Next-to-Next-to-Leading-Order QCD Prediction for the Pion Electromagnetic Form Factor

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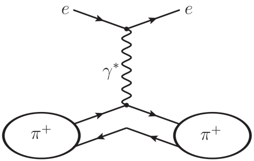
Motivations

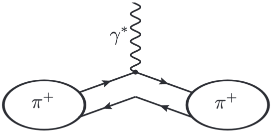
- Many previous reports at this conference have mentioned π meson.
- Process $\pi^+(p) \to \pi^+(p')\gamma^*$ with space-like momentum transfer $Q^2 \equiv -(p'-p)^2$ is theoretically clean.

$$\langle \pi^+(p') | j_\mu^{\mathrm{em}}(0) | \pi^+(p) \rangle = F_\pi(Q^2)\,(p+p')_\mu$$

- Factorization theorem established at leading-power.
- Key targets for the forthcoming EIC experiments at BNL with high precision.

R. Abdul Khalek et al., Nucl. Phys. A 1026, 122447 (2022), arXiv:2103.05419 [physics.ins-det].





Motivations

- Next-to-leading order calculation:
  - R. D. Field et al (1981), F.-M. Dittes et al (1981), R. S. Khalmuradov et al (1985) $\Leftarrow$ wrong hard kernel
  - M. H. Sarmadi (1982), E. Braaten et al (1987) $\Leftarrow$ correct hard kernel, wrong IR-subtraction
  - Melic et al (1999) $\Leftarrow$ correct

- Next-to-next-leading order calculation:
  - L.-B. Chen, W. Chen, F. Feng, and Y. Jia, Phys. Rev. Lett. 132, 201901 (2024), arXiv:2312.17228 [hep-ph] $\Leftarrow$ covariant trace method

- In this work, we calculate the pion electromagnetic form factor (EMFF) within a rigorous factorization framework, which include the complete analysis of the evanescent operator mixing.

Definition & Kinematics

$$\pi^+(p) \to \pi^+(p') + \gamma^* \quad (1)$$

$q = p' - p$ is a spacelike four vector. Then we define $Q^2 = -q^2 \geq 0$.

- (Space-like) Pion EMFF can be defined as

$$\langle \pi^+(p') | j_\mu^{\rm em}(0) | \pi^+(p) \rangle = F_\pi(Q^2)\,(p+p')_\mu + \tilde{F}_\pi(Q^2)\,(p-p')_\mu, \quad (2)$$

$$j_\mu^{\rm em}(x) = \sum_{u,d} e_q\, \bar{q}(x)\gamma_\mu q(x)\,. \quad (3)$$

- Employing the vector-current conservation condition leads to

$$\tilde{F}_\pi(Q^2) = 0 \quad (4)$$

- Isospin symmetry indicates

$$F_{\pi^-}(Q^2) = -F_\pi(Q^2), \qquad F_{\pi^0}(Q^2) = 0. \quad (5)$$

- Electric charge conservation indicates

$$F_\pi(0) = 1. \quad (6)$$

- Since current operator is Hermitian $j_\mu^{{\rm em},\dagger} = j_\mu^{\rm em}$

$$F_\pi(Q^2) = F_\pi^*(Q^2) \quad \Rightarrow \quad F_\pi(Q^2) \text{ is a real valued quantity.} \quad (7)$$

Hard-collinear factorization

$$\langle \pi^+(p')|j_\mu^{\rm em}(0)|\pi^+(p)\rangle = F_\pi(Q^2)\,(p+p')_\mu, \quad Q^2 = -(p'-p)^2 \tag{8}$$

For large Q^2,

$$Q^2 = -q^2 \to \infty \quad \text{or} \quad Q^2 \gg \Lambda^2_{\rm QCD} \tag{9}$$

the pion mass m_π and light quark mass m_u, m_d can be neglected.

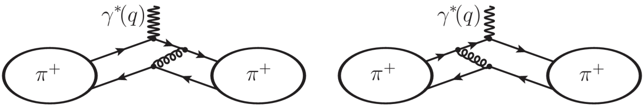
We introduce two light-cone momentum n, $\bar{n}$ $(n \cdot \bar{n} = 2)$

$$p_\mu = (n \cdot p)\frac{\bar{n}_\mu}{2}, \qquad p'_\mu = (\bar{n} \cdot p')\frac{n_\mu}{2}, \qquad n \cdot p \sim \bar{n} \cdot p' \sim \mathcal{O}(\sqrt{Q^2}) \tag{10}$$

The transverse momenta are quite small, quark and anti-quark are almost collinear.

$$k_u = yp', \quad k_{\bar{d}} = (1-y)p', \quad k_u = xp, \quad k_{\bar{d}} = (1-x)p \tag{11}$$

x and y determine the share of the longitudinal momentum.



Hard-collinear factorization

At leading power in the $\frac{\Lambda^2_{\rm QCD}}{Q^2}$ expansion, we obtain the hard-collinear factorization

$$F_\pi(Q^2) = (e_u - e_d)\frac{4\pi\alpha_s(\nu)}{Q^2} f_\pi^2 \int dx \int dy\, T_1(x, y, Q^2, \nu, \mu)\, \phi_\pi(x,\mu)\, \phi_\pi(y,\mu)\,, \quad (12)$$

where x, y are momentum fraction carried by u-quark.

$f_\pi = (130.2 \pm 1.2)$ MeV is pion decay constant.

ν is the renormalization scale and μ is the factorization scale.

Non-perturbative quantity — leading-twist pion LCDA ϕ_π

$$\langle \pi^+(p')|\bar{u}(\tau\,\bar{n})\;[\tau\,\bar{n}, 0]\;\gamma_\mu\,\gamma_5\,d(0)|0\rangle = -i\,f_\pi\,p'_\mu \int_0^1 dx\, e^{ix\,\tau\,\bar{n}\cdot p'}\,\phi_\pi(x,\mu)\,, \quad (13)$$

The hard kernel $T_1(x, y, Q^2, \nu, \mu)$ can be calculated perturbatively in terms of the α_s.

The non-perturbative contributions are absorbed in the leading-twist pion light-cone distribution amplitude (LCDA) ϕ_π, which can be calculated in Lattice QCD or extracted from other process, e.g., pion-photon transition form factor.

In our work, we calculate the hard kernel $T_1(x, y, Q^2, \nu, \mu)$ to NNLO.

Operator basis

In two-loop amplitude, there are three operators generated from bare two-loop calculation,

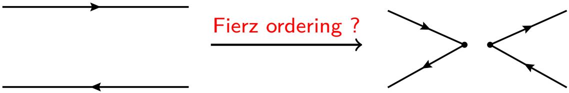
$$\tilde{O}_2(s,t) = \left[(\bar{\chi}_u W_{\bar{n}})(0)\gamma^{\perp}_{\rho 1}(W^{\dagger}_{n}\xi_u)(0)\right]\left[(\bar{\xi}_d W_n)(t\bar{n})\gamma^{\perp}_{\rho 1}(W^{\dagger}_{\bar{n}}\chi_d)(sn)\right] ,$$

$$\tilde{O}_3(s,t) = \left[(\bar{\chi}_u W_{\bar{n}})(0)\gamma^{\perp}_{\rho 1}\gamma^{\perp}_{\rho 2}\gamma^{\perp}_{\rho 3}(W^{\dagger}_{n}\xi_u)(0)\right]\left[(\bar{\xi}_d W_n)(t\bar{n})\gamma^{\perp}_{\rho 3}\gamma^{\perp}_{\rho 2}\gamma^{\perp}_{\rho 1}(W^{\dagger}_{\bar{n}}\chi_d)(sn)\right] ,$$

$$\tilde{O}_4(s,t) = \left[(\bar{\chi}_u W_{\bar{n}})(0)\gamma^{\perp}_{\rho 1}\gamma^{\perp}_{\rho 2}\gamma^{\perp}_{\rho 3}\gamma^{\perp}_{\rho 4}\gamma^{\perp}_{\rho 5}(W^{\dagger}_{n}\xi_u)(0)\right]$$
$$\left[(\bar{\xi}_d W_n)(t\bar{n})\gamma^{\perp}_{\rho 5}\gamma^{\perp}_{\rho 4}\gamma^{\perp}_{\rho 3}\gamma^{\perp}_{\rho 2}\gamma^{\perp}_{\rho 1}(W^{\dagger}_{\bar{n}}\chi_d)(sn)\right] .$$

The operator in the definition of LCDA is

$$O_1(s,t) = \left[(\bar{\chi}_u W_{\bar{n}})(0)\not{n}\gamma_5(W^{\dagger}_{\bar{n}}\chi_d)(sn)\right]\left[(\bar{\xi}_d W_n)(t\bar{n})\not{\bar{n}}\gamma_5(W^{\dagger}_{n}\xi_u)(0)\right] . \qquad (14)$$



Operator basis

$$\tilde{O}_2(s,t) = \left[(\bar{\chi}_u W_{\bar{n}})(0)\gamma^{\perp}_{\rho 1}(W^{\dagger}_{n}\xi_u)(0)\right]\left[(\bar{\xi}_d W_n)(t\bar{n})\gamma^{\perp}_{\rho 1}(W^{\dagger}_{\bar{n}}\chi_d)(sn)\right] ,$$

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$$\left[(\bar{\xi}_d W_n)(t\bar{n})\gamma^{\perp}_{\rho 5}\gamma^{\perp}_{\rho 4}\gamma^{\perp}_{\rho 3}\gamma^{\perp}_{\rho 2}\gamma^{\perp}_{\rho 1}(W^{\dagger}_{\bar{n}}\chi_d)(sn)\right] .$$

$$O_1(s,t) = \left[(\bar{\chi}_u W_{\bar{n}})(0)\not{n}\gamma_5(W^{\dagger}_{\bar{n}}\chi_d)(sn)\right]\left[(\bar{\xi}_d W_n)(t\bar{n})\not{\bar{n}}\gamma_5(W^{\dagger}_{n}\xi_u)(0)\right] \quad (15)$$

$\tilde{O}_2$, $\tilde{O}_3$ and $\tilde{O}_4$ can be reduced to O_1 by Fierz transformation, but valid only in four dimensions.

$$\tilde{O}_2 = \frac{1}{4}O_1, \quad \tilde{O}_3 = O_1, \quad \tilde{O}_4 = 4O_1. \quad (d=4) \quad (16)$$

The computation of the bare amplitude in dimensional regularization with $D = 4 - 2\epsilon$.

So that the above relations cannot be used directly.

There are UV divergence and IR divergence in the amplitudes.

The calculations need to be carried out in $D = 4 - 2\epsilon$ until the all divergences are canceled.

therefore it is necessary to include the evanescent operators,

$$\begin{aligned} O_2 &= \tilde{O}_2 - \frac{1}{4} f_2(\epsilon) O_1 \,, \\ O_3 &= \tilde{O}_3 - f_3(\epsilon) O_1 \,, \\ O_4 &= \tilde{O}_4 - 4 f_4(\epsilon) O_1 \,. \end{aligned} \tag{17}$$

$O_2 = O_3 = O_4 = 0$ if $d = 4$.

Directly reduce the amplitude to O_0 (covariant trace method) may lead to incorrect results, for example,

The one-loop QCD corrections to the hard spectator-scattering kernels for the topological penguin amplitudes in the charmless hadronic B decays.

Beneke Jager, Nucl. Phys. B 768, 51 (2007)

Nucleon Form Factors

Huang, Shi, Wang, Zhao, Phys. Rev. Lett. 135, 061901 (2025)

NNLO QCD Computation

Hard kernel can be expand in $\alpha_s/4\pi$

$$T_1 = T_1^{(0)} + \frac{\alpha_s}{4\pi} T_1^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 T_1^{(2)} \tag{18}$$

The hard-scattering amplitude can written as

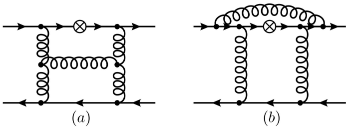
$$\begin{aligned}\Pi_\mu =& \langle u(p_1')\,\bar{d}(p_2')|j_\mu^{\rm em}(0)|u(p_1)\,\bar{d}(p_2)\rangle\,,\\ =&(p+p')_\mu\,(e_u - e_d)\,\frac{(4\pi)^2}{Q^4}\sum_k\sum_l\left[\left(\frac{Z_\alpha \alpha_s}{4\pi}\right)^{l+1} A_k^{(l)} \otimes \langle O_k\rangle^{(0)}\right],\end{aligned} \tag{19}$$

where l is the number of loop, $\langle O_k\rangle^{(0)}$ represents the bare matrix element.
Z_α is the denotes the renormalization constant of the α_s.
$\{A_k^{(l)}\}$ are the coefficients which contain the loop integrals.

After the standard renomarization, there are still IR-divergence in amplitudes, or the UV divergence in the bare operators.

The UV-renormalized O_k can be expanded as

$$\langle O_k\rangle = \sum_i \sum_{l=0}\left(\frac{\alpha_s}{4\pi}\right) Z_{ki}^{(l)} \otimes \langle O_i\rangle^{(0)} \tag{20}$$



- The diagrams are generated by `FeynArts`, and there are 1066 diagrams contribute.
  1889(1602) diagrams generated in $n_f = 3(2)$.
- Target integrals are reduced with `FIRE` and 57 master integrls are solved by canonical differential equations.
- Canonical form is obtained with Lee' s algorithm as implemented in the program `Libra`.
  R. N. Lee, JHEP 04, 108 (2015), arXiv:1411.0911 [hep-ph].
  R. N. Lee, Comput. Phys. Commun. 267, 108058(2021), arXiv:2012.00279 [hep-ph].

$$d\vec{\mathbf{I}}(\vec{\mathbf{x}};\epsilon) = \epsilon\left(\sum_{k=1}^{6} B_k\, d\log W_k(\vec{\mathbf{x}})\right)\vec{\mathbf{I}}(\vec{\mathbf{x}};\epsilon), \tag{21}$$

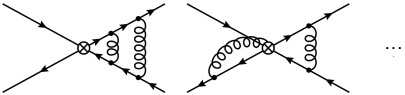
where $W_k \in \{x, y, \bar{x}, \bar{y}, x-y, \bar{x}-y\}$.

- Boundary condition is fixed using the PSLQ algorithm with 100 digits numerical results given by `AMFlow` at three distinct kinematic points.
  X. Liu and Y.-Q. Ma, Comput. Phys. Commun. 283,108565 (2023), arXiv:2201.11669 [hep-ph].

Computation of $Z_{21}^{(2)}$

- Similar to the 2-loop ERBL kernel calculation...

$$Z_{21}^{(2)} = -M_{21}^{\text{off}(2)} + \sum_{j=1}^{3} M_{2j}^{\text{off}(1)} \otimes M_{j1}^{\text{off}(1)} \tag{22}$$



- 33 diagrams that exchange gluon between u- and d-quark contribute.
- δ-function generated by Wilson line Feynman rules is expressed as $\delta(x) = \mathrm{Disc}_x \frac{1}{x}$

$$\mathrm{Disc}_{x'} \int \frac{d^D l_1 \, d^D l_2}{(i\pi^{D/2})^2} \frac{1}{[l_1^2 - m^2][l_2^2 - m^2][(l_1 + l_2 + x\,p)^2 - m^2][(l_1 + l_2 + p)^2 - m^2]} \frac{1}{n \cdot l_1 + x'} \,.$$

- Targets with linear propagators are reduced to about 30 master integrals.
- Master integrals are partly checked by `AMFlow`.
- Some diagrams are independently checked by evanescent mixing from $\gamma\gamma^* \to \pi^0$.

J. Gao, T. Huber, Y. Ji, and Y.-M. Wang, Phys. Rev. Lett. 128, 062003 (2022), arXiv:2106.01390 [hep-ph].

- The final expression of $Z_{21}^{(2)}$ is independent of the IR regulator as expected, providing a consistency check for our results.

Expression of $T_1^{(0)}$

$$T_1^{(0)} = \frac{C_F}{N_c} \frac{1}{2xy} \tag{23}$$

At LO,

$$F_\pi(Q^2) = (e_u - e_d) \frac{4\pi\alpha_s(\nu)}{Q^2} f_\pi^2 \int dx \int dy \, T_1^{(0)} \, \phi_\pi(x,\mu) \, \phi_\pi(y,\mu) \,, \tag{24}$$

Expression of $T_1^{(2)}$

Collecting all pieces together results in a lengthy expression of $T_1^{(2)}$

$$
\begin{aligned}
T_1^{(2)} = {} & 1 \times \frac{1}{1-x} \times \frac{(C_A - 2C_F)C_F(540C_A\zeta_3 - 360C_F\zeta_3 + 630C_F\zeta_4)}{60N_c} \\
& + G(0,x) \times \frac{1}{(1-x)(1-x-y)} \times \frac{(C_A - 2C_F)^2 C_F(-6\zeta_2 + 18\zeta_3)}{12N_c} \\
& + G(0,y)^3 \times \frac{1}{x(x-y)} \times \frac{C_F^3}{6N_c} \\
& + G(0,x)G(1,1,y,x) \times \frac{1}{xy} \times \frac{C_F(-7C_A^2 + 3C_AC_F + 5C_F^2)}{N_c} + \text{more than 3000 terms}
\end{aligned}
$$

where C_A and C_F are color factors and $\zeta(n)$ is the Riemann zeta function.

$G(\ldots)$ is multiple polylogarithms (MPLs), which are defined by $G(x) \equiv 1$ and

$$G(l_1, l_2, \ldots, l_n, x) \equiv \int_0^x \frac{\mathrm{d}t}{t - l_1} G(l_2, \ldots, l_n, t)\,, \tag{25}$$

$$G(\vec{0}_n, x) \equiv \frac{1}{n!} \ln^n x\,. \tag{26}$$

Asymptotic form factor

$$F_\pi(Q^2) = (e_u - e_d)\frac{4\pi\alpha_s(\nu)}{Q^2} f_\pi^2 \int dx \int dy \, T_1(x, y, Q^2, \nu, \mu)\, \phi_\pi(x, \mu)\, \phi_\pi(y, \mu)\,, \quad (27)$$

Asymptotic pion LCDA

$$\phi_\pi^{\rm Asy} = 6\, x(1-x), \quad (28)$$

Performing the two-fold convolution (with `PolyLogTools`) results in

C. Duhr and F. Dulat, JHEP 08, 135 (2019), arXiv:1904.07279 [hep-th].

$$\begin{aligned} F_\pi^{\rm Asy} =& (e_u - e_d)\frac{4\,\pi\,\alpha_s}{Q^2} 2\, f_\pi^2 \Bigg\{ 1 + \left(\frac{\alpha_s}{4\,\pi}\right) \left[9\, \ln\left(\frac{\nu^2}{Q^2}\right) + \frac{79}{3} \right] \\ &+ \left(\frac{\alpha_s}{4\,\pi}\right)^2 \Bigg[81 \ln^2 \frac{\nu^2}{Q^2} + 538 \ln \frac{\nu^2}{Q^2} \\ &- \left(\frac{560}{9}\,\zeta_2 + \frac{128}{9}\,\zeta_3 - 120\right) \ln \frac{\mu^2}{Q^2} \\ &- \frac{1100}{9}\,\zeta_2 - \frac{1736}{3}\,\zeta_3 - 24\,\zeta_4 + \frac{3280}{9}\,\zeta_5 + \frac{13136}{9} \Bigg] \Bigg\}. \end{aligned} \quad (29)$$

where ν is the renomarization scale and μ is the factorization scale.

Numerical analysis

- ϕ_π models needed to calculate $F_\pi(Q^2)$

$$F_\pi(Q^2) = (e_u - e_d)\frac{4\pi\alpha_s(\nu)}{Q^2} f_\pi^2 \int dx \int dy\, T_1(x, y, Q^2, \nu, \mu)\, \phi_\pi(x,\mu)\, \phi_\pi(y,\mu)\,, \tag{30}$$

- Expanding ϕ_π in Gegenbauer polynomials

$$\phi_\pi(x,\mu) = 6\,x(1-x) + \sum_{m=0,2,4,\ldots}^{\infty} a_m(\mu)\, C_m^{3/2}(2x-1)\,. \tag{31}$$

$$\text{Model I}: \phi_\pi(x,\mu_0) = \frac{\Gamma(2+2\alpha_\pi)}{\Gamma^2(1+\alpha_\pi)}(x\bar{x})^{\alpha_\pi}\,, \text{with } \alpha_\pi(\mu_0) = 0.585^{+0.061}_{-0.055}$$

S. J. Brodsky et al, Phys. Rev. D 77, 056007 (2008), arXiv:0707.3859 [hep-ph].

G. S. Bali et al, JHEP 08, 065 (2019), [Addendum: JHEP 11, 037 (2020)], arXiv:1903.08038 [hep-lat].

$$\text{Model II}: \{a_2, a_4, a_6, a_8\}(\mu_0) = \{0.181(32), 0.107(36), 0.073(50), 0.022(55)\}\,,$$

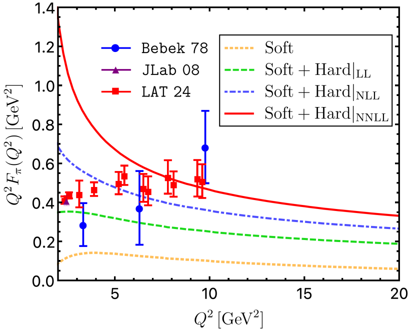
S. Cheng et al, Phys. Rev. D 102, 074022 (2020), arXiv:2007.05550 [hep-ph].

$$\text{Model III}: \{a_2, a_4\}(\mu_0) = \{0.149^{+0.052}_{-0.043}, -0.096^{+0.063}_{-0.058}\}\,,$$

N. G. Stefanis, Phys. Rev. D 102, 034022 (2020), arXiv:2006.10576 [hep-ph].

$$\text{Model IV}: \{a_2, a_4, a_6\}(\mu_0) = \{0.196(32), 0.085(26), 0.056(15)\}, \quad \mu_0 = 2\,\text{GeV}\,.$$

I. Cloet et al, arXiv:2407.00206 [hep-lat].



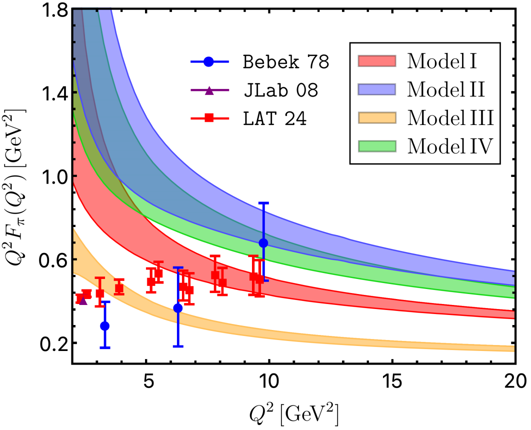
Model I : $\phi_\pi(x,\mu_0) = \frac{\Gamma(2+2\alpha_\pi)}{\Gamma^2(1+\alpha_\pi)}(x\bar{x})^{\alpha_\pi}$, with $\alpha_\pi(\mu_0) = 0.585^{+0.061}_{-0.055}$ determined by the lattice result $a_2(\mu_0) = 0.116^{+0.019}_{-0.020}$ at $\mu_0 = 2\text{GeV}$.

- $\nu^2 = Q^2$, $\mu^2 = 1/2Q^2$.
- NNLO correction: about $30\% \sim 50\%$ at $Q^2 \in [5, 20]\,\mathrm{GeV}^2$.
- Soft contribution: about 25%.
- Bebek 78 (Wilson Synchrotron Laboratory-LEPP,1978) and JLab08 (Jefferson Lab Collaboration,2008) are experiment data.

Bebek et al., Phys. Rev. D 17, 1693 (1978) Jefferson Lab Collaboration Phys. Rev. C 78, 045203 (2008)

- LAT24 is the Lattice date.

Ding, Gao, Hanlon, Mukherjee, Petreczky, Shi, Syritsyn, Zhang, Zhao, Phys.Rev.Lett. 133, 181902 (2024).



- Errors come from $\nu^2 \in [1/2, 2]Q^2$, $\mu^2 \in [1/4, 3/4]Q^2$.

Summary

- We have endeavored to accomplish the two-loop computation of the pion EMFF analytically.
- NNLO QCD correction can bring about a sizable impact.
- Future studies
  - Inclusion of massive quark loops.
  - N3LO QCD corrections.
  - ...

Thank you for your attention!