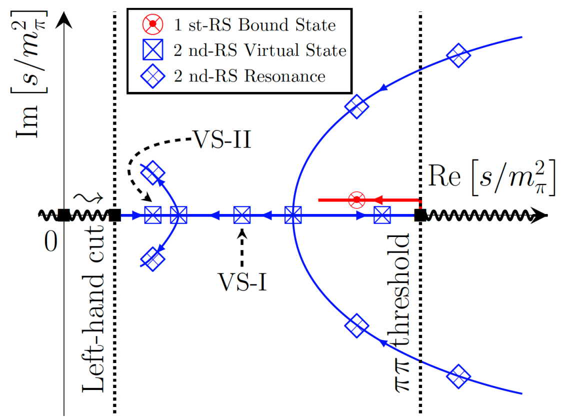
Pion GFFs: pion mass dependence

- Using the scattering phase shifts at unphysical pion masses (239 MeV, 283 MeV, 391 MeV) obtained from Roy equation analyses

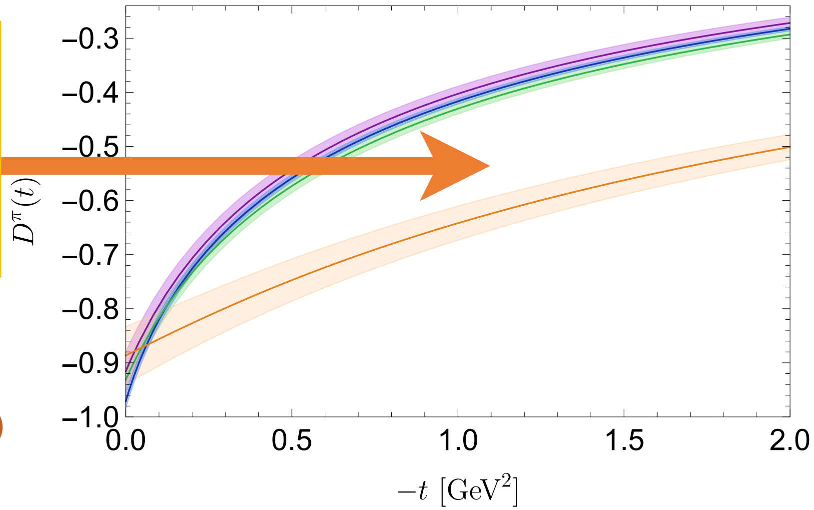
XHC et al. PRD (2023); Rodas et al. [HadSpec], PRD (2024)







Fast change before $m_\pi = 391$ MeV: σ meson becomes a $\pi\pi$ bound state!



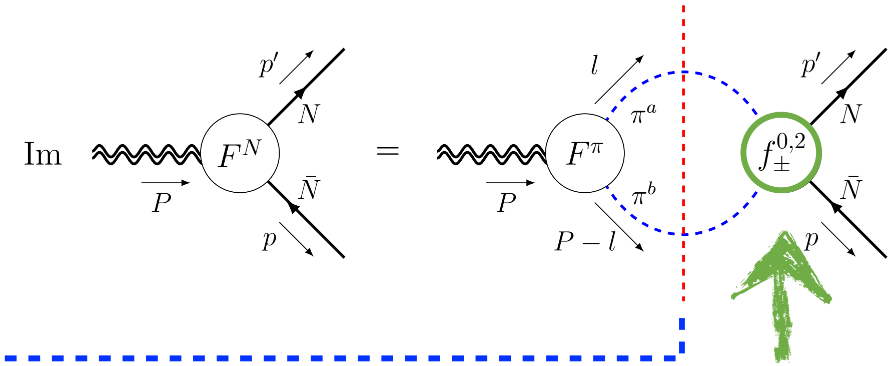
Briceno et al. [HadSpec], PRL (2017)

XHC et al. PRD (2023); Rodas et al. [HadSpec], PRD (2024)

Unitarity relation for nucleon GFFs

- Discontinuity

$$\text{Disc}\left\langle N(p')\bar N(p)\right|\hat T^{\mu\nu}(0)\left|0\right\rangle$$
$$\propto \sum_n \langle N(p')\bar N(p)|n\rangle\langle n|\hat T^{\mu\nu}(0)|0\rangle^*\delta^4(p+p'-p_n)$$



- In the elastic region, only $\pi\pi$ intermediate state

$\pi\pi(K\bar K)\to N\bar N$ scattering

$$\text{Disc}\left\langle N(p')\bar N(p)\right|\hat T^{\mu\nu}(0)\left|0\right\rangle$$
$$=\frac{1}{4m_N}\bar u(p')\left[\text{Disc}\,\hat A(t)\Delta^\mu\Delta^\nu+\text{Disc}\,\hat J(t)\left(i\Delta^{\{\mu}\sigma^{\nu\}\rho}P_\rho\right)+\text{Disc}\,\hat D(t)\left(P^\mu P^\nu-tg^{\mu\nu}\right)\right]v(p)$$
$$=\frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt t}\int \mathrm{d}\Omega_l\left\langle N(p')\bar N(p)\right|\pi^a(l)\pi^b(P-l)\right\rangle\left\langle\pi^a(l)\pi^b(P-l)\right|\hat T^{\mu\nu}(0)\left|0\right\rangle^*$$
$$=\frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt t}\int \mathrm{d}\Omega_l\;\bar u(p')\left[\delta^{ab}\mathbb{1}\left(A^++\frac{(\not P-2\not l)}{2}B^+\right)+i\epsilon_{bac}\tau^c\left(A^-+\frac{(\not P-2\not l)}{2}B^-\right)\right]v(p)$$

Isospin-even   Isospin-odd

$$\times\frac{\delta^{ab}}{2}\left[(A^\pi(t))^*(2l-P)^\mu(2l-P)^\nu+(D^\pi(t))^*\left(P^\mu P^\nu-tg^{\mu\nu}\right)\right]$$
$$=\frac{1}{2}\frac{i}{(4\pi)^2}\frac{p_\pi}{\sqrt t}\int \mathrm{d}\Omega_l\;\bar u(p')\frac{3}{2}\left(A^++\frac{(\not P-2\not l)}{2}B^+\right)v(p)\left[(A^\pi(t))^*(2l-P)^\mu(2l-P)^\nu+(D^\pi(t))^*\left(P^\mu P^\nu-tg^{\mu\nu}\right)\right]$$

$A^\pm, B^\pm$: Lorentz invariant πN scattering amplitudes

Unitarity relation for nucleon GFFs

- Discontinuity of the nucleon GFFs $\Gamma^2 = m_N^2 \sqrt{\frac{2}{3} f_-^2 - f_+^2}$

$$\mathrm{Im}\, A(t) = \frac{3p_\pi^5}{\sqrt{6t}} \left[f_-^2(t) + \sqrt{\frac{3}{2}} \frac{m_N}{p_N^2} \Gamma^2(t) \right]^* A^\pi(t)$$

$$\mathrm{Im}\, J(t) = \frac{3p_\pi^5}{2\sqrt{6t}} \left[f_-^2(t) \right]^* A^\pi(t)$$

$$\mathrm{Im}\, D(t) = -\frac{3m_N p_\pi}{2p_N^2\sqrt{t}} \left[\frac{4p_\pi^2}{3t} \left(\left(f_+^0(t)\right)^* - (p_\pi p_N)^2 \left(f_+^2(t)\right)^* \right) A^\pi(t) + \left(f_+^0(t)\right)^* D^\pi(t) \right]$$

$$\Theta(t) = \frac{1}{4m_N} \left[-4p_N^2 A(t) + 2tJ(t) - 3tD(t) \right]$$

$$\mathrm{Im}\, \Theta(t) = -\frac{3p_\pi}{4p_N^2\sqrt{t}} \left[f_+^0(t) \right]^* \Theta^\pi(t)$$

$f_\pm^J$: $\pi\pi \to K\bar{K}$ partial wave amp. With $+/-$ for parallel/anti-parallel $N\bar{N}$ helicities

Frazer & Fulco (1960); Höhler (1983)

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$, matrix elements

Coupled-channel

$$\left\langle N(p')\, \bar{N}(p) \middle| \hat{T}^{\mu\nu}(0) \middle| 0 \right\rangle = \bar{u}(p') \left(T_S^{\mu\nu} + T_T^{\mu\nu} \right) v(p)$$

$$T_S^{\mu\nu} = \frac{1}{3} \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) \Theta(t)$$

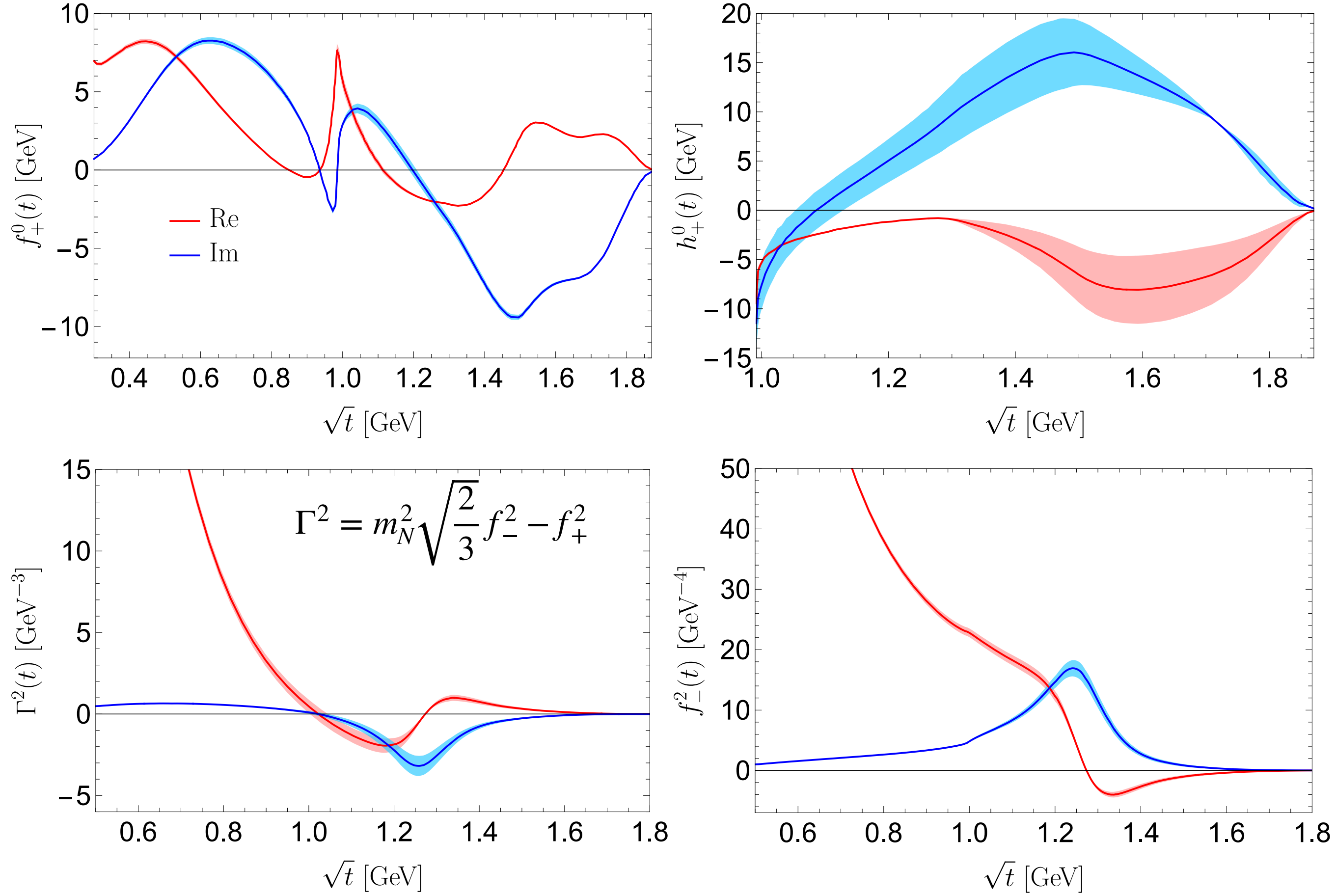
$$\mathrm{Im}\, \Theta(t) = -\frac{3}{4p_N^2\sqrt{t}} \left\{ p_\pi \left[f_+^0(t) \right]^* \Theta^\pi(t) \theta(t - t_\pi) + \frac{4}{3} p_K \left[h_+^0(t) \right]^* \Theta^K(t) \theta(t - t_K) \right\}$$

$$T_T^{\mu\nu} = \frac{1}{4m_N} \left[\Delta^\mu \Delta^\nu + \frac{\Delta^2}{3t} \left(P^\mu P^\nu - t g^{\mu\nu} \right) \right] A(t) + \left[i\Delta^{\{\mu}\sigma^{\nu\}\rho} P_\rho + \frac{2i\sigma^{\rho\kappa}\Delta_\rho P_\kappa}{3t} \left(P^\mu P^\nu - t g^{\mu\nu} \right) \right] J(t)$$

$\pi\pi/K\bar{K} \to N\bar{N}$ partial wave amplitudes

- Inputs: $\pi\pi/K\bar{K} \to N\bar{N}$ partial wave amp. $f_{\pm}^{0,2}, h_{+}^{0}$ from Roy-Steiner analyses

Hite & Steiner, Nuove Cimento A (1973)



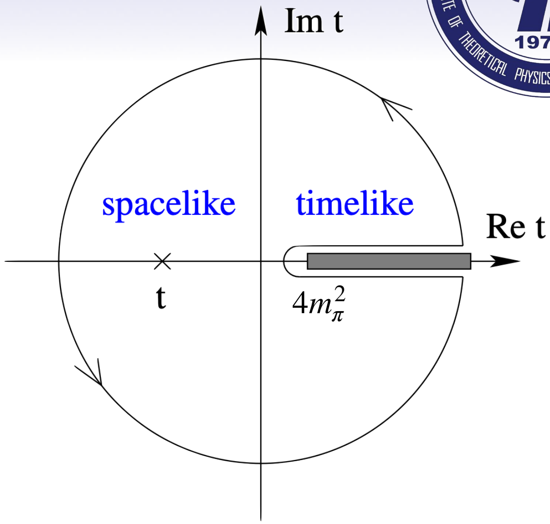
πN amplitudes from modern Roy-Steiner equation analyses

Ditsche, et al., JHEP (2012); Hoferichter, et al., PRL 115, 092301(2015); PRL 115, 192301 (2015); Phys. Rept. (2016);
XHC, Q.-Z. Li & H.-Q. Zheng, JHEP (2022); Hoferichter, et al., PLB (2024)

Gravitational form factors of the nucleon

- Dispersive relations (DRs) for the nucleon GFFs :

$$(A, J, \Theta)(t) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dt' \frac{\mathrm{Im}(A, J, \Theta)(t')}{t' - t}$$



- Constraints:
  - Normalizations: mass m_N, spin $1/2$

    ⇒ sum rules saturated by $\pi\pi, K\bar{K}$ continuum and some higher mass states

    Belushkin et al., PRC (2007); Hoferichter et al., EPJA (2016) …

$$\frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}(A, J, \Theta)(t')}{t'} = \left(1, \frac{1}{2}, m_N\right)$$

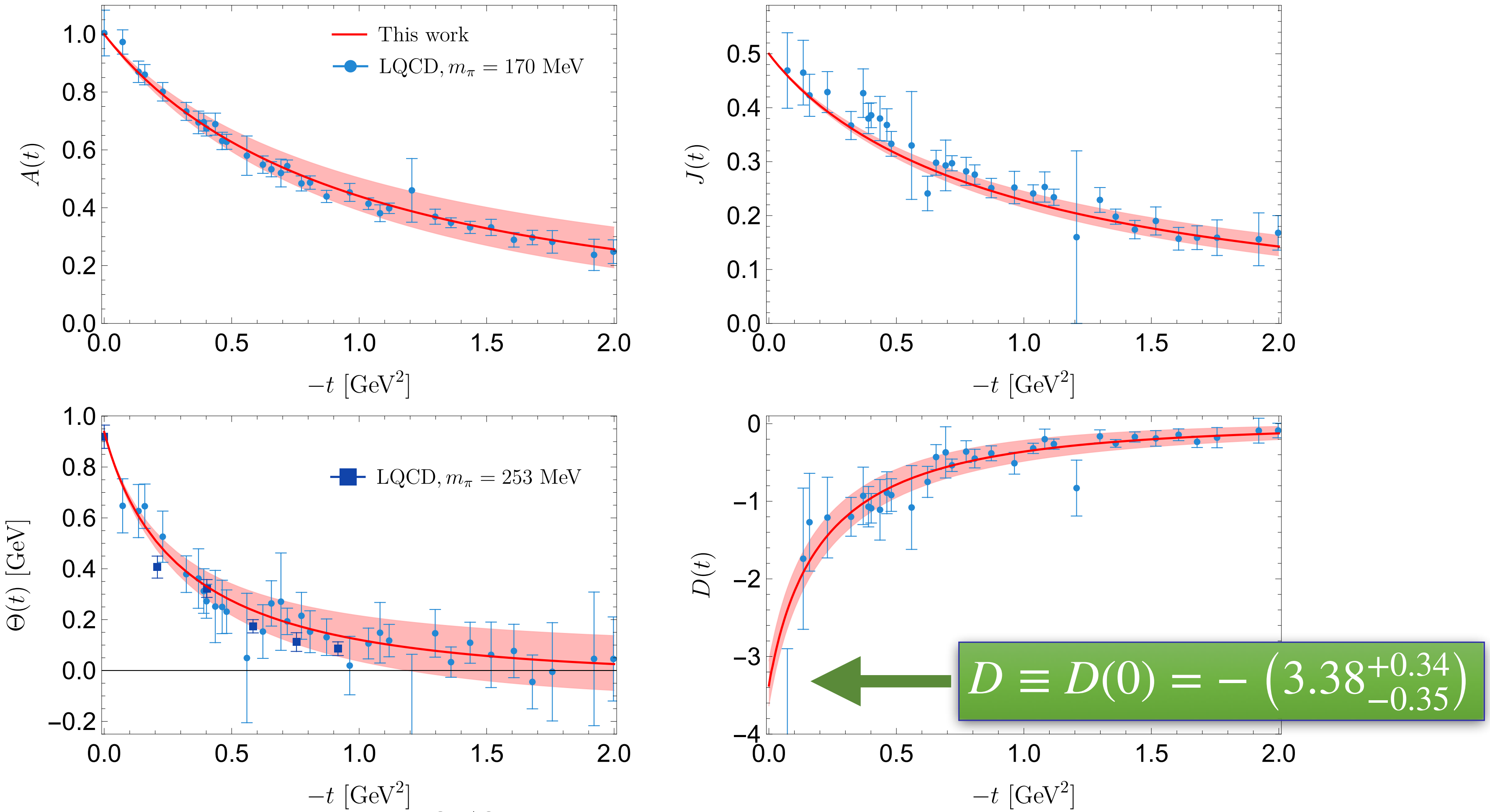
Introduce S-wave $0^{++}$and D-wave (2^{++}) poles to the spectral functions:
$\pi c_{S,D} m_{S,D}^2 \delta\left(t - m_{S,D}^2\right)$ to satisfy the sum rules

✓ 0^{++} : $m_S \in (1.5, 1.8)$ GeV to cover $f_0(1500)$ and $f_0(1710)$
✓ 2^{++} : $m_D \in (1.5, 2.2)$ GeV to cover $f_2(1500)$, $f_2(1950)$ and $f_2(2010)$

Nucleon GFFs

- **Predictions：**

LQCD ($m_\pi \simeq 170$ MeV): Hackett et al., PRL (2024)



LQCD ($m_\pi \simeq 253$ MeV): Wang et al. [χQCD], PRD (2024)

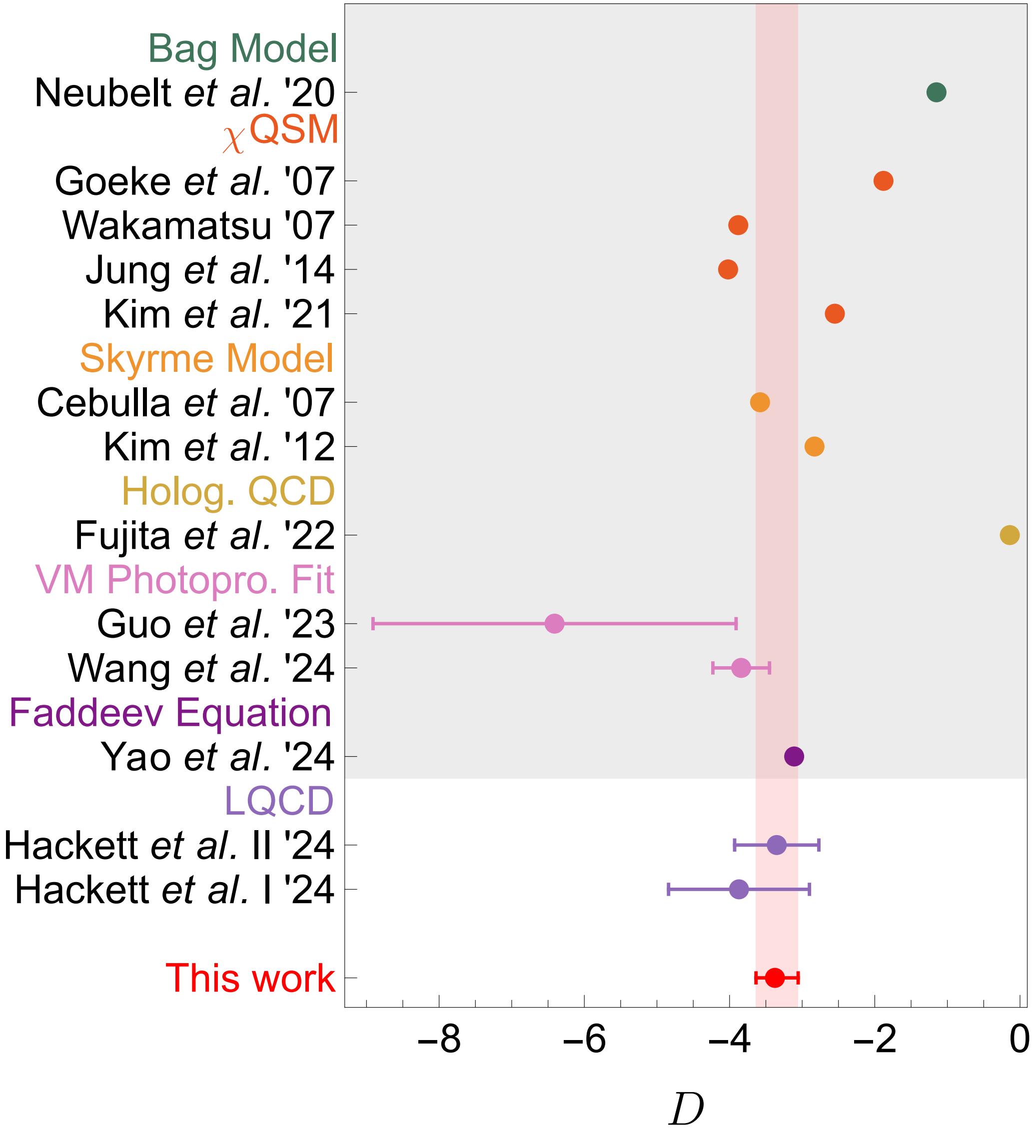
☑ GFFs be model-independently predicted using data-driven dispersion relations + ChPT

☑ Not a fit!

Nucleon D-term

- Nucleon D-term $D \equiv D(0)$:

$$D = -\left(3.38^{+0.34}_{-0.35}\right)$$



- Positivity bound:

$$D < -0.2$$

Gegelia & Polyakov, PLB (2021)

"The stability condition"

normal force $\frac{2}{3}s(r) + p(r) \geq 0$

$$\widetilde{D}(r) = \int \frac{d^3\mathbf{\Delta}}{(2\pi)^3} e^{-i\mathbf{\Delta r}} D\left(-\mathbf{\Delta}^2\right) < 0$$

Nucleon GFFs: D-term & radii

- Various radii in the Breit frame
  - Radius of the scalar trace density

$$\langle r_\Theta^2 \rangle = \frac{6\dot{\Theta}(0)}{m_N} = 6\dot{A}(0) - \frac{9D}{2m_N^2}$$

  - Radius of the mass (energy) density

$$\langle r^2_{\text{Mass}} \rangle = 6\dot{A}(0) - \frac{3D}{2m_N^2}$$

  - Mechanical radius

Polyakov, PLB (2003)
Polyakov, & Schweitzer, IJMPA (2018)

$$\langle r^2_{\text{Mech}} \rangle = \frac{6D}{\int_{-\infty}^{0} \mathrm{d}t D(t)}$$

  - Radius of the angular momentum (spin) density $J(t) + \frac{2}{3}t\frac{\mathrm{d}J(t)}{\mathrm{d}t}$

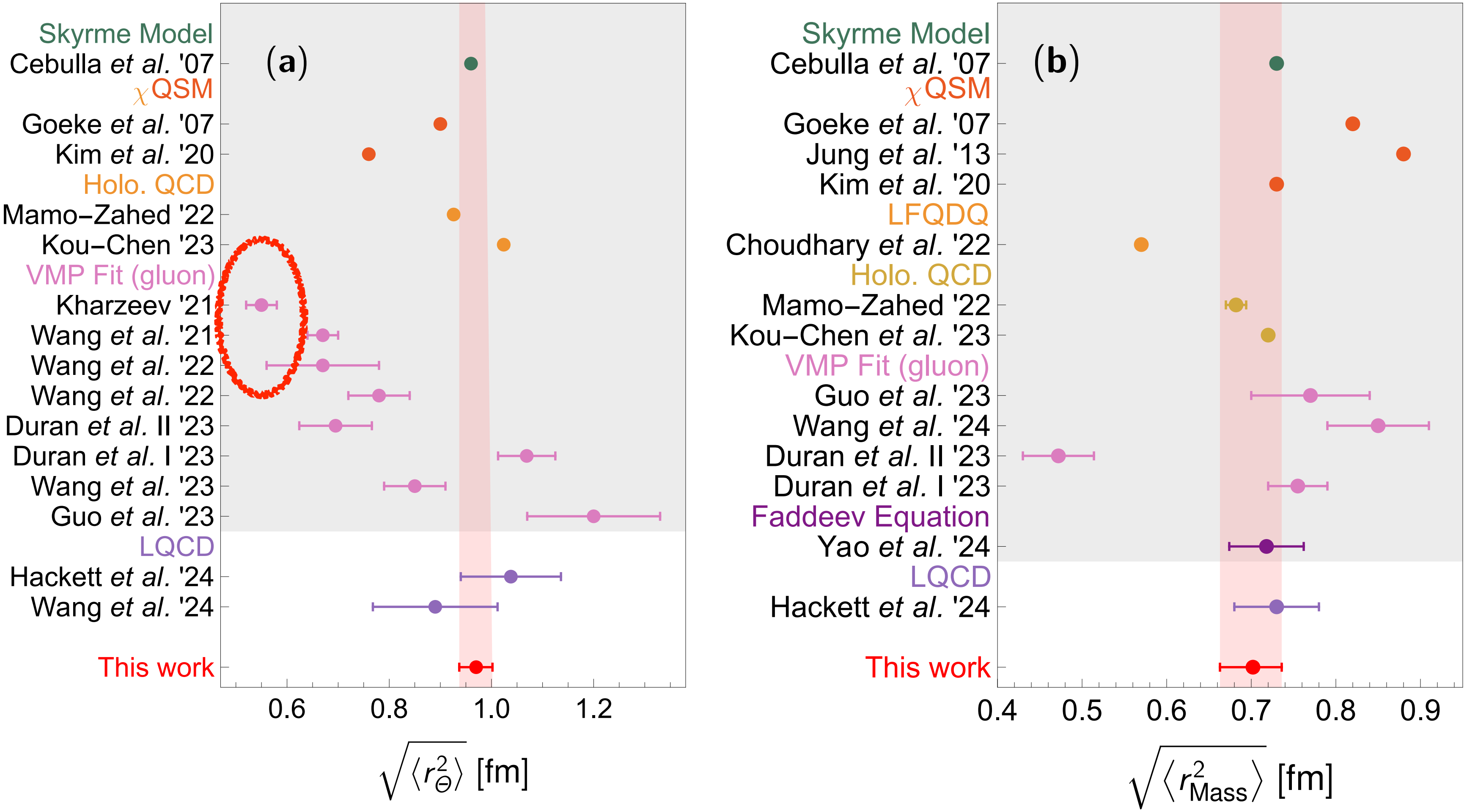
$$\langle r_J^2 \rangle = 20J'(0)$$

Polyakov, PLB (2003)
Lorcé et al., PLB (2018)

Quantity	Result	Error budget
D-term	$-3.38^{+0.34}_{-0.35}$	+(0.18)ChPT(0.12)PWA(0.26)eff -(0.16)ChPT(0.12)PWA(0.29)eff
$\sqrt{\langle r_\Theta^2 \rangle}$ [fm]	$0.97^{+0.03}_{-0.03}$	+(0.01)ChPT(0.01)PWA(0.03)eff -(0.02)ChPT(0.01)PWA(0.26)eff
$\sqrt{\langle r^2_{\text{Mass}} \rangle}$	$0.70^{+0.03}_{-0.04}$	+(0.02)ChPT(0.01)PWA(0.02)eff +(0.02)ChPT(0.01)PWA(0.03)eff
$\sqrt{\langle r^2_{\text{Mech}} \rangle}$	$0.72^{+0.09}_{-0.08}$	+(0.02)ChPT(0.00)PWA(0.09)eff -(0.03)ChPT(0.01)PWA(0.07)eff
$\sqrt{\langle r_J^2 \rangle}$	$0.70^{+0.02}_{-0.02}$	+(0.01)ChPT(0.01)PWA(0.01)eff -(0.01)ChPT(0.00)PWA(0.02)eff

ChPT: NLO ChPT inputs
pwa: $\pi\pi/K\bar{K} \to N\bar{N}$
eff: effective poles m_S, m_D

Comparison with other works



Physics of the scalar (trace) radius

- The scalar (trace) radius is larger than the mass radius as long as $D < 0$

$T_\mu^{\ \mu}$ $\quad\quad\quad\quad$ T^{00}

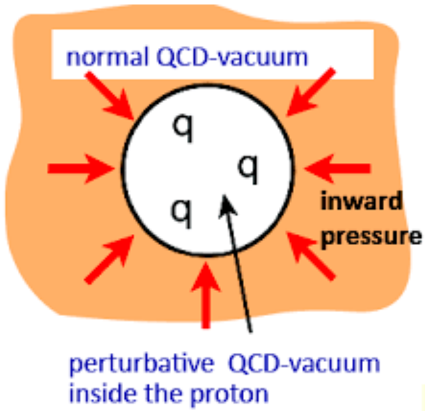
$$\sqrt{\langle r_\Theta^2 \rangle} = 0.97^{+0.03}_{-0.03}\text{fm} \quad > \quad \sqrt{\langle r_{C,p}^2 \rangle} = 0.84\text{fm} \quad > \quad \sqrt{\langle r_{\text{Mass}}^2 \rangle} = 0.70^{+0.03}_{-0.04}\text{fm}$$

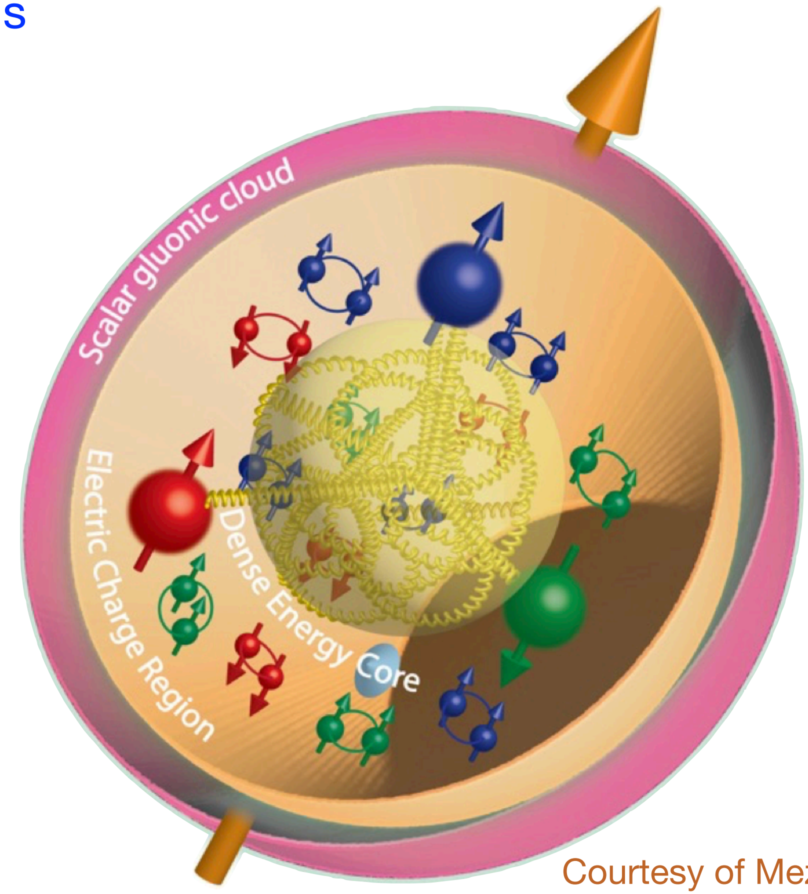
proton charge radius

"the largest radius in strong interaction"

- In the MIT bag model, scalar radius may be considered as the bag (confinement) radius ~1 fm (physical boundary of confinement)

Ji, Front.Phys.(Beijing) (2021)





Courtesy of Meziani

Summary and outlook

- ☑ The pion, kaon and nucleon GFFs are precisely determined using model-independent dispersive method

- ☑ Quantification of systematic and theoretical uncertainties

- ☑ Nucleon static properties:

$$D = -3.38^{+0.34}_{-0.35}, \quad \sqrt{\langle r^2_\Theta \rangle} = 0.97^{+0.03}_{-0.03}\text{fm} \quad > \quad \sqrt{\langle r^2_{C,p} \rangle} = 0.84\text{fm} \quad > \quad \sqrt{\langle r^2_{\text{Mass}} \rangle} = 0.70^{+0.03}_{-0.04}\text{fm}$$

proton charge radius

- ☐ Pion-mass dependence of nucleon GFFs

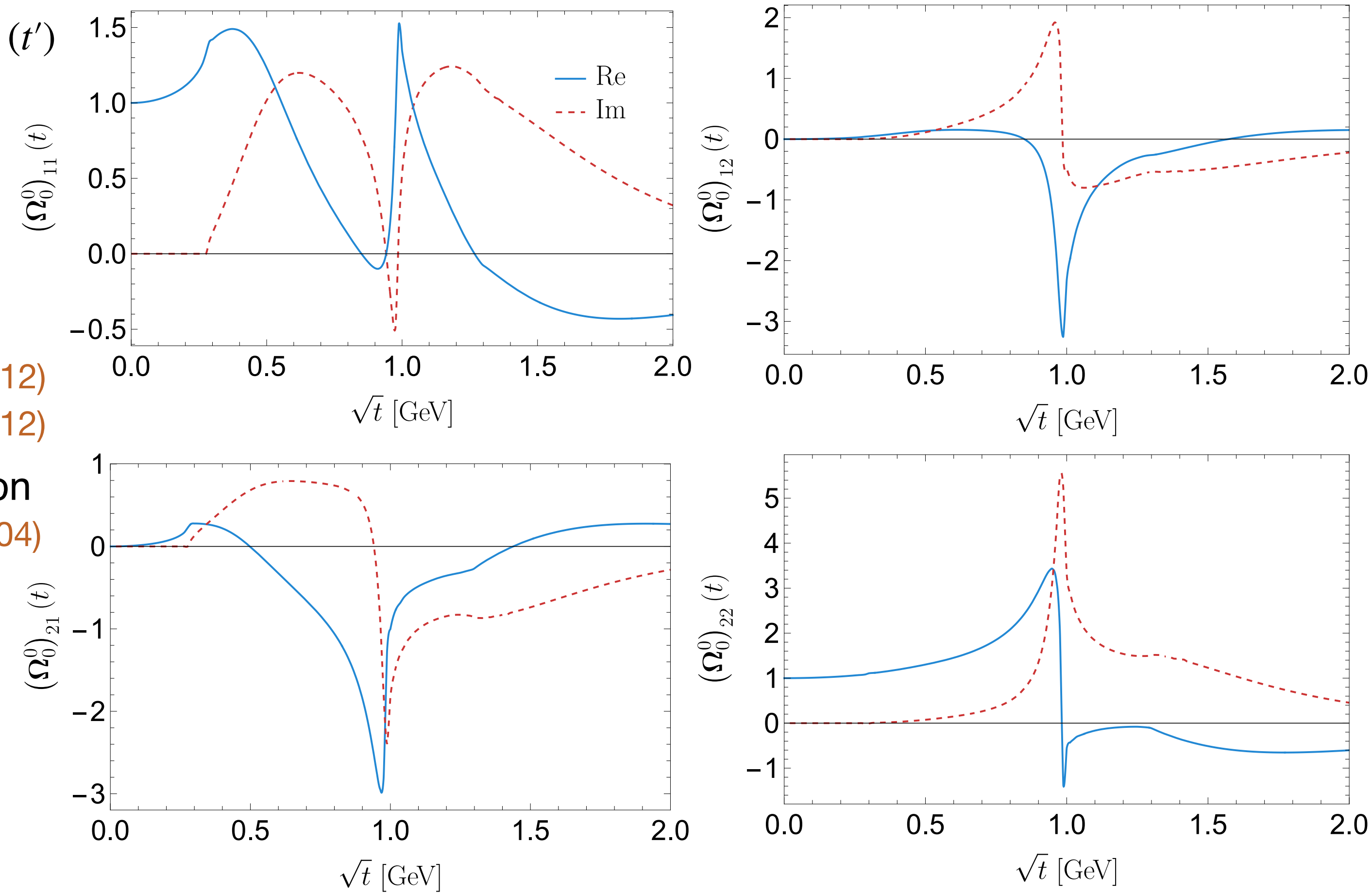
- ☐ Hyperon gravitational structure……

Thank you for your attention!

Muskhelishvili-Omnes representation

- **Coupled-channel**: solution known as the Muskhelishvili-Omnes (MO) representation
  - Take isoscalar-scalar $\pi\pi - K\bar{K}$ as example (matching point~1.3 GeV)

$$\mathbf{\Omega}_0^0(t) = \frac{1}{\pi}\int_{t_\pi}^{\infty} \frac{\mathrm{d}t'}{t'-t} \left[\mathbf{T}_0^0(t)\right]^* \mathbf{\Sigma}_0^0(t)\mathbf{\Omega}_0^0(t')$$



$\pi\pi$ phase shift: Roy equation

Ananthanarayan et al., Phys. Rept. (2012)

Caprini et al., EPJC (2012)

$\pi\pi \to K\bar{K}$: Roy-Steiner equation

Büttiker et al., EPJC (2004)

See also Hoferichter et al., JHEP (2012)

ChPT matching

- Pion and kaon trace GFFs

$$[\boldsymbol{\Theta}(t)]^T = \left[\boldsymbol{P}_0(t)\right]^T \boldsymbol{\Omega}_0^0(t), \quad \boldsymbol{P}_0(t) = \begin{pmatrix} 2m_\pi^2 + \beta_\pi t \\ \frac{2}{\sqrt{3}}\left(2m_K^2 + \beta_K t\right) \end{pmatrix}$$

$$\beta_\pi = \dot{\Theta}^\pi(0) - 2m_\pi^2 \left(\dot{\boldsymbol{\Omega}}_0^0\right)_{11}(0) - \frac{4m_K^2}{\sqrt{3}} \left(\dot{\boldsymbol{\Omega}}_0^0\right)_{12}(0),$$

$$\beta_K = \dot{\Theta}^K(0) - \sqrt{3} m_\pi^2 \left(\dot{\boldsymbol{\Omega}}_0^0\right)_{21}(0) - 2m_K^2 \left(\dot{\boldsymbol{\Omega}}_0^0\right)_{22}(0),$$

- Matching to **NLO ChPT** Donoghue, Leutwyler, ZPC 52 (1991) 343

$$\dot{\Theta}^\pi(0) = 1 - 4L_{12}^r \frac{m_\pi^2}{F_\pi^2} - 24\left(L_{11}^r - L_{13}^r\right)\frac{m_\pi^2}{F_\pi^2} - \frac{3}{2}\frac{m_\pi^2}{F_\pi^2} I_\pi + \frac{m_\pi^2}{2F_\pi^2} I_\eta = 0.98(2)$$

$$\dot{\Theta}^K(0) = 1 - 4L_{12}^r \frac{m_K^2}{F_\pi^2} - 24\left(L_{11}^r - L_{13}^r\right)\frac{m_K^2}{F_\pi^2} - \frac{m_K^2}{F_\pi^2} I_\eta = 0.94(14)$$

$\Rightarrow$ chiral logs: $I_i = \frac{1}{48\pi^2}\left(\ln\frac{\mu^2}{m_i^2} - 1\right)$

- Similar coupled-channel analysis for D-wave

$\Rightarrow$ coupled channel results for A^π, A^K and D^π, D^K