LECTURE 1

$$\mathcal{L} = \mathcal{L}_I + \mathcal{L}_{FPG} + \mathcal{L}_{GF}$$

$$\mathcal{L}_I = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi}_i \left(i\gamma_\mu D^\mu - m \right)_{ij} \psi_j$$

$$\mathcal{L}_{FPG} + \mathcal{L}_{GF} = s \left[\bar{c}^a \mathcal{F}^a - \frac{\xi}{2} \bar{c}^a b^a \right]$$

Nakanishi Lautrup

BRST operator

ghost antifields

gauge fixing function

gauge fixing parameter

$$s A^a_\mu = \partial_\mu c^a + g f^{abc} A^b_\mu c^c$$

$$s\psi_i = i g c^a t^a_{ij} \psi_j \qquad s c^a = -\frac{1}{2} f^{abc} c^b c^c$$

$$s\bar{c}^a = b^a \qquad s b^a = 0 \qquad s\bar{\psi}_i = -i g c^a \bar{\psi}_j t^a_{ji}$$

$$b^a \mathcal{F}^a - \frac{1}{\xi} (b^a)^2 = 0$$

$$b^a = \frac{1}{\xi} \mathcal{F}^a \qquad \mathcal{F}^a = \partial^\mu A^a_\mu$$

$$\mathcal{L}_{GF} + \mathcal{L}_{FPG} = -\bar{c}^a \partial^\mu D^{ab}_\mu c^b + \frac{1}{2\xi} \left(\partial^\mu A^a_\mu \right)^2$$

$$\mathcal{L}_{BV} = \mathcal{L}_I + \sum_\Phi \Phi^* s \Phi$$

$$\boxed{S(\Gamma) = \int d^4x \sum_\Phi \frac{\delta \Gamma}{\delta \Phi^*} \frac{\delta \Gamma}{\delta \Phi} = 0}$$

Quantum Action Principle

$\Phi = A, \psi, \bar{\psi}, c, \cancel{\bar{c}}, \cancel{b}$

SLAVNOV TAYLOR FUNCTIONAL

$gh(A^*) = -1$

$-gh(A)$

$gh(c^*) = -2$

$$\frac{\delta^2}{\delta C^a\,\delta A^b_\mu} S(\Gamma)\Big|_{\Phi=0} = 0$$

$$\frac{\delta}{\delta c^b}\frac{\delta}{\delta A^a_\mu} S(\Gamma) = \frac{\delta\Gamma}{\delta c^b A^*_\rho}\,\frac{\delta^2\Gamma}{\delta A^a_\mu\,\delta A^\rho} + \dots$$

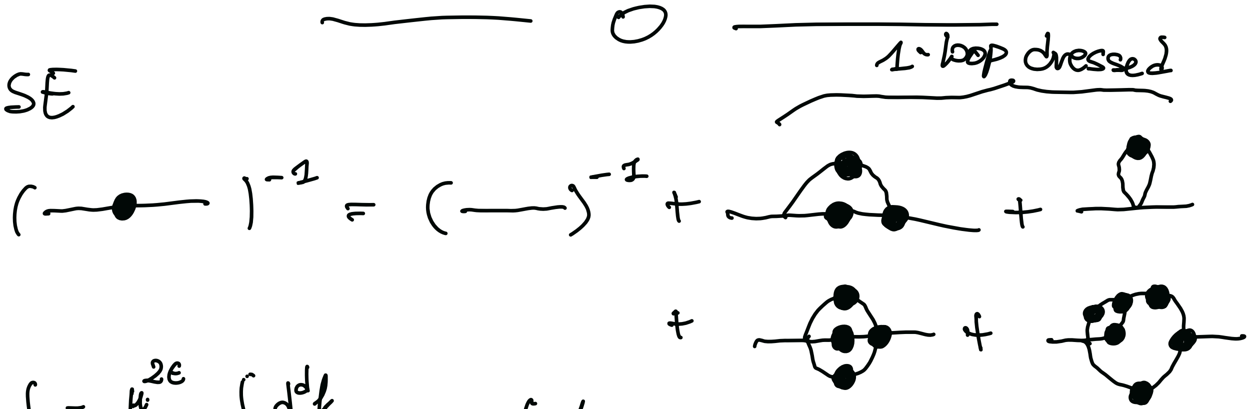
$$= \underbrace{\Gamma_{c^b A^{*m}_\rho}}\;\Gamma_{A^m_\rho A^a_\mu}$$

$$q_\rho\, C(q^2)\, \underbrace{\Gamma_{A^a_\rho A^b_\mu}(q)} = 0$$

$$\Gamma_{A_\rho A_\mu}(q) \propto \underbrace{P_{\rho\mu}(q)}_{g_{\rho\mu} - \frac{q_\rho q_\mu}{q^2}}$$

$$\boxed{\frac{\delta\Gamma}{\delta \bar{c}^m} + i q^\mu \frac{\delta\Gamma}{\delta A^*_\mu} = 0}\qquad \text{FPE}$$

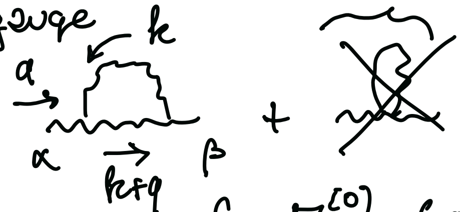
DSE



2-loop dressed

$$\int_k = \frac{\mu^{2\epsilon}}{(2\pi)^d}\int d^dk \qquad \int_k \frac{1}{k^2} = 0$$

In the Feynman gauge

$\Pi^{(1)}_{gl;\alpha\beta} =$ 

$$= \frac{1}{2} g^2 C_A \int_k \frac{\Gamma^{(0)}_{\alpha\mu\nu}(q,k,-k-q)\,\Gamma^{(0)\mu\nu}_{\beta}(-q,k+q,-k)}{k^2(k+q)^2}$$

$\Pi^{(1)}_{gh;\alpha\beta} =$  $= -g^2 C_A \int_k \frac{k_\alpha (k+q)_\beta}{k^2 (k+q)^2}$

$$\Gamma^{(0)}_{\alpha\mu\nu}(q,k_1,k_2) = g_{\mu\nu}(k_1-k_2)_\alpha + g_{\nu\alpha}(k_2-q)_\mu + g_{\alpha\mu}(q-k_1)_\nu$$

$$q^\alpha \Pi^{(1)}_{\alpha\beta}(q) = 0 \qquad \text{(exercise)}$$

However

$$q^\alpha \Pi^{(1)}_{gl;\alpha\beta}(q) \neq 0$$

$$q^\alpha \overline{\Pi}^{(1)}_{gl;\alpha\beta}(q) \neq 0$$

$$\underline{\Gamma_{\alpha\mu\beta}} = \underbrace{\tilde{\Gamma}_{\alpha\mu\nu}}_{\text{background part}} + \underbrace{\Gamma^P_{\alpha\mu\nu}}_{\text{pinch part}} \to k_{2\nu} g_{\alpha\mu} - k_{1\mu} g_{\alpha\nu}$$

$$\tilde{\Gamma}_{\alpha\mu\nu}(q,k_1,k_2) = g_{\mu\nu}(k_1-k_2)_\alpha - 2g_{\nu\alpha}q_\mu + 2g_{\mu\alpha}q_\nu$$

$$\Gamma\Gamma = \underbrace{\tilde{\Gamma}\tilde{\Gamma} + \Gamma^P\Gamma + \Gamma\Gamma^P - \Gamma^P\Gamma^P}_{\tilde{\Pi}^{(1)}_{gl;\alpha\beta}}$$

$$\Pi^{(1)}_{\alpha\beta} = \frac{1}{2} g^2 C_A \int_k \frac{\tilde{\Gamma}^{(0)}_{\alpha\mu\nu}\tilde{\Gamma}^{(0)\mu\nu}_{\beta}}{k^2(k+q)^2}$$

$$\underbrace{-g^2 C_A \int_k \frac{\tilde{\Gamma}_\alpha \tilde{\Gamma}_\beta}{k^2(k+q^2)}}_{\tilde{\Pi}^{(1)}_{gh;\alpha\beta}} \underbrace{- 2g^2 C_A \int \frac{P_{\alpha\beta}(q)}{k^2(k+q)^2}}_{\Pi^{(1)P}_{\alpha\beta}}$$

$$\tilde{\Gamma}_\alpha(q,k_1,k_2) = (k_2-k_1)_\alpha$$

$$\Pi^{(1)}_{\alpha\beta} = \tilde{\Pi}^{(1)}_{\alpha\beta} - 2\,\Pi^{(1)P}_{\alpha\beta}$$

$$q^\alpha \tilde{\Pi}^{(1)}_{gl;\alpha\beta} = 0 \qquad q^\alpha \tilde{\Pi}^{(1)}_{gh;\alpha\beta} = 0 \qquad \text{(exercise)}$$

$$\tilde{\Gamma}_{\alpha\mu\nu} \tilde{\Gamma}_\beta{}^{\mu\nu} = d(2k+q)_\alpha (2k+q)_\beta + 8q^2 P_{\alpha\beta}(q)$$

$$\int_k \frac{\tilde{\Gamma}_\alpha \tilde{\Gamma}_\beta}{k^2(k+q)^2} = \left(-\frac{1}{d-1}\right) q^2 P_{\alpha\beta} \int \frac{1}{k^2(k+q)^2}$$

$$\tilde{\Pi}^{(1)}_{\alpha\beta} = \left(\frac{7d-6}{d-1}\right) \frac{g^2 C_A}{2} q^2 P_{\alpha\beta}(q) \int \frac{1}{k^2(k+q)^2}$$

$$= \tilde{\Pi}^{(1)} P_{\alpha\beta}(q) \qquad P_{\alpha\beta}(q) = g_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}$$

$$\boxed{\tilde{\Pi}^{(1)} = i b g^2 q^2 \left[\frac{2}{\epsilon} + \dots\right] \qquad b = \frac{11 C_A}{48\pi^2}}$$

$$\hat{d}^{(1)} \propto \frac{1}{q^2 + \tilde{\Pi}^{(1)}}$$

Renormalization Group Invariant

1-loop coefficient of the QCD β function:

$$\beta = -g^3 b$$

$$Z_g^{(1)} = \frac{1}{\sqrt{Z_\beta^{(1)}}}$$

Addendum to lecture 1

1) $gh(c) = -gh(\bar{c}) = 1$. $gh(\Phi) = 0$

$\Gamma_{\Phi_1 \dots \Phi_n} \neq 0 \qquad gh(\Gamma_{\Phi_1 \dots \Phi_n}) = 0$

2) $\mathcal{L}_{BV} = \mathcal{L}_I + \sum_{\Phi} \Phi^* s\Phi$ $\qquad \Phi^* = A^*, \bar{\varphi}^*, \varphi^*, c^*$

$gh(\Phi^*) = -1 - gh(\Phi)$

$st(\Phi^*) = -st(\Phi)$

$$\Phi^* = \frac{\delta\Psi}{\delta\Phi}$$

$$\Gamma^{(0)} = \int d^4x\, \mathcal{L}_{BV}$$

$$\Gamma^{(0)}[\Phi, \Phi^*] = \Gamma^{(0)}[\Phi] + \sum_i s\Phi \frac{\delta\Psi}{\delta\Phi}$$

$$\overset{!}{=} \Gamma^{(0)}[\Phi] + s\Psi$$

$$s\Psi = \int d^4x \left(\mathcal{L}_{GF} + \mathcal{L}_{FPG}\right)$$

$$\mathcal{L}_{BV} = \mathcal{L}_I + \mathcal{L}_{GF} + \mathcal{L}_{FPG}$$

SUMMARY LECTURE 1

1) $S(\Gamma) = 0$ $\qquad \int d^4x \sum_i \frac{\delta\Gamma}{\delta\Phi^*} \frac{\delta\Gamma}{\delta\Phi} = 0 \qquad$ STI

$$\frac{\delta\Gamma}{\delta\bar{c}^m} + i\varphi^* \frac{\delta\Gamma}{\delta A^{*m}_\mu} = 0$$

2) $$\left.\frac{\delta^2\Gamma}{\delta A^b_\beta \delta c^a}\right|_{\Phi=0} = 0$$

$$\frac{\delta\Gamma}{\delta c} = \frac{\delta^2\Gamma}{\delta c\, \delta A^*_\rho} \frac{\delta\Gamma}{\delta A^\rho} + \dots$$

$$\left.\frac{\delta^2\Gamma}{\delta A_\beta \delta c}\right|_{\Phi=0} = 0 \qquad \Gamma_{cA^*_\rho} \Gamma_{A_\rho A_\beta} = 0$$

$$\Gamma_{cA^*_\rho} = q_\rho\, C(q^2) \qquad q^\rho \underbrace{\Gamma_{A_\rho A_\beta}}_{} = 0$$

$$P_{\rho\beta}(q)\,\Gamma_{AA}(q^2)$$

$$P_{\rho\beta}(q) = g_{\rho\beta} - \frac{q_\rho q_\beta}{q^2}$$

$$\Pi^{(1)}_{\alpha\beta}(q) = \Pi^{(1)}_{\alpha\beta;gl} + \Pi^{(1)}_{\alpha\beta;gh}$$

$$\approx \widetilde{\Pi}^{(1)}_{\alpha\beta;gl} + \widetilde{\Pi}^{(1)}_{\alpha\beta;gh} - 2P_{\alpha\beta}\,\Pi^{(1)P}$$

$$= \widetilde{\Pi}^{(1)}_{\alpha\beta} - 2P_{\alpha\beta}\,\Pi^{(1)P}$$

Cross talk $\quad q^\alpha \Pi^{(1)}_{\alpha\beta;gl} \neq 0 \qquad q^\alpha \Pi^{(1)}_{\alpha\beta;gh} \neq 0$

block-wise $\quad q^\alpha \widetilde{\Pi}^{(1)}_{\alpha\beta;gl} = 0 \qquad q^\alpha \widetilde{\Pi}^{(1)}_{\alpha\beta;gh} = 0$

3) $\quad \widetilde{\Pi}^{(1)}_{\alpha\beta} = P_{\alpha\beta}\, \underbrace{\widetilde{\Pi}^{(1)}(q^2)}_{g^2 q^2 b\left[\frac{2}{\epsilon} + \cdots\right]}$

$$\beta = -b g^3$$

$$b = \frac{11}{48\pi^2} C_A$$

$$C_A = N$$

Exercise 1

a) $q^\alpha \Pi^{(1)}_{\alpha\beta}(q) = 0 \quad \Rightarrow q^\alpha \widetilde{\Pi}^{(1)}_{\alpha\beta} = 0$

b) block-wise

$$\widetilde{\Gamma}_\alpha(k, q, -k-q) = -(2k+q)_\alpha$$

$$\widetilde{\Gamma}_{\alpha\mu\nu}\, \widetilde{\Gamma}_\beta{}^{\mu\nu} = d\,(2k+q)_\alpha (2k+q)_\beta + 8q^2 P_{\alpha\beta}$$

$$q^\alpha \widetilde{\Pi}^{(1)}_{\alpha\beta;gh} \propto \int_k \frac{(q^2 + 2k\cdot q)(2k+q)_\beta}{k^2 (k+q)^2}$$

$$k^2 (k+q)^2$$

$$= \int_k \frac{(k+q)^2 - k^2}{k^2 (k+q)^2} (2k+q)_\beta$$

$$= \int \frac{(\cancel{2k}+q)_\beta}{k^2} - \int \frac{(\cancel{2k+2q}-q)_\beta}{(k+q)^2}$$

$$= 2q_\beta \int_k \frac{1}{k^2} = 0$$

LECTURE 2

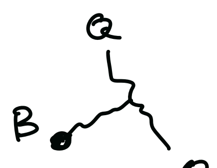
Background Field Method (BFM)

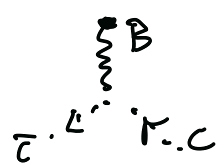
$$A^a_\mu = B^a_\mu + Q^a_\mu \qquad \tilde{\mathcal{F}}^a = \underbrace{\tilde{D}^{ab}_\mu}_{\delta^{ab}\partial_\mu + g f^{abc} B^c_\mu} Q^\mu_b$$

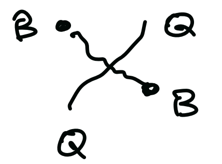
$$s B^a_\mu = 0$$

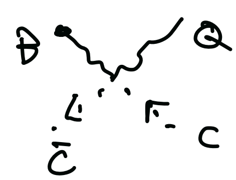
$$s Q^a_\mu = \partial_\mu c^a + g f^{abc} (Q^b_\mu + B^b_\mu) c^c$$

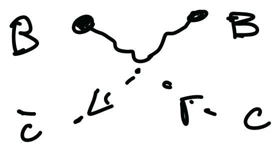
$$\Gamma' = \int d^4x \, \big(\mathcal{L}'_I + \underbrace{\mathcal{L}'_{GF} + \mathcal{L}'_{FPG}}_{s\left(\bar{c}^a \tilde{\mathcal{F}}^a - \frac{\xi}{2} \bar{c}^a b^a\right)} \big)$$











$$\Gamma^{(0)}_{B_\alpha Q_\mu Q_\rho} = \tilde{\Gamma}^{(0)}_{\alpha\mu\rho}$$

$$\Gamma^{(0)}_{B_\alpha c \bar{c}} = \tilde{\Gamma}^{(0)}_{\alpha}$$

$$\tilde{\Pi}^{(1)}_{\alpha\beta} = \Gamma^{(1)}_{B_\alpha B_\beta}$$

THEOREM (2008) $\qquad$ PT $\equiv$ BFM

1) STI

$$sB_\mu^a = \omega_\mu^a \qquad s\omega_\mu^a = 0$$

$$S'(\Gamma') = 0 \qquad 0 = S(\Gamma') + \int d^4x\, \omega_\mu^a \left(\frac{\delta \Gamma'}{\delta B_\mu^a} - \frac{\delta \Gamma'}{\delta Q_\mu^a} \right)$$

ghost equation

2) $$0 = \frac{\delta \Gamma'}{\delta \bar{c}^m} + \left(\tilde{\mathcal{D}}^\mu \frac{\delta \Gamma'}{\delta Q_\mu^*} \right)^m - (\mathcal{D}^\mu \omega_\mu)^m - g f^{mrs} Q_\mu^r \omega_\mu^s$$

3) antighost equation (Landau gauge)

$$0 = \frac{\delta \Gamma'}{\delta c^m} - \left(\tilde{\mathcal{D}}^\mu \frac{\delta \Gamma'}{\delta \omega_\mu} \right)^m - (\mathcal{D}^\mu Q_\mu^*)^m - g f^{mnr} c^{*n} c^r + g f^{mnr} \frac{\delta \Gamma'}{\delta b^n} \bar{c}^r = 0$$

4) Residual gauge invariance wrt B

$$W_\theta(\Gamma') = \int d^4x \sum_\varphi \delta_\theta \varphi \frac{\delta \Gamma'}{\delta \varphi} = 0$$

So what?

1) $$\left. \frac{\delta^2 S'}{\delta \omega_\mu \delta Q_\nu} \right|_{\Phi=0} = 0$$

$$i\Gamma_{B_\mu Q_\nu} = [i g_\mu^\rho + \Gamma_{\omega_\mu Q^{*\rho}}]\, \Gamma_{Q^\rho Q_\nu}$$

$$\left. \frac{\delta^2 S'}{\delta \omega_\mu \delta B_\nu} \right|_{\Phi=0} = 0$$

$$i\Gamma_{B_\mu B_\nu} = [i g_\mu^\rho + \Gamma_{\omega_\mu Q^{*\rho}}]\, \Gamma_{Q^\rho B_\nu}$$

$$\boxed{-i\Gamma_{\omega_\mu Q^*_\nu} = g_{\mu\nu} G + \frac{q_\mu q_\nu}{q^2} L}$$

$$\Gamma_{BQ} = (1+G)\Gamma_{QQ}$$

$$\Gamma_{BB} = (1+G)\Gamma_{QB}$$

$$\Gamma_{BB} = (1+G)^2 \Gamma_{QQ}$$

$$\Gamma^{(1)}_{BB} = \Gamma^{(1)}_{QQ} + 2G^{(1)}\Gamma^{(0)}_{QQ}$$

$$\tilde{\Pi}^{(1)} = \Pi^{(1)} + 2\Pi^{(1)P}$$

2) & 3)

ghost equation

$\frac{\delta}{\delta c}\Big|_{\Phi=0}$ $\qquad \Gamma_{c\bar{c}} + iq^\mu \Gamma_{cQ^*_\mu} = 0$ $\qquad$ ⓐ

$\frac{\delta}{\delta \omega_\mu}\Big|_{\Phi=0}$ $\qquad i\Gamma_{\omega_\mu \bar{c}} = iq_\mu + q^\rho \Gamma_{\omega_\mu Q^*_\rho}$ $\qquad$ ⓑ

antighost equation

$\frac{\delta}{\delta Q^*_\mu}\Big|_{\Phi=0}$ $\qquad i\Gamma_{Q^*_\mu c} = iq_\mu + \Gamma_{\omega_\rho Q^*_\mu} q^\rho$ $\qquad$ ⓒ

$\frac{\delta}{\delta \bar{c}}\Big|_{\Phi=0}$ $\qquad \Gamma_{c\bar{c}} = -iq^\mu \Gamma_{\bar{c}\omega_\mu}$ $\qquad$ ⓓ

$$q^\mu \text{ⓒ} \Rightarrow \qquad \underbrace{q^\mu \Gamma_{Q^*_\mu c}}_{\overset{\text{ⓐ}}{=}\ i\Gamma_{c\bar{c}}} = q^2 - i q^\mu q^\rho \Gamma_{\omega_\rho Q^*_\mu}$$

$$\Gamma_{cQ^*_{\mu}} = q_\mu C(q^2) \qquad C = E = F^{-1}$$

$$\Gamma_{\bar{c}\omega_\mu} = q_\mu E(q^2) \qquad \boxed{F^{-1} = 1 + G + L}$$

$$\Gamma_{c\bar{c}} = -iq^2 \underbrace{F^{-1}(q^2)}_{\text{ghost dressing function}}$$

$$L(q^2) \ll G(q^2)$$

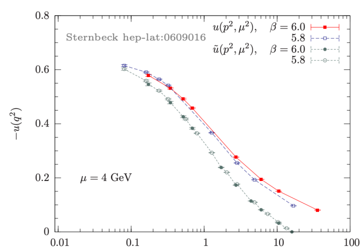
$$L(q^2)\big|_{q^2=0} = 0 = L(0) \qquad 1 + G + L\big|_{q^2=0} = 1 + G(0) = F^{-1}(0)$$

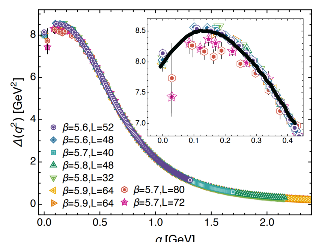
What happens if $G(0) = -1$?

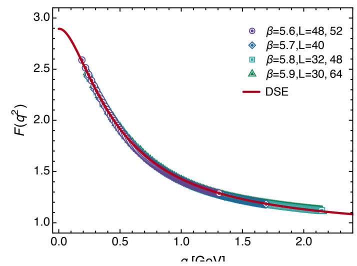
$$F^{-1}(0) = 0 \qquad F(0) = \infty$$

Why should $G(0) = -1$ anyway. Kugo-Ojima confinement criterion

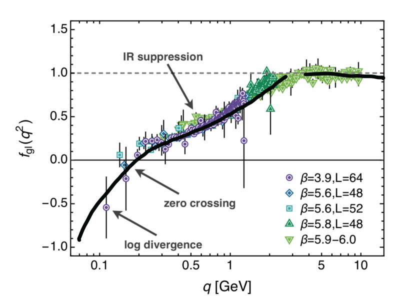
$$G(q^2) = u(q^2) \xrightarrow[q^2 \to 0]{} -1$$







$$q^2 \bar{J}(q^2) = a q^2 \log\frac{q^2+m_{gl}^2}{\mu^2} + b q^2 \log\frac{q^2}{\mu^2}$$



$$\Gamma_{\Phi^a \Phi^b} = i\delta^{ab} P_{\alpha\beta}(q)\, \Delta^{-1}_{\Phi\Phi}(q^2)$$

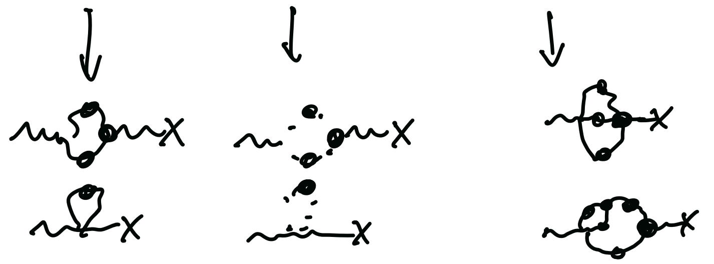
$$\Delta^{-1}_{BB}(q^2) = [1+G(q^2)]\, \Delta^{-1}(q^2)$$

$$\boxed{\hat{d}(q^2) = \frac{\alpha(\mu^2)\,\Delta(q^2;\mu^2)}{[1+G(q^2;\mu^2)]^2}}$$

candidate for describing α_s for all q_s

$$\Delta^{-1}_{\alpha\beta}(q) = \Delta^{-1} P_{\alpha\beta} = \frac{q^2 P_{\alpha\beta} + \tilde{\Pi}_{\alpha\beta}}{1+G}$$

$$\tilde{\Pi}_{\alpha\beta} = \tilde{\Pi}^{\alpha\beta}_{gl} + \tilde{\Pi}^{\alpha\beta}_{gh} + \tilde{\Pi}^{\alpha\beta}_{gl;\,2l}$$



omit them gauge invariantly

4) From the Ward Identi functional W

$$q^\mu \Gamma_{B_\mu Q_\alpha Q_\beta}(q,r,p) = i\,\Delta^{-1}_{\alpha\beta}(r) - i\,\Delta^{-1}_{\alpha\beta}(p)$$

$$q^\mu \Gamma_{B_\mu c\bar{c}}(q,r,p) = i\,D^{-1}(r^2) - i\,D^{-1}(p^2)$$

——— o ———

$$sA^a_\mu = \mathcal{D}^{ab}_\mu c^b \qquad sc^a = -\frac{1}{2} f^{abc} c^b c^c$$

$$s\bar{c}^a = b^a \qquad sb^a = 0$$

anti BRST

$$\bar{s} A^a_\mu = \mathcal{D}^{ab}_\mu \bar{c}^b \qquad \bar{s}\bar{c}^a = -\frac{1}{2} f^{abc} \bar{c}^b \bar{c}^c$$

$$\bar{s} c^a = \bar{b}^a \qquad \bar{s}\bar{b}^a = 0$$

Theorem 2015

$$\underset{\mathcal{F}^a = \partial^\mu A^a_\mu}{\mathrm{BRST} + \overline{\mathrm{BRST}}} \quad \cong \quad \underset{\tilde{\mathcal{F}}^a = \tilde{\mathcal{D}}^\mu_{ab} Q^b_\mu}{\mathrm{BFM}}$$

$$F^{-1} = 1 + G + L + \xi K$$

——— o ———

$$\Delta^{-1}(0) \neq 0$$

It is in fact true that $\Delta^{-1}(0) = 0$ in the absence of a dynamical mechanism

$$\widetilde{\Pi}^{gl}_{\mu\nu} = \frac{1}{2} g^2 C_A \int_k \Gamma^{(0)}_{\mu\alpha\beta}(q, k, -k-q)\, \Delta^{\alpha\rho}(k)\, \Delta^{\beta\sigma}(k+q)\, \widetilde{\Gamma}_{\nu\rho\sigma}(-q, k+q, -k)$$
$$+ g^2 C_A \int_k \left[g_{\mu\nu} (\Delta^{\alpha}_{\alpha}(k)) - \Delta_{\mu\nu}(k) \right]$$

$$\widetilde{\Pi}^{gh}_{\mu\nu} = - g^2 C_A \int_k k_\mu D(k) D(k+q) \widetilde{\Gamma}_\nu (+q, k, -k-q)$$
$$+ g^2 C_A \int_k g_{\mu\nu} D(k)$$

What happens @ $q^2 = 0$

Assume that the vertices are regular as $q^2 \to 0$

$$\widetilde{\Gamma}_{\mu\alpha\beta}(0, k, -k) = -i \frac{\partial}{\partial k^\mu} \Delta^{-1}_{\alpha\beta}(k)$$

$$\widetilde{\Gamma}_\mu (0, k, -k) = -i \frac{\partial}{\partial k^\mu} D^{-1}(k)$$

Insert back to get

$$d\widetilde{\Pi}^{gl} = \frac{1}{2} g^2 C_A \int_k \frac{\partial}{\partial k^\mu} \left[\underline{\Gamma^{(0)}_{\mu\alpha\beta}(0, k, -k)\, \Delta^{\alpha\beta}(k)} \right]$$

$$d\widetilde{\Pi}^{gh} = g^2 C_A \int_k \frac{\partial}{\partial k^\mu} \left[\underline{\Gamma^{(0)}_\mu (0, k, -k) D(k)} \right]$$

$$\widetilde{\Delta}^{-1}(0) = \int_k \frac{\partial}{\partial k^\mu} \mathcal{F}_\mu(k) = 0$$
$$|||$$
$$\Delta^{-1}(0) = 0$$

$$\int_k \mathcal{F}_\mu(k) = 0$$

$$\int_k \mathcal{F}_\mu(k+q) = 0$$

So HOW $\Delta^{-1}(0) = m^2_{gl} > 0$???

Addendum for LECTURE 3

$$\widetilde{\Pi}^{gl;2l}_{\mu\nu}(q) = \frac{ig^4}{6}\Gamma^{(0)\,amnr}_{\mu\alpha\beta\gamma}\int_k\int_l \Delta^{\alpha\rho}(k+l)\,\Delta^{\beta\sigma}(l)\,\Delta^{\gamma\tau}(k+q)$$
$$\widetilde{\Gamma}^{brnm}_{\nu\alpha\beta\gamma}(-q,k+q,l,-l-k)$$
$$+ ig^4 f^{bre} f^{emn}\,\Gamma^{(0)\,amnr}_{\mu\alpha\beta\gamma}\int_k Y^{\alpha\beta}_{\delta}(k)$$
$$\Delta^{\delta\gamma}(k)\,\Delta^{r\pi}(k+q)\,\widetilde{\Gamma}_{\nu\pi\lambda}(-q,k+q,-k)$$

$$Y^{\alpha\beta}_{\delta}(k) = \int_l \Delta^{\alpha\rho}(k+l)\,\Delta^{\beta\sigma}(l)\,\Gamma_{\sigma\rho\delta}(l,k-l,k)$$
$$= (g^{\beta}_{\delta}k^{\alpha} - g^{\alpha}_{\delta}k^{\beta})\,Y(k^2)$$

$$\widetilde{\Pi}^{gl;2l}_{\alpha\beta}(0)$$

$$\widetilde{\Gamma}^{mnrs}_{\mu\alpha\beta\gamma}(0,k,l,-l-k) = \left(f^{mne}f^{esr}\frac{\partial}{\partial k^{\mu}} + f^{mre}f^{ens}\frac{\partial}{\partial l^{\mu}}\right)$$
$$\Gamma_{\alpha\beta\gamma}(k,l,-l-k)$$

$$d\,\Pi^{gl;2l}(0) = -i(d-1)\,g^4 C_A^2\int_k \frac{\partial}{\partial k_{\mu}}\Big[\underbrace{k_{\mu}\,\Delta(k^2)\,Y(k^2)}_{\mathcal{F}_{\mu}}\Big]$$
$$= 0$$

How $\Delta^{-1}(0) \neq 0$?

$$\Delta(q^2) = \frac{1}{q^2 + \Pi(q^2)} = \frac{1}{q^2[1 + \widetilde{\Pi}(q^2)]}$$

dimensionful   dimensionless

Schwinger '60s   Higgs

$$\Pi(q^2) \underset{q^2\to 0}{=} \frac{m^2_{gl}}{q^2}$$

$$\Delta(q^2) \xrightarrow[q^2\to 0]{} m^2_{gl}$$

$\boxed{\text{Dynamical}}$

Dynamical realization of SM
we need the presence of $\widehat{C}$ terms such that:

1) The seagull identity should be evaded
2) The form of STIs/WIs should be maintained
3) They decouple from on-shell amplitudes

We will consider only 3-gluon & ghost-gluon vertices

$$\widetilde{\Gamma}_{\mu\alpha\beta}(q,r,p) = \widetilde{\Gamma}^{np}_{\mu\alpha\beta}(q,r,p) + \widetilde{\Gamma}^{p}_{\mu\alpha\beta}(q,r,p)$$

$$\widetilde{\Gamma}_{\mu} = \widetilde{\Gamma}^{np}_{\mu} + \widetilde{\Gamma}^{p}_{\mu}$$

$$P^{\mu\rho}_{(q)}\, P^{\alpha\sigma}_{(r)}\, P^{\beta\nu}_{(p)}\, \widetilde{\Gamma}^{p}_{\mu\alpha\beta} = 0$$

$$P^{\mu\rho}(q)\, \widetilde{\Gamma}^{p}_{\mu} = 0$$

$$\widehat{\Gamma}^{p}_{\mu\alpha\beta} = \frac{q^\mu}{q^2}\, \widetilde{C}_{\alpha\beta}(q,r,p)$$

$$\widetilde{\Gamma}^{p}_{\mu} = \frac{q^\mu}{q^2}\, \widetilde{C}(q,r,p)$$

$$q^\mu \tilde{\Gamma}_{\mu\alpha\beta} = q^\mu \tilde{\Gamma}^{np}_{\mu\alpha\beta} + \tilde{C}_{\alpha\beta} = i\Delta^{-1}_{\alpha\beta}(r) - i\Delta^{-1}_{\alpha\beta}(p)$$

$$q^\mu \tilde{\Gamma}_{\mu} = q^\mu \tilde{\Gamma}^{np}_{\mu} + \tilde{C} = iD^{-1}(r) - iD^{-1}(p)$$

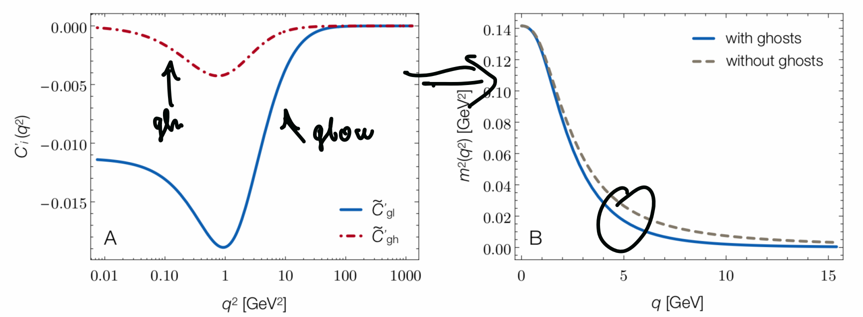
Send $q \to 0$ & solve order by order

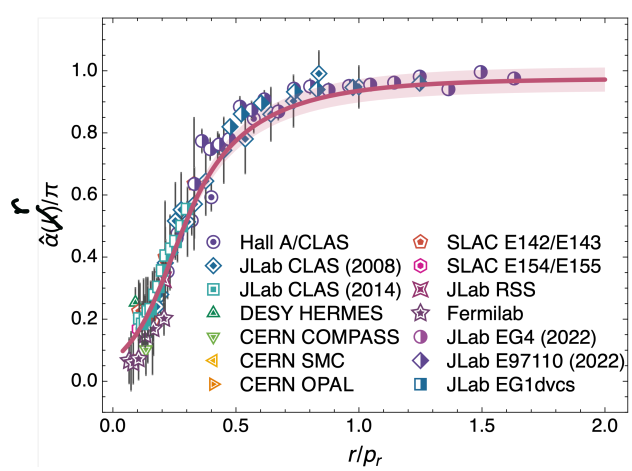
0th : $\tilde{C}_{\alpha\beta}(0, r, -r) = 0 \qquad \tilde{C}(0, r, -r) = 0$

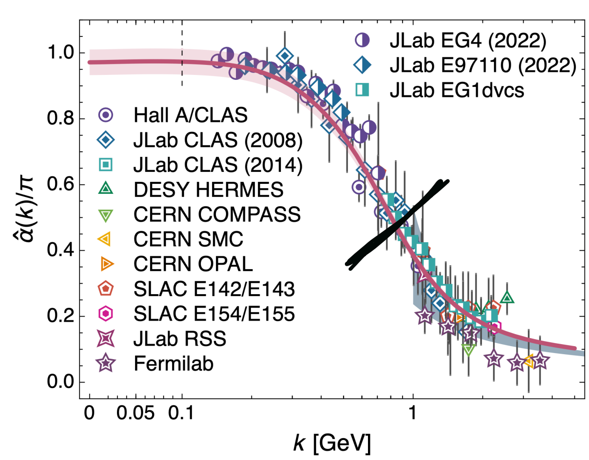
1st :
$$\tilde{\Gamma}^{np}_{\mu\alpha\beta}(0, r, -r) = -i\frac{\partial}{\partial r^\mu}\Delta^{-1}_{\alpha\beta}(r) - \frac{\partial}{\partial q^\mu}\tilde{C}_{\alpha\beta}(q, r, -r-q)\Big|_{q=0}$$

$$\tilde{\Gamma}^{np}_{\mu}(0, r, -r) = -i\frac{\partial}{\partial r^\mu}D^{-1}(r^2) - \frac{\partial}{\partial q^\mu}\tilde{C}_{gh}(q, r, -r-q)\Big|_{q=0}$$

$$\frac{\text{gluon}}{\text{gh}} = 5$$





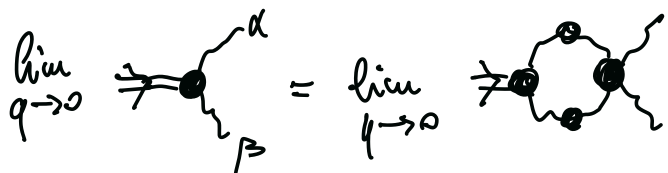


$$\Delta^{-1}(0) = \frac{3}{2} g^2 C_A F(0) \left\{ \int_k k^2 \Delta(k^2) \left[\underline{\underline{1}} - \frac{3}{2} g^2 C_A Y(k^2) \right] \tilde{C}'_{gl} \right.$$
$$\left. - \frac{1}{3} \int_k k^2 D(k^2)\, \tilde{C}'_{gh}(k^2) \right\}$$

$$\tilde{C}'_{gl,gh} = \lim_{q \to 0} \left\{ \frac{\partial \tilde{C}_{gl,gh}(q, k, -k-q)}{\partial (k+q)} \right\}$$

Looking only to the three gluon vertex

$$f^{amn} \lim_{q \to 0} \tilde{C}_{\alpha\beta}(q, r, p) = f^{abc} \lim_{q \to 0} \left\{ \int_k \tilde{C}_{\gamma\delta}(q, k, -k-q) \right.$$
$$\left. \Delta^{\gamma\rho}(k)\, \Delta^{\delta\sigma}(k+q)\, K^{bmnc}_{\rho\alpha\beta\sigma}(-k, r, k+q) \right\}$$



Now we know that $\Delta^{-1}(0) \neq 0$

$$\Delta^{-1}(q^2) = \underset{\sim\sim\sim}{q^2 J(q^2)} + \underset{\sim\sim}{m^2(q^2)}$$

Insert back into WI

$$\tilde{C}_{\alpha\beta} = m^2(p^2)\, P_{\alpha\beta}(p) - m^2(r^2)\, P_{\alpha\beta}(r)$$

As $q \to 0$
$$\tilde{C}_{gl} = m^2(p^2) - m^2(r^2)$$

$$\boxed{\tilde{C}'_{gl} = \frac{d}{dr^2} m^2(r)}$$

$$m^2(x) = \Delta^{-1}(0) + \int_0^x dy\, \tilde{C}'_{gl}(y)$$

$m^2(\infty) = 0$
$$\Delta^{-1}(0) = -\int_0^\infty dy\, \tilde{C}'_{gl}(y)$$
modulo a normalization constant

Recall

$$\hat{d}(q^2) = \frac{\Delta(q^2;\mu^2)\,\alpha(\mu^2)}{[1+G(q^2;\mu^2)]^2} \qquad \begin{matrix} \text{RGI} \\ q^2\in[0,\infty) \end{matrix}$$

As the gluon propagator saturates

$$\hat{d}(0) = \frac{\alpha(\mu^2)}{\hat{m}^2_{gl}(\mu^2)} \stackrel{!}{=} \frac{\alpha_0}{m_0^2} \qquad \hat{m}^2_{gl} = \frac{m^2_{gl}(0;\mu^2)}{F^2(0;\mu^2)}$$

$$\mathbb{D} = \frac{\Delta(q^2;\mu^2)\,m^2_{gl}(0;\mu^2)}{m_0^2}$$

Look @ the lattice & produce an interpolator $\mathcal{D}\approx\mathbb{D}$ in the region where $q^2\lesssim\mu^2$ & it is such that

$$\frac{1}{\mathcal{D}} = \begin{cases} m_0^2 + \mathcal{O}(k^2\log k^2) & k^2 \ll \Lambda^2_{QCD} \\ k^2 + \mathcal{O}(1) & k^2 \gg \Lambda^2_{QCD} \end{cases}$$

Then

$$\hat{d}(q^2) = \hat{\alpha}(q^2)\,\mathcal{D}(q^2)$$

$$\hat{\alpha}(q^2) = \frac{\alpha_0}{F^2(0;\mu^2)}\underbrace{\frac{\mathbb{D}(q^2)}{\mathcal{D}(q^2)}}_{\sim 1 \text{ in the region where } q^2\lesssim\mu^2}\left[\frac{F(q^2;\mu^2)/F(0;\mu^2)}{1-L(q^2;\mu^2)F(q^2;\mu^2)}\right]^2$$

$$\hat{\alpha}(q^2) = \frac{\gamma_m\pi}{\log\frac{K^2(q^2)}{\Lambda^2_{QCD}}} \qquad K^2(q^2) = \frac{a_0^2+a_1q^2+q^4}{b_0+q^2}$$

$a_0 = 1.9\,\Lambda^2_{QCD}$

$a_1 = 1.7805\,\Lambda^2_{QCD}$

$b_0 = 2.2010\,\Lambda^2_{QCD}$

Define

$$\zeta_H = K(q^2)\big|_{q^2=\Lambda^2_{QCD}} = 1.413\,\Lambda_{QCD}$$