

Non-holomorphic modular symmetry

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arXiv: 2406.02527; JHEP11(2025)140 ,arXiv: 2506.19822

Modular invariant SUSY theory

➤ Modular action

[Lauer, Mas, Nilles, 1989; Ferrara, Lust et al, 1989; Feruglio, 1706.08749]

$$\tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

➤ Field **non-linear** transformation

$$\psi \rightarrow (c\tau + d)^{-k} \rho(\gamma) \psi$$

weight $k \in \mathbb{Z}$ ρ is a unitary representation of Γ_N or Γ'_N

➤ Superpotential

$$\mathcal{W} = \sum Y_{I_1 I_2 \dots I_n}(\tau) \psi_{I_1} \psi_{I_2} \dots \psi_{I_n}$$

Yukawa coupling $Y_{I_1 I_2 \dots I_n}(\tau)$ is a **holomorphic** modular form of level N :

$$Y_{I_1 I_2 \dots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 I_2 \dots I_n}(\tau)$$

Modular invariance requires

weights balance: $k_Y = k_{I_1} + k_{I_2} + \dots + k_{I_n}$

invariant singlet: $\rho_Y \otimes \rho_{I_1} \otimes \dots \otimes \rho_{I_n} \supset \mathbf{1}$



$SL(2, \mathbb{Z})$ on torus T^2



finite modular groups as G_f

$$G_f = \begin{cases} \Gamma_N \equiv SL(2, \mathbb{Z}) / \pm\Gamma(N) \\ \Gamma'_N \equiv SL(2, \mathbb{Z}) / \Gamma(N) \end{cases}$$

$$\Gamma(N) = \{ \gamma \in SL(2, \mathbb{Z}) \mid \gamma = 1_2 \pmod{N} \}$$



Is SUSY unique?

➤ Modular symmetry standalone is not enough!



$Y(\tau)$: Yukawa couplings = modular form of level N and weight k_Y

ATLAS SUSY Searches* - 95% CL Lower Limits
August 2023

ATLAS Preliminary
 $\sqrt{s} = 13$ TeV

Model	Signature	$\mathcal{L} \cdot dt [fb^{-1}]$	Mass limit	Reference						
Inclusive Searches	$0, \mu, \mu$	2-8 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$ (See Diagram)	1.0	1.85	$m(\tilde{g}) > 400$ GeV	2018.14293	
	mono-jet	1-3 jets	$A_{\mu\mu}^{j\cancel{E}}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$ (See Diagram)	0.9		$m(\tilde{g}) > 350$ GeV	2103.10074	
	$0, \mu, \mu$	2-6 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	Forbidden	1.15-1.95	$m(\tilde{g}) > 600$ GeV	2018.14293	
	$1, \mu, \mu$	2-6 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	Forbidden		$m(\tilde{g}) > 600$ GeV	2101.01029	
	$0, \mu, \mu$	2-9 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	Forbidden		$m(\tilde{g}) > 700$ GeV	2004.13077	
	$0, \mu, \mu$	7-11 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	Forbidden		$m(\tilde{g}) > 800$ GeV	2008.06032	
3 σ jet, missing transverse momentum	$0, \mu, \mu$	2-6 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	1.15	1.97	$m(\tilde{g}) > 200$ GeV	2007.01094	
	$0, \mu, \mu$	3-6 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	1.25	2.45	$m(\tilde{g}) > 500$ GeV	2011.06026	
	$0, \mu, \mu$	6 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	1.25		$m(\tilde{g}) > 300$ GeV	1909.08407	
	$0, \mu, \mu$	2-6 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.68	1.20	$m(\tilde{g}) > 400$ GeV	2101.13227	
	$0, \mu, \mu$	6-8 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	Forbidden	0.13-0.85	$10 \text{ GeV} < m(\tilde{g}) < 200 \text{ GeV}$	2101.13227	
	$2, \tau, \tau$	2-6 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	Forbidden	0.13-0.85	$100 \text{ GeV} < m(\tilde{g}) < 100 \text{ GeV}$	1908.03122	
EW Direct	$0, \mu, \mu$	≥ 1 jet	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	1.25		$m(\tilde{g}) > 100$ GeV	2004.14063, 2010.03799	
	$1, \mu, \mu$	3 jets/1 b	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	Forbidden	1.05	$m(\tilde{g}) > 500$ GeV	2012.02799, ATLAS-COHE-2023-043	
	$1, 2, \tau$	2 jets/1 b	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	Forbidden	1.4	$m(\tilde{g}) > 1000$ GeV	2103.01109	
	$0, \mu, \mu$	2 τ	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.55	0.85	$m(\tilde{g}, \tilde{u}, \tilde{d}) > 100$ GeV	1805.01649	
	$0, \mu, \mu$	mono-jet	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.55	0.85	$m(\tilde{g}, \tilde{u}, \tilde{d}) > 100$ GeV	2102.10074	
	$1, 2, \mu, \mu$	1-4 b	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	Forbidden	0.067-1.18	$m(\tilde{g}) > 500$ GeV	2006.05080	
Long-lived particles	Direct $\tilde{g}, \tilde{u}, \tilde{d}$ prod., long-lived $\tilde{g}, \tilde{u}, \tilde{d}$	Disapp. bk	1 jet	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.205	0.36	$m(\tilde{g}) > 0$, wino-bino	2105.01676, 2108.07596
	Stable g R hadron	pixel dE/dx	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.42		$m(\tilde{g}) > 0$, wino-bino	1911.12006	
	Metastable g R hadron, $g \rightarrow \text{spit}$	pixel dE/dx	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	Forbidden	1.06	$m(\tilde{g}) > 70$ GeV, wino-bino	2004.10094, 2108.07596	
	$\tilde{g}, \tilde{u}, \tilde{d}$	Disapp. bk	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.34	0.40	$m(\tilde{g}, \tilde{u}, \tilde{d}) > 0$	1908.00215	
	$\tilde{g}, \tilde{u}, \tilde{d}$	≥ 1 jet	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.26		$m(\tilde{g}) > 10$ GeV	1911.12006	
	$0, \mu, \mu$	≥ 1 b	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.55	0.94	$m(\tilde{g}) \rightarrow \tilde{W}_2 \rightarrow 1$	1911.12006	
RPV	Direct $\tilde{g}, \tilde{u}, \tilde{d}$ prod., long-lived $\tilde{g}, \tilde{u}, \tilde{d}$	Disapp. bk	1 jet	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.21	0.66	$m(\tilde{g}) \rightarrow \tilde{W}_2 \rightarrow 1$	2102.11004
	Stable g R hadron	pixel dE/dx	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	2.05		$m(\tilde{g}) \rightarrow \tilde{W}_2 \rightarrow 1$	2108.07596	
	Metastable g R hadron, $g \rightarrow \text{spit}$	pixel dE/dx	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	2.2		$m(\tilde{g}) \rightarrow \tilde{W}_2 \rightarrow 1$	2004.13077	
	$\tilde{g}, \tilde{u}, \tilde{d}$	Disapp. bk	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.34	0.7	$m(\tilde{g}) > 0.1$ ms	2011.07812	
	$\tilde{g}, \tilde{u}, \tilde{d}$	pixel dE/dx	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.36		$m(\tilde{g}) > 0.1$ ms	2005.06013	
	$\tilde{g}, \tilde{u}, \tilde{d}$	≥ 2 jets	$A_{\mu\mu}^{jj}$	140	$\tilde{g}, \tilde{u}, \tilde{d}$	0.77		$m(\tilde{g}) > 10$ ms	2005.06013	

A few independent modular forms at weight k_Y

[PDG,2024]

low energy SUSY is not observed so far!

Non-SUSY



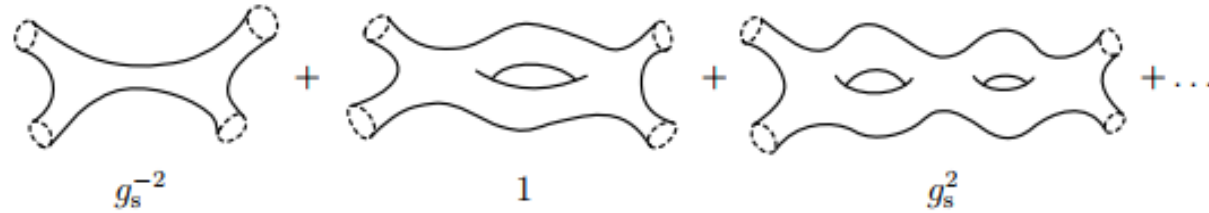
$Y(\tau, \bar{\tau})$: non-holomorphic Yukawa couplings

A finite choice of $Y(\tau, \bar{\tau})$ to have prediction power

*Only a selection of the available mass limits on new states or phenomena is shown. Many of the limits are based on simplified models, c.f. refs. for the assumptions made.

Harmonic condition replacing holomorphicity

- From top-down, the effective interactions in the low energy expansion of the four-graviton amplitude are **non-holomorphic automorphic** functions satisfying harmonic eigenvalue equations.



[Green, Gutperle, hep-th/9701093; Green, Russo, Vanhove, 1001.2535; Hoker, Kaidi, 2208.07242]

- Modular invariance based on automorphic forms naturally have **multiple moduli** [Ding, Feruglio, Liu, 2010.07952]
 - Automorphic forms coincide with the harmonic Maaß forms for single modulus τ
- Harmonic Maaß forms of level N and weight k [Qu, Ding, 2406.02527]

① **modularity:** $Y(\gamma\tau) = (c\tau + d)^k Y(\tau), \quad \gamma \in \Gamma(N)$

② **harmonic condition (eigenfunction of Casimir operator Δ_k):**

$$\Delta_k Y(\tau) = 0, \quad \Delta_k \equiv -4y^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \bar{\tau}} + 2iky \frac{\partial}{\partial \bar{\tau}}$$

top-down evidence of SUSY unnecessary: Almumin, Chen et al, 2102.11286

③ **growth condition:** $Y(\tau) = \mathcal{O}(y^\alpha), \quad \text{Im}(\tau) \equiv y \rightarrow +\infty$

Modular form

VS.

polyharmonic Maaß form

$$\frac{\partial}{\partial \bar{\tau}} Y(\tau) = 0$$

$$Y(\tau) = \underbrace{\sum_{n=0}^{+\infty} a_n q_N^n}_{\text{holomorphic}}, \quad q_N \equiv e^{2\pi i \tau / N}$$

$$\left[-4y^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \bar{\tau}} + 2iky \frac{\partial}{\partial \bar{\tau}} \right] Y(\tau) = 0$$

$$Y(\tau) = \underbrace{\sum_{n=0}^{+\infty} c_n^+ q_N^n}_{\text{holomorphic}} + \underbrace{c_0^- y^{1-k} + \sum_{n=-1}^{-\infty} c_n^- \Gamma(1-k, -4\pi n y / N) q_N^n}_{\text{non-holomorphic}}$$

Weight: $k = 0, 1, 2, 3, \dots$

$k = \dots, -3, -2 - 1, 0, 1, 2, 3, \dots$

Basis: $Y_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} Y_j(\tau)$

$Y_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} Y_j(\tau)$

- A familiar non-holomorphic polyharmonic Maaß form

modified Eisenstein series $\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi y} = 1 - \frac{3}{\pi y} - 24q - 72q^2 - 96q^3 + \dots$

Formalism of non-holomorphic modular flavor symmetry

- Modular action of τ

$$\tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

τ is a scalar field not a chiral superfield

[Qu, Ding, 2406.02527]

- **Non-linear** transformations of SM fields

$$\psi_i \rightarrow (c\tau + d)^{-k_\psi} [\rho_\psi(\gamma)]_{ij} \psi_j \quad \psi_i^c \rightarrow (c\tau + d)^{-k_{\psi^c}} [\rho_{\psi^c}(\gamma)]_{ij} \psi_j^c \quad H \rightarrow (c\tau + d)^{-k_H} \rho_H(\gamma) H$$

- Modular invariant kinetic terms

$$\begin{aligned} \mathcal{L}_K &= (-i\tau + i\bar{\tau})^{-k_\psi} i\psi^\dagger \bar{\sigma}^\mu D_\mu \psi + (-i\tau + i\bar{\tau})^{-k_{\psi^c}} i\psi^{c\dagger} \bar{\sigma}^\mu D_\mu \psi^c + \dots \\ &\xrightarrow{\langle \tau \rangle} \langle -i\tau + i\bar{\tau} \rangle^{-k_\psi} i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + \langle -i\tau + i\bar{\tau} \rangle^{-k_{\psi^c}} i\psi^{c\dagger} \bar{\sigma}^\mu \partial_\mu \psi^c + \dots \end{aligned}$$

$$D_\mu \equiv \partial_\mu + \frac{k\pi i}{6} E_2(\tau) \partial_\mu \tau$$

- Modular invariant Yukawa interactions

$$\mathcal{L}_Y = Y^{(k_Y)}(\tau) \psi^c \psi H + \text{h.c.}$$

Yukawa couplings are harmonic Maass forms of weight k_Y and level N

$$Y^{(k_Y)}(\tau) \rightarrow Y^{(k_Y)}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y^{(k_Y)}(\tau)$$

with

$$\begin{cases} k_Y = k_{\psi^c} + k_\psi + k_H \\ \rho_Y \otimes \rho_{\psi^c} \otimes \rho_\psi \otimes \rho_H \supset \mathbf{1} \end{cases}$$

CP and non-holomorphic modular symmetry

- A unique CP transformation up to modular transformations

$$\tau \rightarrow -\tau^*$$

[Baur, Nilles et al, 1901.03251; Novichkov, Petcov et al, 1905.11970]

- CP action on matter field and harmonic Maaß form

matter field: $\psi(x) \xrightarrow{CP} X_\psi \psi^*(x_P)$

X_ψ and X_Y are unitary matrix in flavor space

Maaß form: $Y(\tau) \xrightarrow{CP} Y(-\tau^*) = X_Y Y^*(\tau)$

$$u(S) = S^{-1}, \quad u(T) = T^{-1}$$

- Consistency condition $CP \rightarrow \gamma \rightarrow CP^{-1}$

modulus: $\tau \xrightarrow{CP} -\tau^* \xrightarrow{\gamma} -\frac{a\tau^*+b}{c\tau^*+d} \xrightarrow{CP^{-1}} \frac{a\tau-b}{-c\tau+d} \longrightarrow u(\gamma) = CP \circ \gamma \circ CP^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$

matter field & Maaß form: $X_r \rho_r^*(\gamma) X_r^{-1} = \rho_r(u(\gamma))$ fixing X_r for each irrep r

symmetric basis: $\rho_r(S) = \rho_r^T(S), \quad \rho_r(T) = \rho_r^T(T)$

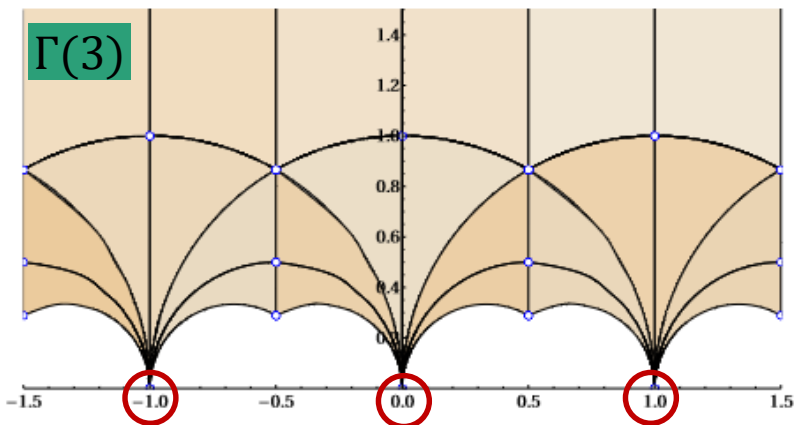
$$\begin{cases} X_r \rho_r^*(S) X_r^{-1} = \rho_r(S^{-1}) = \rho_r^*(S) \\ X_r \rho_r^*(T) X_r^{-1} = \rho_r(T^{-1}) = \rho_r^*(S) \end{cases} \longrightarrow X_r = 1 \longrightarrow g_i^* = g_i \text{ real couplings}$$

[Novichkov, Petcov et al, 1905.11970]

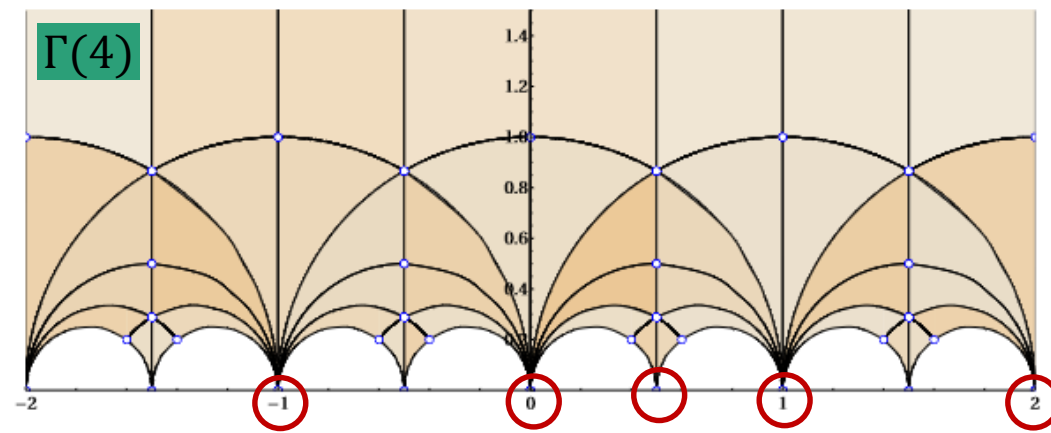
Non-holomorphic Eisenstein series

➤ Cusp of $\Gamma(N)$: a representative of the equivalence classes of rational numbers $\mathbb{Q} \cup \{i\infty\}$ under $\Gamma(N)$

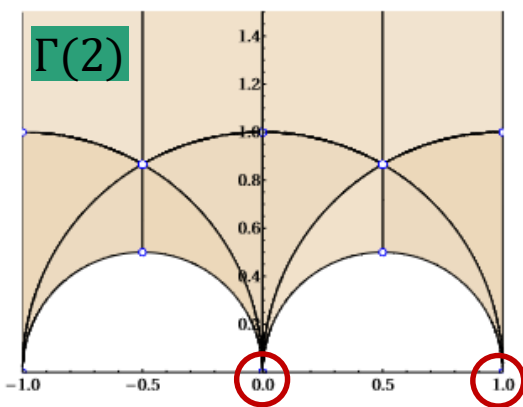
- Two **coprime** integers A and C to represent a cusp $\overline{A/C}$
- $\overline{A/C}$ and $\overline{A'/C'}$ describe the same cusp iff $(A, C) = \pm(A', C') \pmod{N}$
- $\Gamma(1) \cong SL(2, \mathbb{Z})$ has a single cusp $i\infty = \overline{1/0}$



$$\mathcal{C}(3) = \{\overline{1/0}, \overline{0/1}, \overline{1/1}, \overline{-1/1}\}$$

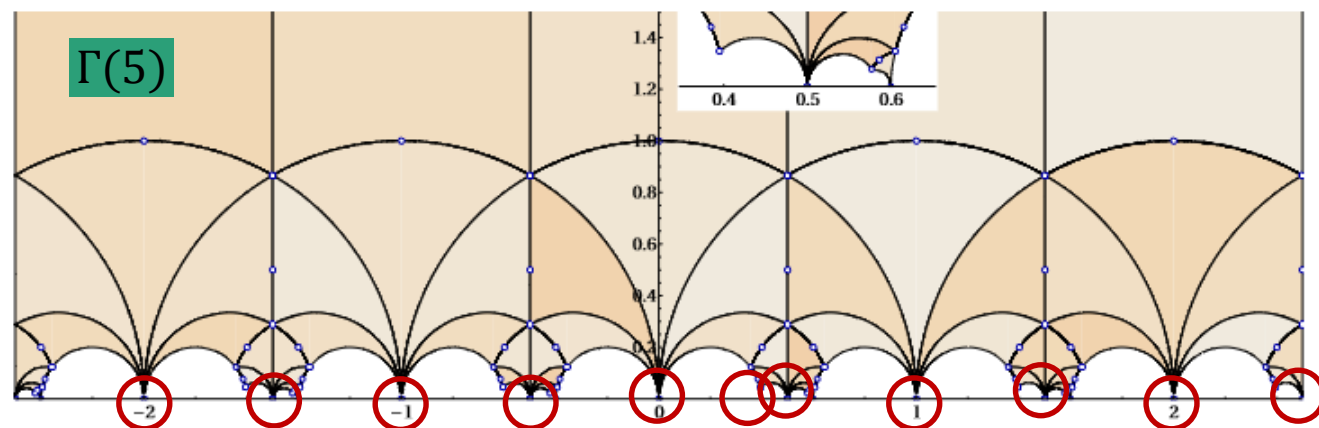


$$\mathcal{C}(4) = \{\overline{1/0}, \overline{0/1}, \overline{1/1}, \overline{2/1}, \overline{1/2}, \overline{-1/1}\}$$



$$\mathcal{C}(2) = \{\overline{1/0}, \overline{0/1}, \overline{1/1}\}$$

[figures from
2008.05329 by Ivo,
Miguel, Ye-Ling]



$$\mathcal{C}(5) = \{\overline{1/0}, \overline{0/1}, \overline{1/1}, \overline{2/1}, \overline{-2/1}, \overline{-1/1}, \overline{1/2}, \overline{3/2}, \overline{5/2}, \overline{-3/2}, \overline{-1/2}, \overline{2/5}\}$$

➤ Non-holomorphic Eisenstein series of weight k : defined at each cusp of $\Gamma(N)$ [Qu, Lu, Ding, 2506.19822]

$$E_k(N; \tau; s; \overline{A/C}) = \sum_{\substack{(c,d) \equiv (-C,A) \pmod{N} \\ \gcd(c,d)=1}} \frac{y^s}{(c\tau + d)^k |c\tau + d|^{2s}} \longrightarrow \text{non-holomorphicity}$$

• Modular transformation of E_k

$$E_k(N; \gamma\tau; s; \overline{A/C}) = (c\tau + d)^k E_k(N; \tau; s; \overline{\gamma^{-1}(A/C)}) \longrightarrow \text{Modularity preserved for } \gamma \in \Gamma(N)$$

• Harmonic condition

$$\Delta_k E_k(N; \tau; s; \overline{A/C}) = -s(s + k - 1) E_k(N; \tau; s; \overline{A/C}) \longrightarrow s = 0 \text{ or } s = 1 - k$$

— The linear space of weight k polyharmonic Maaß form is spanned by the Eisenstein series $E_k(N; \tau; s; \overline{A/C})|_{s=1-k}$, its dimension equals to the number of cusp for $k \leq 0$.

— At weight $k = 1$, $E_1(N; \tau; s; \overline{A/C})|_{s=0}$ are holomorphic and linearly dependent. The 1st derivative

$$E_1^{(1)}(N; \tau; 0; \overline{A/C}) \equiv \frac{\partial E_1(N; \tau; s; \overline{A/C})}{\partial s} \Big|_{s=0} \text{ generates weight } k = 1 \text{ polyharmonic Maaß form.}$$

Polyharmonic Maaß form of level N

Finite modular group $\Gamma'_3 \cong T'$: $S^2 = R$, $(ST)^3 = T^3 = R^2 = 1$, $RT = TR$

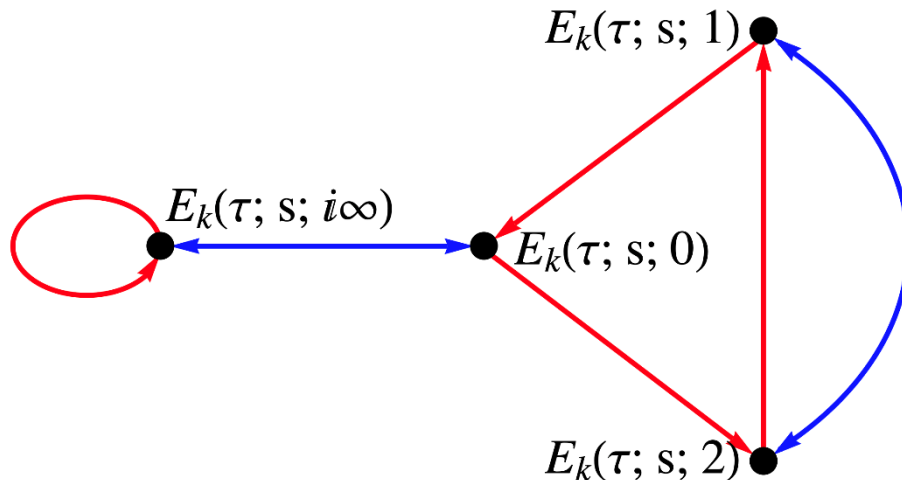
➤ Non-holomorphic Eisenstein series of level $N = 3$

$E_k(\tau; s; 0)$, $E_k(\tau; s; 1)$, $E_k(\tau; s; 2)$, $E_k(\tau; s; i\infty)$

Cusp: $C(3) = \{\overline{0/1}, \overline{1/1}, \overline{2/1}, \overline{1/0}\} \equiv \{0, 1, 2, i\infty\}$

$$S: \begin{cases} E_k(S\tau; s; i\infty) \xrightarrow{S} (-\tau)^k E_k(\tau; s; 0), \\ E_k(S\tau; s; 0) \xrightarrow{S} (-1)^k (-\tau)^k E_k(\tau; s; i\infty), \\ E_k(S\tau; s; 1) \xrightarrow{S} (-\tau)^k E_k(\tau; s; 2), \\ E_k(S\tau; s; 2) \xrightarrow{S} (-1)^k (-\tau)^k E_k(\tau; s; 1). \end{cases}$$

$$T: \begin{cases} E_k(\tau; s; i\infty) \xrightarrow{T} E_k(\tau; s; i\infty), \\ E_k(\tau; s; 0) \xrightarrow{T} E_k(\tau; s; 2), \\ E_k(\tau; s; 1) \xrightarrow{T} E_k(\tau; s; 0), \\ E_k(\tau; s; 2) \xrightarrow{T} E_k(\tau; s; 1). \end{cases}$$



S transformation distinguishes odd from even weights

[Qu, Lu, Ding, 2506.19822]

➤ Multiplets of polyharmonic Maaß forms of level N=3

Even weight $2k$ ($k \leq 0$):

$$Y_1^{(2k)}(\tau) = E_{2k}(\tau; 1 - 2k; i\infty) + E_{2k}(\tau; 1 - 2k; 0) + E_{2k}(\tau; 1 - 2k; 1) + E_{2k}(\tau; 1 - 2k; 2),$$

$$Y_3^{(2k)}(\tau) = \begin{pmatrix} E_{2k}(\tau; 1 - 2k; i\infty) - \frac{1}{3} \left(E_{2k}(\tau; 1 - 2k; 0) + E_{2k}(\tau; 1 - 2k; 1) + E_{2k}(\tau; 1 - 2k; 2) \right) \\ \frac{2}{3} \left(E_{2k}(\tau; 1 - 2k; 0) + e^{2\pi i/3} E_{2k}(\tau; 1 - 2k; 1) + e^{4\pi i/3} E_{2k}(\tau; 1 - 2k; 2) \right) \\ \frac{2}{3} \left(E_{2k}(\tau; 1 - 2k; 0) + e^{4\pi i/3} E_{2k}(\tau; 1 - 2k; 1) + e^{2\pi i/3} E_{2k}(\tau; 1 - 2k; 2) \right) \end{pmatrix}$$

singlet 1

triplet 3

Odd weight $2k+1$ ($k < 0$):

$$Y_{\hat{2}}^{(2k+1)}(\tau) = \begin{pmatrix} i \sqrt{\frac{2}{3}} \left(E_{2k+1}(\tau; -2k; 0) + \omega E_{2k+1}(\tau; -2k; 1) + \omega^2 E_{2k+1}(\tau; -2k; 2) \right) \\ E_{2k+1}(\tau; -2k; i\infty) - \frac{i}{\sqrt{3}} \left(E_{2k+1}(\tau; -2k; 0) + E_{2k+1}(\tau; -2k; 1) + E_{2k+1}(\tau; -2k; 2) \right) \end{pmatrix}$$

$$Y_{\hat{2}''}^{(2k+1)}(\tau) = \begin{pmatrix} E_{2k+1}(\tau; -2k; i\infty) + \frac{i}{\sqrt{3}} \left(E_{2k+1}(\tau; -2k; 0) + E_{2k+1}(\tau; -2k; 1) + E_{2k+1}(\tau; -2k; 2) \right) \\ i \sqrt{\frac{2}{3}} \left(E_{2k+1}(\tau; -2k; 0) + \omega^2 E_{2k+1}(\tau; -2k; 1) + \omega E_{2k+1}(\tau; -2k; 2) \right) \end{pmatrix}$$

doublet $\hat{2}$

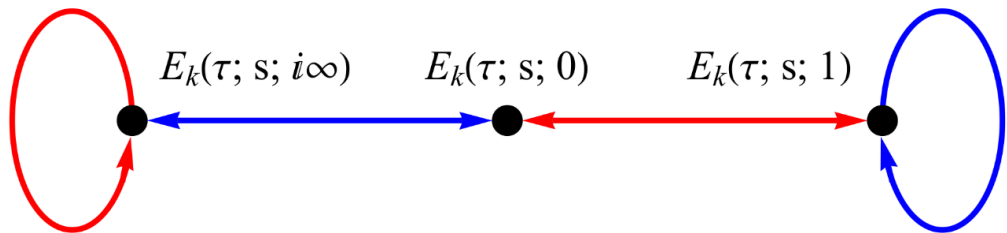
doublet $\hat{2}''$

Odd weight 1: replace E_k with 1st derivative

$$E_1(\tau; 0; \overline{A/C}) \rightarrow E_1^{(1)}(\tau; 0; \overline{A/C}) \equiv \left. \frac{\partial E_1(\tau; s; \overline{A/C})}{\partial s} \right|_{s=0}$$

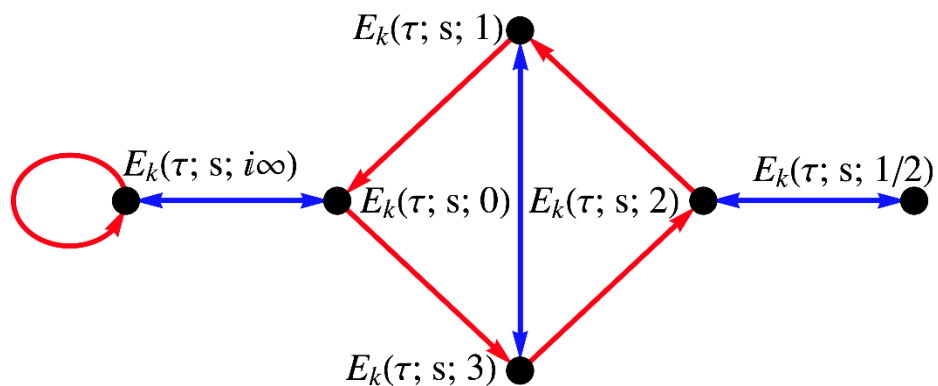
k	-3	-2	-1	0	1	2	3
T'	$\hat{2} + \hat{2}''$	$1 + 3$	$\hat{2} + \hat{2}''$	$1 + 3$	$\hat{2} + \hat{2}''$	$1 + 3$	$\hat{2} + \hat{2}''$

➤ **Level $N = 2$:** $S^2 = T^2 = (ST)^3 = 1 \rightarrow \Gamma'_2 = \Gamma_2 \cong S_3$



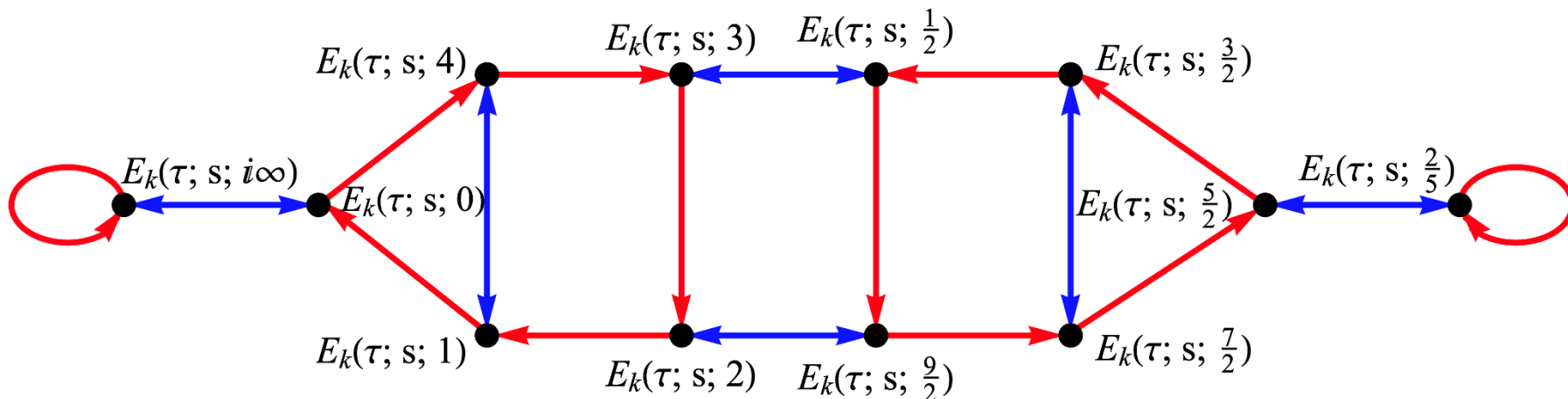
k	-3	-2	-1	0	1	2	3
S_3	--	1 + 2	--	1 + 2	--	1 + 2	--

➤ **Level $N = 4$:** $S^2 = R, (ST)^3 = T^4 = R^2 = 1, RT = TR \rightarrow \Gamma'_4 \cong S'_4$



k	-3	-2	-1	0	1	2	3
S'_4	$\hat{3} + \hat{3}'$	1 + 2 + 3	$\hat{3} + \hat{3}'$	1 + 2 + 3	$\hat{3} + \hat{3}'$	1 + 2 + 3	$\hat{1}' + \hat{3}$ + $\hat{3}'$

➤ **Level $N = 5$:** $S^2 = R, (ST)^3 = T^5 = R^2 = 1, RT = TR \rightarrow \Gamma'_5 \cong A'_5$



k	-3	-2	-1	0
A'_5	$\hat{6} + \hat{6}$	1 + 3 + 3' + 5	$\hat{6} + \hat{6}$	1 + 3 + 3' + 5
k	1	2	3	4
A'_5	$\hat{6} + \hat{6}$	1 + 3 + 3' + 5	$\hat{4}'$ + $\hat{6}$ + $\hat{6}$	1 + 3 + 3' + 4 + 5 + 5

Non-holomorphic modular model based on $\Gamma'_3 \cong T'$

➤ Minimality as a guiding principle

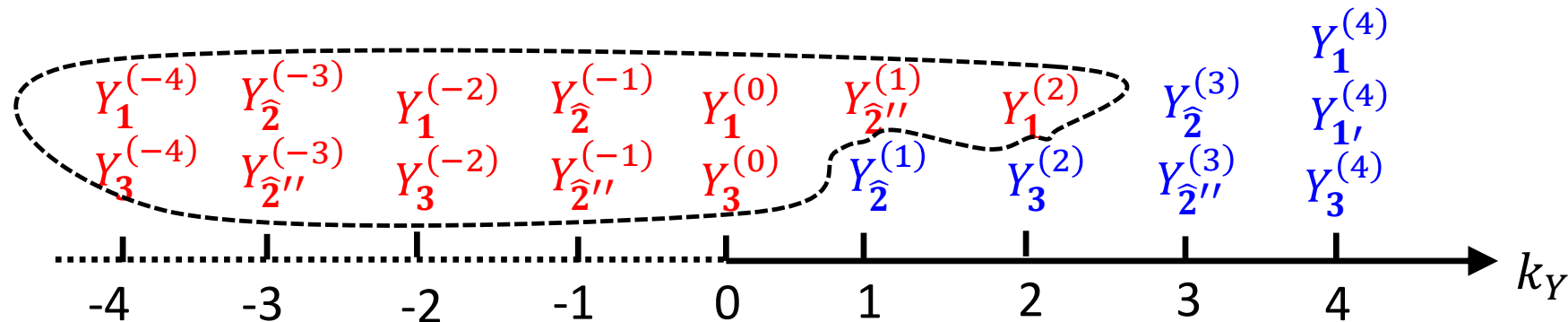
[Qu, Lu, Ding, 2506.19822]

- **No flavons other than τ** are introduced
- Three generations of LH lepton doublets transform as a T' **triplet**
- RH lepton singlets transform as T' **singlets**
- Higgs field is invariant under T'
- **Minimal** type-I seesaw with **two RH neutrinos**



$$-\mathcal{L} = \sum_i \alpha_i [E_i^c L f_i(Y)]_1 H^* + g [N^c L f_N(Y)]_1 H + \Lambda [N^c N^c f_M(Y)] + \text{h.c.}$$

➤ Non-holomorphic polyharmonic Maaß forms appear at weights $k \leq 2$



Lepton sector

	L	e^c	μ^c	τ^c	N^c	H
T'	$\mathbf{3}$	$\mathbf{1}'$	$\mathbf{1}'$	$\mathbf{1}$	$\widehat{\mathbf{2}}$	$\mathbf{1}$
k_I	-1	1	3	5	2	0

gCP symmetry is included so that all couplings are real in the working basis

[Qu, Lu, Ding, 2506.19822]

➤ Charged leptons

$$-\mathcal{L}_l^Y = \alpha(e^c L Y_{\mathbf{3}}^{(0)} H^*)_1 + \beta(\mu^c L Y_{\mathbf{3}}^{(2)} H^*)_1 + \gamma(\tau^c L Y_{\mathbf{3}}^{(4)} H^*)_1 + \text{h.c.}$$

➔ $M_l = \begin{pmatrix} \alpha Y_{\mathbf{3},3}^{(0)} & \alpha Y_{\mathbf{3},2}^{(0)} & \alpha Y_{\mathbf{3},1}^{(0)} \\ \beta Y_{\mathbf{3},3}^{(2)} & \beta Y_{\mathbf{3},2}^{(2)} & \beta Y_{\mathbf{3},1}^{(2)} \\ \gamma Y_{\mathbf{3},1}^{(4)} & \gamma Y_{\mathbf{3},3}^{(4)} & \gamma Y_{\mathbf{3},2}^{(4)} \end{pmatrix} v$

➤ Neutrino mass : seesaw mechanism with 2RHN

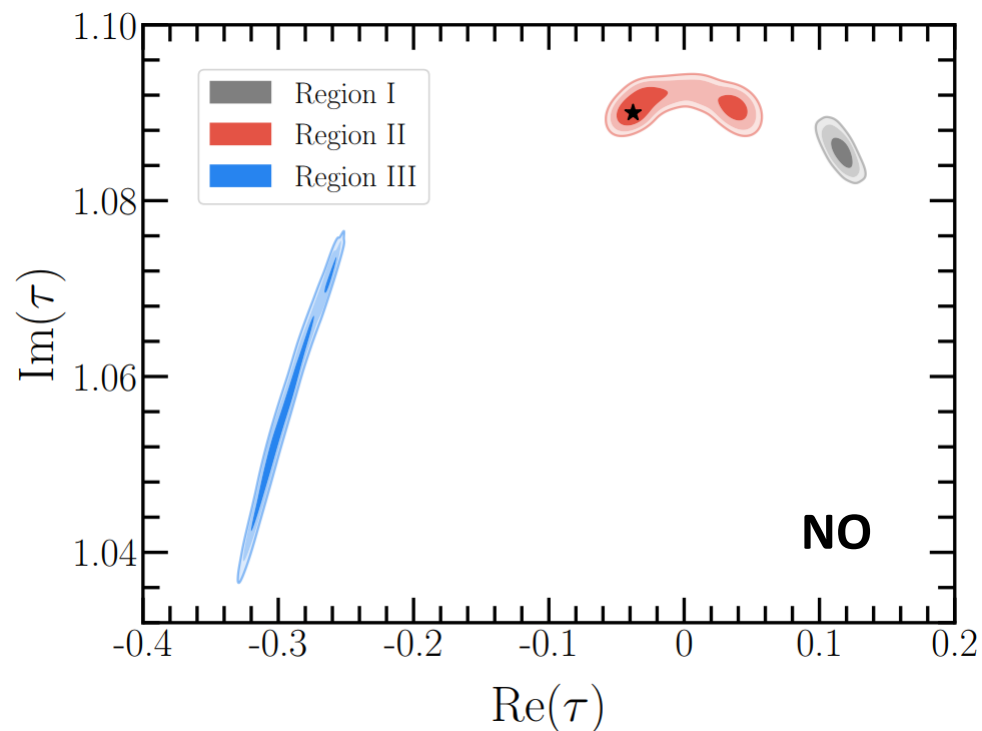
$$-\mathcal{L}_\nu = g_1(N^c L H Y_{\widehat{\mathbf{2}}''}^{(1)})_1 + g_2(N^c L H Y_{\widehat{\mathbf{2}}}^{(1)})_1 + \frac{1}{2}\Lambda \left(N^c N^c Y_{\mathbf{3}}^{(4)} \right)_1 + \text{h.c.}$$

➔ $M_D = \begin{pmatrix} g_1 Y_{\widehat{\mathbf{2}}'',2}^{(1)} & \sqrt{2} g_2 Y_{\widehat{\mathbf{2}},1}^{(1)} & -\sqrt{2} g_1 Y_{\widehat{\mathbf{2}}'',1}^{(1)} - g_2 Y_{\widehat{\mathbf{2}},2}^{(1)} \\ g_1 Y_{\widehat{\mathbf{2}}'',1}^{(1)} - \sqrt{2} g_2 Y_{\widehat{\mathbf{2}},2}^{(1)} & \sqrt{2} g_1 Y_{\widehat{\mathbf{2}}'',2}^{(1)} & -\sqrt{2} g_1 Y_{\widehat{\mathbf{2}},1}^{(1)} \end{pmatrix} v, \quad M_N = \Lambda \begin{pmatrix} -Y_{\mathbf{3},2}^{(4)} & \frac{Y_{\mathbf{3},3}^{(4)}}{\sqrt{2}} \\ \frac{Y_{\mathbf{3},3}^{(4)}}{\sqrt{2}} & Y_{\mathbf{3},1}^{(4)} \end{pmatrix}$

Input pars	NO	IO
$\text{Re}(\tau)$	-0.03777	4.388×10^{-3}
$\text{Im}(\tau)$	1.090	1.083
β/α	17.70	16.00
γ/α	284.6	257.1
g_2/g_1	0.1490	-0.08106
$\alpha v/\text{GeV}$	4.696×10^{-3}	5.156×10^{-3}
$\frac{g_1^2 v^2}{\Lambda}/\text{meV}$	191.6	603.8

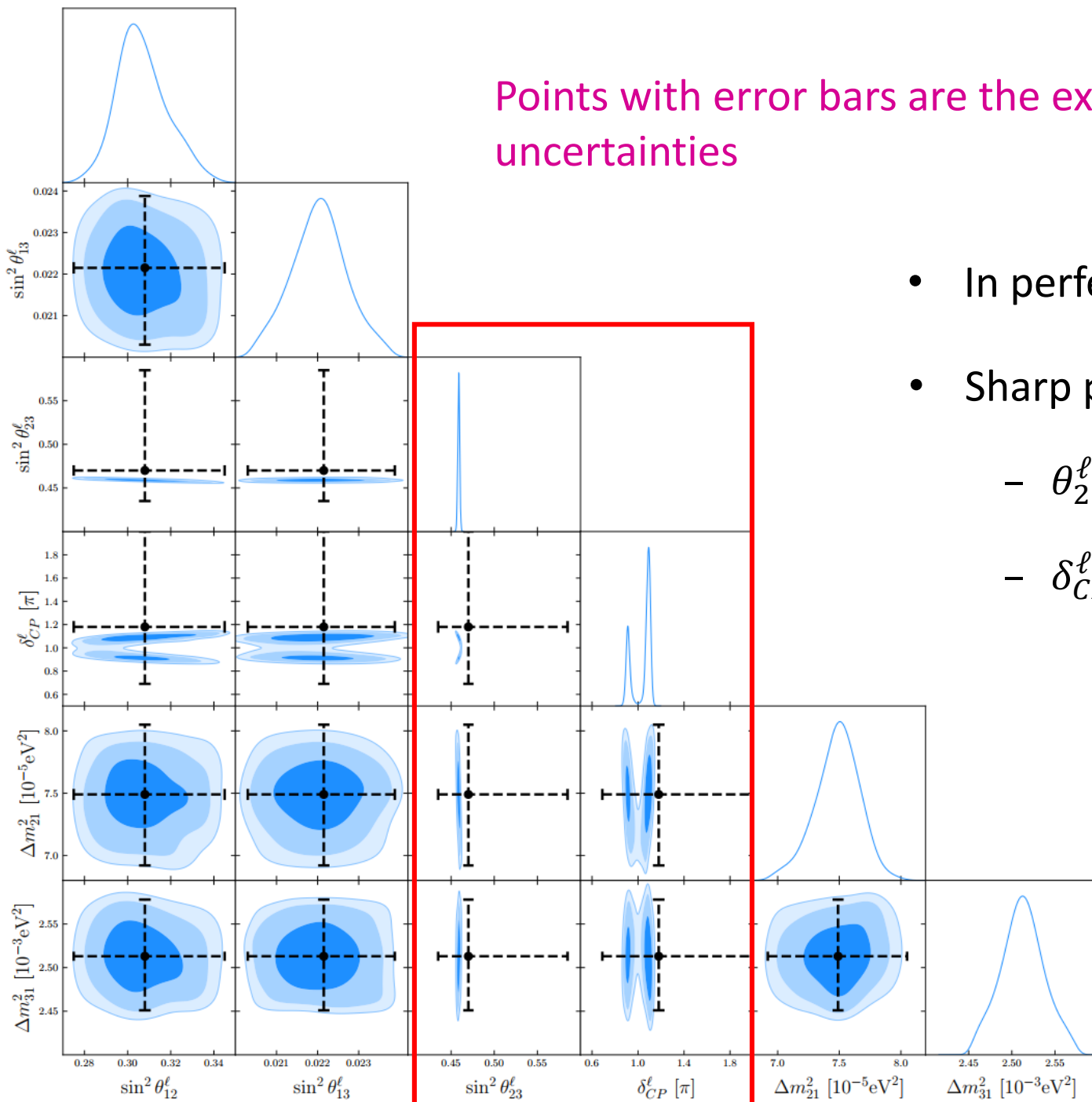
Predictions	NO	IO
$\sin^2 \theta_{12}^\ell$	0.308	0.308
$\sin^2 \theta_{13}^\ell$	0.02215	0.02236
$\sin^2 \theta_{23}^\ell$	0.459	0.561
δ_{CP}^ℓ	1.09π	1.56π
$\Delta m_{21}^2/10^{-5}\text{eV}^2$	7.48	7.49
$\Delta m_{31}^2/10^{-3}\text{eV}^2$	2.513	-2.484
m_e/m_μ	0.004737	0.004737
m_μ/m_τ	0.05882	0.05882
m_e/MeV	0.469652	0.469652

ϕ	1.04π	0.35π
m_1/meV	0	49.08
m_2/meV	8.65	49.84
m_3/meV	50.13	0
$\sum_i m_i/\text{meV}$	58.78	98.92
$m_{\beta\beta}/\text{meV}$	1.91	42.34
m_β/meV	8.84	48.76
χ_ℓ^2	0.90	0.92



7 parameters vs 12 observables

Points with error bars are the experimental best values and the 3σ uncertainties



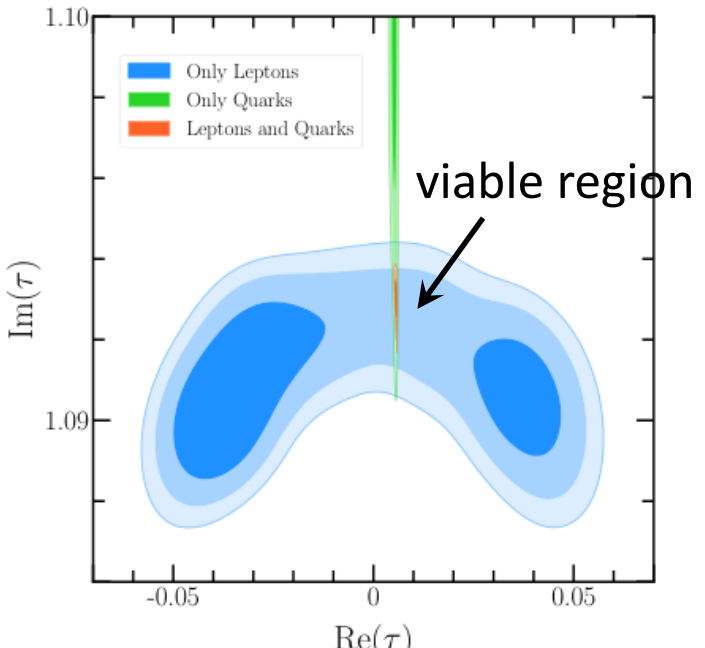
- In perfect agreement with neutrino data
- Sharp predictions for θ_{23}^l and Dirac CP phase δ_{CP}^l
 - θ_{23}^l in the **1st octant**: $\sin^2 \theta_{23}^l \in [0.454, 0.463]$
 - $\delta_{CP}^l \in [0.84\pi, 1.16\pi]$

Extension to quark sector → minimal modular model

	Q_L	u^c	c^c	t^c	d^c	s^c	b^c
T'	3	1	1'	1''	1	1''	1'
k_I	k_Q	$-2 - k_Q$	$-k_Q$	$-k_Q$	$-4 - k_Q$	$-4 - k_Q$	$-k_Q$

→ $M_u = \begin{pmatrix} \alpha_u Y_{\mathbf{3},1}^{(-2)} & \alpha_u Y_{\mathbf{3},3}^{(-2)} & \alpha_u Y_{\mathbf{3},2}^{(-2)} \\ \beta_u Y_{\mathbf{3},3}^{(0)} & \beta_u Y_{\mathbf{3},2}^{(0)} & \beta_u Y_{\mathbf{3},1}^{(0)} \\ \gamma_u Y_{\mathbf{3},2}^{(0)} & \gamma_u Y_{\mathbf{3},1}^{(0)} & \gamma_u Y_{\mathbf{3},3}^{(0)} \end{pmatrix} v, \quad M_d = \begin{pmatrix} \alpha_d Y_{\mathbf{3},1}^{(-4)} & \alpha_d Y_{\mathbf{3},3}^{(-4)} & \alpha_d Y_{\mathbf{3},2}^{(-4)} \\ \beta_d Y_{\mathbf{3},2}^{(-4)} & \beta_d Y_{\mathbf{3},1}^{(-4)} & \beta_d Y_{\mathbf{3},3}^{(-4)} \\ \gamma_d Y_{\mathbf{3},3}^{(0)} & \gamma_d Y_{\mathbf{3},2}^{(0)} & \gamma_d Y_{\mathbf{3},1}^{(0)} \end{pmatrix} v$

- Minimal modular invariant model for quarks and leptons: **13 parameters VS. 22 observables**



5 + 6 + 2 = **13** parameters

- lepton couplings: $\alpha, \beta, \gamma, g_1, g_2$
- quark couplings: $\alpha_u, \beta_u, \gamma_u, \alpha_d, \beta_d, \gamma_d$
- quark and lepton shared: $\text{Re}(\tau), \text{Im}(\tau)$

- The complex modulus τ common in quark and lepton sectors, is mostly fixed by the quark parameters

$\langle \tau \rangle = 0.00506 + 1.093i$ close to imaginary axis

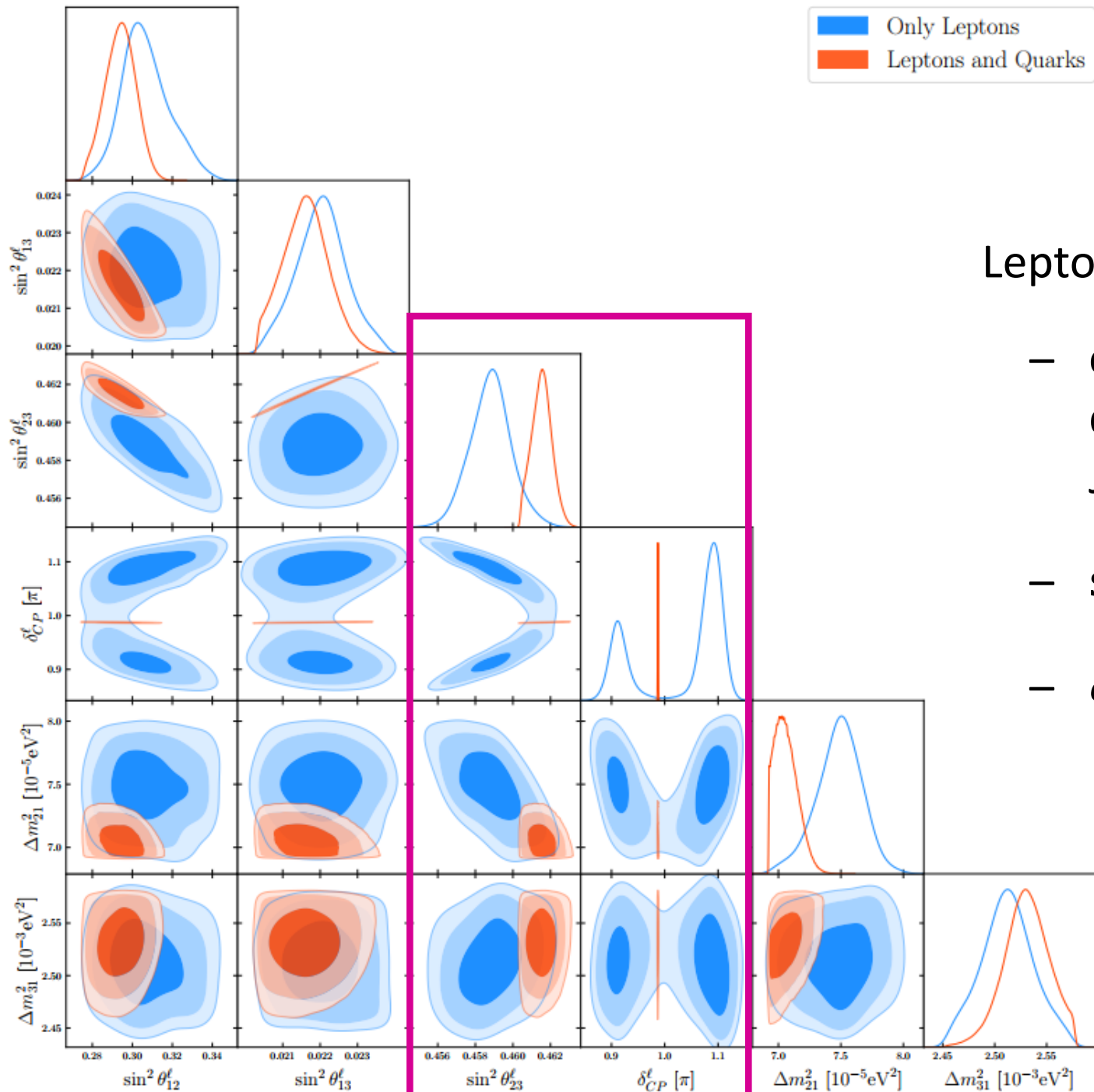
- Tiny region of τ after combining quarks & leptons

- The model can well describe the masses and mixing of both quarks and leptons: **12** masses+**6** mixing angles+**4** CP phases. **The neutrino spectrum can only be NO after combined with quark sector.**

Observable	Combined		Leptons only		Quarks only	
	Best fit	χ^2 breakdown	Best fit	χ^2 breakdown	Best fit	χ^2 breakdown
$\sin^2 \theta_{12}^\ell$	0.291	2.385	0.308	0.001	—	—
$\sin^2 \theta_{13}^\ell$	0.02158	0.974	0.02215	6.88×10^{-5}	—	—
$\sin^2 \theta_{23}^\ell$	0.462	0.424	0.459	0.751	—	—
δ_{CP}^ℓ/π	0.99	0.690	1.09	0.143	—	—
$\Delta m_{21}^2/10^{-5} \text{eV}^2$	7.11	3.995	7.48	0.001	—	—
$\Delta m_{31}^2/10^{-3} \text{eV}^2$	2.53	0.387	2.51	2.28×10^{-5}	—	—
m_e/m_μ	0.004737	0.099	0.004737	0.099	—	—
m_μ/m_τ	0.05882	0.008	0.05882	0.008	—	—
$\min(\chi_{\text{leptons}}^2)$	8.86		0.90		-	
θ_{12}^q	0.2263	0.033	—	—	0.2257	0.492
θ_{13}^q	0.0037	0.631	—	—	0.0037	0.449
θ_{23}^q	0.0374	12.330	—	—	0.0398	2.969
$\delta_{CP}^q/^\circ$	70.52	0.072	—	—	73.32	1.253
m_u/m_c	0.00204	1.252×10^{-8}	—	—	0.00204	2.260×10^{-7}
m_c/m_t	0.00318	7.770×10^{-8}	—	—	0.00318	1.057×10^{-7}
m_d/m_s	0.0588	0.066	—	—	0.0950	2.522
m_s/m_b	0.018	0.659	—	—	0.019	0.430
$\min(\chi_{\text{quarks}}^2)$	13.79		—		8.11	
$\min(\chi_{\text{comb}}^2)$	22.65		—		—	

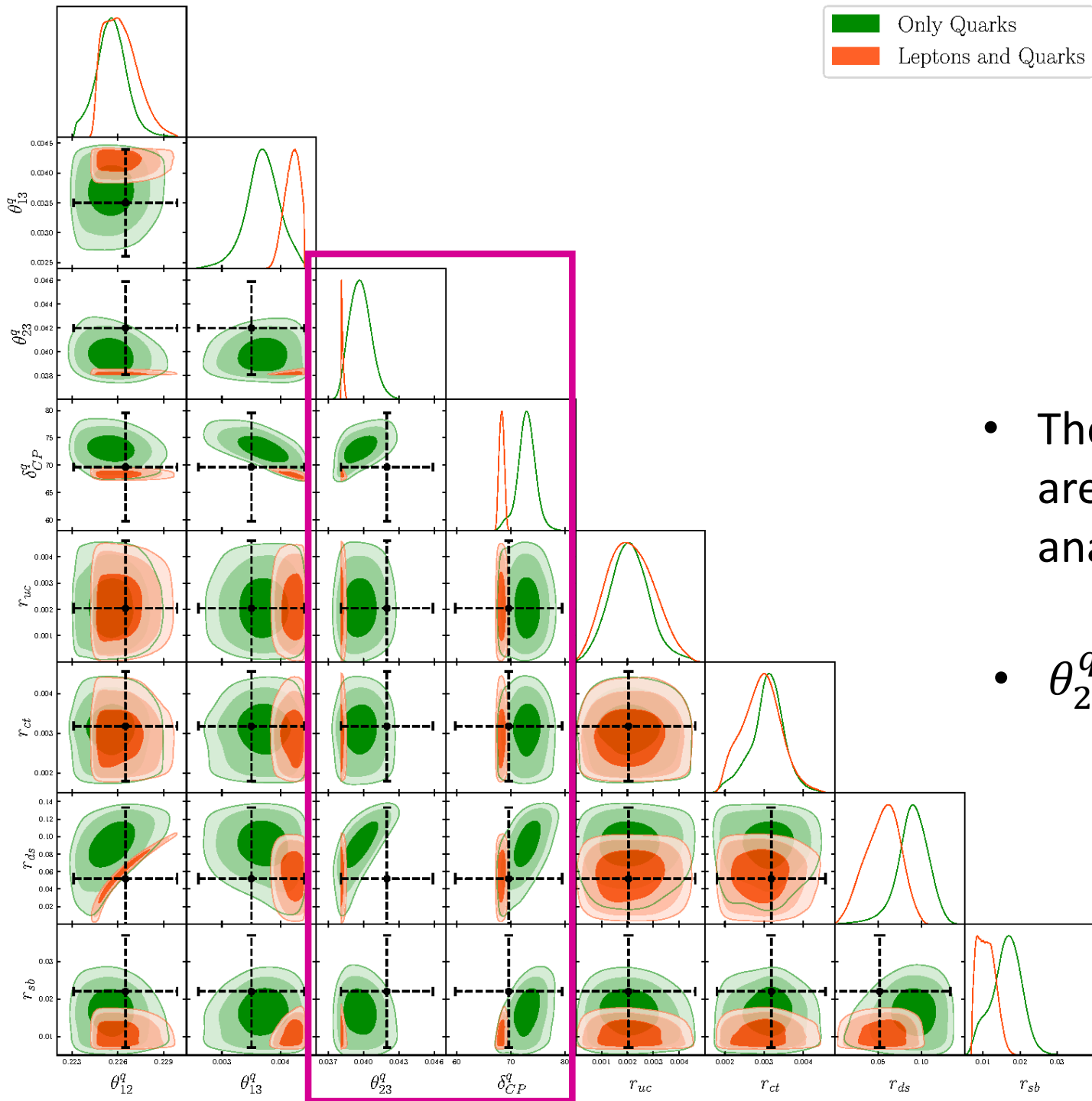
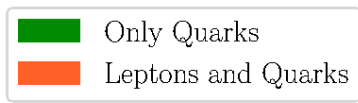
[Qu, Lu, Ding, 2506.19822]

- Slightly tension for θ_{23}^q in the combined analysis



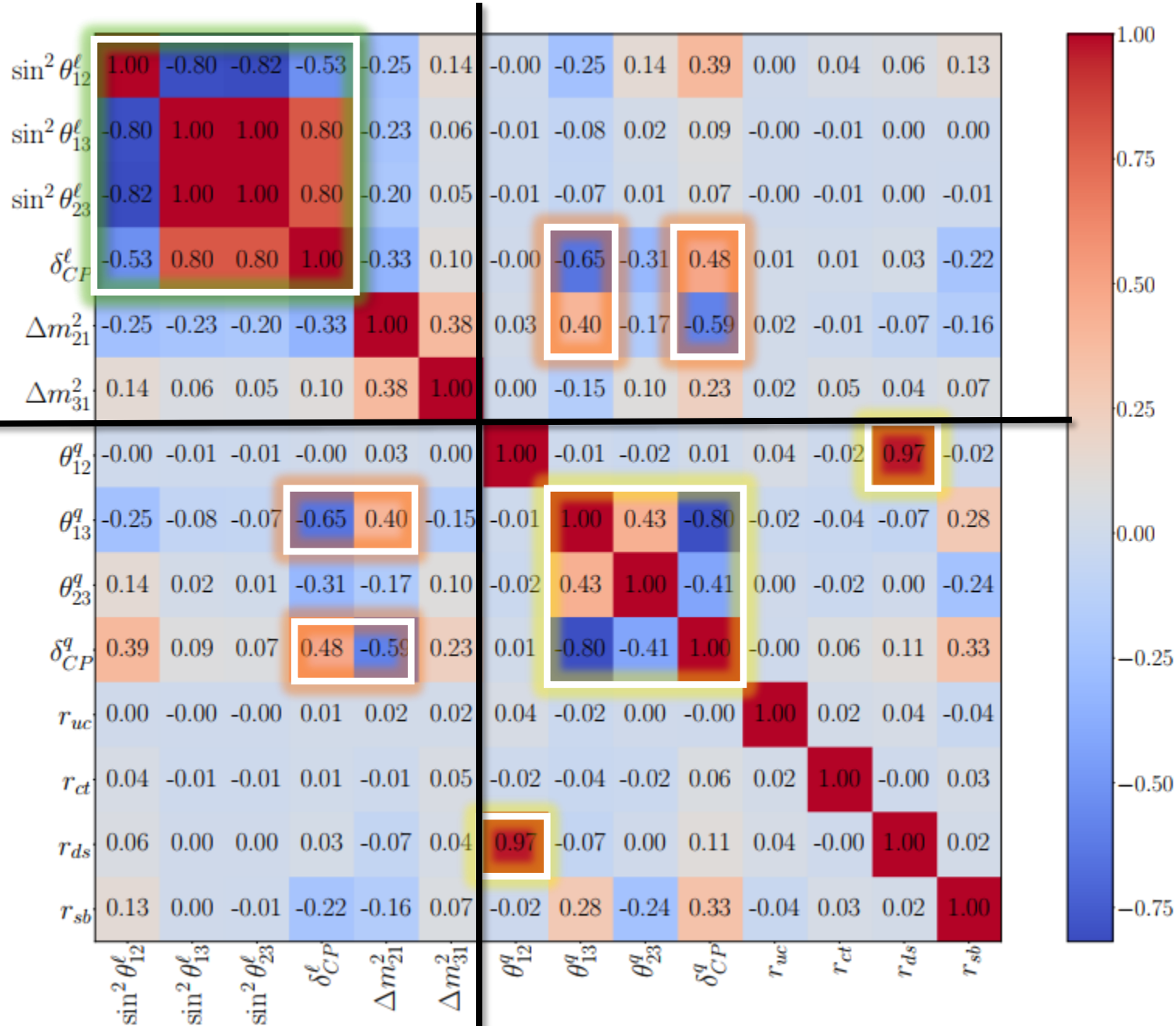
Lepton observables:

- only **NO** neutrino mass is viable after quark sector is combined, testable at JUNO
- $\sin^2 \theta_{23}^l \sim 0.46$ in the 1st octant
- $\delta_{CP}^l \sim \pi$



- The allowed regions of quark observables are significantly reduced in the combined analysis!
- θ_{23}^q in marginal agreement with data

- Correlations between flavor observables, τ is the portal connecting quarks and leptons



- Lepton observables:** strong correlations among $\theta_{12}^\ell, \theta_{13}^\ell, \theta_{23}^\ell$ and δ_{CP}^ℓ
- Quark observables:** strong correlations among $\theta_{13}^q, \theta_{23}^q$ and δ_{CP}^q , strong correlation between θ_{12}^q and m_d/m_s (GST relation?)
- Quark & lepton observables:** strong correlations between $\theta_{13}^q, \delta_{CP}^q$ and $\delta_{CP}^\ell, \Delta m_{21}^2$

Summary

- Formalism of non-holomorphic modular flavor symmetry: **modular form** → **polyharmonic Maaß form**, supersymmetry is unnecessary!
- Polyharmonic Maaß forms of weight $k \leq 1$ can be constructed from **non-holomorphic Eisenstein series**, the dimension equals the number of cusp.
- **Minimal** modular invariant model for quarks and leptons: **13 real parameters describe 22 observables**.
- Testable predictions: **NO neutrino mass**, $\sin^2 \theta_{23}^\ell \sim 0.46$, $\delta_{CP}^\ell \sim \pi \dots$
- Open questions:
 - origin of harmonic condition for polyharmonic Maaß forms ?
 - quantum corrections?
 - possible top-down connection?

.....

Backup

Polyharmonic Maaß forms of level $N=2,3,4,5$

k	-4	-3	-2	-1	0	1	2	3	4
$\Gamma_2 = S_3$	1 + 2	--	1 + 2	--	1 + 2	--	1 + 2	--	1 + 2
$\Gamma'_3 = T'$	1 + 3	$\hat{2} + \hat{2}''$	1 + 3	$\hat{2} + \hat{2}''$	1 + 3	$\hat{2} + \hat{2}''$	1 + 3	$\hat{2} + \hat{2}''$	1 + 1' + 3
$\Gamma'_4 = S'_4$	1 + 2 + 3	$\hat{3} + \hat{3}'$	1 + 2 + 3	$\hat{3} + \hat{3}'$	1 + 2 + 3	$\hat{3} + \hat{3}'$	1 + 2 + 3	$\hat{1}' + \hat{3} + \hat{3}'$	1 + 2 + 3 + 3'
$\Gamma'_5 = A'_5$	1 + 3 + 3' + 5	$\hat{6} + \hat{6}$	1 + 3 + 3' + 5	$\hat{6} + \hat{6}$	1 + 3 + 3' + 5	$\hat{6} + \hat{6}$	1 + 3 + 3' + 5	$\hat{4}' + \hat{6} + \hat{6}$	1 + 3 + 3' + 4 + 5 + 5

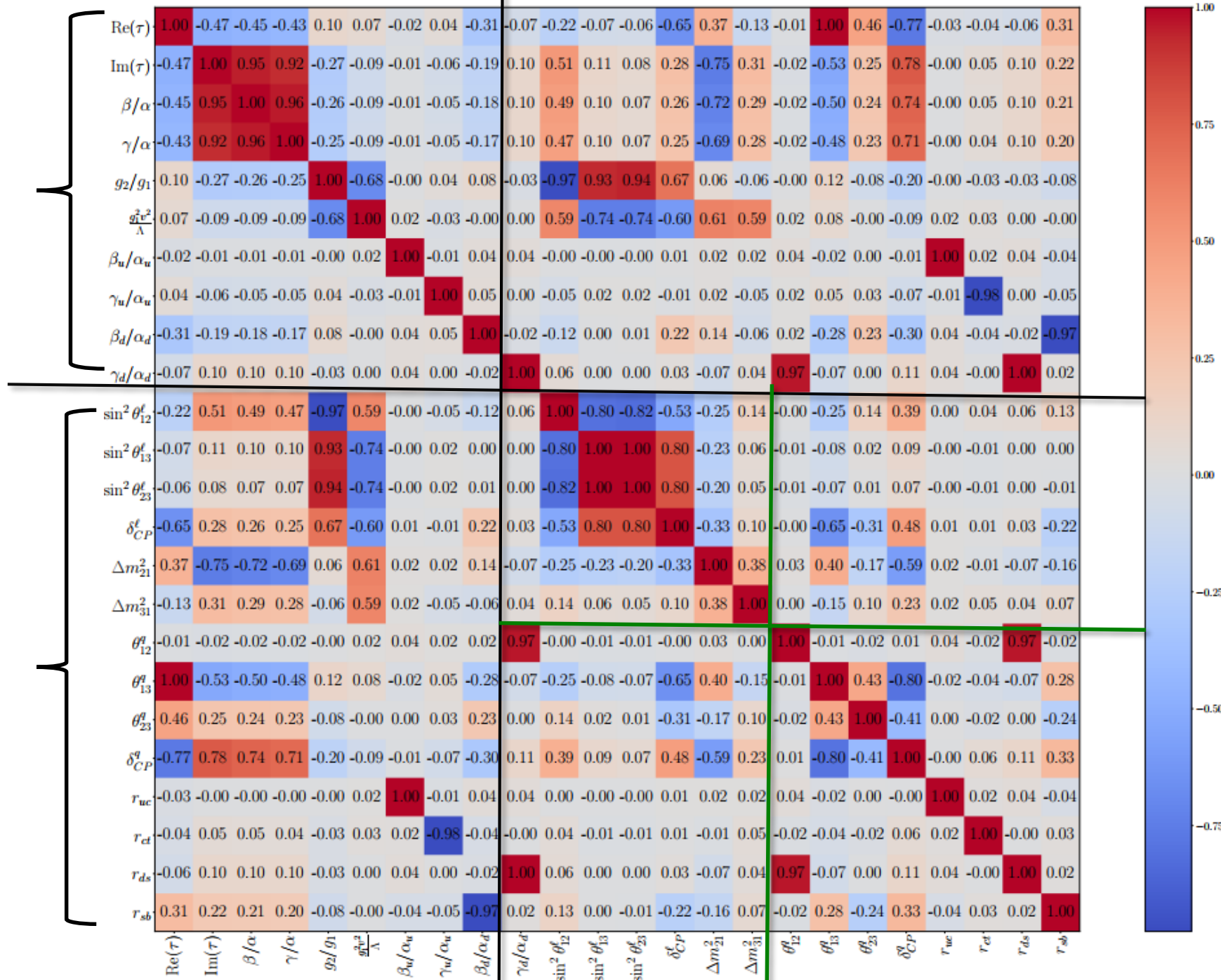
➤ The best fit values of free parameters and flavor observables

Inputs	Without gCP	With gCP
$\text{Re}(\tau)$	0.005315	0.005056
$\text{Im}(\tau)$	1.104	1.093
β/α	5.084	17.77
γ/α	0.004373	286.8
$ g_2/g_1 $	0.1647	0.1535
$\arg(g_2/g_1)$	5.799	0
$\alpha v/\text{GeV}$	0.2789	0.004679
$\frac{g_1^2 v^2}{\Lambda}/\text{meV}$	164.7	185.3
β_u/α_u	0.006165	0.006747
γ_u/α_u	497.3	496.2
β_d/α_d	30.11	29.12
γ_d/α_d	0.1131	0.06111
$\alpha_u v/\text{GeV}$	0.3806	0.3869
$\alpha_d v/\text{GeV}$	0.09457	0.1018

Observables	Without gCP	With gCP
$\sin^2 \theta_{12}^\ell$	0.308	0.291
$\sin^2 \theta_{13}^\ell$	0.02205	0.02158
$\sin^2 \theta_{23}^\ell$	0.443	0.462
δ_{CP}^ℓ/π	1.09	0.99
$\Delta m_{21}^2/10^{-5}\text{eV}^2$	7.57	7.11
$\Delta m_{31}^2/10^{-3}\text{eV}^2$	2.51	2.53
m_e/m_μ	0.004737	0.004737
m_μ/m_τ	0.05882	0.05882
ϕ/π	1.10	0.99
m_1/meV	0	0
m_2/meV	8.70	8.43
m_3/meV	50.11	50.26
$\sum m_i/\text{meV}$	58.81	58.69
$m_{\beta\beta}/\text{meV}$	2.13	1.32
m_β/meV	8.84	8.65
θ_{12}^q	0.2256	0.2263
θ_{13}^q	0.0037	0.0037
θ_{23}^q	0.0406	0.0374
$\delta_{CP}^q/^\circ$	74.33	70.52
m_u/m_c	0.00204	0.00204
m_c/m_t	0.00318	0.00318
m_d/m_s	0.1066	0.0588
m_s/m_b	0.019	0.018
χ_ℓ^2	4.57	8.86
χ_q^2	8.77	13.79
χ_{tot}^2	13.33	22.65

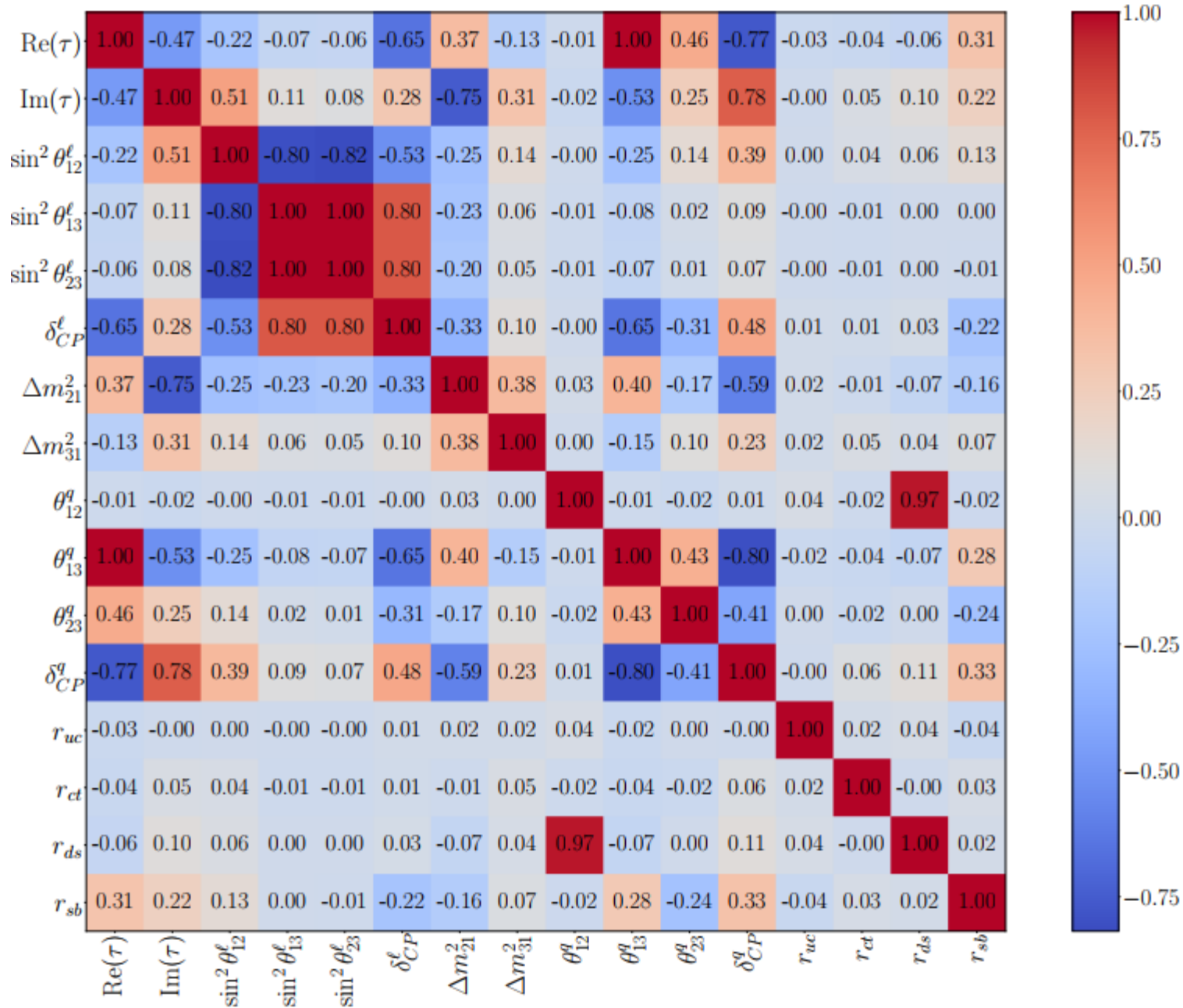
Parameters

Observables

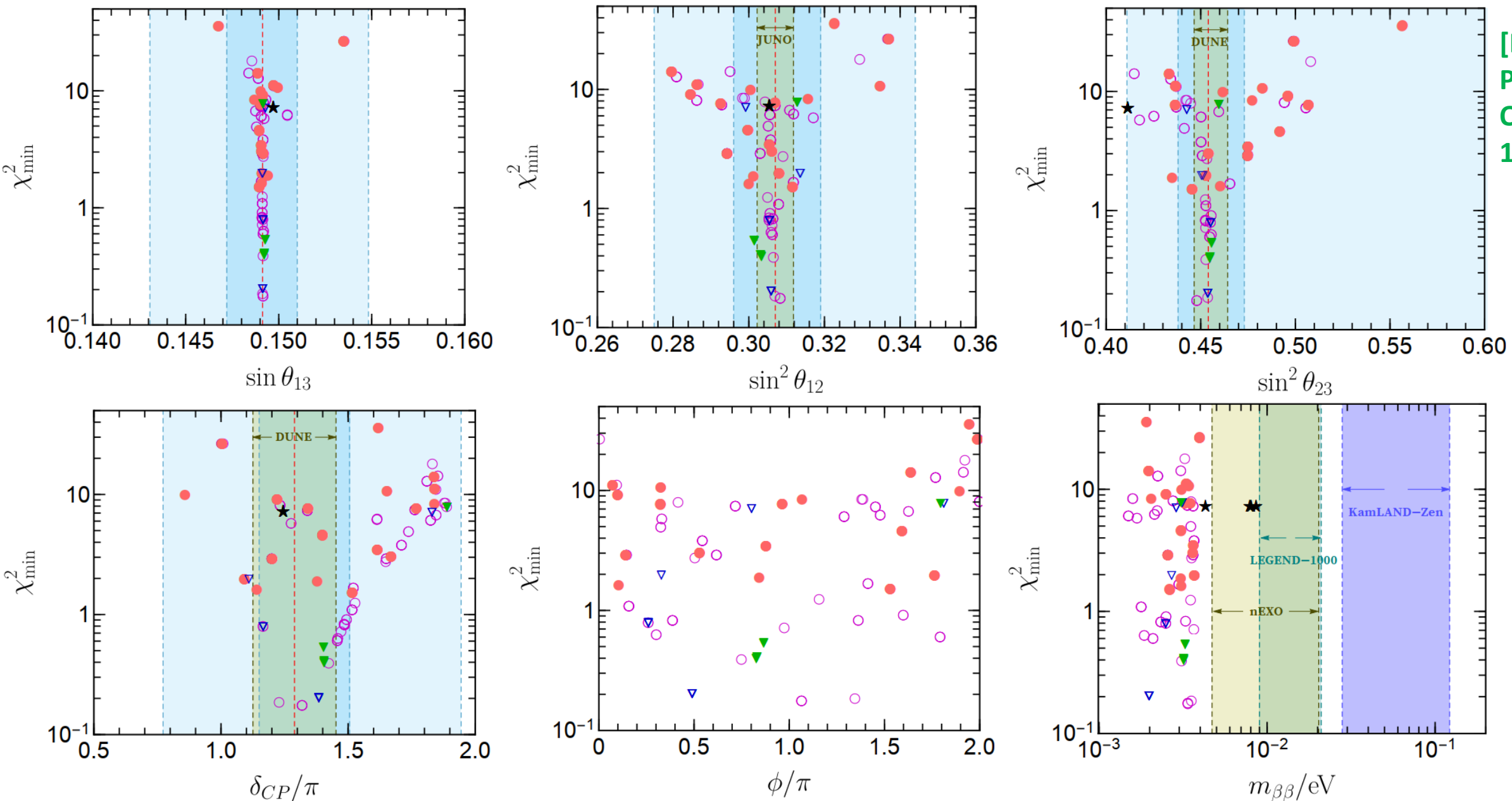


- Correlations among the free parameters and flavor observables

- Correlations among flavor observables and $\text{Re}(\tau)$, $\text{Im}(\tau)$



Confronting non-holomorphic S4 models with experiments



[Ding, Lu,
Petcov,
Qu, 2408.
15988]