All-orders evolution: principle and practice

FÍSICA MATEMÁTICAS COMPUTACIÓN

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2025 International Workshop and School on Hadron Structure and Strong Interactions

Nanjing, October 13th - 17th, 2025

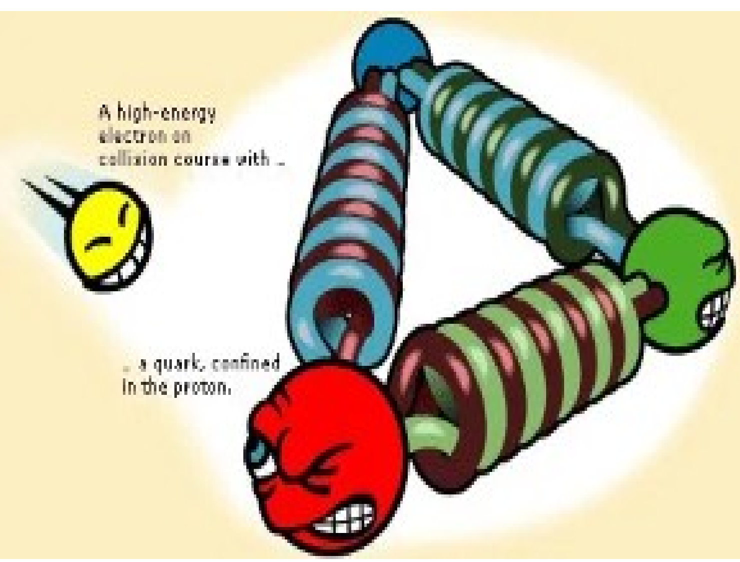
QCD: Basic Facts

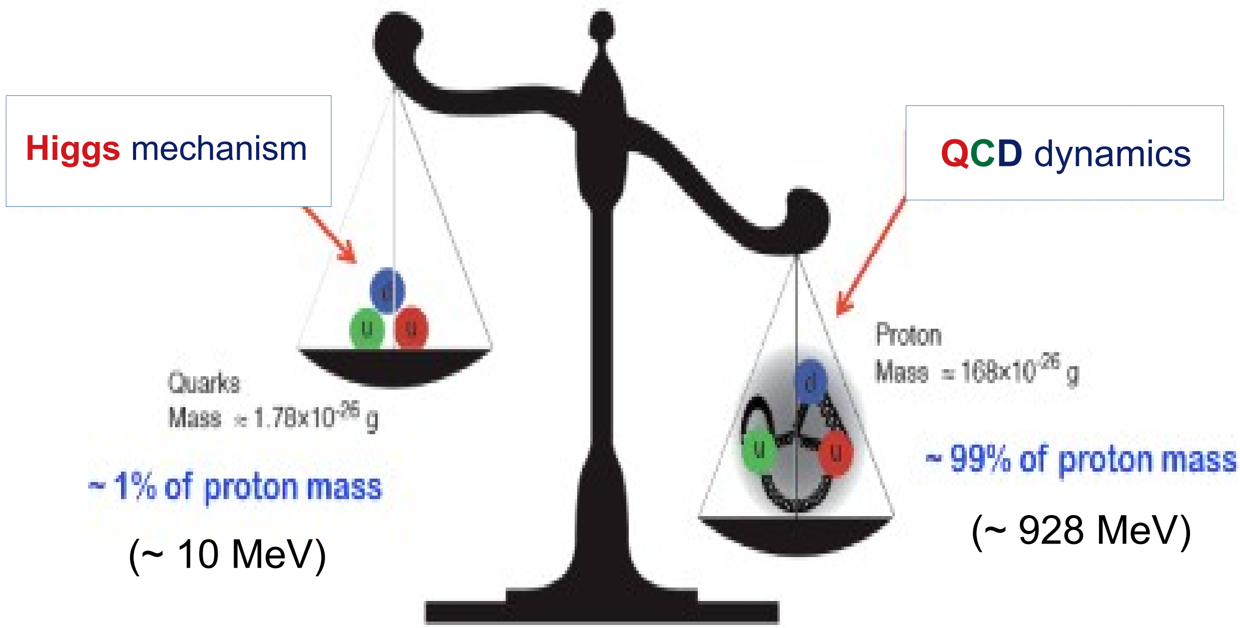
- **QCD** is characterized by two **emergent** phenomena: confinement and dynamical generation of mass (DGM).

$$\mathcal{L}_{\rm QCD} = \sum_{j=u,d,s,\ldots} \bar{q}_j[\gamma_\mu D_\mu + m_j]q_j + \tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu},$$
$$D_\mu = \partial_\mu + ig\tfrac{1}{2}\lambda^a A^a_\mu,$$
$$G^a_{\mu\nu} = \partial_\mu A^a_\nu + \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu,$$

- Quarks and gluons not *isolated* in nature.
  - Formation of colorless bound states: "**Hadrons**"
  - **1-fm scale** size of hadrons?

- Emergence of hadron masses (EHM) from QCD **dynamics**





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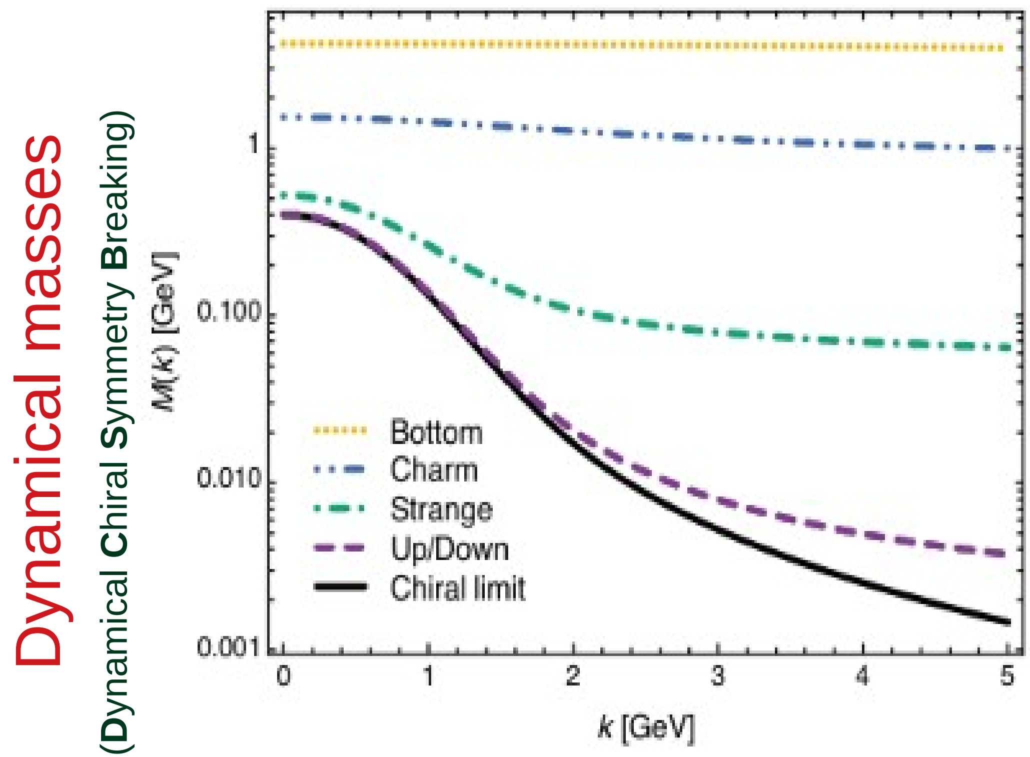
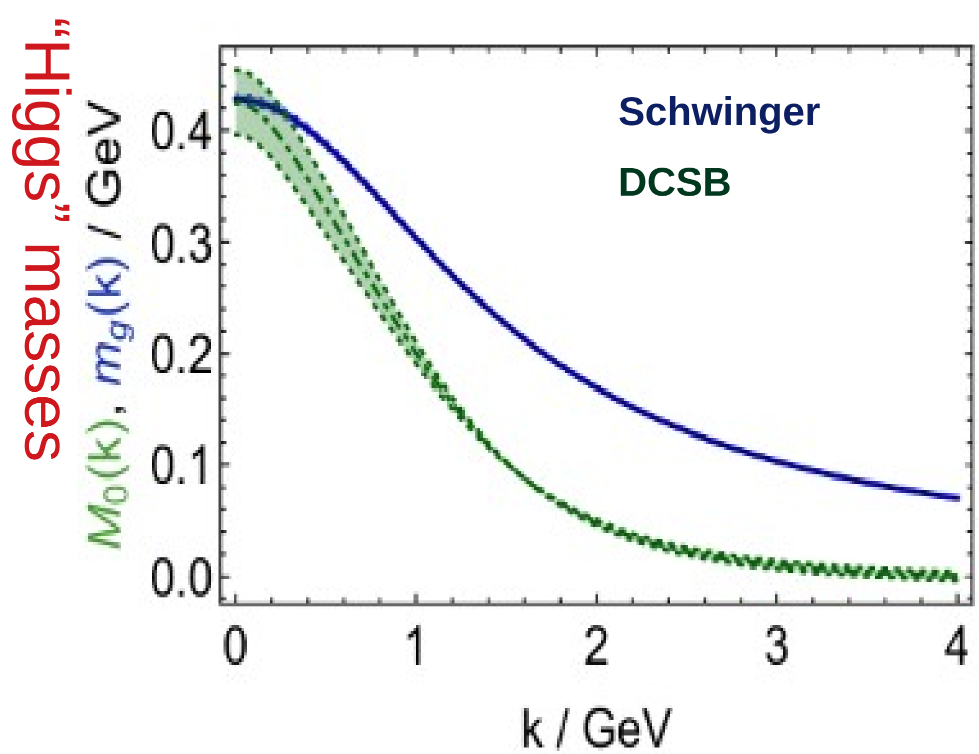
Can we trace them down to fundamental d.o.f ?

- Emergence of hadron masses (EHM) from QCD **dynamics**



$$S_f^{-1}(p) = Z_f^{-1}(p^2)(i\gamma\cdot p + \mathrm{M_f(p^2)})$$

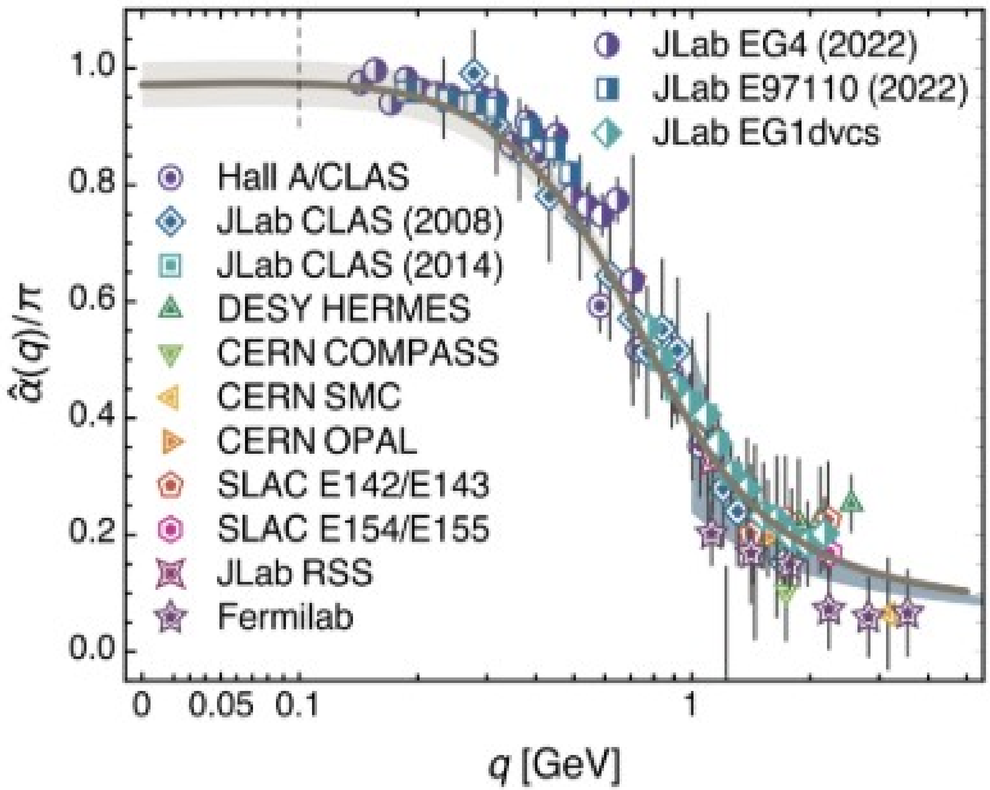


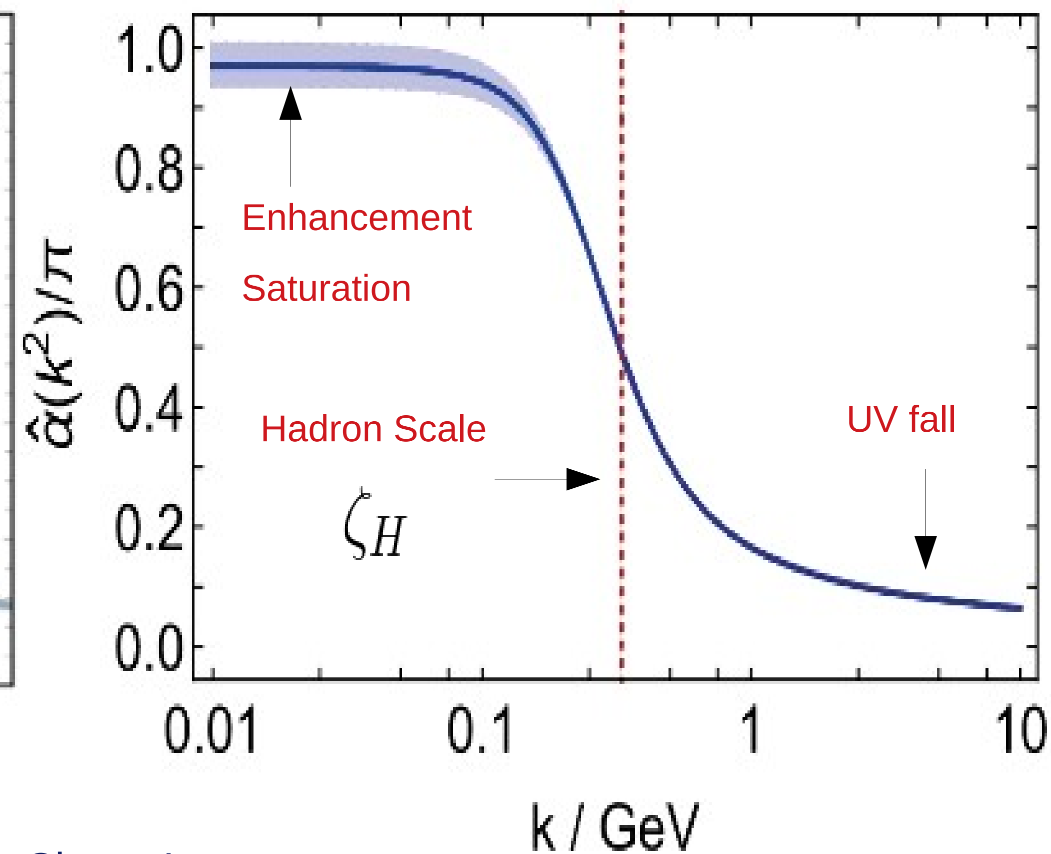
Gluon and quark *running masses*

QCD: Basic Facts

- **Confinement** and the **EHM** are tightly connected with **QCD's running coupling**.

(figure: D. Binosi's courtesy!)





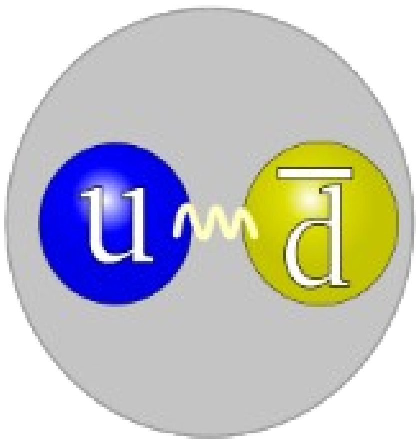
Modern picture of **QCD** coupling. 'Effective Charge'

Combined continuum + QCD lattice analysis

ζ_H : Fully **dressed valence** quarks express all hadron's properties

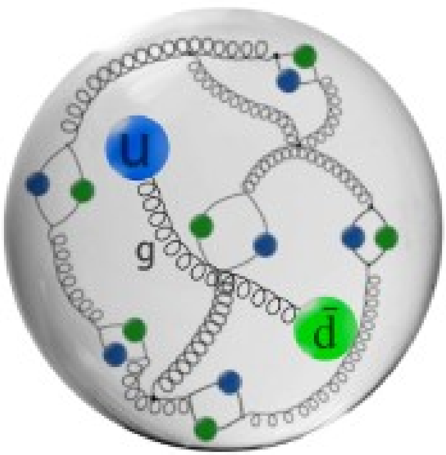
Parton distributions: energy scales

ζ_H



Resolution Scale

$\zeta > \zeta_H$



- **Fully-dressed valence quarks**

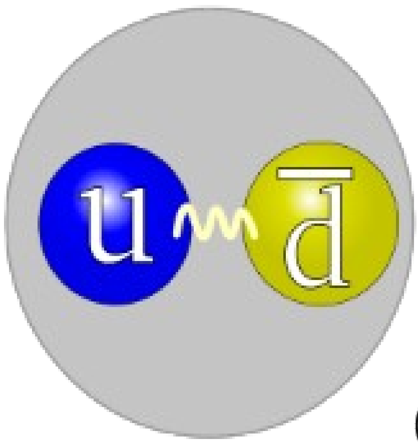
(quasiparticles)

- **Unveiling of glue and sea d.o.f.**

(partons)

Parton distributions: energy scales

ζ_H



$(M_u = M_d)$

- **Fully-dressed valence quarks**
  - At this scale, **all properties** of the hadron are contained within their valence quarks.
  - **QCD constraints** are defined from here (e.g. large-x behavior of the PDF)

$$u^\pi(x;\zeta) \overset{x\simeq 1}{\approx} (1-x)^{\beta\,=\,2+\gamma(\zeta)}$$

Parton distributions: energy scales

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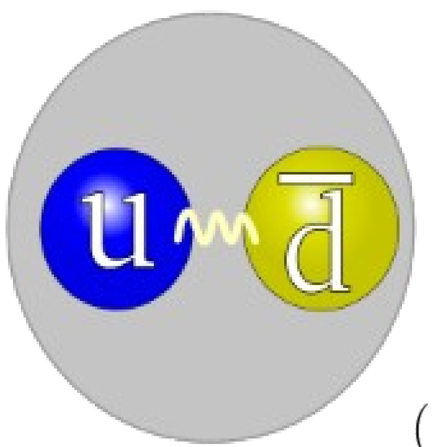

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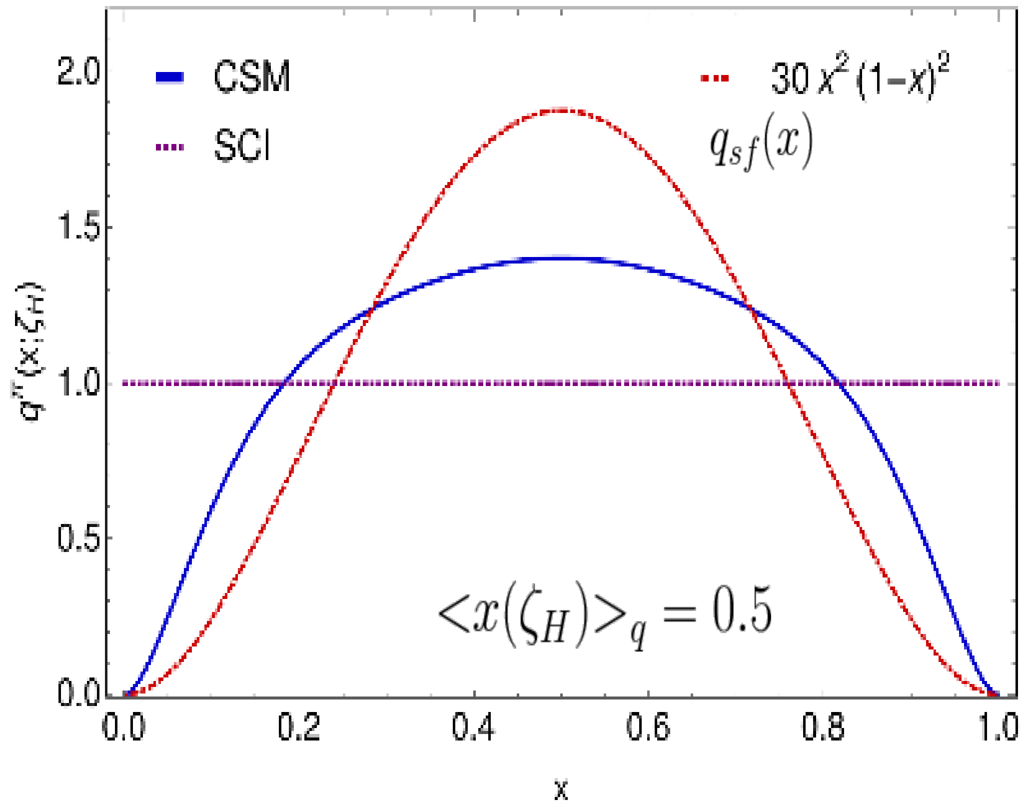

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- **CSM** results produce:
  - **EHM-induced** dilated distributions
  - Soft end-point behavior

Cui:2020tdf

Parton distributions: energy scales

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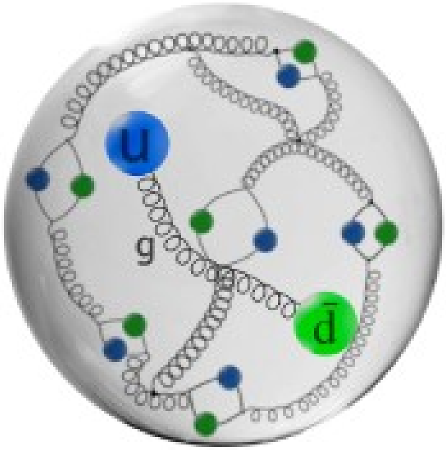
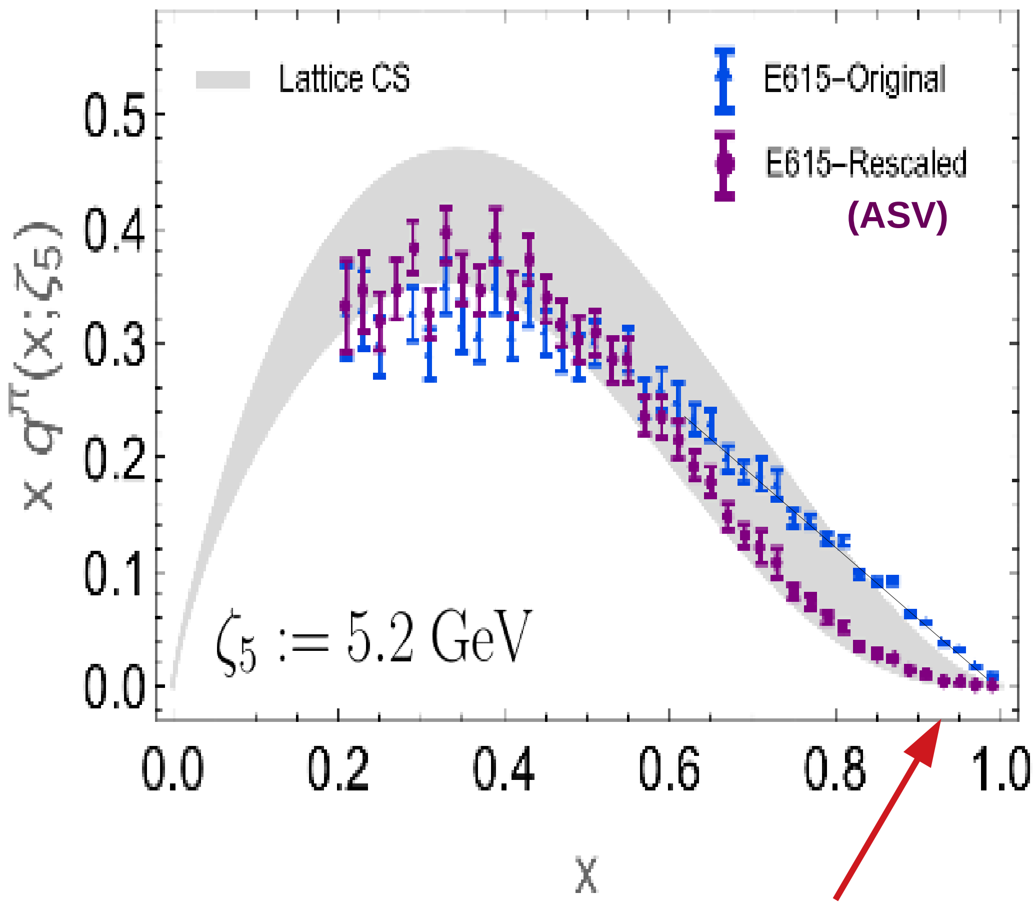


- **Unveiling of glue and sea d.o.f.**

- **Experimental** data is given **here**.
- The interpretation of parton distributions from cross sections demands **special care**.
- In addition, the synergy with **lattice QCD** and phenomenological approaches is welcome.

Parton distributions: energy scales

Conway:1989fs Aicher:2010cb Sufian:2020vzb



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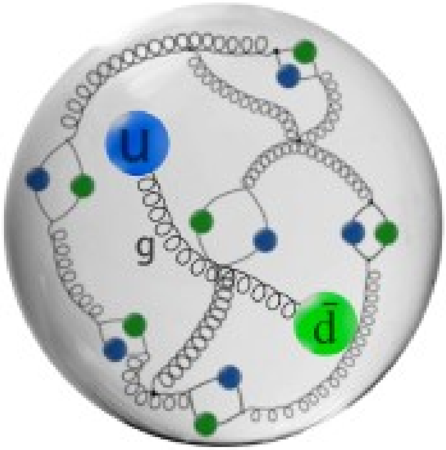

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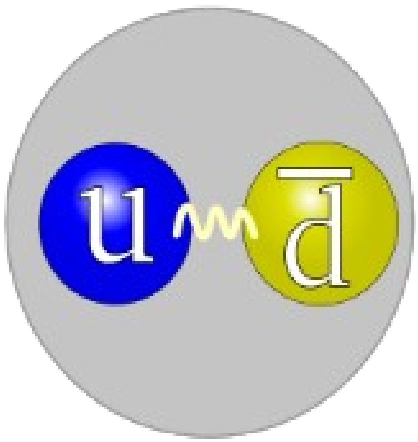
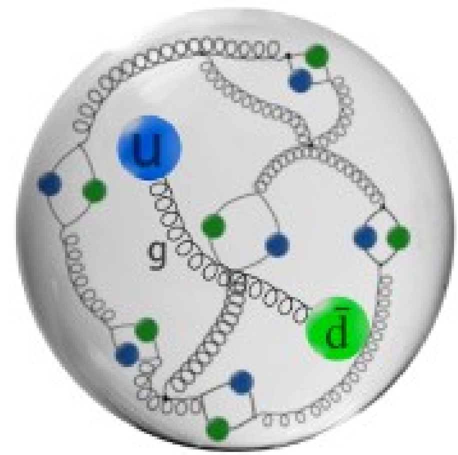


Resolution Scale

Evolution equations



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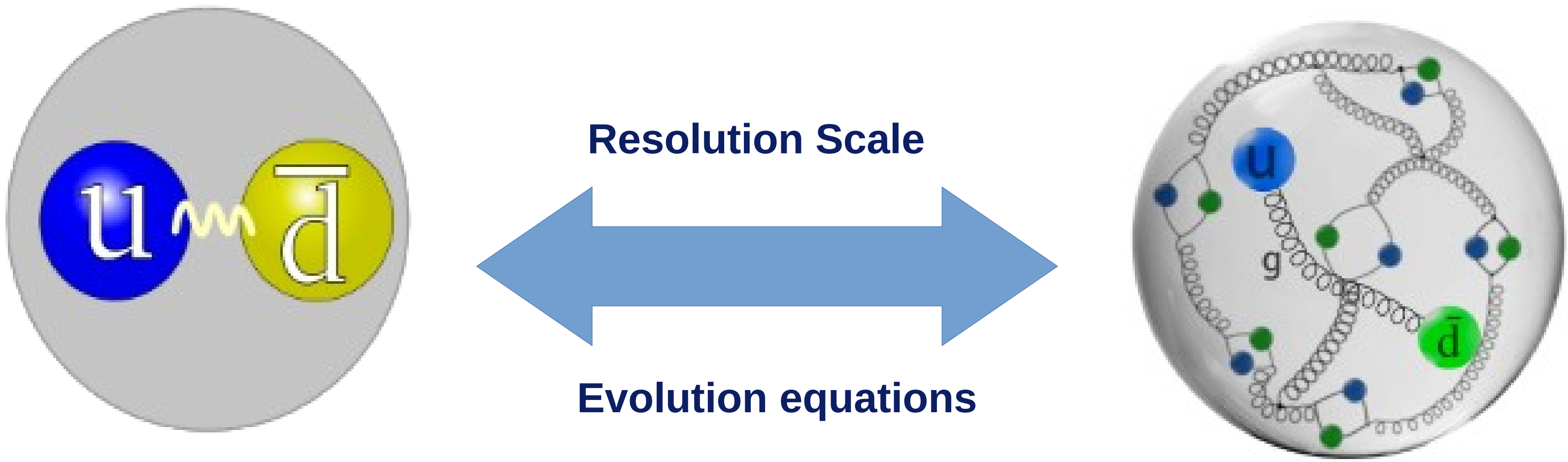

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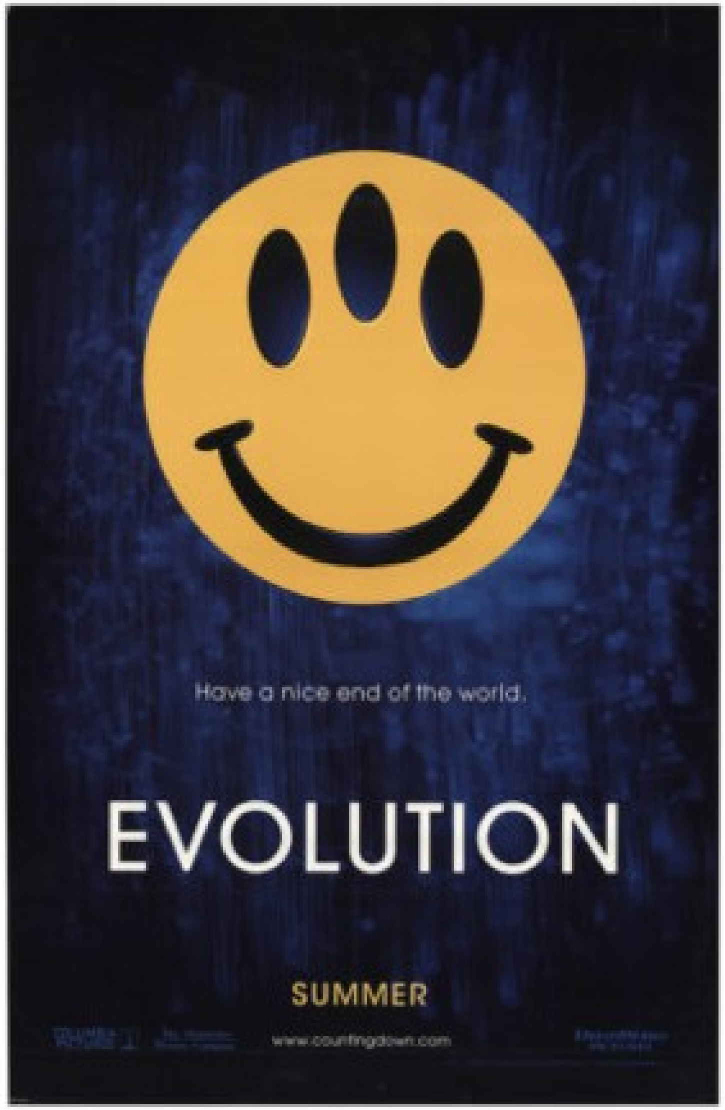
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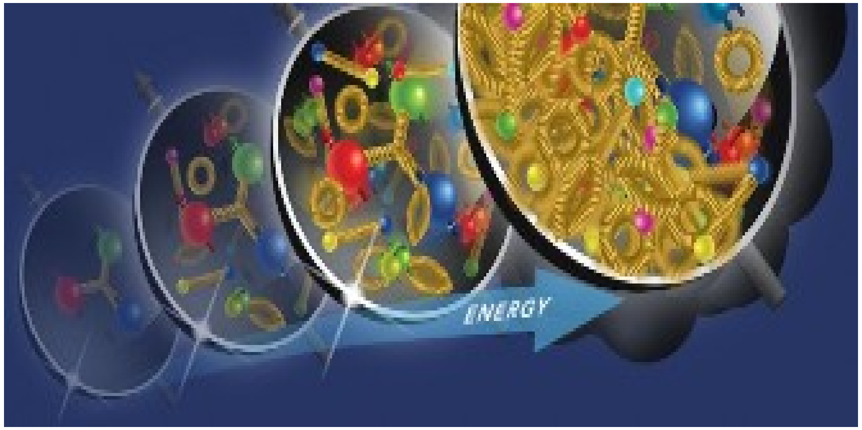
DGLAP: All orders evolution

Raya:2021zrz
Cui:2020tdf

$$\left\{\zeta^2 \frac{d}{d\zeta^2}\int_0^1 dy\delta(y-x) - \frac{\alpha(\zeta^2)}{4\pi}\int_x^1 \frac{dy}{y}\begin{pmatrix} P_{qq}^{\rm NS}\left(\frac{x}{y}\right) & 0 \\ 0 & \mathbf{P}^{\rm S}\left(\frac{\mathbf{x}}{\mathbf{y}}\right)\end{pmatrix}\right\}\begin{pmatrix} H_\pi^{\rm NS,+}(y,t;\zeta) \\ \mathbf{H}_\pi^{\rm S}(y,t;\zeta)\end{pmatrix} = 0$$

DGLAP leading-order evolution equations



DGLAP: All orders evolution

Assumption: define an **effective** charge such that

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Starting from fully-dressed **quasiparticles**, at ζ_H → **Sea** and **Gluon** content unveils, as prescribed by **QCD**

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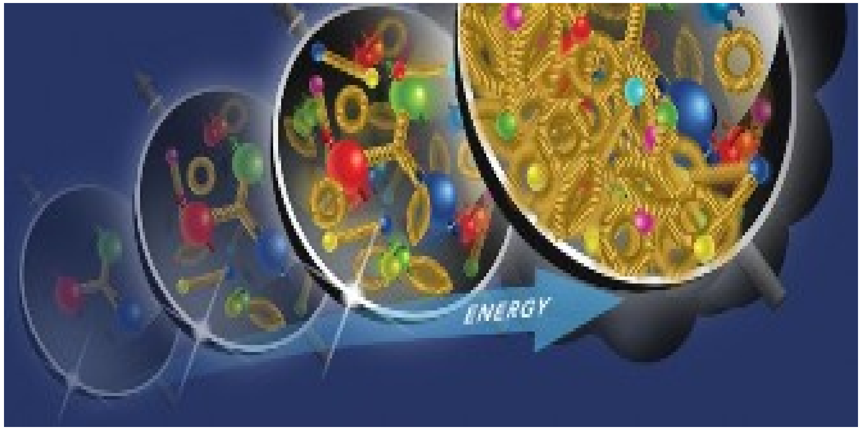
DGLAP ~~leading order~~ evolution equations



- **Not** the **LO** QCD coupling but an **effective** one.
- Making this equation **exact**.
- Connecting with the **hadron scale**, at which the **fully-dressed** valence-**quarks** express **all** of the hadron's properties.

(thus carrying all the momentum)

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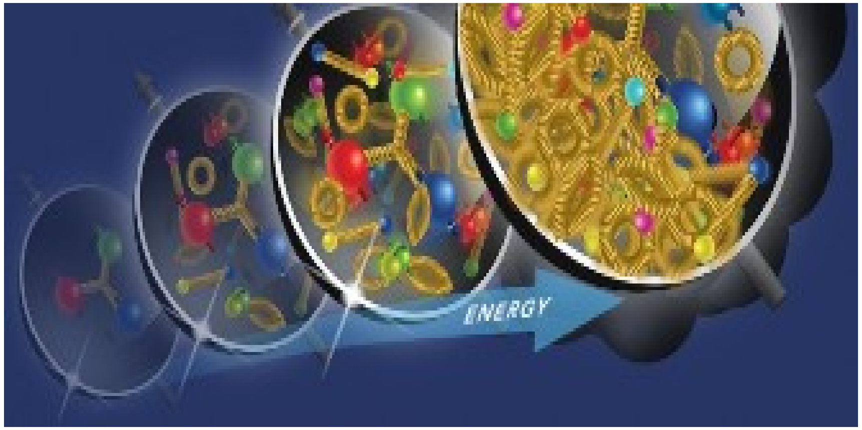
$$\left\{ \zeta^2 \frac{d}{d\zeta^2} \mathbb{1} + \frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma_{qq}^{(n)} & 0 & 0 \\ 0 & \gamma_{qq}^{(n)} & 2n_f \gamma_{qg}^{(n)} \\ 0 & \gamma_{gq}^{(n)} & \gamma_{gg}^{(n)} \end{pmatrix} \right\} \begin{pmatrix} \langle x^n \rangle_{\rm NS}(\zeta) \\ \langle x^n \rangle_{\rm S}(\zeta) \\ \langle x^n \rangle_g(\zeta) \end{pmatrix} = 0$$

DGLAP ~~leading-order~~ evolution equations

$$\gamma_{AB}^{(n)} = -\int_0^1 dx\, x^n P_{AB}^C(x)$$



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PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\mathrm{H}}(x) = q_{\mathrm{H}}(x) + \bar{q}_{\mathrm{H}}(x)$$

singlet combination

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Valence-quark PDF in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_H}$$

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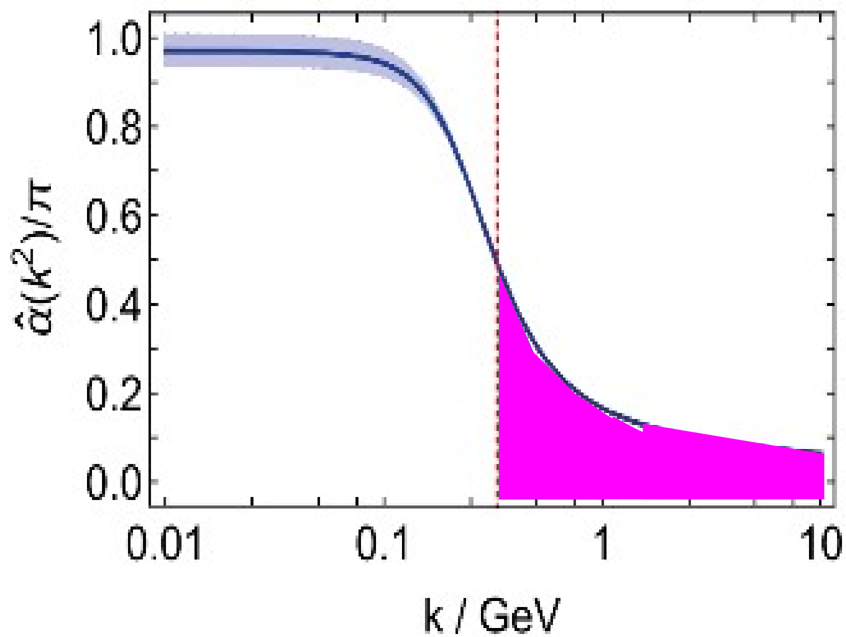

Moments' evolution is controlled by the integrated "strength" of the coupling beyond the hadron scale

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$$\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

The ratio of lightcone momentum fractions encodes the required information of the charge

DGLAP: All orders evolution

Cui:2020tdf

Implication 1: valence-quark PDF

$$\langle x^n \rangle_{q_H}^{\zeta} = \langle x^n \rangle_{q_H}^{\zeta_H} \exp\left(-\frac{\gamma_{qq}^n}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right) = \langle x^n \rangle_{q_H}^{\zeta_H} \left[\frac{\langle x \rangle_{q_H}^{\zeta}}{\langle x \rangle_{q_H}^{\zeta_H}} \right]^{\gamma_{qq}^n / \gamma_{qq}}$$

This ratio encodes the information of the charge

Direct connection bridging from hadron to experimental scale: only one input is needed to evolve "all" the Mellin moments up and reconstruct the PDF.

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Direct connection bridging from hadron to experimental scale: only one input is needed to evolve "all" the Mellin moments up and reconstruct the PDF.

This ratio encodes the information of the charge and use isospin symmetry (pion case)

$$\langle x \rangle_{u_\pi}^{\zeta_H} = \langle x \rangle_{d_\pi}^{\zeta_H} = \frac{1}{2}$$

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Capitalizing on the Mellin moments of asymptotically large order:

$$q(x;\zeta) \underset{x\to 1}{\sim} (1-x)^{\beta(\zeta)}\left(1+\mathcal{O}(1-x)\right)$$

$$\beta(\zeta) = \beta(\zeta_H) + \frac{3}{2}\ln\frac{\langle x(\zeta_H)\rangle}{\langle x(\zeta)\rangle}$$

DGLAP: All orders evolution

Cui:2020tdf

Implication 1: valence-quark PDF

$$\langle x^n \rangle^{\zeta}_{q_\pi} = \langle x^n \rangle^{\zeta_H}_{q_\pi} \exp\left(-\frac{\gamma^n_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right) = \langle x^n \rangle^{\zeta_H}_{q_\pi} \left[\langle 2x \rangle^{\zeta}_{q_\pi} \right]^{\gamma^n_{qq}/\gamma_{qq}}$$

Direct connection bridging from hadron to experimental scale: only one input is needed to evolve "all" the Mellin moments up and reconstruct the PDF.

This ratio encodes the information of the charge and use isospin symmetry (pion case)

$$\langle x \rangle^{\zeta_H}_{u_\pi} = \langle x \rangle^{\zeta_H}_{d_\pi} = \frac{1}{2}$$

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Under a sensible assumption at large momentum scale:

$$q(x;\zeta) \underset{x\to 0}{\sim} x^{\alpha(\zeta)} \left(1 + \mathcal{O}(x)\right)$$

$$1 + \alpha(\zeta) = \frac{3}{2} \langle x(\zeta) \rangle \ln \frac{\langle x(\zeta_H) \rangle}{\langle x(\zeta) \rangle} + \beta(\zeta_H) \langle x(\zeta) \rangle + \mathcal{O}\left(\frac{\langle x(\zeta) \rangle}{\ln \frac{\langle x(\zeta_H) \rangle}{\langle x(\zeta) \rangle}} \right)$$

DGLAP: All orders evolution

Cui:2020tdf

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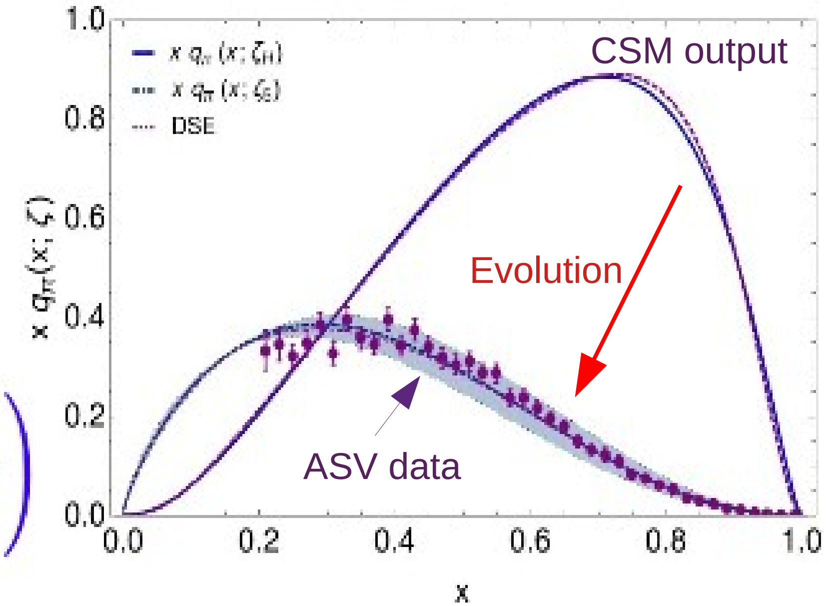
Reconstruction after evolving:



DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta_H}_{u_\pi} = \frac{1}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \begin{pmatrix} 2(n+1) \\ j \end{pmatrix} \langle x^j\rangle^{\zeta_H}_{u_\pi}$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{(\langle 2x\rangle^{\zeta}_{u_\pi})^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} (\langle 2x\rangle^{\zeta}_{u_\pi})^{-\gamma_0^j/\gamma_0^1}.$$

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DGLAP: All orders evolution

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- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.
- Thus arriving at the recurrence **relation** on the left which is satisfied if, and only if, the source distribution is related by evolution to a symmetric one at the initial scale .

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{(\langle 2x\rangle^{\zeta}_{u_\pi})^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} (\langle 2x\rangle^{\zeta}_{u_\pi})^{-\gamma_0^j/\gamma_0^1}.$$

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Reported lattice moments agree very well with the recursion formula

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7		0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

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Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.

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1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7	0.0065(24)	0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

DGLAP: All orders evolution

Implication 2: recursion of Mellin moments (pion case)

$$\langle x^{2n+1}\rangle^{\zeta}_{u_\pi} = \frac{\left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{\gamma_0^{2n+1}/\gamma_0^1}}{2(n+1)} \times \sum_{j=0,1,\ldots}^{2n} (-)^j \binom{2(n+1)}{j} \langle x^j\rangle^{\zeta}_{u_\pi} \left(\langle 2x\rangle^{\zeta}_{u_\pi}\right)^{-\gamma_0^j/\gamma_0^1}.$$

- Since **isospin symmetry** limit implies:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Odd** moments can be expressed in terms of previous **even** moments.
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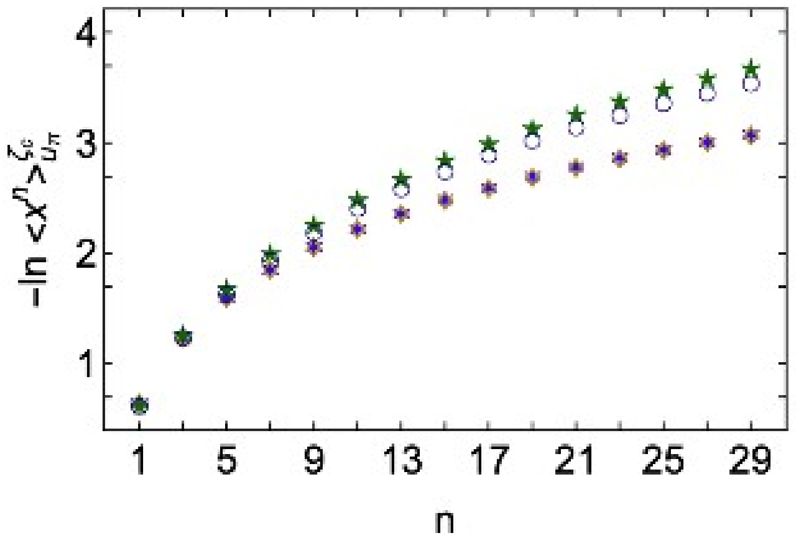
Reported lattice moments agree very well with the recursion formula and so also does and estimate for the 7-th moment from lattice reconstruction.

Moments from global fits can be also compared to the estimated from recursion !

n	$\langle x^n\rangle^{\zeta_5}_{u_\pi}$ Ref. [99]	Eq. (17)
1	0.230(3)(7)	0.230
2	0.087(5)(8)	0.087
3	0.041(5)(9)	0.041
4	0.023(5)(6)	0.023
5	0.014(4)(5)	0.015
6	0.009(3)(3)	0.009
7	0.0065(24)	0.0078

[99] C. Alexandrou et al., PRD104(2021)054504

Moments computed from: P. Barry et al., PRL127(2021)232001



DGLAP: All orders evolution

Implication 3: physical bounds (pion case).

$$\langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1}$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

DGLAP: All orders evolution

Implication 3: physical bounds (pion case).

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1}$$

↑

$$q(x;\zeta_H) = \delta(x - 1/2)$$

- Keeping **isospin symmetry,** implying:

$$q(x;\zeta_H) = q(1-x;\zeta_H)$$

- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.

DGLAP: All orders evolution

Implication 3: physical bounds (pion case).

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

$$q(x;\zeta_H) = \delta(x-1/2) \qquad q(x;\zeta_H) = 1$$

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- **Lower bound** is imposed by considering the limit of a system of two strongly massive and maximally correlated) partons: both carry half of the momentum.
- **Upper bound** comes out from considering the opposite limit of a weekly interacting system of two (then fully decorrelated) partons: all the momentum fractions are equally probable.

DGLAP: All orders evolution

Implication 3: physical bounds (pion case).

$$\frac{1}{2^n} \leq \langle x^n \rangle^{\zeta}_{u_\pi} \left(\langle 2x \rangle^{\zeta}_{u_\pi} \right)^{-\gamma_0^n/\gamma_0^1} \leq \frac{1}{1+n}$$

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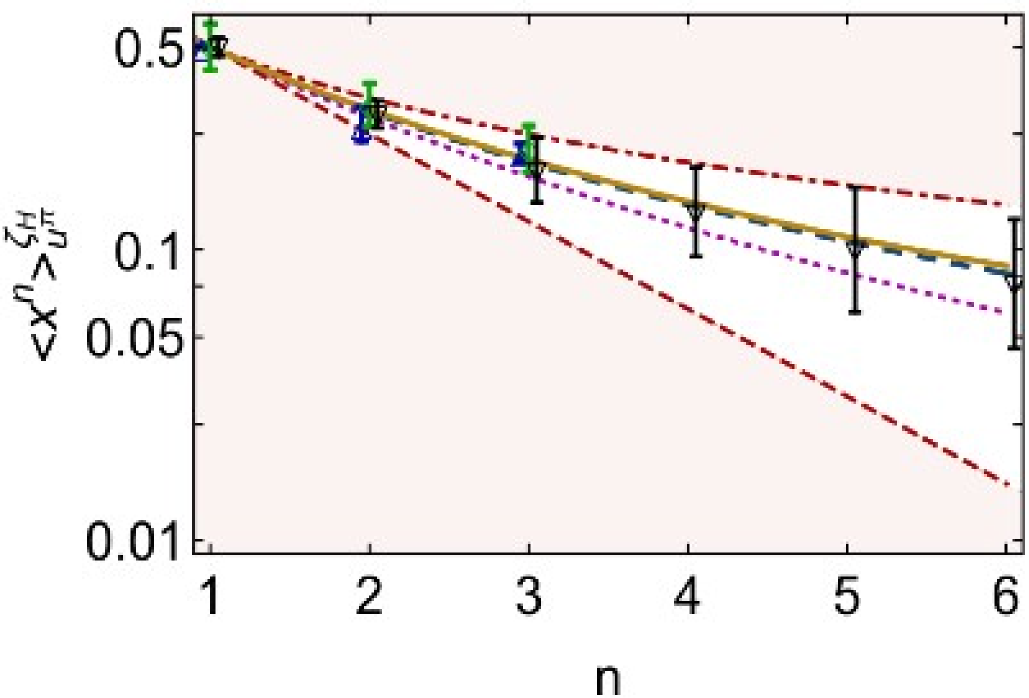

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	Joo:2019bzr	Sufian:2019bol	Alexandrou:2021mmi
n	[61]	[62]	[63]
1	0.254(03)	0.18(3)	0.23(3)(7)
2	0.094(12)	0.064(10)	0.087(05)(08)
3	0.057(04)	0.030(05)	0.041(05)(09)
4			0.023(05)(06)
5			0.014(04)(05)
6			0.009(03)(03)

Lattice moments verifying the **recurrence relation** too.

DGLAP: All orders evolution

Cui:2020tdf

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

$$\Sigma^q_{\mathrm{H}}(x) = q_{\mathrm{H}}(x) + \bar{q}_{\mathrm{H}}(x)$$

singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\mathrm{H}}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathrm{H}}(y) + 2P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathrm{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}$$

Hard-wall threshold

$$\mathcal{P}^{\zeta}_q = \theta(\zeta - M_q)$$

Quark singlet and glue PDFs in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + 2\mathcal{P}^{\zeta}_q \gamma^n_{qg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

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$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + 2 P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathsf{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\mathsf{H}}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathsf{H}}(y) \right\}$$

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Sea-quark PDF

$$\langle x^n \rangle^{\zeta}_{S^q_H} = \langle x^n \rangle^{\zeta}_{\Sigma^q_H} - \langle x^n \rangle^{\zeta}_{q_H}$$

DGLAP: All orders evolution

Cui:2020tdf

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions:

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singlet combination

$$\zeta^2 \frac{d}{d\zeta^2} q_{\rm H}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\rm H}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma^q_{\rm H}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\rm H}(y) + 2 P^{\zeta}_{q\leftarrow g}\left(\frac{x}{y}\right) g_{\rm H}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\rm H}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma^q_{\rm H}(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\rm H}(y) \right\}$$

Hard-wall threshold

$$\mathcal{P}^{\zeta}_q = \theta(\zeta - M_q)$$

Quark singlet and glue PDFs in Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^{\zeta}_{\Sigma_H} + 2 n_f \mathcal{P}^{\zeta}_q \gamma^n_{ug} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \langle x^n \rangle^{\zeta}_{\Sigma_H} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_H} \right\}$$

Sea-quark PDF

$$\langle x^n \rangle^{\zeta}_{S^q_H} = \langle x^n \rangle^{\zeta}_{\Sigma^q_H} - \langle x^n \rangle^{\zeta}_{q_H}$$

Full singlet and sea

$$\langle x^n \rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x^n \rangle^{\zeta}_{\Sigma^q_H}, \quad \langle x^n \rangle^{\zeta}_{S_H} = \sum_q \langle x^n \rangle^{\zeta}_{S^q_H}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H, \ \forall q$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

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$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\beta^n_{\Sigma g} = -\frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

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$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

$$\lambda^n_\pm = \frac{1}{2} \mathrm{Tr}\left(\Gamma^n\right) \pm \sqrt{\frac{1}{4} \mathrm{Tr}^2\left(\Gamma^n\right) - \mathrm{Det}\left(\Gamma^n\right)}$$

$$\beta^n_{\Sigma g} = -\frac{2n_f \gamma^n_{ug}}{\lambda^n_+ - \lambda^n_-}$$

$$S^n_\pm = \left[S(\zeta_H, \zeta) \right]^{\lambda^n_\pm / \gamma_{uu}}$$

$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

DGLAP: All orders evolution

Implication 4: glue and sea from valence

$M_q = \zeta_H, \ \forall q$

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$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+\right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+\right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ 0 \end{pmatrix}$$

$$\alpha^n_\pm = \pm \frac{\lambda^n_\pm - \gamma^n_{uu}}{\lambda^n_+ - \lambda^n_-}$$

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DGLAP: All orders evolution

Implication 4: glue and sea from valence

$$M_q = \zeta_H, \; \forall q$$

All quarks active

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \gamma^n_{uu} & 2n_f \gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix}$$

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In terms of the moments for the sum of all valence-quark distributions at hadronic scale

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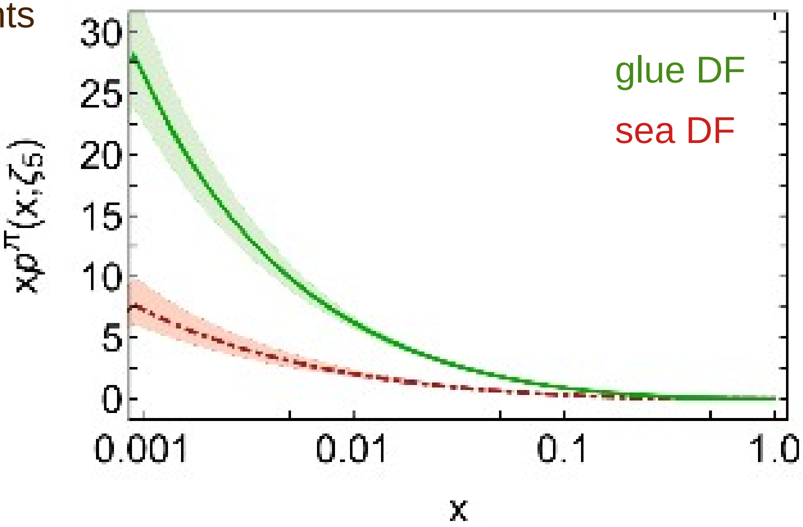
Compute all the moments and reconstruct:



DGLAP: All orders evolution

Implication 4: glue and sea from valence

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$$\beta^n_{g\Sigma} = \frac{(\lambda^n_+ - \gamma^n_{uu})(\lambda^n_- - \gamma^n_{uu})}{2n_f \gamma^n_{ug} (\lambda^n_+ - \lambda^n_-)}$$

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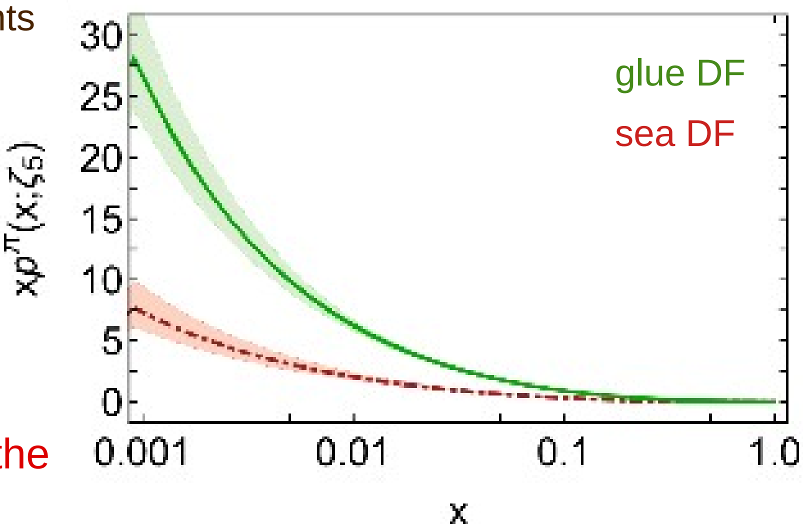

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DGLAP: All orders evolution

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n=1 case

$n_f = 4$

$$\langle x \rangle^{\zeta}_{\Sigma_H} = \sum_q \langle x \rangle^{\zeta}_{q_H} + \langle x \rangle^{\zeta}_{S_H} = \frac{3}{7} + \frac{4}{7} \left[S(\zeta_H, \zeta) \right]^{7/4}$$

$$\langle x \rangle^{\zeta}_{g_H} = \frac{4}{7} \left(1 - \left[S(\zeta_H, \zeta) \right]^{7/4} \right)$$

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n=1 case

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$$\langle x \rangle^{\zeta}_{\Sigma_\pi} = \langle 2x \rangle^{\zeta}_{q_\pi} + \langle x \rangle^{\zeta}_{S_\pi} = \frac{3}{7} + \frac{4}{7} \left[S(\zeta_H, \zeta) \right]^{7/4}$$

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ζ_5	$\langle 2x \rangle^{\pi}_{q}$	$\langle x \rangle^{\pi}_{g}$	$\langle x \rangle^{\pi}_{\text{sea}}$
Ref.[55]	0.412(36)	0.449(19)	0.138(17)
Herein	0.40(4)	0.45(2)	0.14(2)

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Z-F. Cui et al., arXiv:2006.1465

DGLAP: All orders evolution

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Z-F. Cui et al., arXiv:2006.1465
R.S. Sufian et al., arXiv:2001.04960

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n=1 case

$n_f = 4$

$$\langle x \rangle^{\zeta}_{\Sigma_H} \underset{\zeta^2 \to \infty}{=} \langle x \rangle^{\zeta}_{S_H} \underset{\zeta^2 \to \infty}{=} \frac{3}{7}$$

$$\langle x \rangle^{\zeta}_{g_H} \underset{\zeta^2 \to \infty}{=} \frac{4}{7}$$

Asymptotic limit: G. Altarelli, Phys. Rep. 81, 1 (1982)

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n=1 case
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owing to $\lambda^n_\pm > 0$

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n=1 case
$n_f = 4$

$$\Sigma_H(x) \underset{\zeta^2 \to \infty}{=} \frac{3}{7} \frac{\delta(x)}{x}$$

$$g_H(x) \underset{\zeta^2 \to \infty}{=} \frac{4}{7} \frac{\delta(x)}{x}$$

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DGLAP: All orders evolution

Implication 5: correlating glue and sea

$M_q = \zeta_H, \ \forall q$

All quarks active

$$\begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = \begin{pmatrix} \alpha^n_+ S^n_- + \alpha^n_- S^n_+ & \beta^n_{\Sigma g} \left(S^n_- - S^n_+ \right) \\ \beta^n_{g\Sigma} \left(S^n_- - S^n_+ \right) & \alpha^n_- S^n_- + \alpha^n_+ S^n_+ \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta_H}_{\Sigma_H} \\ \langle x^n \rangle^{\zeta_H}_{g_H} \end{pmatrix}$$

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All quarks active

$$\begin{pmatrix} \alpha_+^n \left[S_-^n\right]^{-1} + \alpha_-^n \left[S_+^n\right]^{-1} & \beta_{\Sigma g}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) \\ \beta_{g\Sigma}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) & \alpha_-^n \left[S_-^n\right]^{-1} + \alpha_+^n \left[S_+^n\right]^{-1} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta_H} \\ \langle x^n \rangle_{g_H}^{\zeta_H} \end{pmatrix}$$

The equation can be easily inverted

DGLAP: All orders evolution

Implication 5: correlating glue and sea

$M_q = \zeta_H, \ \forall q$

All quarks active

$$\begin{pmatrix} \alpha_+^n \left[S_-^n\right]^{-1} + \alpha_-^n \left[S_+^n\right]^{-1} & \beta_{\Sigma g}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) \\ \beta_{g\Sigma}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) & \alpha_-^n \left[S_-^n\right]^{-1} + \alpha_+^n \left[S_+^n\right]^{-1} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta_H} \\ 0 \end{pmatrix}$$

The equation can be easily inverted and, relying on the hadronic scale definition, delivers a constraint for all Mellin moments of glue and sea at any experimental scale:

$$\frac{\langle x^n \rangle_{\Sigma_\pi}^{\zeta}}{\langle x^n \rangle_{g_\pi}^{\zeta}} = \frac{\langle x^n \rangle_{\mathcal{S}_\pi}^{\zeta} + \langle 2x^n \rangle_{u_\pi}^{\zeta}}{\langle x^n \rangle_{g_\pi}^{\zeta}} = \frac{\alpha_-^n S_+^n + \alpha_+^n S_-^n}{\beta_{g\Sigma}^n \left(S_-^n - S_+^n\right)}$$

DGLAP: All orders evolution

Implication 5: correlating glue and sea

$$M_q = \zeta_H,\ \forall q$$

All quarks active

$$\begin{pmatrix} \alpha_+^n \left[S_-^n\right]^{-1} + \alpha_-^n \left[S_+^n\right]^{-1} & \beta_{\Sigma g}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) \\ \beta_{g\Sigma}^n \left(\left[S_-^n\right]^{-1} - \left[S_+^n\right]^{-1}\right) & \alpha_-^n \left[S_-^n\right]^{-1} + \alpha_+^n \left[S_+^n\right]^{-1} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta} \\ \langle x^n \rangle_{g_H}^{\zeta} \end{pmatrix} = \begin{pmatrix} \langle x^n \rangle_{\Sigma_H}^{\zeta_H} \\ 0 \end{pmatrix}$$

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$$\frac{\langle x \rangle_{\Sigma_\pi}^{\zeta}}{\langle x \rangle_{g_\pi}^{\zeta}} = \frac{\langle x \rangle_{S_\pi}^{\zeta} + \langle 2x \rangle_{u_\pi}^{\zeta}}{\langle x \rangle_{g_\pi}^{\zeta}} = \frac{\frac{3}{4} + \left(\langle 2x \rangle_{u_\pi}^{\zeta}\right)^{7/4}}{1 - \left(\langle 2x \rangle_{u_\pi}^{\zeta}\right)^{7/4}}$$

	$\langle 2x \rangle_{u_\pi}^{\zeta}$	$\langle x \rangle_{S_\pi}^{\zeta}$	$\langle x \rangle_{g_\pi}^{\zeta}$	$n_f = 4$ $\langle x \rangle_{\Sigma_\pi}^{\zeta_H}$	$\langle x \rangle_{g_\pi}^{\zeta_H}$
NLO	0.53(2)	0.14(4)	0.34(6)	1.15(14)	-0.14(13)
NLL-Cos	0.47(2)	0.14(5)	0.39(6)	1.11(16)	-0.11(16)
NLL-Exp	0.46(2)	0.16(5)	0.38(6)	1.15(12)	-0.14(13)
NLL-dM	0.46(3)	0.15(7)	0.40(5)	1.12(22)	-0.11(18)

All-orders DGLAP: hard-wall thresholds

Let us now solve generally the hard-wall model of **massless partons** with **hard-wall thresholds** for each flavor activation, that can be analytically solved!

$$\gamma^n_{qq} = \gamma^n_{uu},\ \gamma^n_{gq} = \gamma^n_{gu},\ \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{uu} \langle x^n \rangle^\zeta_{q_H}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^\zeta_{\Sigma^q_H} + 2\theta(\zeta - M_q) \gamma^n_{ug} \langle x^n \rangle^\zeta_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \sum_q \langle x^n \rangle^\zeta_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^\zeta_{g_H} \right\}$$

Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$

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$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma^{u+d}_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma^n_{uu} & 4\gamma^n_{ug} \\ \gamma^n_{gu} & \gamma^n_{gg} \end{pmatrix} \begin{pmatrix} \langle x^n \rangle^{\zeta}_{\Sigma^{u+d}_H} \\ \langle x^n \rangle^{\zeta}_{g_H} \end{pmatrix} - \frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} 0 \\ \gamma^n_{gu} \langle x^n \rangle^{\zeta}_{\Sigma^s_H} \end{pmatrix}$$

Consider, for the sake of simplicity, three flavors and $\zeta \leq M_s$, such that the singlet combinations can be rearranged and the strange decoupled from the light flavors.

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In pion's (proton's) case $\langle x \rangle^{\zeta_H}_{s_\pi} = 0$

$$\langle x \rangle^{\zeta}_{\Sigma^s_\pi} \equiv 0$$

$$\begin{pmatrix} \langle x \rangle^{\zeta}_{\Sigma^{u+d}_\pi} \\ \langle x \rangle^{\zeta}_{g_\pi} \end{pmatrix} = \begin{pmatrix} \frac{3}{11} + \frac{8}{11} [S(\zeta_H, \zeta]^{11/8} \\ \frac{8}{11} \left(1 - [S(\zeta_H, \zeta]^{11/8}\right) \end{pmatrix}$$

$$S(\zeta_H, \zeta) = \exp\left(-\frac{\gamma_{uu}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2)\right)$$

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In kaon's case (after some algebra) $\langle x \rangle^{\zeta_H}_{s_K} = s_0$

$$\langle x \rangle^{\zeta}_{\Sigma^s_K} = s_0\, S(\zeta_H, \zeta)$$

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In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

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\end{aligned}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

$$\langle x \rangle^\zeta_{q_H} = \langle x \rangle^{\zeta_H}_{q_H} S(\zeta_H, \zeta)$$

$$\langle x \rangle^\zeta_{g_H} = \frac{4}{7} - \tau(M_s, M_c) \left[\langle 2x \rangle^\zeta_{u_\pi} \right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-7/4} - \frac{24}{275} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-3/16} \left[\langle 2x \rangle^{M_s}_{u_\pi} \right]^{-25/16} + \frac{8}{11} \left[\langle 2x \rangle^{M_c}_{u_\pi} \langle 2x \rangle^{M_s}_{u_\pi} \right]^{-3/16}$$

Capitalizing on the universality of the effective charge, **all hadrons'** momentum fraction averages can be expressed in terms of **pion's** ones.

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3 (always) active flavors

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$$q = u, d, s, c$$

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\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle_{g_H}^{\zeta} &= -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma_{gu}^n \sum_q \langle x^n \rangle_{\Sigma_H^q}^{\zeta} + \gamma_{gg}^n \langle x^n \rangle_{g_H}^{\zeta} \right\}
\end{aligned}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

$$\langle x \rangle_{q_H}^{\zeta} = \langle x \rangle_{q_H}^{\zeta_H} S(\zeta_H, \zeta) \qquad \langle x \rangle_{g_H}^{\zeta} = \frac{4}{7} - \tau(M_s, M_c) \left[\langle 2x \rangle_{u_\pi}^{\zeta} \right]^{7/4}$$

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3 (always) active flavors →

$$\tau(\zeta_H, M_c) = -\frac{12}{175} \left[\langle 2x \rangle_{u}^{M_c} \right]^{-7/4} + \frac{16}{25} \left[\langle 2x \rangle_{u}^{M_c} \right]^{-3/16}$$

4 (always) active flavors →

$$\tau(\zeta_H, \zeta_H) = \frac{4}{7}$$

Thus recovering the previous result!

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$$\langle x \rangle^\zeta_{S^q_H} = \langle x \rangle^\zeta_{\Sigma^q_H} - \langle x \rangle^\zeta_{q_H} = \theta(\zeta - M_q) \frac{1}{3\pi} \int_{M_q}^{\zeta} \frac{dz}{z} \alpha(z^2) \langle x \rangle^z_{g_H} S(z, \zeta)$$

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$$\gamma^n_{qq} = \gamma^n_{uu},\ \gamma^n_{gq} = \gamma^n_{gu},\ \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s, c$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{uu} \langle x^n \rangle^\zeta_{q_H}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^\zeta_{\Sigma^q_H} + 2\theta(\zeta - M_q) \gamma^n_{ug} \langle x^n \rangle^\zeta_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \sum_q \langle x^n \rangle^\zeta_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^\zeta_{g_H} \right\}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

$$\langle x \rangle^\zeta_{q_H} = \langle x \rangle^{\zeta_H}_{q_H} S(\zeta_H, \zeta)$$

$$\langle x \rangle^\zeta_{g_H} = \frac{4}{7} - \tau(M_s, M_c) \left[\langle 2x \rangle^\zeta_{u_\pi} \right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-7/4} - \frac{24}{275} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-3/16} \left[\langle 2x \rangle^{M_s}_{u_\pi} \right]^{-25/16} + \frac{8}{11} \left[\langle 2x \rangle^{M_c}_{u_\pi} \langle 2x \rangle^{M_s}_{u_\pi} \right]^{-3/16}$$

$$\langle x \rangle^\zeta_{S^q_H} = \langle x \rangle^\zeta_{\Sigma^q_H} - \langle x \rangle^\zeta_{q_H} = \theta(\zeta - M_q) \frac{1}{3\pi} \int_{M_q}^{\zeta} \frac{dz}{z} \alpha(z^2) \langle x \rangle^z_{g_H} S(z, \zeta)$$

$$\sum_q \langle x \rangle^\zeta_{S^q_H} = \frac{3}{7} + \tau(M_s, M_c) \left[\langle 2x \rangle^\zeta_{u_\pi} \right]^{7/4} - \sum_q \langle x \rangle^\zeta_{q_H}$$

All-orders DGLAP: hard-wall thresholds

Let us now solve generally the hard-wall model of **massless partons** with **hard-wall thresholds** for each flavor activation, that can be analytically solved!

$$\gamma^n_{qq} = \gamma^n_{uu},\ \gamma^n_{gq} = \gamma^n_{gu},\ \gamma^n_{qg} = \gamma^n_{ug}$$

$$q = u, d, s, c$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{uu} \langle x^n \rangle^\zeta_{q_H}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{\Sigma^q_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{uu} \langle x^n \rangle^\zeta_{\Sigma^q_H} + 2\theta(\zeta - M_q) \gamma^n_{ug} \langle x^n \rangle^\zeta_{g_H} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^\zeta_{g_H} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{gu} \sum_q \langle x^n \rangle^\zeta_{\Sigma^q_H} + \gamma^n_{gg} \langle x^n \rangle^\zeta_{g_H} \right\}$$

In general, at any momentum scale $\zeta \geq M_c$ and again specializing for the averaged momentum fraction, the solutions are:

$$\langle x \rangle^\zeta_{q_H} = \langle x \rangle^{\zeta_H}_{q_H} S(\zeta_H, \zeta)$$

$$\langle x \rangle^\zeta_{g_H} = \frac{4}{7} - \tau(M_s, M_c) \left[\langle 2x \rangle^\zeta_{u_\pi} \right]^{7/4}$$

$$\tau(M_s, M_c) = -\frac{12}{175} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-7/4} - \frac{24}{275} \left[\langle 2x \rangle^{M_c}_{u_\pi} \right]^{-3/16} \left[\langle 2x \rangle^{M_s}_{u_\pi} \right]^{-25/16} + \frac{8}{11} \left[\langle 2x \rangle^{M_c}_{u_\pi} \langle 2x \rangle^{M_s}_{u_\pi} \right]^{-3/16}$$

$$\langle x \rangle^\zeta_{S^q_H} = \langle x \rangle^\zeta_{\Sigma^q_H} - \langle x \rangle^\zeta_{q_H} = \theta(\zeta - M_q) \frac{1}{3\pi} \int_{M_q}^{\zeta} \frac{dz}{z} \alpha(z^2) \langle x \rangle^z_{g_H} S(z, \zeta)$$

Momentum conservation

$$\sum_q \langle x \rangle^\zeta_{S^q_H} = \frac{3}{7} + \tau(M_s, M_c) \left[\langle 2x \rangle^\zeta_{u_\pi} \right]^{7/4} - \sum_q \langle x \rangle^\zeta_{q_H}$$

All-orders DGLAP: Pauli blocking

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions

$$\zeta^2 \frac{d}{d\zeta^2} q_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} P_{q\leftarrow q}\left(\frac{x}{y}\right) q_{\mathrm{H}}(y)$$

$$\zeta^2 \frac{d}{d\zeta^2} \Sigma_{\mathrm{H}}^q(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q\leftarrow q}\left(\frac{x}{y}\right) \Sigma_{\mathrm{H}}^q(y) + 2 P_{q\leftarrow g}^{\zeta}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} g_{\mathrm{H}}(x) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g\leftarrow q}\left(\frac{x}{y}\right) \Sigma_{\mathrm{H}}^q(y) + P_{g\leftarrow g}\left(\frac{x}{y}\right) g_{\mathrm{H}}(y) \right\}$$

Modeling the Pauli-blocking contribution:

$$P_{q\leftarrow g}^{\zeta}(z) = \left[P_{q\leftarrow g}(z) + \delta_q \sqrt{3}(1-2z)\mathcal{D}\left(\frac{\zeta}{\zeta_H}\right)\right] \theta(\zeta - M_q)$$

$$\mathcal{D}(t) = \frac{1}{1+(t-1)^2}$$

All-orders DGLAP: Pauli blocking

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_\pi}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} + 2\theta(\zeta - M_q) \left[\gamma^n_{qg} + \delta_q a_n \mathcal{D}\left(\frac{\zeta}{\zeta_H} \right) \right] \langle x^n \rangle^{\zeta}_{g_\pi} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_\pi} \right\} ;$$

Modeling the Pauli-blocking contribution:

$$a_n = \frac{\sqrt{3}n}{2 + 3n + n^2}$$

$$\mathcal{D}(t) = \frac{1}{1 + (t-1)^2}$$

Momentum conservation

$$\gamma_{qq} + \gamma_{gq} = 2\sum_q \gamma_{qg} + \gamma_{gg} = \sum_q \delta_q = 0$$

All-orders DGLAP: Pauli blocking

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_\pi}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} + 2\theta(\zeta - M_q) \left[\gamma^n_{qg} + \delta_q a_n \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x^n \rangle^{\zeta}_{g_\pi} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_\pi} \right\} ;$$

Modeling the Pauli-blocking contribution:

$$a_n = \frac{\sqrt{3}n}{2 + 3n + n^2}$$

$$\mathcal{D}(t) = \frac{1}{1 + (t-1)^2}$$

Momentum conservation

$$\gamma_{qq} + \gamma_{gq} = 2 \sum_q \gamma_{qg} + \gamma_{gg} = \sum_q \delta_q = 0$$

Equations and solutions for $\sum_q \langle x \rangle^{\zeta}_{S^q_H}$ and $\langle x \rangle^{\zeta}_{g_H}$ remain the same, while:

$$\langle x^n \rangle^{\zeta}_{\mathcal{S}_q} = -\frac{1}{\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \left[\gamma^n_{ug} + g_q a_n \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x^n \rangle^{z}_{g_\pi} \left[S(z,\zeta) \right]^{\gamma^n_{uu}/\gamma_{uu}}$$

All-orders DGLAP: Pauli blocking

PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{q_\pi}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \gamma^n_{qq} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} + 2\theta(\zeta - M_q) \left[\gamma^n_{qg} + \delta_q a_n \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x^n \rangle^{\zeta}_{g_\pi} \right\}$$

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^n \rangle^{\zeta}_{g_\pi} = -\frac{\alpha(\zeta^2)}{4\pi} \left\{ \sum_q \gamma^n_{gq} \langle x^n \rangle^{\zeta}_{\Sigma^q_\pi} + \gamma^n_{gg} \langle x^n \rangle^{\zeta}_{g_\pi} \right\} ;$$

Modeling the Pauli-blocking contribution:

$$a_n = \frac{\sqrt{3}n}{2 + 3n + n^2}$$

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Momentum conservation

$$\gamma_{qq} + \gamma_{gq} = 2\sum_q \gamma_{qg} + \gamma_{gg} = \sum_q \delta_q = 0$$

Particularly, for the pion momentum fractions, and with:

$$\delta_u = \delta_d = \delta = -\delta_s/2 \, ; \delta_c = 0 \, .$$

$$\langle x \rangle^{\zeta}_{S^{u+d}_\pi} = \frac{2}{3\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \left[1 - \frac{\sqrt{3}}{2} \delta \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x \rangle^{z}_{g_\pi} S(z,\zeta) \, ,$$

$$\langle x \rangle^{\zeta}_{S^s_\pi} = \theta(\zeta - M_s) \frac{1}{3\pi} \int_{M_s}^{\zeta} \frac{dz}{z} \alpha(z^2) \left[1 + \sqrt{3} \delta \mathcal{D}\left(\frac{\zeta}{\zeta_H}\right) \right] \langle x \rangle^{z}_{g_\pi} S(z,\zeta)$$

$$\langle x \rangle^{\zeta}_{S^c_\pi} = \theta(\zeta - M_c) \frac{1}{3\pi} \int_{M_c}^{\zeta} \frac{dz}{z} \alpha(z^2) \langle x \rangle^{z}_{g_\pi} S(z,\zeta) \, .$$

All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

$$\left(\zeta^2 \frac{d}{d\zeta^2} + \tilde{\gamma}^0_{gg}(n_f) \frac{\alpha(\zeta^2)}{4\pi} \right) \langle x^0 \rangle^{\zeta}_{\tilde{g}_H} = 4 \frac{\alpha(\zeta^2)}{4\pi} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H}$$

All-orders DGLAP: Polarized distributions

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In general, at any momentum scale $\zeta \geq M_c$:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} ,$$

$$\Delta G^{\zeta}_{H} = \langle x^0 \rangle^{\zeta}_{\tilde{g}^q_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} \left\{ \frac{12}{29} \left([S(\zeta_H, M_s)]^{-87/32} - 1 \right) [S(M_s, M_c)]^{-81/32} [S(M_c, \zeta)]^{-75/32} + \frac{4}{9} \left([S(M_s, M_c)]^{-81/32} - 1 \right) [S(M_c, \zeta)]^{-75/32} + \frac{12}{25} \left([S(M_c, \zeta)]^{-75/32} - 1 \right) \right\}$$

$$S(\zeta_H, \zeta) = \frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

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In general, at any momentum scale $M_s \leq \zeta \leq M_c$:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H} ,$$

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$$S(\zeta_H,\zeta) = \frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

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In general, at any momentum scale $\zeta_H \leq \zeta \leq M_s$:

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All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

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In general, at any momentum scale $\zeta_H \leq \zeta$, and neglecting the mass thresholds:

$$a^{\zeta}_{0H} = \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = \langle x^0 \rangle^{\zeta_H}_{\tilde{\Sigma}_H}$$

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$$S(\zeta_H, \zeta) = \frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations, expressed by the corresponding massless splitting functions, after converting to Mellin space and specializing for 0-th order

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A [CT18]+ no thresholds

B [Ya2022]+no thresholds

C [Ya2022]+[Chen2022]

D [Ya2022]+ thresholds

	A	B	C	D
a^{ζ}_{0p}	0.74(11)	0.74(11)	0.65(02)	0.65(02)
ΔG^{ζ}_p	2.27(30)	1.50(25)	1.33(15)	1.41(16)
$\tilde{a}^{\zeta}_{0p}$	0.20(11)	0.38(11)	0.33(04)	0.32(04)

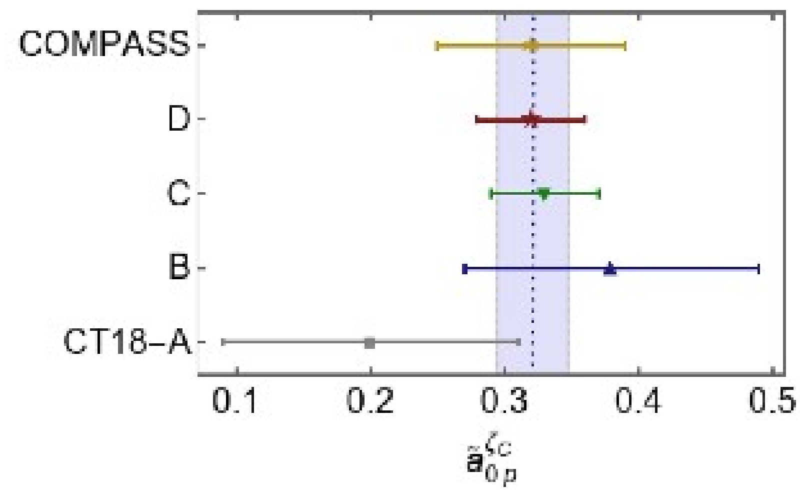
Abelian anomaly corrected:

$$\tilde{a}^{\zeta}_{0p} = a^{\zeta}_{0p} - n_f \frac{\hat{\alpha}(\zeta)}{2\pi} \Delta G^{\zeta}_p$$

$$\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

All-orders DGLAP: Polarized distributions

Polarized PDFs DGLAP evolutions equations



$$\zeta^2 \frac{d}{d\zeta^2} \langle x^0 \rangle^{\zeta}_{\tilde{\Sigma}_H} = 0$$

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In general, at any momentum scale $\zeta_H \leq \zeta$, and neglecting the mass thresholds:

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$$\frac{\langle x \rangle^{\zeta}_{q_H}}{\langle x \rangle^{\zeta_H}_{q_H}} = \exp\left(-\frac{\gamma_{qq}}{2\pi} \int_{\zeta_H}^{\zeta} \frac{dz}{z} \alpha(z^2) \right)$$

Reverse engineering the PDF data

Pion PDF

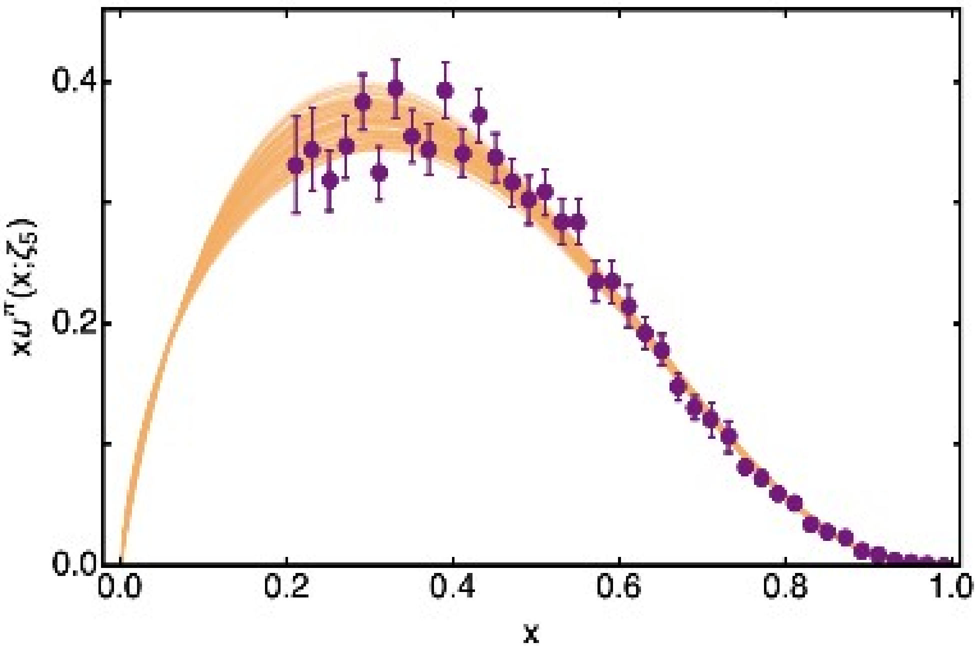
➢ Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^\pi(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

Normalization

$$\{\alpha_i^\zeta | i = 1,2,3\}$$

Free parameters



Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

Pion PDF

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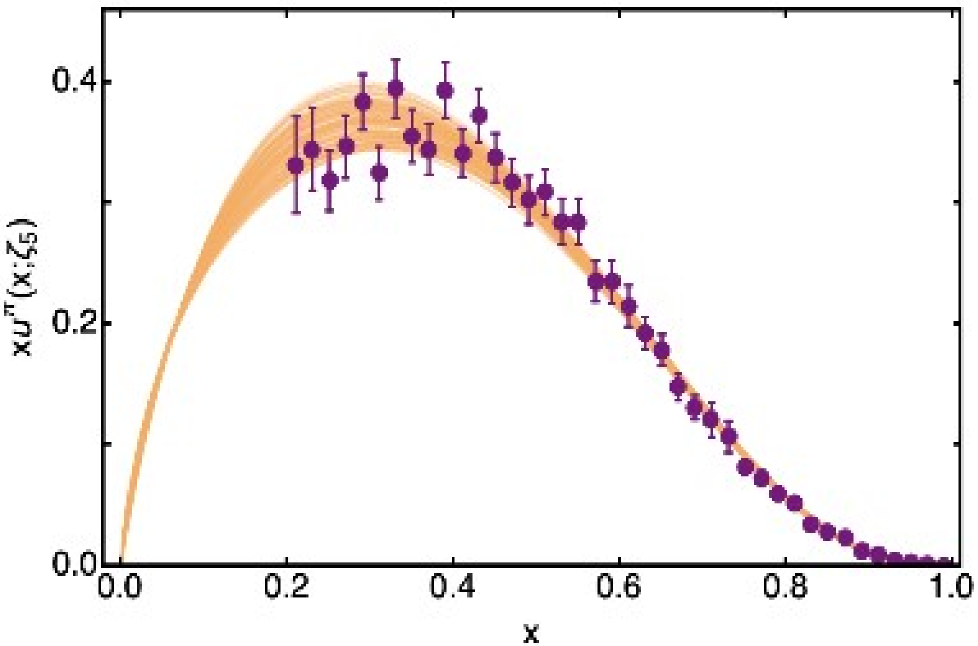

- Then, we proceed as follows:

1) Determine the **best values α_i** via least-squares fit to the data.

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

Pion PDF

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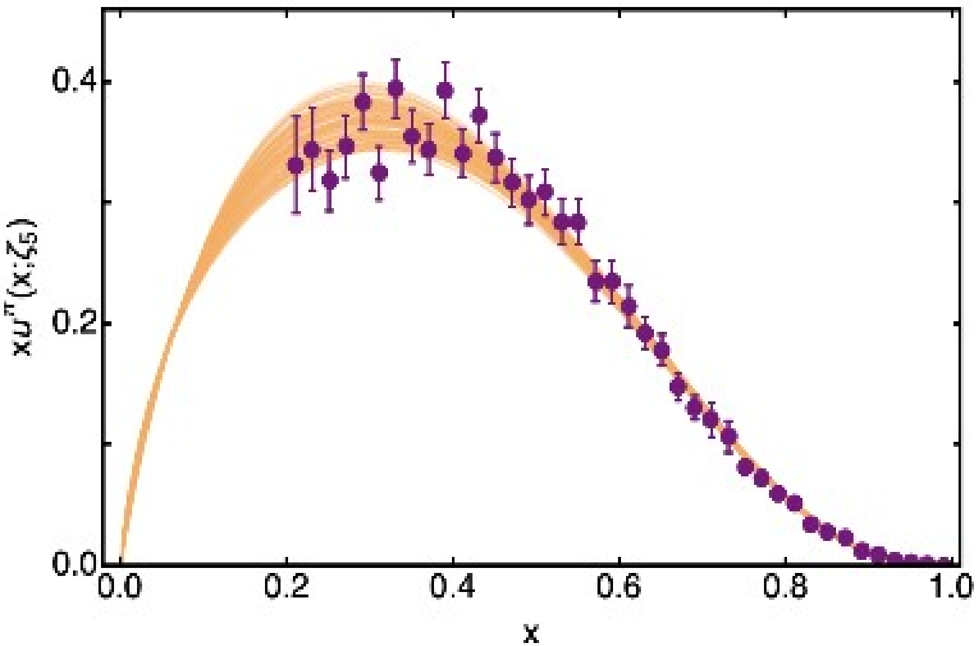


- Then, we proceed as follows:

  1) Determine the **best values α_i** via least-squares fit to the data.

  2) Generate new **values α_i,** distributed randomly around the best fit.

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

Pion PDF

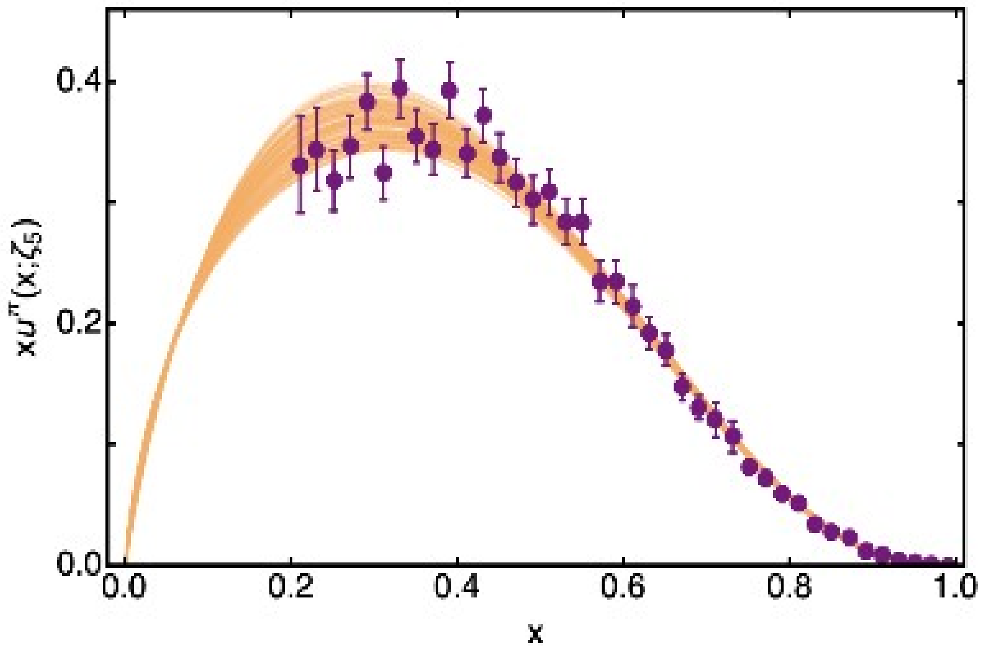
- Let us assume the data can be parameterized with a certain functional form, i.e.:

$$u^{\pi}(x;[\alpha_i];\zeta) = n_u^{\zeta} x^{\alpha_1^{\zeta}} (1-x)^{\alpha_2^{\zeta}} (1+\alpha_3^{\zeta} x^2)$$

Normalization

$$\{\alpha_i^{\zeta} | i = 1,2,3\}$$

Free parameters



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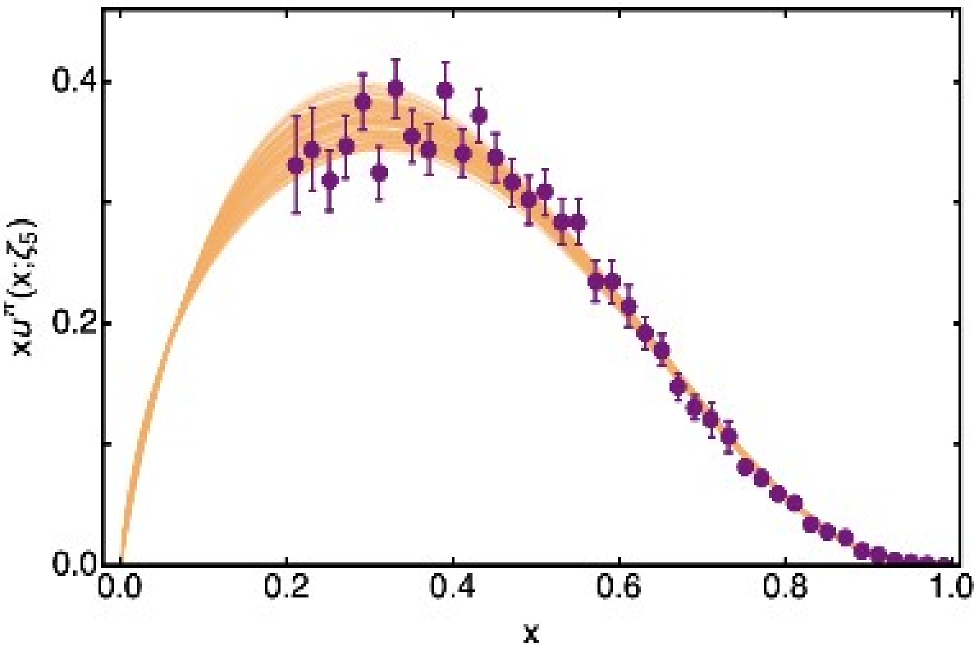

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Data point with error

4) Accept a replica with probability:

$$\mathcal{P} = \frac{P(\chi^2;d)}{P(\chi_0^2;d)}\,,\; P(y;d) = \frac{(1/2)^{d/2}}{\Gamma(d/2)} y^{d/2-1} e^{-y/2}$$

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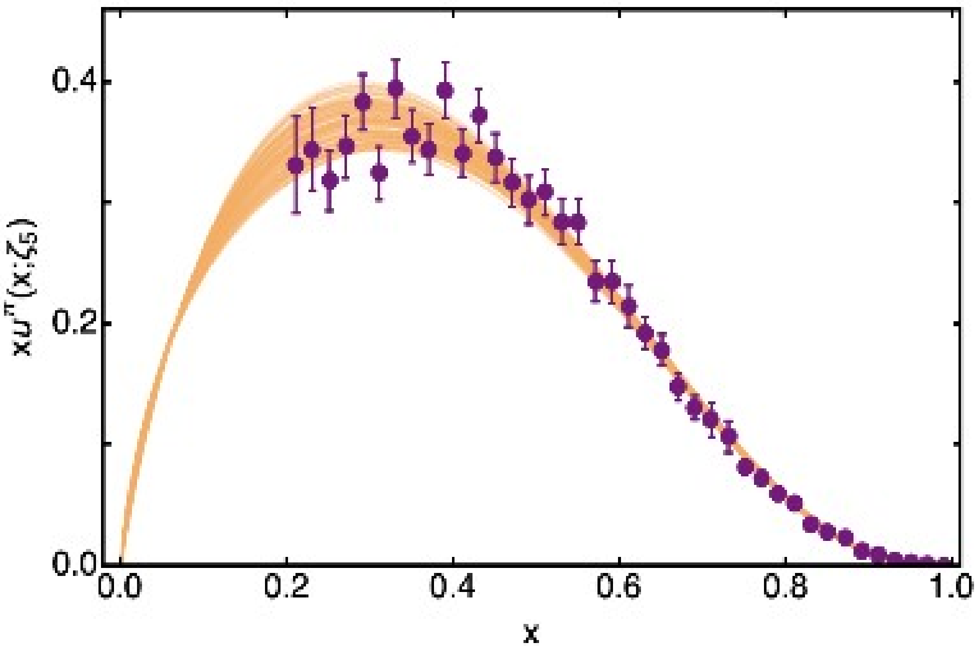

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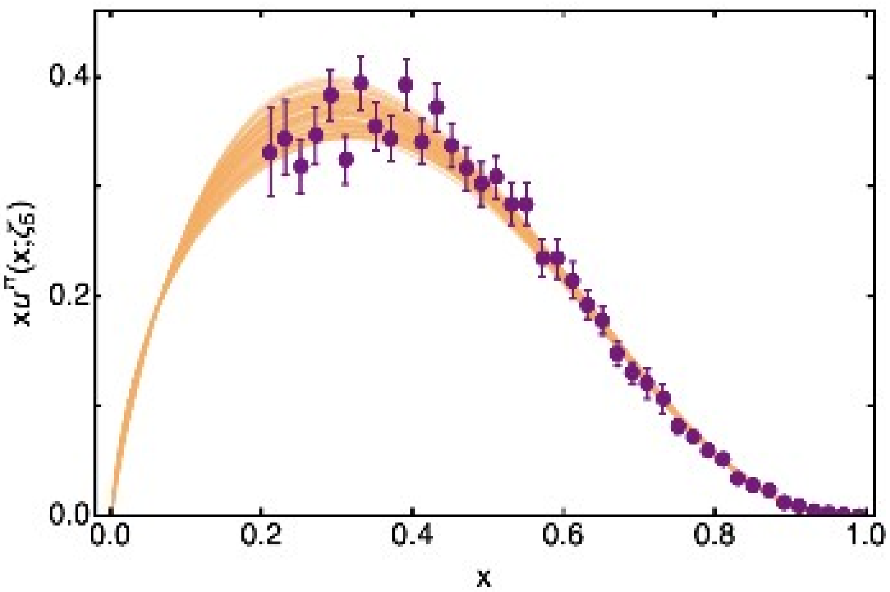
5) Evolve back to ζ_H

Repeat (2-5).

Data from [Aicher et al. Phys. Rev. Lett. 105, 252003 (2010)]

Pion PDF: ASV analysis of E615 data

- Applying this algorithm to the **ASV data** yields:



Mean values (of moments) and errors

{{0.5, 2.75144 × 10^-17}, {0.299833, 0.00647045}, {0.199907, 0.00735448}, {0.142895, 0.0069623},
{0.107274, 0.00608759}, {0.0835168, 0.00532834}, {0.0668711, 0.0046596},
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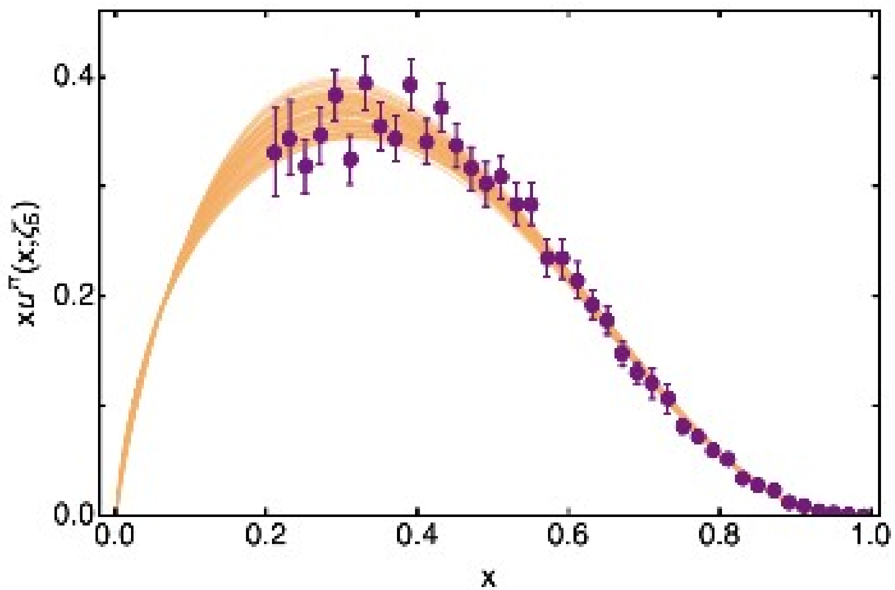

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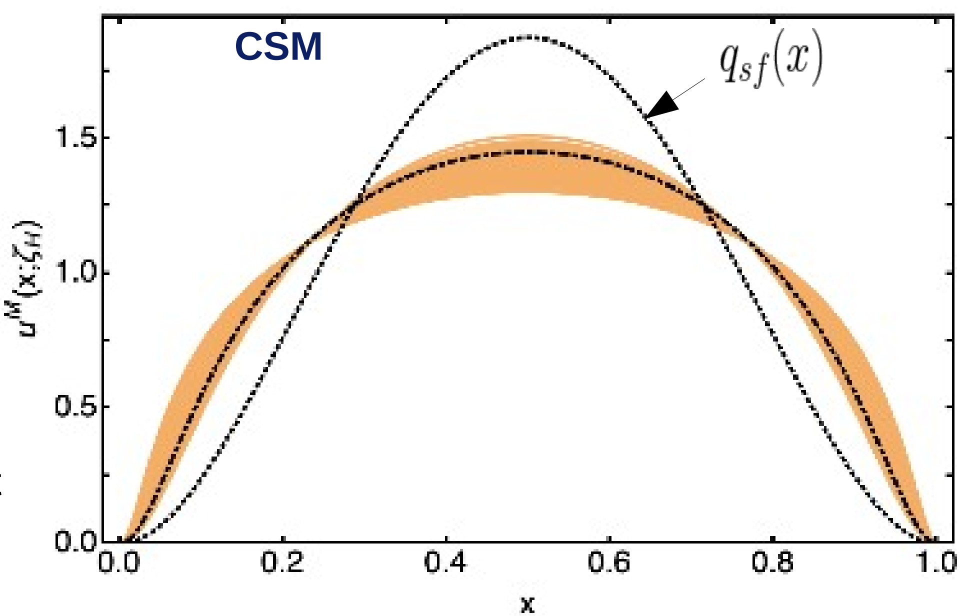
✔ Then, we can **reconstruct** the moments produced by each replica, using the **single-parameter Ansatz**:

$$u^{\pi}(x;\zeta_{\mathcal{H}}) = n_0 \ln(1 + x^2(1-x)^2/\rho^2)$$

✔ The produced moments are compatible with a **symmetric PDF** at the **hadronic scale**.

✔ It seems it favors a **soft end-point** behavior just like the **CSM result**.



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Kaon PDF

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$$u^{K,\pi}(x;[\alpha_i];\zeta) = n_u^\zeta x^{\alpha_1^\zeta}(1-x)^{\alpha_2^\zeta}(1+\alpha_3^\zeta x^2)$$

Pion's free parameters: $\{\alpha_i^\zeta | i = 1,2,3\}$

Kaon's : α_{3K}^ζ

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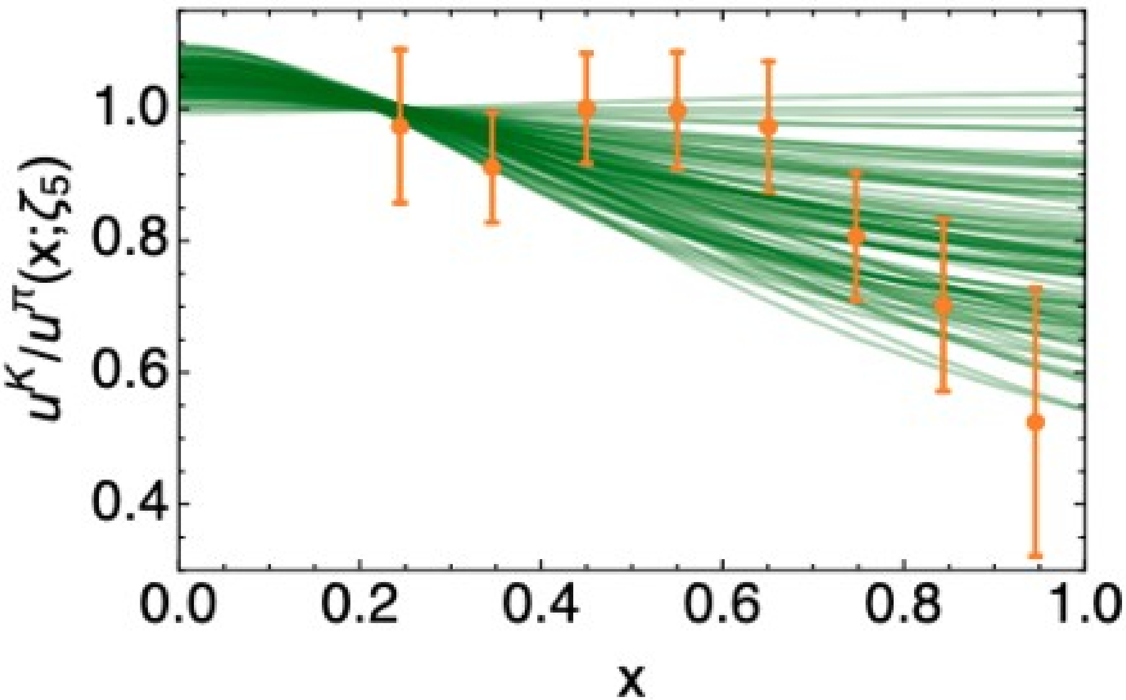

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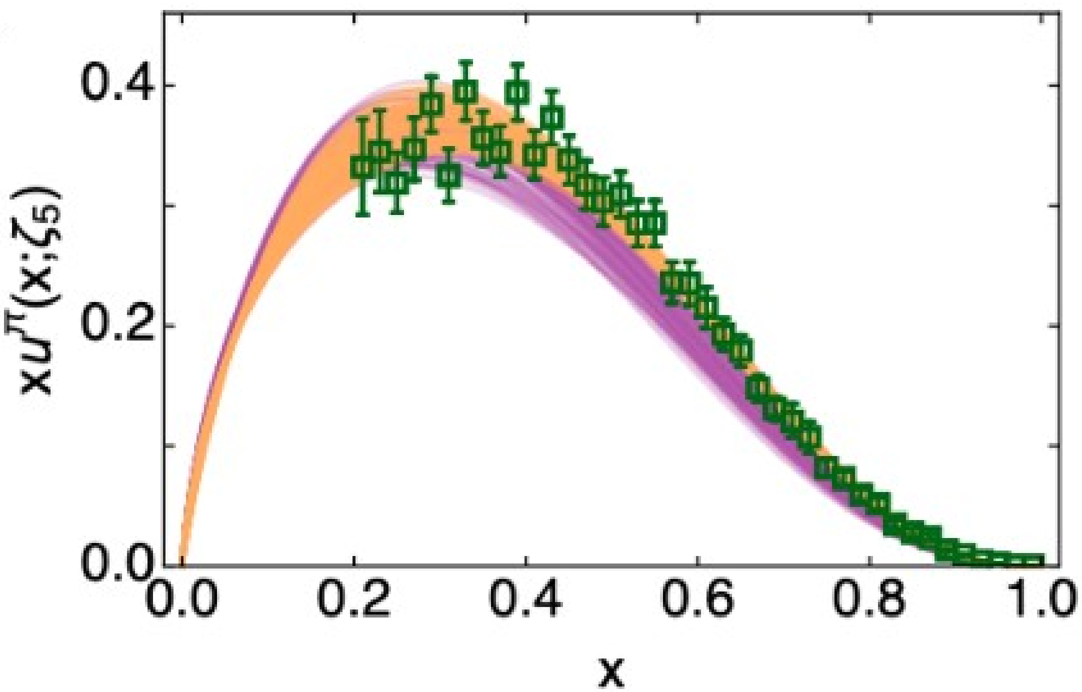

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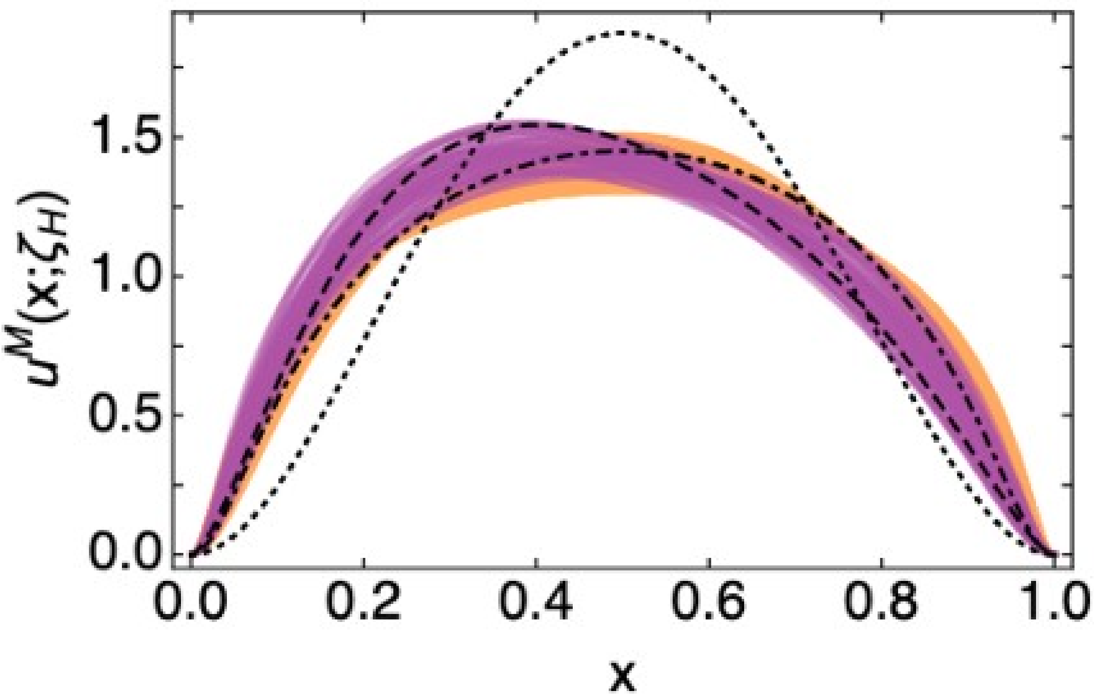

$$q^M(x;\zeta_H) = n_0 \ln\left[1 + x^2(1-x)^2/\rho_M^2\right][1 \pm \gamma_M(1-2x)]$$

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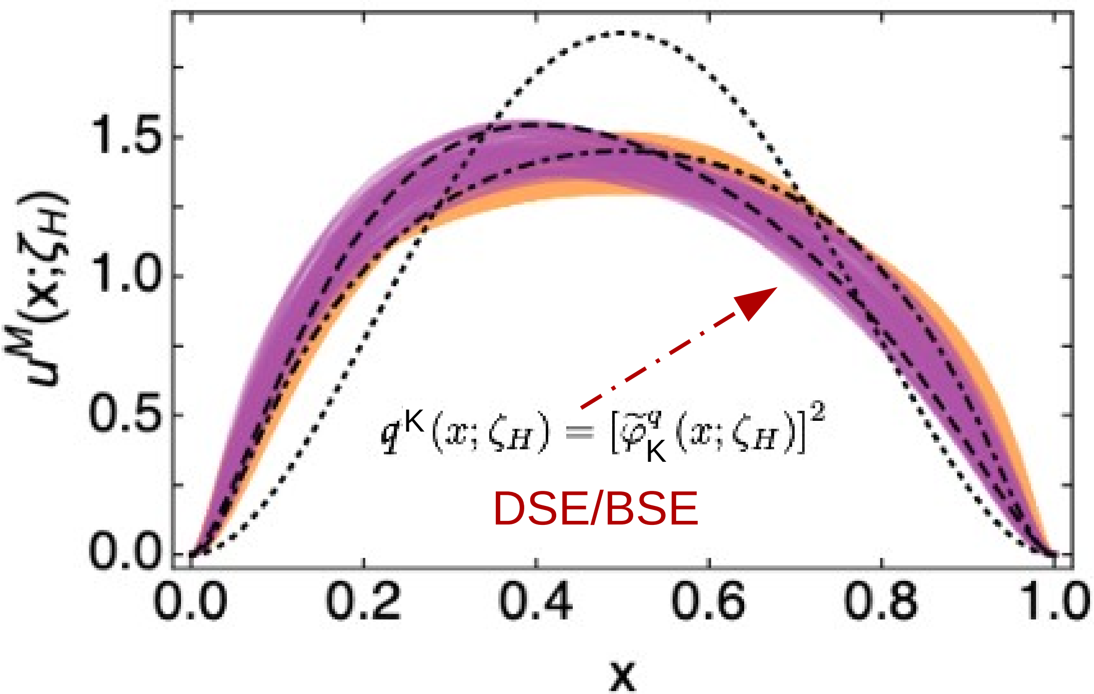

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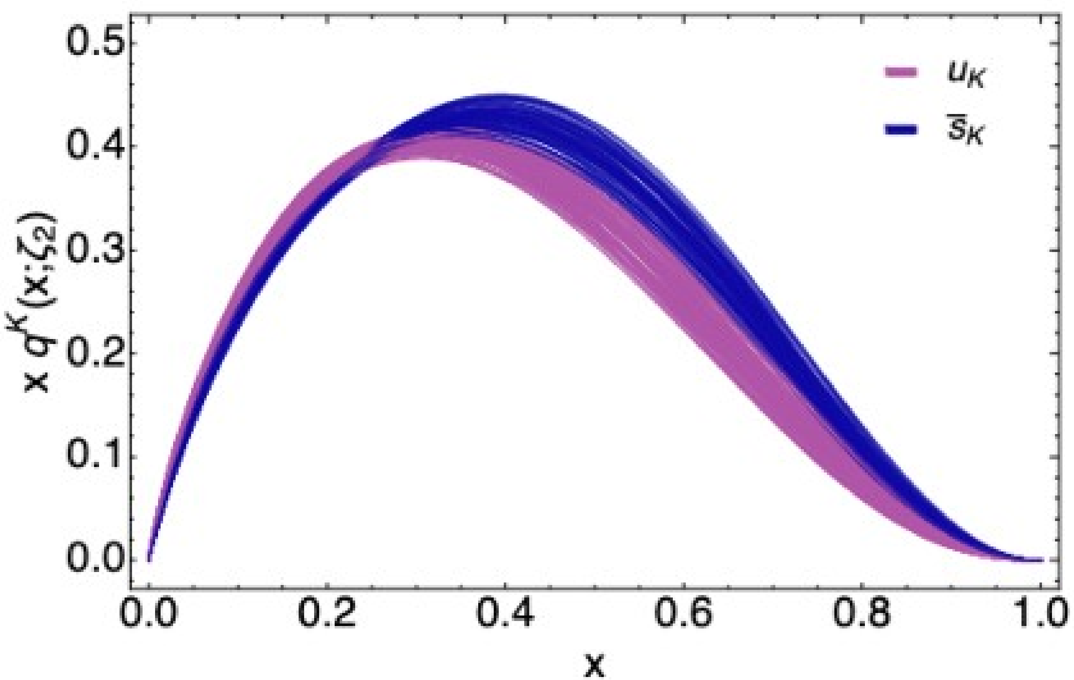

Capitalizing on $\bar{s}^K(x;\zeta_\mathcal{H}) = u^K(1-x;\zeta_\mathcal{H})$, antiquark DF can be derived for each replica and be evolved up again

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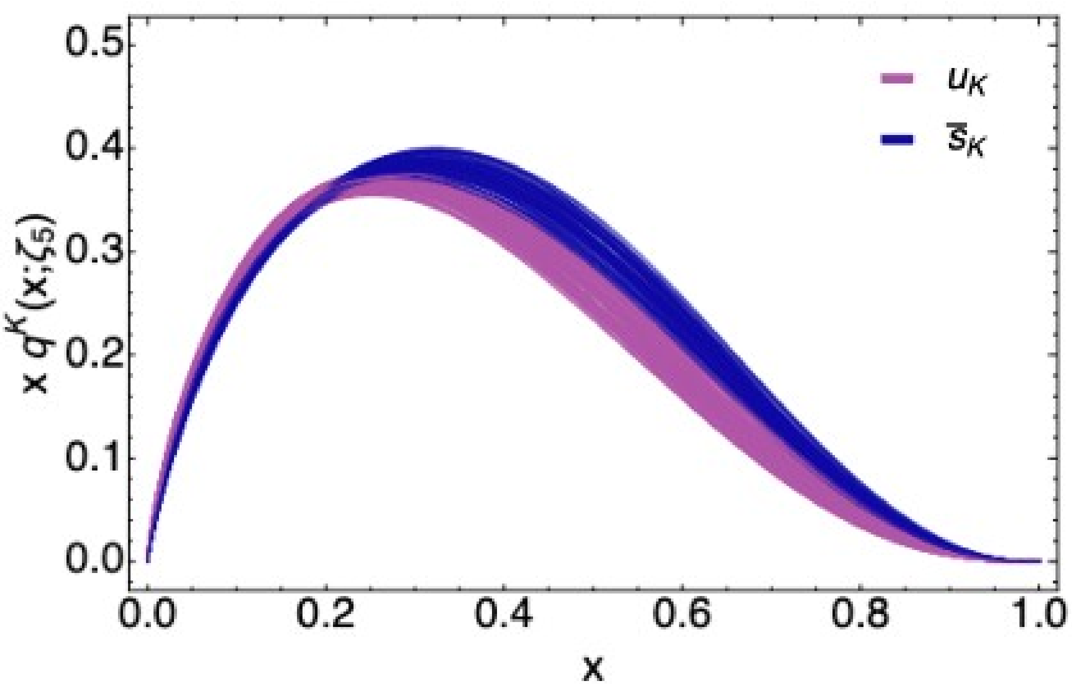

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Kaon PDF: glue and quark singlet

Z-N. Xu et al., arXiv:2411.15376v2

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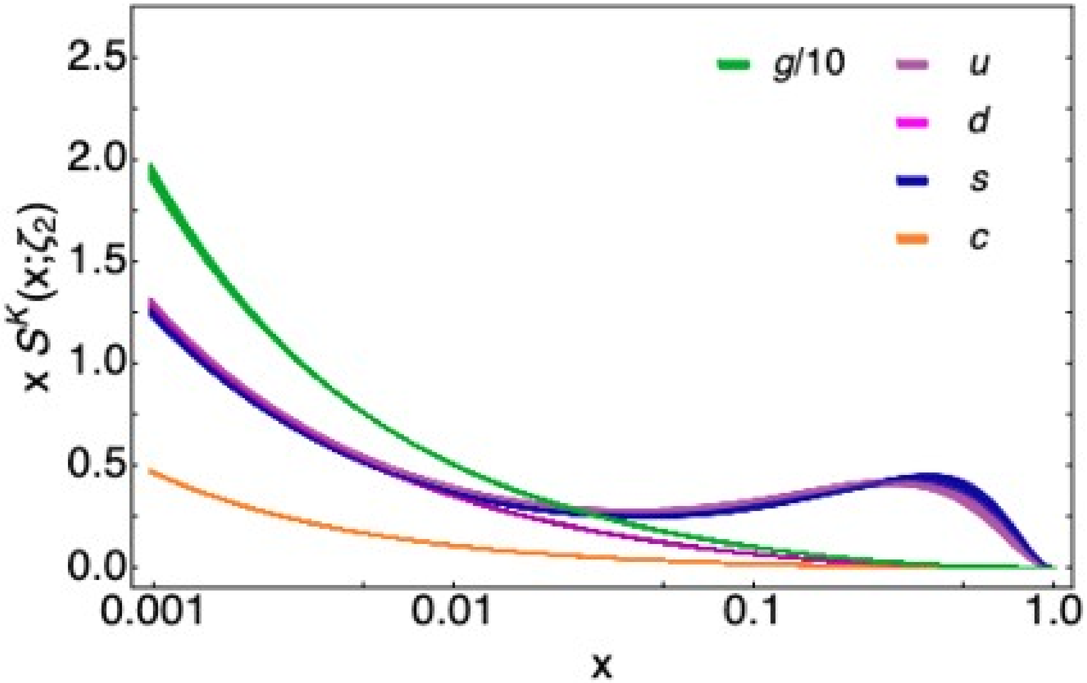

… and, similarly, glue and sea-quark DFs can be also obtained, at different empirical scales!

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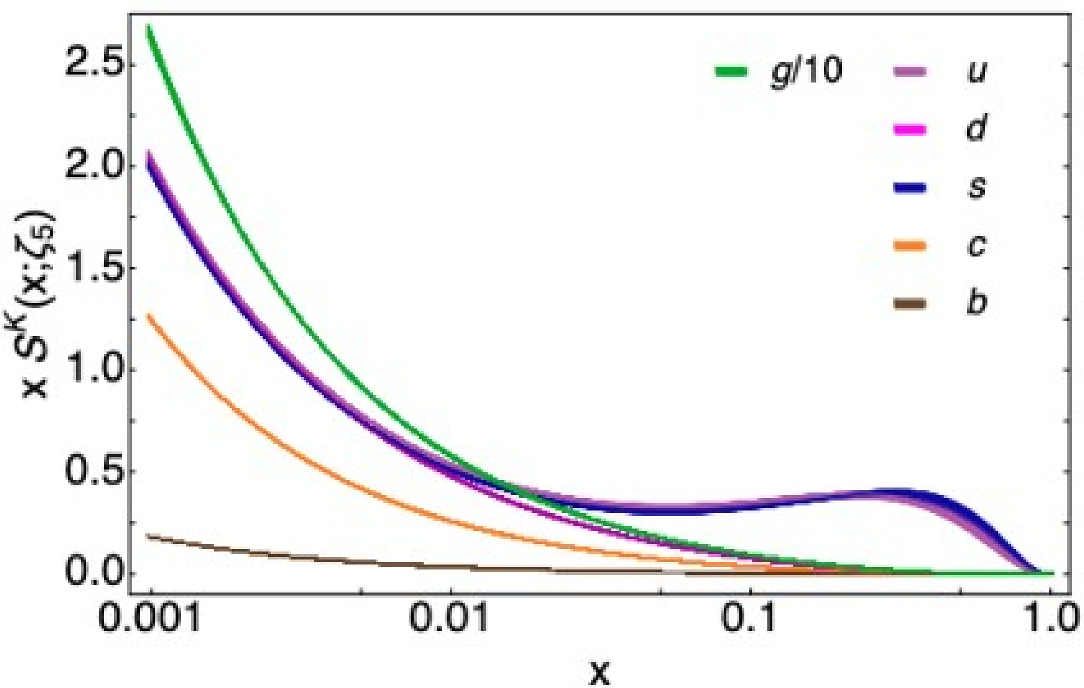

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Kaon PDF: Momentum fractions and comparisons

	$\langle x \rangle^{\zeta}_{q^K}$	$\langle x \rangle^{\zeta}_{S^K_q}$	$\langle x \rangle^{\zeta}_{g^K}$	$\langle x \rangle^{\zeta}_{q^\pi}$
ζ_2				
u	0.230(6)(10)	0.028(2)		0.241(5)(10)
d	0	0.028(2)		0.241(5)(10)
s	0.252(6)(11)	0.026(1)		
c	0	0.008(1)		
b	0	0		
g			0.428(18)	
ζ_5				
u	0.197(5)(9)	0.036(2)		0.207(4)(9)
d	0	0.036(2)		0.207(4)(9)
s	0.216(5)(9)	0.034(2)		
c	0	0.019(1)		
b	0	0.003(1)		
g			0.461(20)	

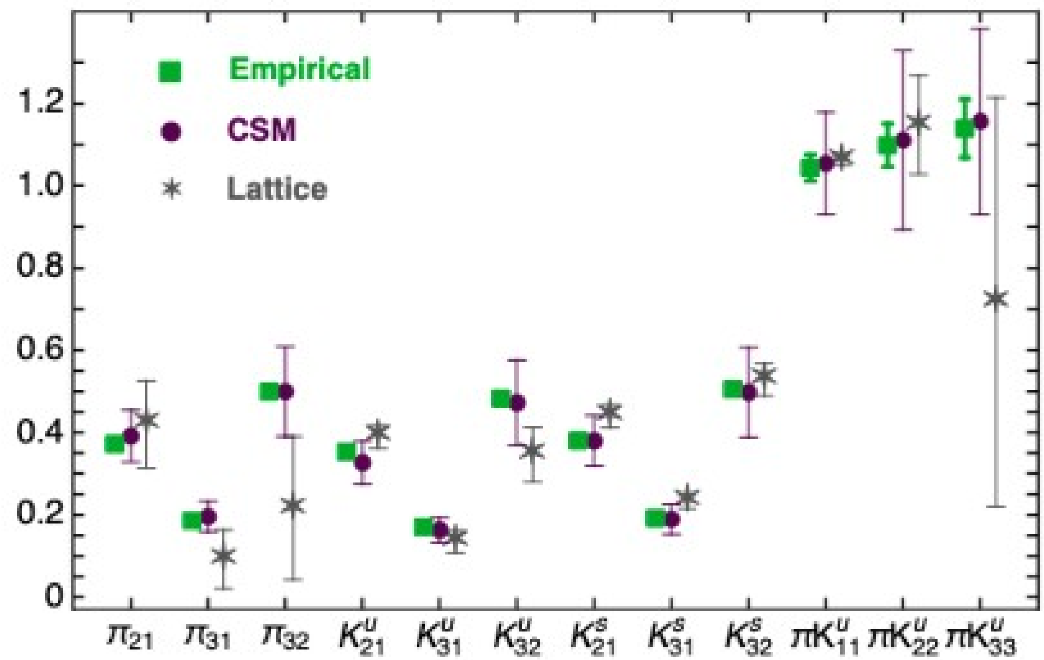
	empirical		[32, lQCD]	
M	π	K	π	K
l	0.538(15)	0.286(12)	0.499(55)	0.317(19)
s	0.026(01)	0.278(13)	0.036(15)	0.339(11)
c	0.008(01)	0.008(01)	0.013(16)	0.028(21)
q	0.572(15)	0.572(18)	0.575(79)	0.683(50)
g	0.428(18)	0.428(18)	0.402(53)	0.422(67)

[32] Alexandrou, et al., arXiv:2405.08529 [hep-lat]

Kaon PDF: Momentum fractions and comparisons

	$\langle x\rangle^{\zeta}_{q^K}$	$\langle x\rangle^{\zeta}_{S_q^K}$	$\langle x\rangle^{\zeta}_{g^K}$	$\langle x\rangle^{\zeta}_{q^\pi}$
ζ_2				
u	0.230(6)(10)	0.028(2)		0.241(5)(10)
d	0	0.028(2)		0.241(5)(10)
s	0.252(6)(11)	0.026(1)		
c	0	0.008(1)		
b	0	0		
g			0.428(18)	
ζ_5				
u	0.197(5)(9)	0.036(2)		0.207(4)(9)
d	0	0.036(2)		0.207(4)(9)
s	0.216(5)(9)	0.034(2)		
c	0	0.019(1)		
b	0	0.003(1)		
g			0.461(20)	

	empirical		[32, lQCD]	
M	π	K	π	K
l	0.538(15)	0.286(12)	0.499(55)	0.317(19)
s	0.026(01)	0.278(13)	0.036(15)	0.339(11)
c	0.008(01)	0.008(01)	0.013(16)	0.028(21)
q	0.572(15)	0.572(18)	0.575(79)	0.683(50)
g	0.428(18)	0.428(18)	0.402(53)	0.422(67)



$$\pi_{ij} = \langle x^i\rangle_{u_S^\pi} / \langle x^j\rangle_{u_S^\pi}$$

$$K^q_{ij} = \langle x^i\rangle_{q_S^K} / \langle x^j\rangle_{q_S^K}$$

$$\pi K^u_{ij} = \langle x^i\rangle_{u_S^\pi} / \langle x^j\rangle_{u_S^K}$$

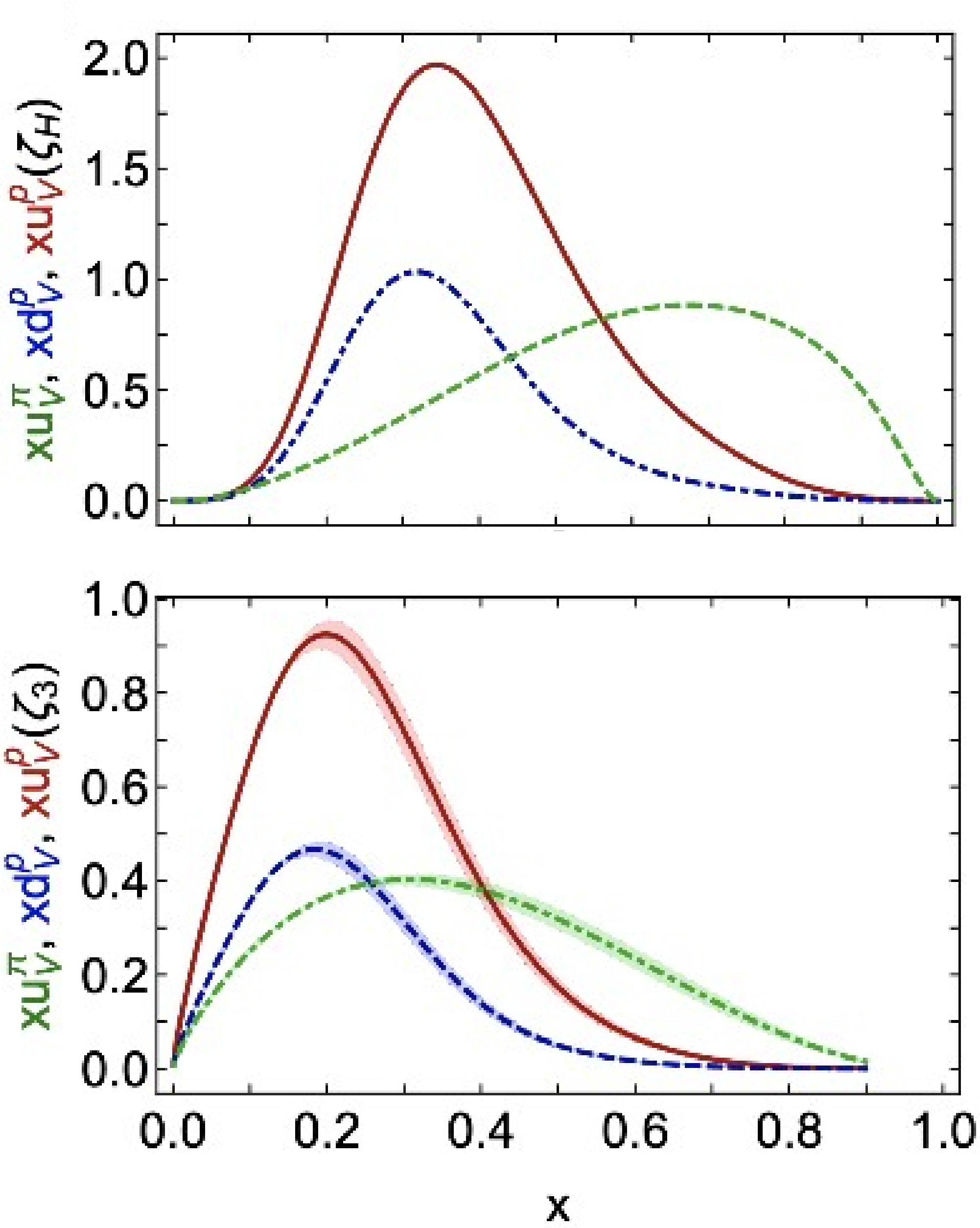
CSM = Z-F Cui, et al., Eur. Phys. J. C80 (2020) 1064.

Lattice = C. Alexandrou, et al., Phys. Rev. D 103 (1) (2021) 014508; Phys. Rev. D 104 (5) (2021) 054504.

[32] Alexandrou, et al., arXiv:2405.08529 [hep-lat]

Proton PDF: from CSM (DSEs) to the experiment

An analogous symmetry-preserving DSE computation of the valence-quark PDFs within a proton, based on diquark-quark approach:
[L. Chang et al., Phys.Lett.B, arXiv:2201.07870]



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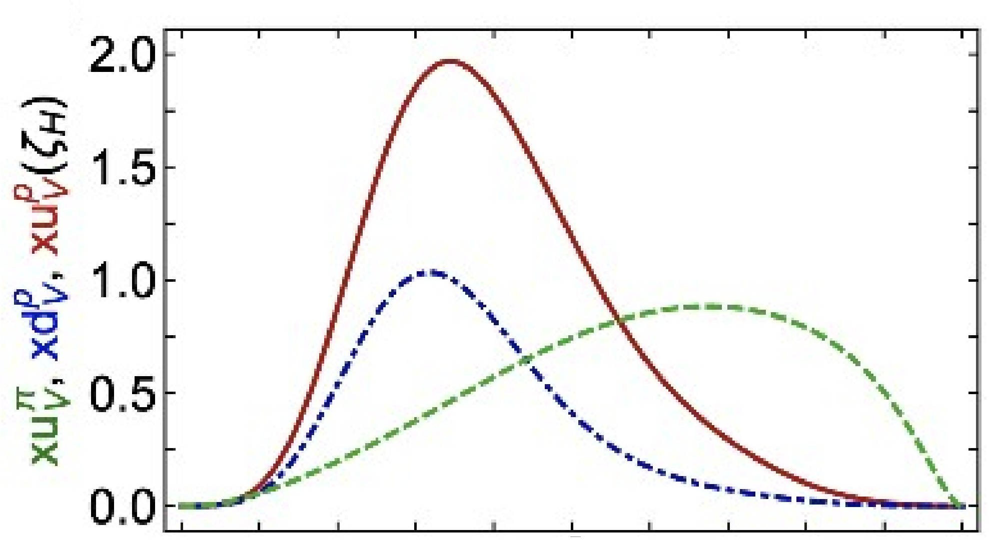
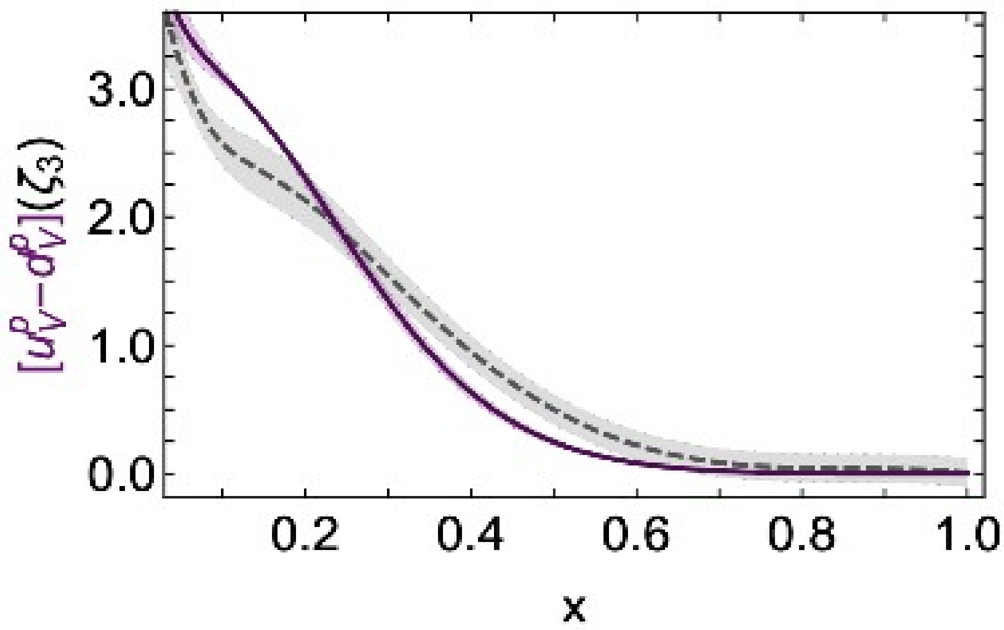
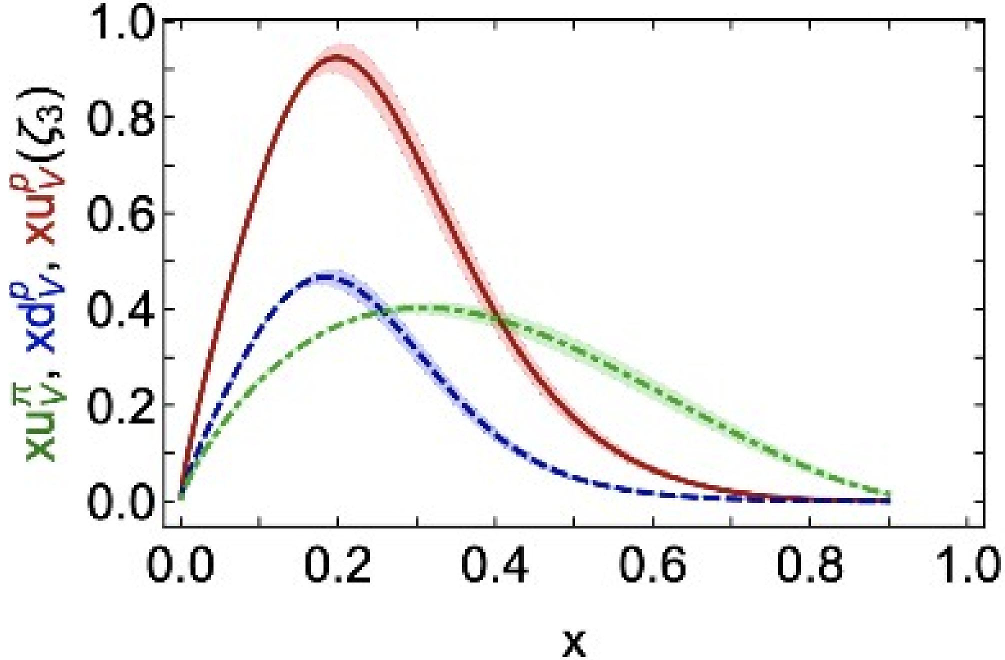
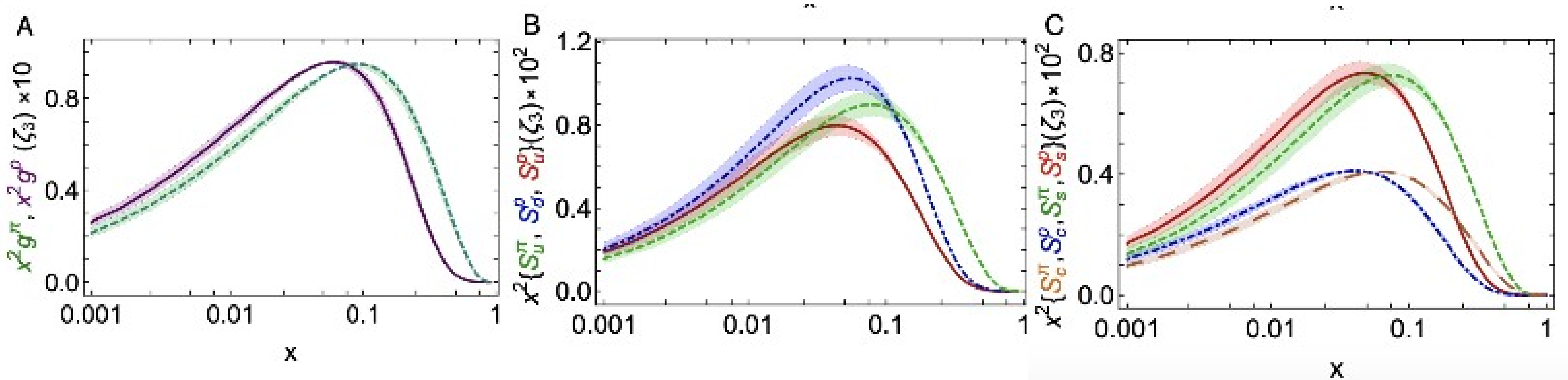


Producing an isovector distribution in fair agreement with lattice results
[H-W. Lin et al., arXiv:2011.14791]





Proton PDF: pion and proton in counterpoint



pion	u^π	$\bar{d}^\pi$	g^π	$\mathcal{S}_\pi^u$	$\mathcal{S}_\pi^{\bar{d}}$	$\mathcal{S}_\pi^s$	$\mathcal{S}_\pi^c$
$\langle x\rangle^{\zeta_2}$	24.0(1.1)	24.0(1.1)	41.0(1.2)	3.3(3)	3.3(3)	2.65(22)	1.33(5)
$\langle x^2\rangle^{\zeta_2}$	9.5(7)	9.5(7)	3.7(1)	0.27(1)	0.27(1)	0.21(1)	0.092(2)
$\langle x^3\rangle^{\zeta_2}$	4.7(4)	4.7(4)	0.92(6)	0.057(1)	0.057(1)	0.044(0)	0.018(1)
$\langle x\rangle^{\zeta_3}$	22.1(1.0)	22.1(1.0)	42.9(1.0)	3.7(3)	3.7(3)	3.0(2)	1.83(6)
$\langle x^2\rangle^{\zeta_3}$	8.4(6)	8.4(6)	3.5(1)	0.27(1)	0.27(1)	0.22(1)	0.120(3)
$\langle x^3\rangle^{\zeta_3}$	4.0(3)	4.0(3)	0.82(5)	0.056(0)	0.056(0)	0.044(0)	0.022(1)

proton	u^p	d^p	g^p	$\mathcal{S}_p^u$	$\mathcal{S}_p^d$	$\mathcal{S}_p^s$	$\mathcal{S}_p^c$
$\langle x\rangle^{\zeta_2}$	32.9(1.4)	15.0(0.7)	40.9(1.1)	2.9(2)	3.7(3)	2.64(22)	1.32(5)
$\langle x^2\rangle^{\zeta_2}$	8.7(6)	3.6(2)	2.4(1)	0.14(1)	0.21(1)	0.13(0)	0.059(2)
$\langle x^3\rangle^{\zeta_2}$	2.9(3)	1.1(1)	0.39(2)	0.019(0)	0.030(1)	0.019(0)	0.008(0)
$\langle x\rangle^{\zeta_3}$	30.4(1.3)	13.8(0.6)	42.8(1.0)	3.3(3)	4.1(3)	3.0(2)	1.82(6)
$\langle x^2\rangle^{\zeta_3}$	7.7(5)	3.2(2)	2.2(1)	0.15(1)	0.21(1)	0.14(0)	0.075(2)
$\langle x^3\rangle^{\zeta_3}$	2.5(2)	0.9(1)	0.35(2)	0.019(0)	0.028(0)	0.019(0)	0.010(1)

Summary

Summary

- The **EHM** is argued to be intimately connected to a **PI effective** charge which enters a conformal regime, below a given momentum scale, where gluons acquiring a dynamical mass decouple from interaction.

- Capitalizing on the latter, two main ideas emerge: (I) the identification of that decoupling with a **hadronic scale** at which the structure of hadrons can be expressed only in terms of valence dressed partons; and (ii) the reliability of an **all-orders** evolution scheme to describe the splitting of valence into more partons, generating thus the glue and sea, when the resolution scale decreases.

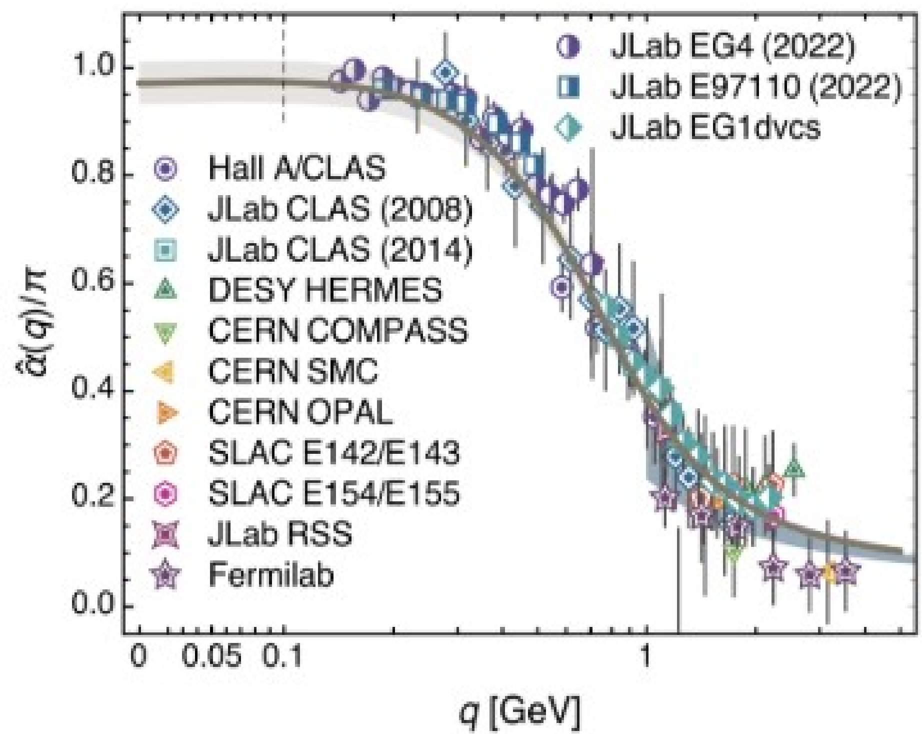
- Key implications stemming from both ideas have been derived and tested for the pion PDFs. Grounding on them, **Lattice QCD** and experimental data have been shown to confirm **CSM** results.

- The robustness of the approach based on **all-orders** evolution from **hadronic** to experimental scale has been proved with its application to the pion, kaon and proton cases. A model featuring massless evolution for quark flavors activated after a hard-wall threshold and accounrting for Pauli blocking has been solved analytically, and seen to expose some of the main results implied by the approach.

To be continued...

Backslides

QCD effective charge



Modern continuum & lattice QCD analysis in the gauge sector delivers an analogue "Gell-Mann-Low" running charge, from which one obtains a process-independent, parameter-free prediction for the low-momentum saturation

- No landau pole
- Below a given mass scale, the interaction become scale-independent and QCD practically conformal again (as in the lagrangian).

Then, we define:

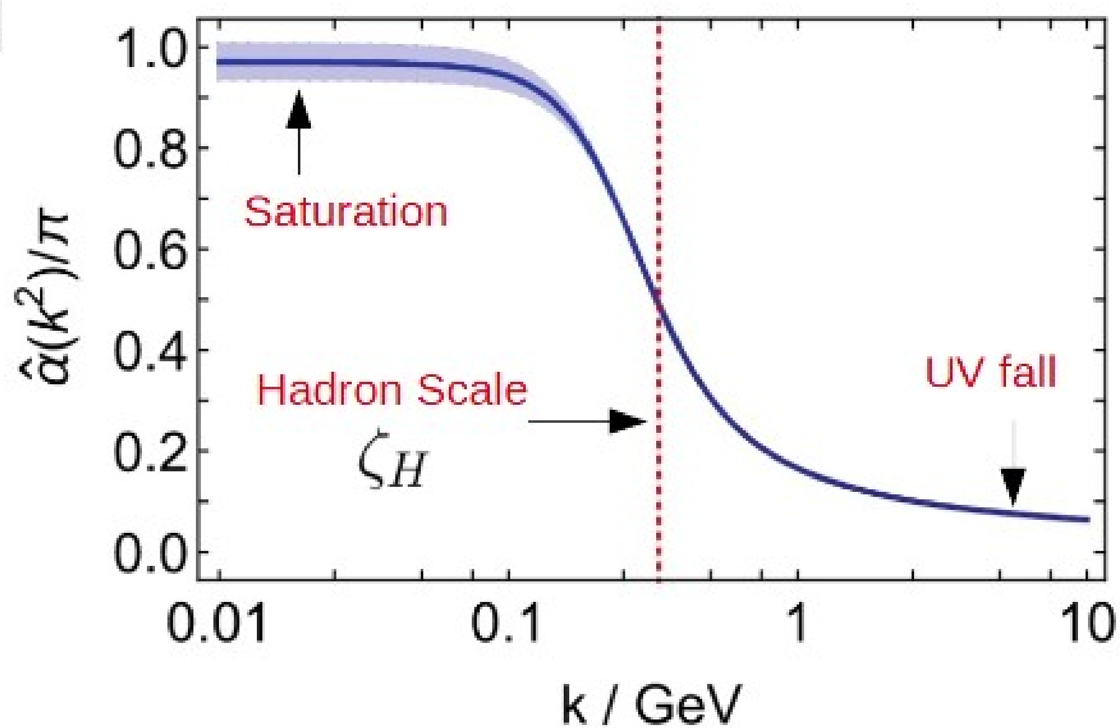
$$\alpha(k^2) = \frac{\gamma_m \pi}{\ln\left[\frac{\mathcal{M}^2(k^2)}{\Lambda^2_{\rm QCD}}\right]}\,;\;\; \alpha(0) = 0.97(4)$$

where

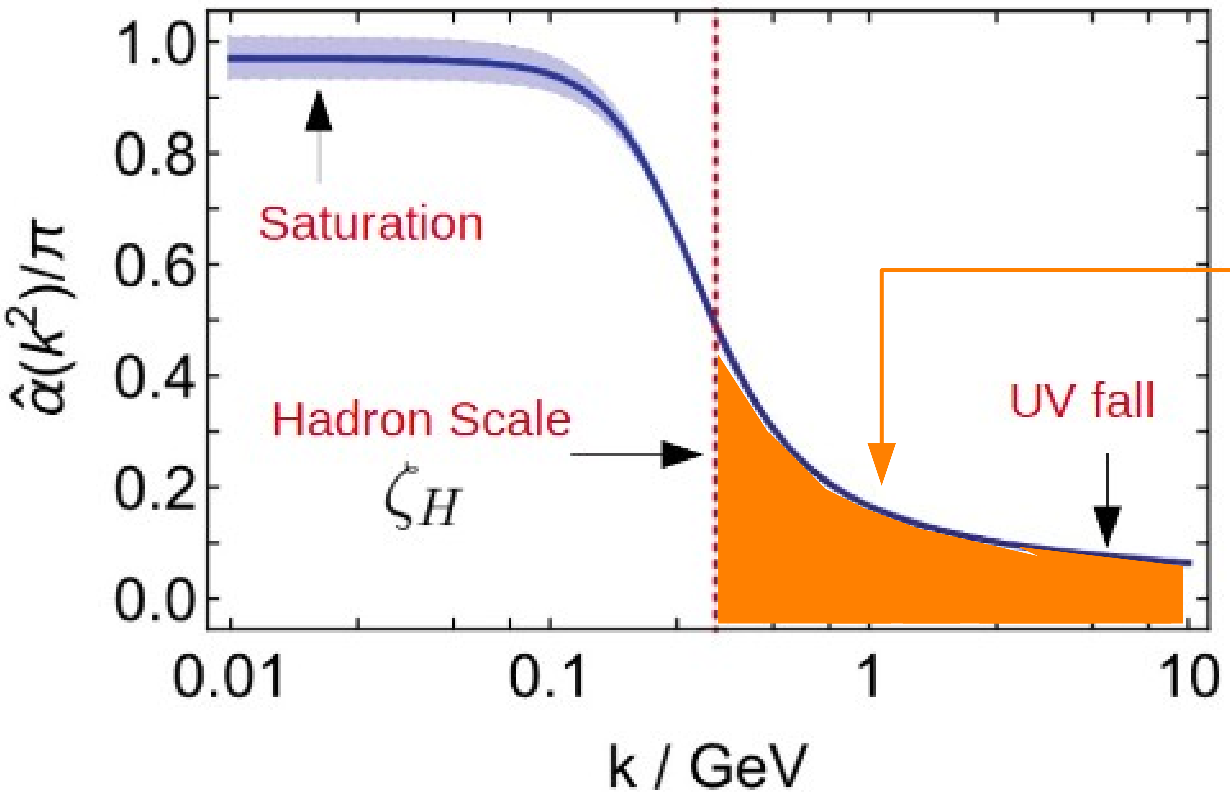
$$\mathcal{M}(k^2 = \Lambda^2_{\rm QCD}) := m_G = 0.331(2)\ \mathrm{GeV}$$

defines the screening mass and an associated wavelength, such that larger gluon modes decouple.

Then, we identify: $\zeta_H := m_G(1 \pm 0.1)$



QCD effective charge

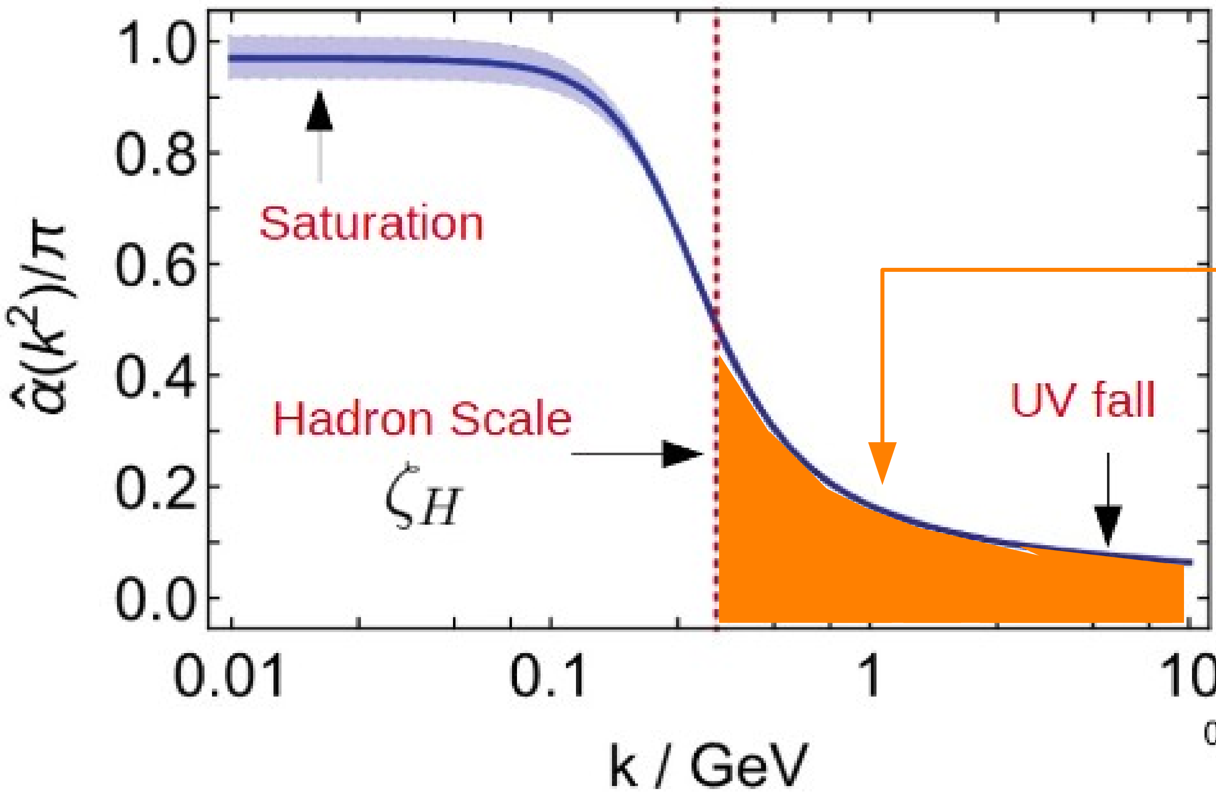


The strength of the charge defines de input for the evolution

$$S(\zeta_H, \zeta_f) = \int_{2\ln(\zeta_H/\Lambda_{QCD})}^{2\ln(\zeta_f/\Lambda_{QCD})} dt\, \widehat{\alpha}(t)$$

$$\langle x(\zeta_5)\rangle_q^\pi = \frac{1}{2}\exp\left(-\frac{8}{9\pi}S(\zeta_H, \zeta_5)\right) = 0.20(2)$$

QCD effective charge



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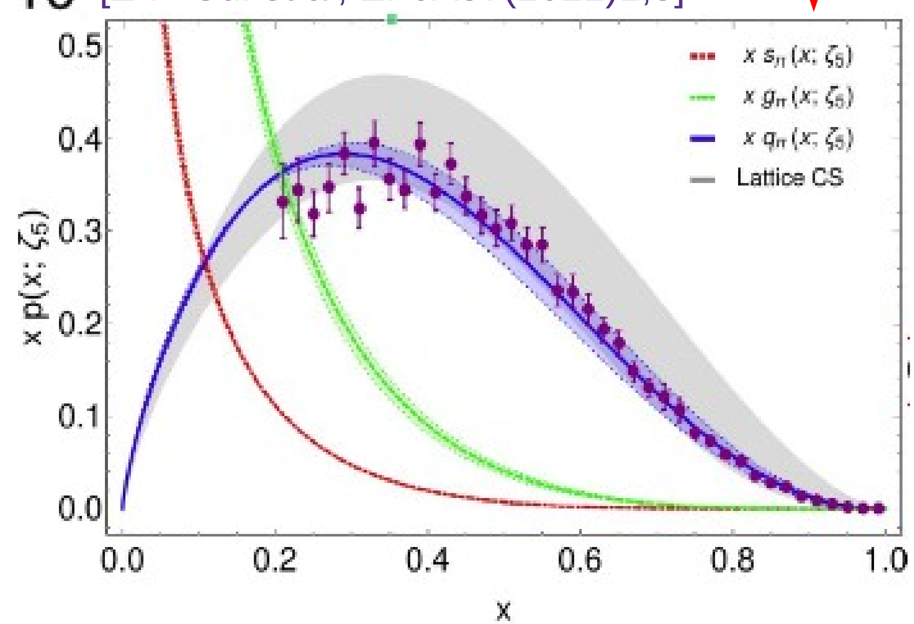
[Z-F. Cui et al, EPJC80(2020)11,1064]
[Z-F. Cui et al, EPJA57(2021)1,5]



Then, the glue, valence- and sea-quark DFs can be predicted, with no tuned parameter, on the ground of the effective charge definition, from the LFWF (or, equivalently, from a symmetry-preserving DSE/BSE computation of the valence-quarks Mellin moments

[M. Ding et al, CPC44(2020)3,031002]