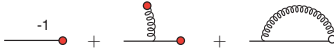


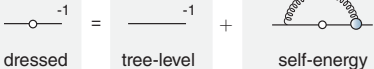
Derivation of DSEs: QCD

- Classical action: $S =$ 

- Classical EoM (Dirac equation) for **quark**: $\frac{\delta S}{\delta \bar{\psi}} =$  $= 0$

- Quantum EoM for **quark**: $\frac{\delta \Gamma}{\delta \bar{\psi}} =$ 

- Quark DSE:**

$$\left. \frac{\delta^2 \Gamma}{\delta \bar{\psi} \delta \psi} \right|_{A, \psi, \bar{\psi}=0} = \text{dressed}^{-1} = \text{tree-level}^{-1} + \text{self-energy}$$


Easy!

Derivation of DSEs: QCD

Classical action:

$$S = \text{---}^{-1} + \text{---} \text{---} + \text{---}^{-1} + \text{---} + \text{---} + \text{---}$$

(ignoring prefactors & i's)

(ignoring gauge fixing
⇒ ghosts)

Gluon: classical equation of motion (Maxwell equation)

$$\frac{\delta S}{\delta A} = \text{---} + \text{---}^{-1} + \text{---} + \text{---} = 0$$

Gluon: quantum equation of motion

$$\frac{\delta \Gamma}{\delta A} = \text{---} + \text{---} + \text{---} + \text{---} + \text{---}$$

Gluon DSE:

$$\left. \frac{\delta^2 \Gamma}{\delta A^2} \right|_{A, \psi, \bar{\psi}=0} = \text{dressed}^{-1} = \text{tree-level}^{-1} + \text{quark loop} + \text{gluon loop} + \text{tadpole} + \text{self-energy} + \text{sunset} + \text{squint}$$

DSEs

- **Quantum equations of motion:**

exact equations in QFT

Roberts, Williams, Prog. Part. Nucl. Phys. 33 (1994),
Alkofer, Smekal, Phys. Rept. 353 (2001), ...

- Like Feynman diagrams, but ingredients are dressed $\Rightarrow$ **integral equations**

- **Nonperturbative**, return perturbation theory if coupling is small

- Set of all n-point CFs determines QFT

- Each DSE contains higher n-point CFs $\Rightarrow$ **infinitely coupled system**

- But do we need to know *every* CF (e.g., 127-gluon vertex) to make predictions? Higher n-point CFs are suppressed by momentum powers, higher loops by propagators

$\Rightarrow$ **truncations:** solve **closed subset** of equations by neglecting higher n-point CFs or using ansätze

$\Rightarrow$ **systematically improvable:** can always enlarge system to get closer to full tower of exact equations

Quark DSE

$$\text{Dressed Quark Propagator}^{-1} = \text{Bare Quark Propagator}^{-1} + \text{Self-energy Diagram}$$

Gluon DSE

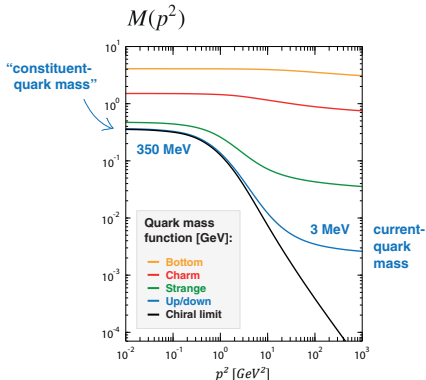
$$\text{Dressed Gluon Propagator}^{-1} = \text{Bare Gluon Propagator}^{-1} + \text{Ghost Loop} + \text{Quark Loop} + \text{Gluon Loops}$$

Three-gluon vertex DSE

$$\text{Dressed Three-gluon Vertex} = \text{Bare Three-gluon Vertex} + \text{Gluon Loops} + \text{Quark Loops} + \text{Ghost Loops}$$

- Similar: **functional renormalization group**

Berges, Tetradis, Wetterich,
Phys. Rept. 363 (2002),
Pawlowski, Annals. Phys. 322 (2007),
Dupuis et al., Phys. Rept. 910 (2021)



$$\text{---} \bigcirc \text{---}^{-1} = \text{---} \text{---}^{-1} + \text{---} \bullet \text{---} \bigcirc \text{---} + \text{---} \bullet \text{---} \bigcirc \text{---} \bullet \text{---}$$

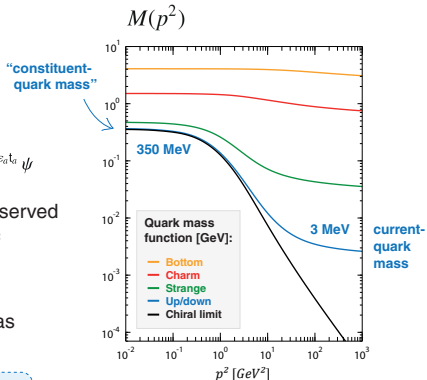
- Remember **chiral symmetry**:

$$SU(N_f)_V: \quad \psi' = e^{i \sum_a \varepsilon_a \mathbf{t}_a} \psi, \quad SU(N_f)_A: \quad \psi' = e^{i \gamma_5 \sum_a \varepsilon_a \mathbf{t}_a} \psi$$

- If chiral symmetry in classical Lagrangian were preserved in QFT, all n-point CFs would be chirally symmetric

$$\{\gamma_5, S(p)\} = 0 \quad \Rightarrow \quad M(p^2) = 0$$

- This is true in perturbation theory: each diagram has odd # gamma matrices $\Rightarrow \text{Trace} = 0 \Rightarrow M(p^2) = 0$

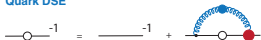


- So it must be a **nonperturbative effect**. Quark DSE:

$$S(p)^{-1} = Z_2 (i \not{p} + m_0) + \Sigma(p), \quad \Sigma(p) = -\frac{4g^2}{3} Z_\Gamma \int \frac{d^4 q}{(2\pi)^4} i \gamma^\mu S(q) D^{\mu\nu}(k) \Gamma^\nu(l, k)$$

Quark DSE

Quark DSE



$$S(p)^{-1} = Z_2 (i\not{p} + m_0) + \Sigma(p), \quad \Sigma(p) = -\frac{4g^2}{3} Z_1 \int \frac{d^4 q}{(2\pi)^4} i\gamma^\mu S(q) D^{\mu\nu}(k) \Gamma^\nu(l, k)$$

- An illustrative model: gluon propagator = δ distribution [Munczek, Nemirovsky, Phys. Rev. D 28 \(1983\)](#)

$$\frac{4}{3} \frac{g^2}{(2\pi)^4} D^{\mu\nu}(k) \rightarrow \Lambda^2 \delta^4(k) \delta^{\mu\nu} \Rightarrow S(p)^{-1} = i\not{p} + m_0 + \Lambda^2 \gamma^\mu S(p) \gamma^\mu \Rightarrow A(p^2) = 1 + \frac{2\Lambda^2}{A(p^2)(p^2 + M(p^2)^2)},$$

$$A(p^2)M(p^2) = m_0 + 2M(p^2) \frac{2\Lambda^2}{A(p^2)(p^2 + M(p^2)^2)}$$

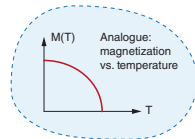
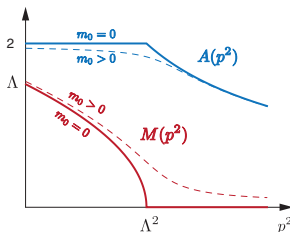
- Chiral limit ($m_0 = 0$):

Chirally symmetric solution:

$$A(p^2) = \frac{1}{2} \left(1 + \sqrt{1 + \frac{8\Lambda^2}{p^2}} \right), \quad M(p^2) = 0$$

Chiral-symmetry breaking solution:

$$A(p^2) = 2, \quad M(p^2) = \sqrt{\Lambda^2 - p^2}$$

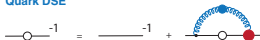


- In realistic DSE solution, no sharp phase transition in chiral limit, but otherwise similar



Quark DSE

Quark DSE



$$S(p)^{-1} = Z_2 (i\not{p} + m_0) + \Sigma(p), \quad \Sigma(p) = -\frac{4g^2}{3} Z_1 \int \frac{d^4 q}{(2\pi)^4} i\gamma^\mu S(q) D^{\mu\nu}(k) \Gamma^\nu(l, k)$$

• Renormalization:

$$\Sigma(p) = i\not{p} \Sigma_A(p^2) + \Sigma_M(p^2) \quad \Rightarrow \quad \begin{aligned} A(p^2) &= Z_2 + \Sigma_A(p^2), \\ M(p^2)A(p^2) &= Z_2 m_0 + \Sigma_M(p^2) \end{aligned}$$

Self-energy integrals are logarithmically UV-divergent

⇒ **regularize** by integrating up to cutoff,

⇒ **renormalize** by setting 2 boundary conditions: $A(\mu^2) \stackrel{!}{=} 1$, $M(\mu^2) \stackrel{!}{=} m$

$$Z_2 = 1 - \Sigma_A(\mu^2), \quad m_0 = \frac{m - \Sigma_M(\mu^2)}{1 - \Sigma_A(\mu^2)}, \quad A(p^2) = 1 + \Sigma_A(p^2) - \Sigma_A(\mu^2),$$

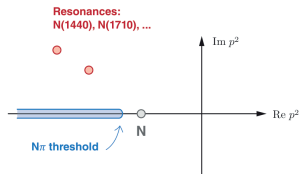
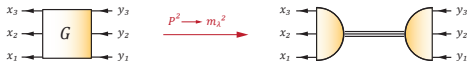
$$M(p^2)A(p^2) = m + \underbrace{\Sigma_M(p^2) - \Sigma_M(\mu^2)}_{\text{finite}}$$

Divergences cancel in subtraction!
Renormalized $A(p^2)$ now depends on renormalization scale μ , but $M(p^2)$ does not

- DSE can be solved **iteratively**: Start with guess for $A(p^2)$, $M(p^2)$, e.g., set them to 1. Calculate self-energy integrals & evaluate them at renormalization scale. Determine Z_2 . Calculate $A(p^2)$, $M(p^2)$ again. Proceed until converged.

Hadrons?

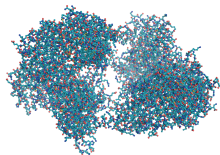
Baryon poles appear in elementary quark 6-point function:



How to calculate this? In principle from DSEs:

The diagram shows the Dyson-Schwinger equations for the quark and gluon propagators. The first equation is for the quark propagator: a bare quark line with a circle and $^{-1}$ equals a dressed quark line with a circle and $^{-1}$ plus a loop diagram with a quark line and a gluon line. The second equation is for the gluon propagator: a bare gluon line with a circle and $^{-1}$ equals a dressed gluon line with a circle and $^{-1}$ plus a series of loop diagrams involving quarks and gluons. Vertical dots indicate higher-order terms.

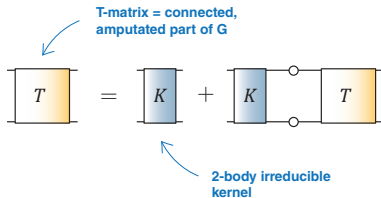
The diagram shows the Bethe-Salpeter equations for the quark-antiquark and quark-gluon vertices. On the left, two boxes are shown: the top one labeled $G_{q\bar{q}}$ and the bottom one labeled G_{3q} . Each box has two external lines. To the right of each box is an equals sign followed by an ellipsis. A large curly brace groups these equations.



complicated structure & equations
for higher n-point functions,
more efficient: solve
Bethe-Salpeter equations

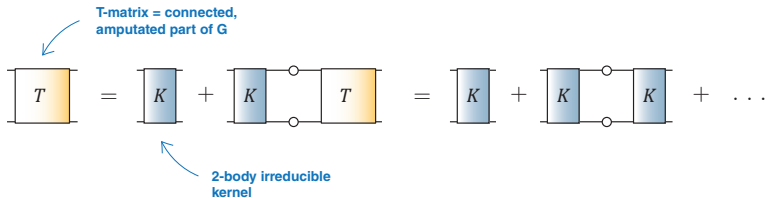
Bethe-Salpeter equations

Write down **inhomogeneous BSE**:



Bethe-Salpeter equations

Write down **inhomogeneous BSE**:



Analogy: **geometric series**

$$\begin{aligned}
 f(x) &= 1 + x f(x) \\
 &= 1 + x + x^2 f(x) \\
 &= 1 + x + x^2 + x^3 f(x)
 \end{aligned}
 \Rightarrow f(x) = \frac{1}{1-x}$$

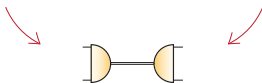
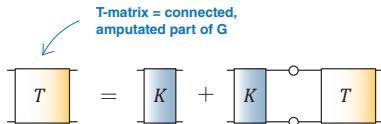
“non-perturbative”

$$f(x) \approx 1 + x + x^2 + x^3 + \dots \quad \text{only for } |x| < 1$$

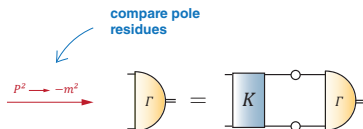
“perturbative”

Bethe-Salpeter equations

Write down **inhomogeneous BSE**:



Homogeneous BSE at pole:



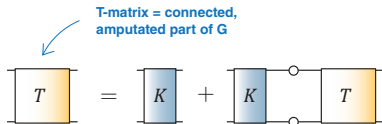
- $q\bar{q}$ irreducible kernel
- chiral symmetry constraints ($V + AV$ WT1)
- can be systematically derived from effective action, depends on QCD's n-point functions



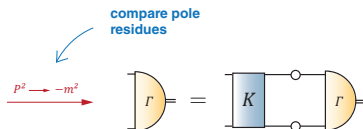
- Analogue of Schrödinger equation in QFT!
- Γ = **Bethe-Salpeter amplitude**

Bethe-Salpeter equations

Write down **inhomogeneous BSE**:

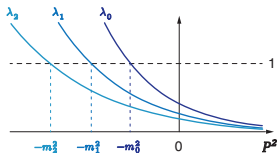


Homogeneous BSE at pole:



BSE = eigenvalue equation,
pole in T $\Leftrightarrow$ **eigenvalue = 1**

$$KG_0 \Gamma_i = \lambda_i \Gamma_i$$



Explicitly:

$$\Gamma(p, P) = \int \frac{d^4 q}{(2\pi)^4} K_{\alpha\gamma, \delta\beta}(p, q, P) [S(q_+) \Gamma(q, P) S(q_-)]_{\gamma\delta}$$

Basis decomposition:

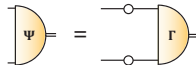
$$\Gamma(p, P) = \sum_{i=1}^n f_i(p^2, \hat{p} \cdot \hat{P}, P^2) \tau_i(p, P)$$

$\Rightarrow$ Coupled Lorentz-invariant equations
for the dressing functions f_i

Chiral symmetry and the pion

Maris, Roberts, Tandy, PLB 420 (1998)

- **Bethe-Salpeter WF** of pion: $[i\Psi(p, P)]_{\alpha\beta} = \int d^4x e^{-ipx} \langle 0 | T \psi_\alpha(x) \bar{\psi}_\beta(0) | P, n \rangle$



- **Axialvector and pseudoscalar currents:**

↙ pion electroweak decay constant

$$f_5^\mu(x) = Z_2 \bar{\psi}(x) i\gamma^\mu \gamma_5 \psi(x), \quad \langle 0 | f_5^\mu(x) | P \rangle = \langle 0 | f_5^\mu(0) | P \rangle e^{ix \cdot P} = i f_\pi P^\mu e^{ix \cdot P},$$

$$j_5(x) = Z_2 Z_m \bar{\psi}(x) i\gamma_5 \psi(x), \quad \langle 0 | j_5(x) | P \rangle = \langle 0 | j_5(0) | P \rangle e^{ix \cdot P} = r_\pi e^{ix \cdot P}.$$

$$\Rightarrow f_\pi P^\mu = Z_2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \{ i\gamma_5 \gamma^\mu \Psi(p, P) \}, \quad r_\pi = Z_2 Z_m \int \frac{d^4p}{(2\pi)^4} \text{Tr} \{ \gamma_5 \Psi(p, P) \}$$



- Now plug in **PCAC relation** from classical Lagrangian:

$$\partial_\mu j_5^\mu = 2m j_5 \Rightarrow f_\pi m_\pi^2 = 2m r_\pi \Rightarrow \text{if } f_\pi \text{ does not vanish in chiral limit, } m_\pi \text{ must be zero!}$$

- How can one see this? From **axialvector WTI** (= PCAC for 3-point functions) in chiral limit:

$$f_\pi \Gamma(p, 0) = A(p^2) M(p^2) \gamma_5 \quad \text{We know that } M(p^2) \neq 0 \text{ in chiral limit} \Rightarrow f_\pi \neq 0 \Rightarrow m_\pi = 0. \text{ Goldstone theorem!}$$

$$\text{Take trace with } S(p) \gamma_5 S(p) \Rightarrow f_\pi r_\pi = -\frac{\langle \bar{\psi} \psi \rangle}{N_f} = Z_2 Z_m N_c \int \frac{d^4p}{(2\pi)^4} \text{Tr} S(p) = Z_2 Z_m \frac{N_c}{(2\pi)^2} \int d^4p \frac{p^2}{A(p^2)} \frac{M(p^2)}{p^2 + M(p^2)^2}$$

$$\text{Plug this back into PCAC relation} \Rightarrow f_\pi^2 m_\pi^2 = -2m \langle \bar{\psi} \psi \rangle / N_f \quad \text{Gell-Mann-Oakes-Renner relation}$$

Chiral symmetry and the pion

So far, everything exact $\Rightarrow$ pion from BSE is automatically **massless in chiral limit** if

- **BSE kernel respects axialvector WT**
- Chiral symmetry is dynamically broken such that $M(p^2) \neq 0$ in chiral limit (automatic in DSE)

... axialvector WT

... vector WT

quark self-energy

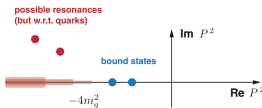
BSE kernel

$\Rightarrow$ BSE kernel and quark propagator **must be related!**

- Let's try a gluon **ladder kernel**:

free propagators

$$\frac{-i\not{p} + m}{p^2 + m^2}$$

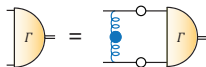


- breaks chiral symmetry ⚡
- generates bound-state poles in T, possibly also resonances
- but also quark thresholds & cuts: “hadrons” decay into quarks, ⚡
- would be ok if elementary d.o.f. were not quarks but **hadrons** ($\rightarrow$ EFTs)

Chiral symmetry and the pion

- Better: **rainbow-ladder**

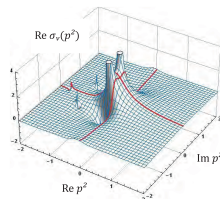
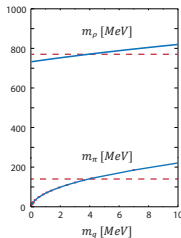
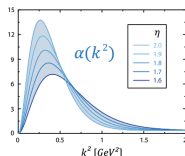
Maris, Roberts, PRC 56 (1997),
Maris, Tandy, PRC 60 (1999),
Qin, Chang et al., PRC 84 (2011)



dressed propagators
from quark DSE

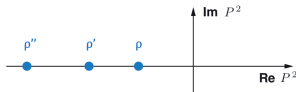
$$\frac{1}{A(p^2)} \frac{-i\not{p} + M(p^2)}{p^2 + M(p^2)^2}$$

Kernel = effective gluon exchange
with effective interaction



► chiral symmetry automatic,
massless pion in
chiral limit ✓

► quark propagators do not
develop real poles
⇒ no quark thresholds ✓



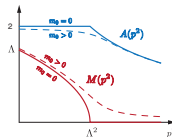
► but also no resonances, instead **bound states**
(need to go beyond rainbow-ladder)

Chiral symmetry and the pion

Consider again our earlier model
(gluon propagator = δ distribution)

Munczek, Nemirovsky, Phys. Rev. D 28 (1983)

$$\frac{4}{3} \frac{g^2}{(2\pi)^4} D^{\mu\nu}(k) \rightarrow \Lambda^2 \delta^4(k) \delta^{\mu\nu}$$



$$A(p^2) = 2,$$

$$M(p^2) = \sqrt{\Lambda^2 - p^2}$$

- Quark condensate:

$$-\langle \bar{u}u \rangle = \frac{N_c}{(2\pi)^2} \int_0^{\Lambda^2} dp^2 p^2 \frac{\sqrt{\Lambda^2 - p^2}}{2\Lambda^2} = \frac{2}{15} \frac{N_c}{(2\pi)^2} \Lambda^3 \approx (220 \text{ MeV})^3$$

$\Lambda = 1 \text{ GeV}$



not bad!

- Meson BSE for $P^2 = 0 \Rightarrow P^\mu = 0 \Rightarrow$ pion amplitude has only 1 tensor: $\Gamma(p) = f_1(p^2) \gamma_5$

$$\Gamma(p) = \frac{4g^2}{3} \int \frac{d^4 q}{(2\pi)^4} i\gamma^\mu S(q) \Gamma(q) S(q) i\gamma^\nu D^{\mu\nu}(k) \rightarrow -\Lambda^2 \gamma^\mu S(p) \Gamma(p) S(p) \gamma^\mu$$

$$\Rightarrow 1 = \Lambda^2 \gamma^\mu \gamma_5 S(p) \gamma_5 S(p) \gamma^\mu = \Lambda^2 \gamma^\mu S(-p) S(p) \gamma^\mu = \frac{4\Lambda^2}{A(p^2)^2 (p^2 + M(p^2)^2)}$$

yes!

BSE has eigenvalue 1
at $P^2 = 0 \Rightarrow$ massless
pion in chiral limit

- Pion decay constant: use $f_1(p^2) = A(p^2)M(p^2)/f_\pi$

$$f_\pi^2 = \frac{N_c Z_2}{4\pi^2} \int dp^2 p^2 \frac{M(p^2)}{A(p^2)} \frac{1}{(p^2 + M(p^2)^2)^2} \left(M(p^2) - \frac{p^2}{4} \frac{dM(p^2)}{dp^2} \right) \quad \dots \text{Pagels-Stokar formula, PRD 20 (1979)}$$

Munczek-Nemirovsky model: $f_\pi = \sqrt{\frac{N_c}{2}} \frac{\Lambda}{4\pi} \approx 97 \text{ MeV}$ not bad! **GMOR:** $m_\pi^2 = \frac{32}{15} \Lambda m \approx 2\Lambda m$

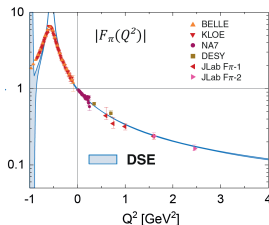
$\uparrow$
 $\Lambda = 1 \text{ GeV}$

Rainbow-ladder results

Rainbow-ladder has been
very useful tool, many calculations
over past > 25 years

Maris, Tandy, Nucl. Phys. B Proc. Suppl. 161 (2006),
Bashir, Chang, Cloet, El-Bennich, Liu, Roberts, Tandy,
Comm. Th. Phys. 58 (2012),
GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
Prog. Part. Nucl. Phys. 91 (2016),
....

• Pion electromagnetic form factor

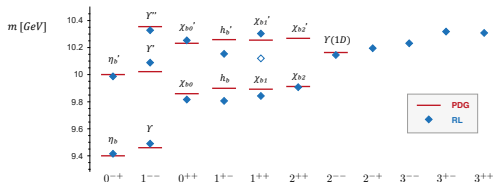


GE, Fischer,
Williams,
PRD 101
(2020)

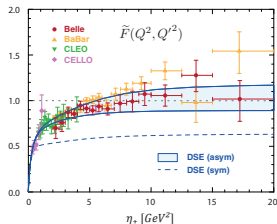
↖ **ρ pole:** vector-meson dominance
comes out automatically!

• Bottomonium spectrum

Fischer, Kubrak, Williams, EPJ A 51 (2015)



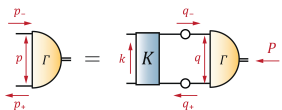
• Pion $\rightarrow \gamma\gamma$ transition form factor



GE, Fischer,
Weil, Williams,
PLB 774 (2017)

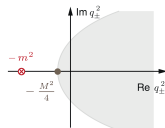
Quark in complex plane

For BSE, we usually need quark propagator for **complex momenta** ($p^2 \in \mathbb{C}$).



$$P^2 = -M^2 \Rightarrow \text{total momentum } P^\mu \text{ complex}$$

$$q_\pm^\mu = q^\mu \pm \frac{P^\mu}{2} \Rightarrow \text{quark momenta } q_\pm^\mu \text{ also complex}$$



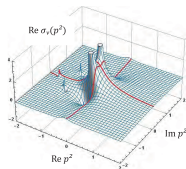
sampled on parabola in complex plane, limited by nearest quark singularities

How to do this?

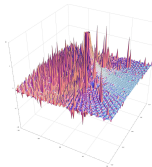
- For some interactions (e.g., Maris-Tandy) easy: put $p^2 \in \mathbb{C}$ in quark DSE and “iterate once more”

$$\text{---} \bigcirc^{-1} = \text{---}^{-1} + \text{---} \text{---} \bigcirc^{-1}$$

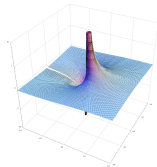
$$S(p)^{-1} = Z_2 (i\not{p} + m_0) - \frac{4g^2}{3} Z_\Gamma \int \frac{d^4 q}{(2\pi)^4} i\gamma^\mu S(q) D^{\mu\nu}(k) \Gamma^\nu(l, k)$$



- In general not possible: pole in gluon propagator produces cuts in integration. Naive integration over cuts yields nonsense
- Workaround: **analytic continuation** from real axis
- Solution: **contour deformations** or **Cauchy method**



Naive calculation



Contour deformations

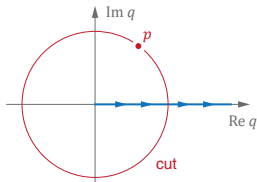
Quark in complex plane

Contour deformations:

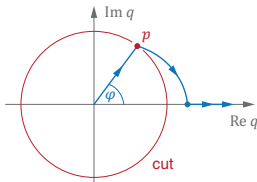
$$\text{---}\bigcirc\text{---}^{-1} = \text{---}\text{---}^{-1} + \text{---}\bigcirc\text{---}$$

$$S(p)^{-1} = Z_2 (i\not{p} + m_0) - \frac{4g^2}{3} Z_\Gamma \int \frac{d^4 q}{(2\pi)^4} i\gamma^\mu S(q) D^{\mu\nu}(k) \Gamma^\nu(l, k)$$

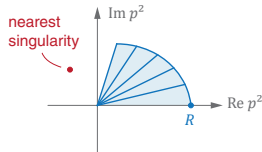
- Integration: $\int d^4 q (\dots) \rightarrow \int_0^\Lambda dq^2 \int_{-1}^1 dz (\dots) \Rightarrow q \text{ is real, } z \in [-1, 1]$
- Gluon momentum: $k^2 = (p - q)^2 = p^2 + q^2 - 2pqz \Rightarrow q^2 - 2pqz + p^2 - k^2 = 0 \Rightarrow q = p \left(z \pm i \sqrt{1 - z^2 - \frac{k^2}{p^2}} \right)$
 $k^2 = 0 \Rightarrow \text{circle in complex } q \text{ plane with opening at } q = p.$



If integrand is singular at $k^2 = 0$:
circular branch cut after z integration,
cannot integrate q along real axis



Deform contour such that it passes through p .
Because DSE is integral equation, we need
quark propagator from previous iteration for
these points in $q \Rightarrow$ solve DSE along this path!
Ensures that you will never cross a cut



Can solve DSE in whole
complex plane, up to angle
where one hits the first
dyn. generated singularity

Quark in complex plane

Contour deformations:

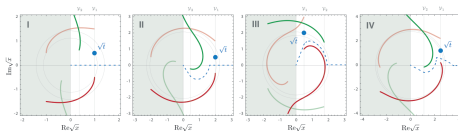
Fermion, gluon, ghost propagators,
glueball correlator, rare pion decay,
 ρ and σ pole position, spectral functions,
quark-photon vertex, timelike pion
form factor, light-front wave functions,
...

Maris, PRD 52 (1995)
Alkofer, Detmold, Fischer, Maris, PRD 70 (2004)
GE, 0909.0703 [hep-ph]
Strauss, Fischer, Kellermann, PRL 109 (2012)
Windisch, Alkofer, Haase, Liebmann, CPC 184 (2013)
Windisch, Huber, Alkofer, PRD 87 (2013)
Weil, GE, Fischer, Williams, PRD 96 (2017)
Pawlowski, Strodthoff, Wink, PRD 98 (2018)
Williams, PLB 798 (2019)
Miramontes, Sanchis-Alepuz, EPJ A 55 (2019)
GE, Duarte, Pena, Stadler, PRD 100 (2019)
Fischer, Huber, PRD 102 (2020)
Santowsky, GE, Fischer, Wallbott, Williams, PRD 102 (2020)
Miramontes, Sanchis-Alepuz, Alkofer, PRD 103 (2021)
Santowsky, Fischer, EPJ C 82 (2022)
GE, Ferreira, Stadler, PRD 105 (2022)
...

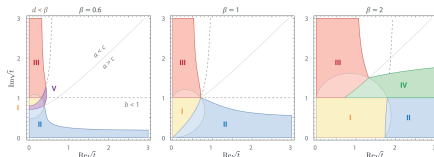
- **Scalar scattering equation:**
direct access to 2nd sheet!

$$\text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5}$$

Cuts from K and G_0 in complex plane of integration variable:

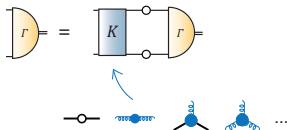


Can still cover **parts** of complex t plane:



Towards ab-initio

Go towards ab-initio calculations
by calculating **higher n-point functions**



Various approaches on the market, e.g.:

- **Gauging quark DSE** yields vertex equation $\Rightarrow$ meson poles

Chang, Roberts, PRL 103 (2009)

- Expansion of BSE kernel via **nPI effective action**

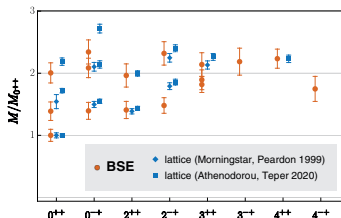
Williams, Fischer, Heupel, PRD 93 (2016)

- Including $\pi\pi$ decay channel generates **resonances**

Santowsky et al., PRD 102 (2020)

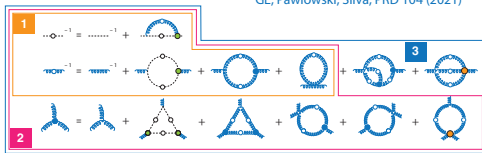
- **Glueball spectrum** agrees with lattice QCD

Huber, Fischer, Sanchis-Alepuz, EPJ C 80 (2020), EPJ C 81 (2021)



- Coupled **Yang-Mills DSEs**

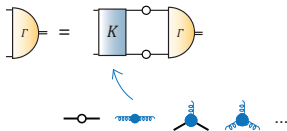
Huber, PRD 101 (2020),
GE, Pawłowski, Silva, PRD 104 (2021)



Truncation error: **1** 60% **2** 10% **3** 4%

Towards ab-initio

Go towards ab-initio calculations
by calculating **higher n-point functions**



Various approaches on the market, e.g.:

- **Gauging quark DSE** yields vertex equation $\Rightarrow$ meson poles

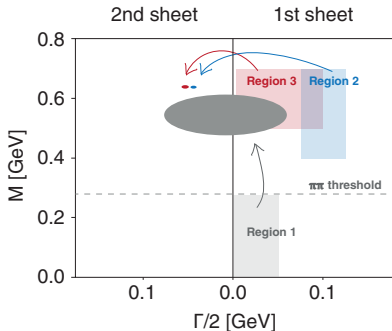
[Chang, Roberts, PRL 103 \(2009\)](#)

- Expansion of BSE kernel via **nPI effective action**

[Williams, Fischer, Heupel, PRD 93 \(2016\)](#)

- Including $\pi\pi$ decay channel generates **resonances**

[Santowsky et al., PRD 102 \(2020\)](#)



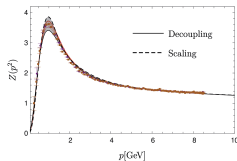
ρ resonance extracted from **homogeneous BSE**:

Solve ρ -meson BSE for complex total momenta on 1st sheet, analytically continue to 2nd sheet to determine pole resonance pole location

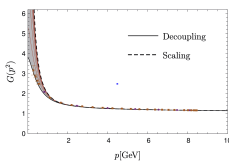
Towards precision

Markus Huber, ACHT 2025; Phys. Rept. 879 (2020) & PRD 101 (2020)

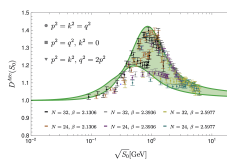
- Gluon propagator



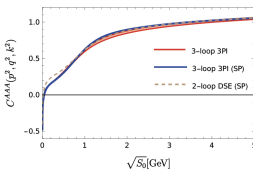
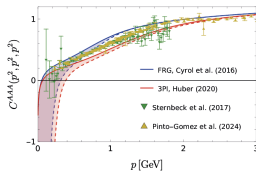
- Ghost propagator



- Ghost-gluon vertex

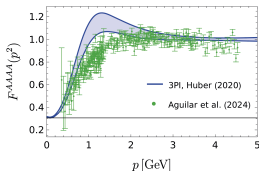


- Three-gluon vertex: lattice vs. FRG vs. DSE



Lattice: Pinto-Gómez, de Soto, Rodríguez-Quintero, PLB 838 (2023), Sternbeck et al., Pos LATTICE 2016; **FRG:** Cyrol et al., PRD 94 (2016); **DSE:** Huber, PRD 101 (2020), Huber, Fischer, Sanchis-Alepuz, 2503.03821

- Four-gluon vertex

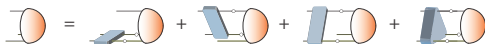


Lattice: Aguilar et al., PLB 858 (2024)
DSE: Huber, PRD 101 (2020)

Baryons

Three-quark BSE (Faddeev equation) for baryons:

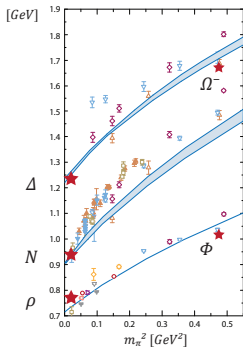
GE, Alkofer, Nicmorus, Krassnigg, PRL 104 (2010)



2-body kernel:
fixed in meson sector

3-body kernel:

Leading diagram (3-gluon vertex)
vanishes by color trace,
higher-order diagrams small (?)
2-quark correlations dominant?



- Analogous results for many **form factors**

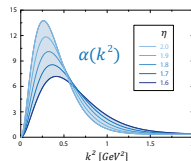
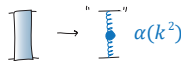
Review: GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
Prog. Part. Nucl. Phys. 91 (2016)

- Relativistically, nucleon also has **p waves!**

L = 0

L = 1

Rainbow-ladder



Scale set by f_π ,
shape parameter $\rightarrow$ bands

Maris, Tandy, PRC 60 (1999)

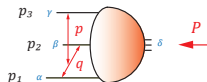
see also:

Qin, Roberts, Schmidt,
PRD 97 (2018)

Baryons

Baryon's Faddeev amplitude:

GE, Alkofer, Nicmorus, Krassnigg, PRL 104 (2010)



$$\Psi_{\alpha\beta\gamma\delta}(p, q, P) = \sum_i f_i(p^2, q^2, p \cdot q, p \cdot P, q \cdot P) \tau_i(p, q, P)_{\alpha\beta\gamma\delta} \otimes \text{Flavor} \otimes \text{Color}$$

Lorentz-invariant
dressing functions

Dirac-Lorentz
tensors:

64 for J = 1/2,
128 for J = 3/2

group theory
(same as before)

s	l	T_{ij}
$1/2$	0	$\mathbb{1} \otimes \mathbb{1}$
$1/2$	0	$\gamma_5^{\mu} \otimes \gamma_5^{\mu}$
$1/2$	1	$\mathbb{1} \otimes \frac{1}{2} [\hat{p}, \hat{q}]$
$1/2$	1	$\mathbb{1} \otimes \hat{p}$
$1/2$	1	$\mathbb{1} \otimes \hat{q}$
$1/2$	1	$\gamma_5^{\mu} \otimes \gamma_5^{\mu} \frac{1}{2} [\hat{p}, \hat{q}]$
$1/2$	1	$\gamma_5^{\mu} \otimes \gamma_5^{\mu} \hat{p}$
$1/2$	1	$\gamma_5^{\mu} \otimes \gamma_5^{\mu} \hat{q}$
$3/2$	1	$3(\hat{p} \otimes \hat{q} - \hat{q} \otimes \hat{p}) - \gamma_5^{\mu} \otimes \gamma_5^{\mu} [\hat{p}, \hat{q}]$
$3/2$	1	$3\hat{p} \otimes \mathbb{1} - \gamma_5^{\mu} \otimes \gamma_5^{\mu} \hat{p}$
$3/2$	1	$3\hat{q} \otimes \mathbb{1} - \gamma_5^{\mu} \otimes \gamma_5^{\mu} \hat{q}$
$3/2$	2	$3\hat{p} \otimes \hat{p} - \gamma_5^{\mu} \otimes \gamma_5^{\mu}$
$3/2$	2	$\hat{p} \otimes \hat{p} + 2\hat{q} \otimes \hat{q} - \gamma_5^{\mu} \otimes \gamma_5^{\mu}$
$3/2$	2	$\hat{p} \otimes \hat{q} + \hat{q} \otimes \hat{p}$
$3/2$	2	$\hat{q} \otimes [\hat{p}, \hat{p}] - \frac{1}{2} \gamma_5^{\mu} \otimes [\gamma_5^{\mu}, \hat{p}]$
$3/2$	2	$\hat{p} \otimes [\hat{p}, \hat{q}] - \frac{1}{2} \gamma_5^{\mu} \otimes [\gamma_5^{\mu}, \hat{q}]$

$$T_{ij}(\Lambda_{\pm} \gamma_5 C \otimes \Lambda_{+})$$

$$(\gamma_5 \otimes \gamma_5) T_{ij}(\Lambda_{\pm} \gamma_5 C \otimes \Lambda_{+})$$

Positive-energy
projector: $\Lambda_{+} u = u$

- can be arranged into eigenstates of **S** and **L** in baryon's rest frame: s, p, d, f, waves

- nonrel. quark model:**

only 2 survive ($l = 0$)

$$\left[\frac{1}{\sqrt{3}} (\uparrow\uparrow\uparrow + \uparrow\downarrow\downarrow - 2\downarrow\downarrow\downarrow) \right], \left[\frac{1}{\sqrt{6}} (\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow - 2\downarrow\uparrow\uparrow) \right]$$

- Relativistically, nucleon also has **p waves**!

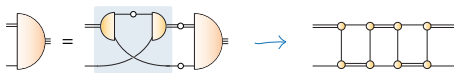
s waves

p waves

Diquark correlations

- Quark-diquark** (two-body) equation

Oettel et al., PRC 58 (1998), GE et al., Ann. Phys. 323 (2008), Cloet et al., FBS 46 (2009), Segovia et al., PRL 115 (2015)



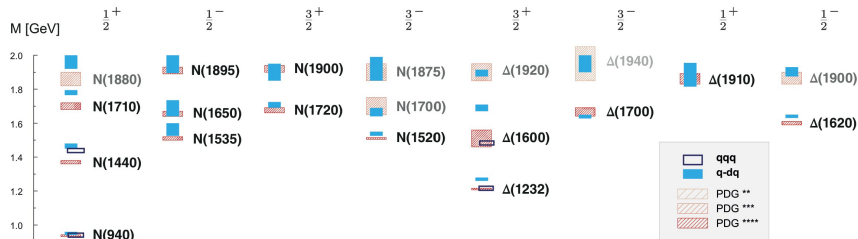
Diquark clustering in baryons?

Barabanov et al., Prog. Part. Nucl. Phys. 116 (2021)



- Three-quark** and **quark-diquark** results very similar

GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)



Diquarks 101



Wave function: Dynamics $\otimes$ $\underbrace{\text{Flavor}}_{A, S}$ $\otimes$ $\underbrace{\text{Color}}_A$

attractive
 $3 \otimes 3 = \bar{3} \oplus 6$
 $\bar{3}$ ←

Flavor:

$$2 \otimes 2 = 1_A \oplus 3_S$$

$$3 \otimes 3 = \bar{3}_A \oplus 6_S$$

$$4 \otimes 4 = 6_A \oplus 10_S$$

$[ud]$
 $[us], [ds]$
 $[uc], [dc], [sc]$
 "good"

$\{uu\}, \{ud\}, \{dd\}$
 $\{us\}, \{ds\}, \{ss\}$
 $\{uc\}, \{dc\}, \{sc\}, \{cc\}$
 "bad"

$I = 0$
 $I = 1$
 $I = 1/2$

Quark content	q-dq	Isospin	contributes to
sss	s {ss}	0	Ω
ccc	c {cc}	0	Ω_c
uud	u [ud] u {ud} d {uu}	1/2 1/2, 3/2 1/2, 3/2	p p, Δ^+ p, Δ^+
nns	n [ns] n {ns} s [nn] s {nn}	0, 1 0, 1 0 1	Λ, Σ Λ, Σ Λ Σ

$\uparrow$
 $n = u, d$

- Nucleon has "good" and "bad" diquarks, Delta only "bad"
- Λ : n [ns], n {ns}, s [nn]
 Σ : n [ns], n {ns}, s {nn}

Diquarks 101



Wave function: $\underbrace{\text{Dynamics}}_{\mathcal{A}, \mathcal{S}} \otimes \underbrace{\text{Flavor}}_{\mathcal{A}, \mathcal{S}} \otimes \underbrace{\text{Color}}_{\mathcal{A}} \stackrel{!}{=} \mathcal{A}$

- $\gamma_5 C, \dots$ **scalar** $\mathcal{A}: (\gamma_5 C)^T = -\gamma_5 C \Leftrightarrow \text{"good"}$
 - $\gamma^\mu C, \dots$ **axialvector** $\mathcal{S}: (\gamma^\mu C)^T = \gamma^\mu C \Leftrightarrow \text{"bad"}$
 - $C, \dots$ **pseudoscalar** $\mathcal{A}$
 - $\gamma^\mu \gamma_5 C, \dots$ **vector** $\mathcal{A}$
 - ...
- } "ugly"

Total wave function:

$$\Gamma^{(\mu)}(q, P) = \sum_i \underbrace{f(q^2, q \cdot P, P^2 = -m^2)}_{\substack{\text{Dressing functions} \\ \text{calculated from BSE}}} \underbrace{\tau_i^{(\mu)}(q, P)}_{\substack{\text{Lorentz/Dirac} \\ \text{tensors}}} \otimes \text{Flavor} \otimes \text{Color}$$

↖

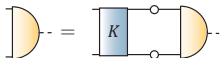
Diquark correlations

Mesons and diquarks are closely related:

$$3 \otimes \bar{3} = \overset{\text{attractive}}{\color{red}1} \oplus 8$$

$$3 \otimes 3 = \overset{\text{attractive}}{\color{red}\bar{3}} \oplus 6$$

In BSE this comes out naturally: [Maris, FBS 32 \(2002\)](#)

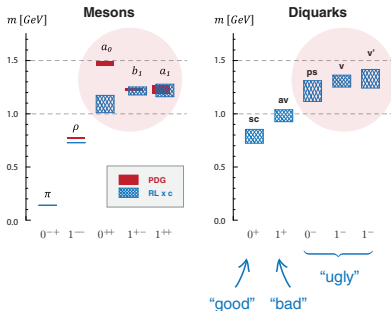


Lowest-lying diquarks are dominant for ground-state octet & decuplet baryons

pseudoscalar mesons $\Leftrightarrow$ **scalar diquarks** (~ 0.8 GeV)
 vector mesons $\Leftrightarrow$ **axialvector diquarks** (~ 1 GeV)

Higher-lying diquarks are subleading, but contribute to excited states & remaining channels

scalar mesons $\Leftrightarrow$ **pseudoscalar diquarks** (~ 1.2 GeV)
 axialvector mesons $\Leftrightarrow$ **vector diquarks** (~ 1.3 GeV)



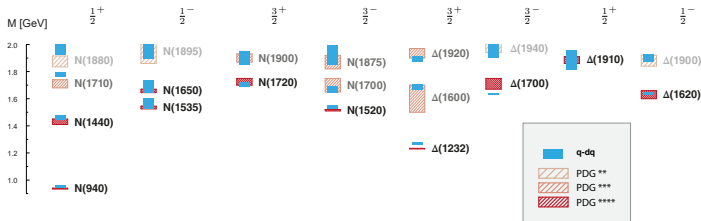
In RL, these are too strongly bound; simulate beyond-RL effects by (one) strength parameter c

[Roberts, Chang, Cloet, Roberts, FBS 51 \(2011\)](#)
[GE, Fischer, Sanchis-Alepuz, PRD 94 \(2016\)](#)

Diquark correlations

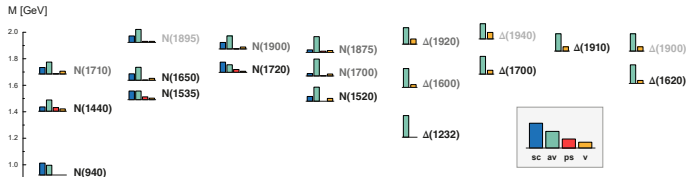
Light baryon spectrum

GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)



Diquark content:

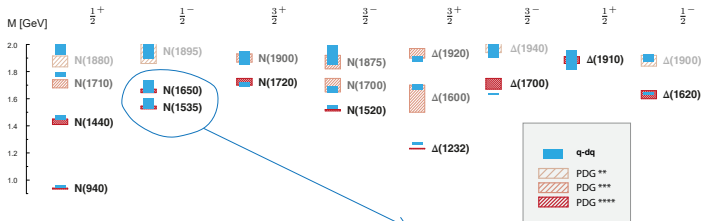
Barabanov et al., PNP 116 (2021)



Diquark correlations

Light baryon spectrum

GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)



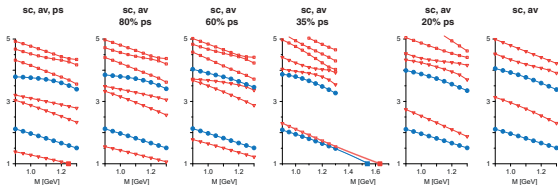
RL, all diquarks:
"N(1535)" too low

"Beyond RL":
N(1535), N(1650)

RL, sc+av only:
"N(1650)" too high

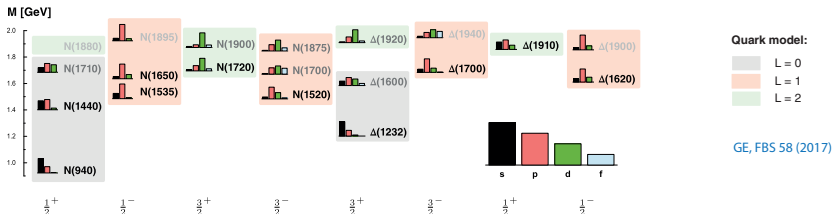
- **Level ordering** determined by diquark dynamics
- Diquarks are not pointlike, also here **rich spectrum!**

Barabanov et al., PNP 116 (2021)



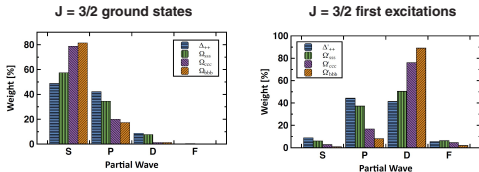
Relativistic effects

Orbital angular momentum: clear traces of nonrelativistic quark model, but strong relativistic effects (in some cases even dominant)

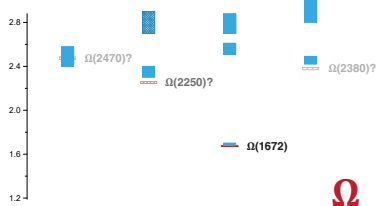
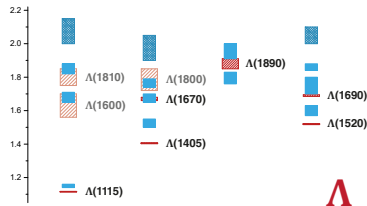
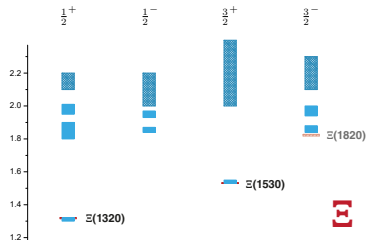
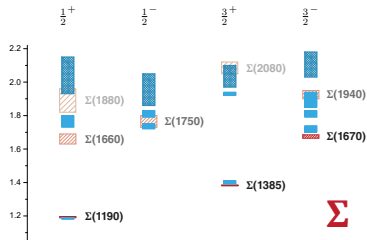


Relativistic contributions
even up to bottom baryons!

[Qin, Roberts, Schmidt, PRD 97 \(2018\)](#)

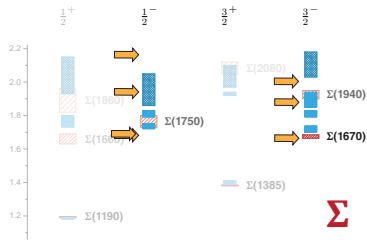


Strange baryons



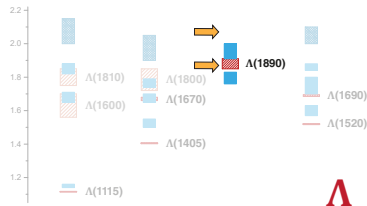
GE, Fischer, FBS 60 (2019), Fischer, GE, PoS Hadron 2017

Strange baryons



New states from Bonn-Gatchina

Sarantsev, Matveev, Nikonov, Anisovich, Thoma,
EPJA 55 (2019)

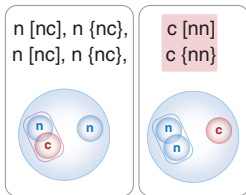


GE, Fischer, FBS 60 (2019), Fischer, GE, PoS Hadron 2017

Heavy baryons

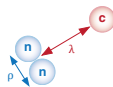
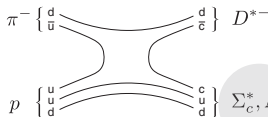
Quark content	q-dq	Isospin	contributes to
nnc	n [nc]	0, 1	Λ_c, Σ_c
$\uparrow$	n {nc}	0, 1	Λ_c, Σ_c
$n = u, d$	c [nn]	0	Λ_c
	c {nn}	1	Σ_c

$\Rightarrow \Lambda_c:$ n [nc], n {nc},
 $\Sigma_c:$ n [nc], n {nc},



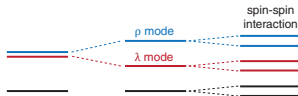
Sometimes these are assumed as dominant components, e.g. **J-PARC** charm baryon spectroscopy program (high-p)

Kim, Hosaka, Kim, Nouni, Shirotori, PTEP 2014 (2014), 10, 103D01,
 Shim, Hosaka, Kim, PTEP 2020 (2020) 5, 053D01



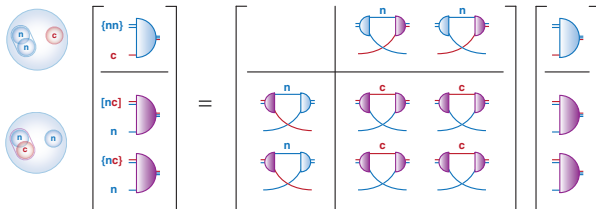
Decay of ρ mode $\rightarrow \Sigma_c \pi$
 Decay of λ mode $\rightarrow p D$

Assumptions
 on spectrum &
 production rates



Heavy baryons

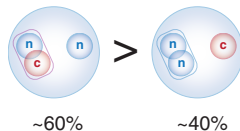
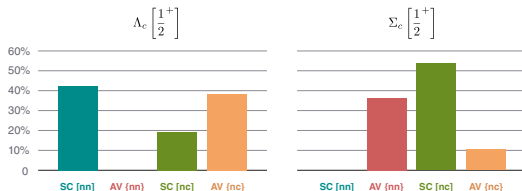
Quark-diquark BSE would not work under this assumption, e.g., Σ_c :



$n[nc]$, $n\{nc\}$ necessary, otherwise no equation. Presumably these are also dominant: cannot switch off $n[nc]$, $n\{nc\}$, but $c\{nn\}$

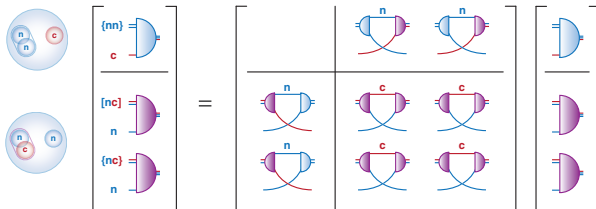
Analogous for Λ_c and hyperons

Results: **wave function contributions** [Torcato, Arriaga, GE, Peña, FBS 64 \(2023\)](#)



Heavy baryons

Quark-diquark BSE would not work under this assumption, e.g., Σ_c :

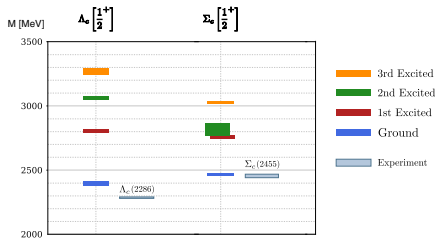


$n[nc]$, $n\{nc\}$ necessary, otherwise no equation. Presumably these are also dominant: cannot switch off $n[nc]$, $n\{nc\}$, but $c\{nn\}$

Analogous for Λ_c and hyperons

Results: spectrum

Torcato, Arriaga, GE, Peña, FBS 64 (2023)

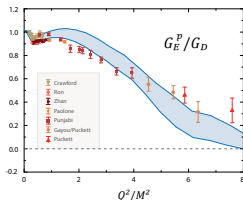


see also:
Yin, Chen, Krein, Roberts, Segovia, PRD 100 (2019)

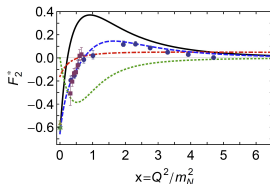
Form factors

Many **form factor calculations** in qq̄ or q(qq) approaches available:

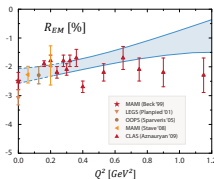
Nucleon electromagnetic FFs
GE, PRD 84 (2011)



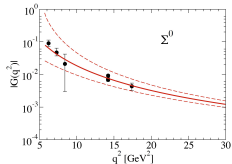
Roper em. transition FFs
Segovia et al., PRL 115 (2015)



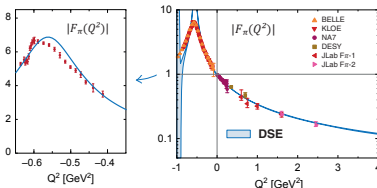
Δ -baryon em. transition FFs
GE, Nicmorus, PRD 85 (2012)



Timelike em. strangeness FFs
Ramalho, Peña, PRD 101 (2020)



Timelike pion FF dynamically develops ρ pole with $\pi\pi$ decay channel
Miramontes, Sanchis-Alepuz, Alkofer, PRD 103 (2021)



GE, Fischer, Williams,
PRD 101 (2020),

1. Introduction

2. Spectrum & symmetries

3. Quark models

4. QFT toolbox

5. Functional methods

6. n-point functions & tensor bases

7. Multiquark states

8. Light-front wave functions

Lecture notes “QCD and hadron physics,”
<https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei>

GE, “*Hadron physics with functional methods*,”
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

Burkert, GE, Klempt, “*The impact of γN and $\gamma^* N$ interactions on our understanding of nucleon excitations*,” arXiv:2506.16482

GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
“*Baryons as relativistic three-quark bound states*”,
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

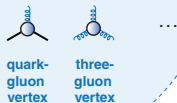
Barabanov, Bedolla, Brooks, Cates, Chen et al.,
“*Diquark correlations in hadron physics: Origin, impact and evidence*”, Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

Correlation functions

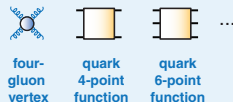
Two-point functions



Three-point functions

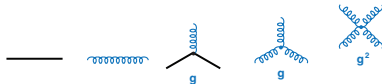


Higher n-point functions



QFT

Classical action of QCD:



Bethe-Salpeter amplitudes

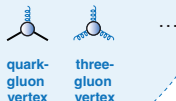


Correlation functions

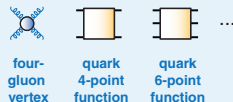
Two-point functions



Three-point functions



Higher n-point functions



Information is contained in Lorentz-invariant **dressing functions:**

$$G_{\alpha\beta\dots}^{\mu\nu\dots}(p_1, \dots, p_n) = \sum_{i=1}^N \underbrace{f_i(p_1^2, p_2^2, \dots)}_{\text{Kinematics?}} \underbrace{\tau_i(p_1, \dots, p_n)}_{\text{Tensor bases?}}^{\mu\nu\dots}_{\alpha\beta\dots}$$

What can we learn from **symmetries?**

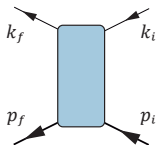
- Lorentz & parity invariance
- Crossing symmetry, charge conjugation
- Gauge invariance

Bethe-Salpeter amplitudes



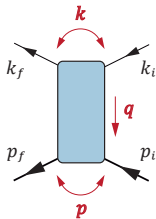
- 1. Example: $2 \rightarrow 2$ scattering**
- 2. Euclidean metric**
- 3. QCD's n-point functions**
- 4. Tensor bases**
- 5. Kinematics**

2→2 scattering (1)



- **Scalar** $\Rightarrow$ only one function $\Gamma(p_i, k_i, p_f, k_f)$
- 4 momenta + momentum conservation $\Rightarrow$ **3 independent momenta**

2→2 scattering (1)

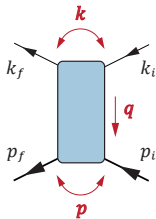


- **Scalar** $\Rightarrow$ only one function $\Gamma(p_i, k_i, p_f, k_f)$
- 4 momenta + momentum conservation $\Rightarrow$ **3 independent momenta**, define:

$$p = \frac{p_i + p_f}{2}, \quad k = \frac{k_i + k_f}{2}, \quad q = p_f - p_i = k_i - k_f$$

$$\begin{aligned} \Rightarrow \quad p_i &= p - \frac{q}{2} & k_i &= k + \frac{q}{2} \\ p_f &= p + \frac{q}{2} & k_f &= k - \frac{q}{2} \end{aligned} \quad \Rightarrow \quad p_i + k_i = p_f + k_f \quad \checkmark$$

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- 3 momenta $\Rightarrow$ **6 Lorentz invariants**:

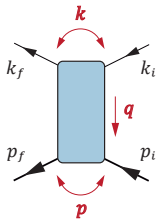
$$p^2, \quad q^2, \quad k^2, \quad p \cdot q, \quad k \cdot q, \quad p \cdot k$$

- Assume all particles are **onshell**: $p_i^2 = p_f^2 = M^2, \quad k_i^2 = k_f^2 = m^2$

$$\left. \begin{aligned} p_i^2 &= p^2 + \frac{q^2}{4} - p \cdot q = M^2 \\ p_f^2 &= p^2 + \frac{q^2}{4} + p \cdot q = M^2 \end{aligned} \right\} \quad p \cdot q = 0, \quad p^2 = M^2 - \frac{q^2}{4}$$

$$\left. \begin{aligned} k_i^2 &= k^2 + \frac{q^2}{4} + k \cdot q = m^2 \\ k_f^2 &= k^2 + \frac{q^2}{4} - k \cdot q = m^2 \end{aligned} \right\} \quad k \cdot q = 0, \quad k^2 = m^2 - \frac{q^2}{4}$$

2→2 scattering (1)



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- 3 momenta $\Rightarrow$ **6 Lorentz invariants**:

$$\cancel{p^2}, \quad q^2, \quad \cancel{k^2}, \quad \cancel{p \cdot q}, \quad \cancel{k \cdot q}, \quad p \cdot k$$

- Assume all particles are **onshell**: $p_i^2 = p_f^2 = M^2$, $k_i^2 = k_f^2 = m^2$

$\Rightarrow$ **2 variables** left:

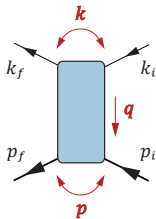
$$\tau = -\frac{q^2}{4M^2}$$

$$\lambda = \frac{p \cdot k}{M^2}$$

$$\left. \begin{aligned} p_i^2 &= p^2 + \frac{q^2}{4} - p \cdot q = M^2 \\ p_f^2 &= p^2 + \frac{q^2}{4} + p \cdot q = M^2 \end{aligned} \right\} \quad p \cdot q = 0, \quad p^2 = M^2 - \frac{q^2}{4}$$

$$\left. \begin{aligned} k_i^2 &= k^2 + \frac{q^2}{4} + k \cdot q = m^2 \\ k_f^2 &= k^2 + \frac{q^2}{4} - k \cdot q = m^2 \end{aligned} \right\} \quad k \cdot q = 0, \quad k^2 = m^2 - \frac{q^2}{4}$$

2→2 scattering (2)



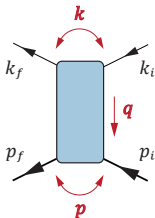
- 3 independent momenta, 2 Lorentz invariants:

$$\begin{aligned} p_i &= p - \frac{q}{2} & k_i &= k + \frac{q}{2} \\ p_f &= p + \frac{q}{2} & k_f &= k - \frac{q}{2} \end{aligned}$$

$$\begin{aligned} \tau &= -\frac{q^2}{4M^2} \\ \lambda &= \frac{p \cdot k}{M^2} \end{aligned}$$

$$\begin{aligned} p \cdot q &= 0, & p^2 &= M^2 - \frac{q^2}{4} \\ k \cdot q &= 0, & k^2 &= m^2 - \frac{q^2}{4} \end{aligned}$$

2→2 scattering (2)



- 3 independent momenta, 2 Lorentz invariants:

$$p_i = p - \frac{q}{2}$$

$$p_f = p + \frac{q}{2}$$

$$k_i = k + \frac{q}{2}$$

$$k_f = k - \frac{q}{2}$$

$$\tau = -\frac{q^2}{4M^2}$$

$$\lambda = \frac{p \cdot k}{M^2}$$

$$p \cdot q = 0, \quad p^2 = M^2 - \frac{q^2}{4}$$

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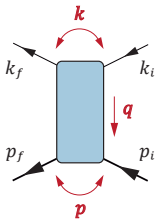
- Define

$$\epsilon = \frac{M^2 - m^2}{2M^2}$$

$$\Rightarrow m^2 = M^2(1 - 2\epsilon)$$

$$\Rightarrow p^2 + k^2 = M^2 + m^2 - \frac{q^2}{2} = 2M^2(1 + \tau - \epsilon)$$

2→2 scattering (2)



- 3 independent momenta, 2 Lorentz invariants:

$$p_i = p - \frac{q}{2}$$

$$k_i = k + \frac{q}{2}$$

$$p_f = p + \frac{q}{2}$$

$$k_f = k - \frac{q}{2}$$

$$\tau = -\frac{q^2}{4M^2}$$

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$$p \cdot q = 0, \quad p^2 = M^2 - \frac{q^2}{4}$$

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- Mandelstam variables:

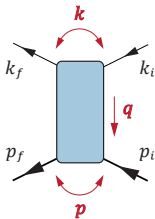
$$s = (p_i + k_i)^2 = (p_f + k_f)^2 = (p + k)^2 = p^2 + k^2 + 2p \cdot k = 2M^2(1 + \tau + \lambda - \epsilon)$$

$$u = (p_i - k_f)^2 = (p_f - k_i)^2 = (p - k)^2 = p^2 + k^2 - 2p \cdot k = 2M^2(1 + \tau - \lambda - \epsilon)$$

$$t = (p_f - p_i)^2 = (k_i - k_f)^2 = q^2 = -4M^2\tau$$

$$\Rightarrow s + t + u = 4M^2(1 - \epsilon) = 2M^2 + 2m^2 \quad \checkmark$$

2→2 scattering (2)



- 3 independent momenta, 2 Lorentz invariants:

$$p_i = p - \frac{q}{2}$$

$$k_i = k + \frac{q}{2}$$

$$p_f = p + \frac{q}{2}$$

$$k_f = k - \frac{q}{2}$$

$$\tau = -\frac{q^2}{4M^2}$$

$$\lambda = \frac{p \cdot k}{M^2}$$

$$p \cdot q = 0, \quad p^2 = M^2 - \frac{q^2}{4}$$

$$k \cdot q = 0, \quad k^2 = m^2 - \frac{q^2}{4}$$

- Define

$$\varepsilon = \frac{M^2 - m^2}{2M^2}$$

$$\Rightarrow m^2 = M^2(1 - 2\varepsilon)$$

$$\Rightarrow p^2 + k^2 = M^2 + m^2 - \frac{q^2}{2} = 2M^2(1 + \tau - \varepsilon)$$

- Mandelstam variables:

$$s = (p_i + k_i)^2 = (p_f + k_f)^2 = (p + k)^2 = p^2 + k^2 + 2p \cdot k = 2M^2(1 + \tau + \lambda - \varepsilon)$$

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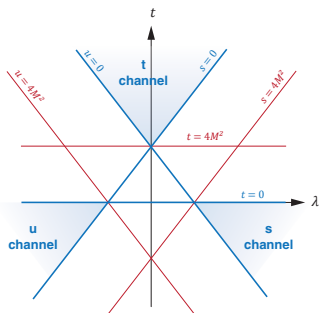
$$\Rightarrow s + t + u = 4M^2(1 - \varepsilon) = 2M^2 + 2m^2 \quad \checkmark$$

$$\begin{aligned} \Rightarrow \Gamma(p_i, k_i, p_f, k_f) \\ &= f(s, t, u) \\ &= f(\lambda, \tau) \end{aligned}$$

$$\Rightarrow \text{Momentum transfer: } \tau = -\frac{t}{4M^2} = \frac{s+u}{4M^2} - 1 + \varepsilon$$

$$\text{Crossing variable: } \lambda = \frac{s-u}{4M^2}$$

Mandelstam plane

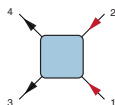


$$\tau = -\frac{t}{4M^2} = \frac{s+u}{4M^2} - 1 + \epsilon$$

$$\lambda = \frac{s-u}{4M^2}$$

here: identical masses ($\epsilon = 0$)

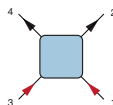
- Different processes described by same amplitude, but with different physical domains:



s channel:

$$1 + 2 \rightarrow 3 + 4$$

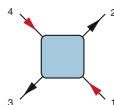
$$N\pi^+ \rightarrow N\pi^+$$



t channel:

$$1 + \bar{3} \rightarrow \bar{2} + 4$$

$$N\bar{N} \rightarrow \pi^-\pi^+$$



u channel:

$$1 + \bar{4} \rightarrow \bar{2} + 3$$

$$N\pi^- \rightarrow \pi^-N$$

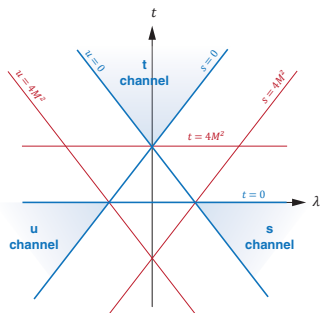
e.g. $N\pi$ scattering

- Dressing functions are determined by (all) their singularities:

$$\text{Loop} = \text{t-channel} + \text{t-channel} + \text{u-channel} + \text{cuts} + (\dots)$$

The diagram shows a blue square loop equal to the sum of three diagrams: a t-channel exchange (red line labeled 't'), a t-channel exchange (red line labeled 't'), and a u-channel exchange (red line labeled 'u').

Mandelstam plane



$$\tau = -\frac{t}{4M^2} = \frac{s+u}{4M^2} - 1 + \epsilon$$

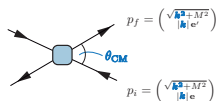
$$\lambda = \frac{s-u}{4M^2}$$

Amplitudes are **Lorentz-invariant**,
no need to go into specific frame
unless to relate with experiments:

- CM Frame:

$$\lambda = \frac{s-u}{M^2} = \frac{1}{M^2} \left[\frac{k^2}{2} (1 + \cos \theta_{\text{CM}}) + \sqrt{k^2 + M^2} \sqrt{k^2 + m^2} \right]$$

$$\tau = \frac{k^2}{2M^2} (1 - \cos \theta_{\text{CM}})$$



- Lab Frame:

$$\left. \begin{array}{l} \lambda = \dots \\ \tau = \dots \end{array} \right\} \Leftrightarrow k_{\text{lab}}, \theta_{\text{lab}}$$

λ and τ are the same in any frame!

But ...



$$\Gamma^{\mu\nu\rho\sigma}$$



$$\Gamma_{\alpha\beta\gamma\delta}$$



$$\Gamma^{\mu\nu}_{\alpha\beta}$$

- Amplitudes are offshell:
 - complicated **kinematic phase space** (here: 6 variables)
 - how to find most convenient set of Lorentz invariants?
- Amplitudes have Dirac and/or Lorentz structure
 - complicated **tensor bases**
 - many **dressings functions**
 - complicates problem both algebraically and numerically
 - use **symmetries** to find efficient tensor bases:
gauge invariance & permutation symmetries
(crossing symmetry, charge conjugation)

After all that, we can
talk about **dynamics!**

1. Example: $2 \rightarrow 2$ scattering
- 2. Euclidean metric**
3. QCD's n -point functions
4. Tensor bases
5. Kinematics

Euclidean conventions

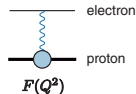
Why Euclidean?

- We're often dealing with **spacelike** momenta (virtual particles)
- **Loop momenta** are spacelike
- Minkowski metric (+, -, -, -) is inconvenient: upper & lower indices, cumbersome to write code
- QFT is already "Euclidean":
To make QFT well-defined, one needs **imaginary-time boundary conditions**:

$$\int d^4x = \int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dx_0 \int d^3x \Leftrightarrow \int d^4p = \int_{-\infty(1+i\epsilon)}^{\infty(1+i\epsilon)} dp_0 \int d^3p$$

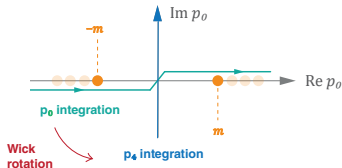
As long as amplitudes fall off fast enough at complex infinity, this is equivalent to Euclidean QFT!

$$\tau = -\frac{q^2}{4M^2} = \frac{Q^2}{4M^2}$$



Corresponds to $i\epsilon$ prescription in Feynman propagator:

$$\int_{-\infty}^{\infty} dp_0 \frac{1}{p^2 - m^2 + i\epsilon} \Rightarrow p_0^2 = p^2 + m^2 - i\epsilon$$



Euclidean conventions

So, let's employ a **Euclidean metric** (+, +, +, +):

$$a_E^\mu = \begin{bmatrix} a \\ ia_0 \end{bmatrix} \Rightarrow \begin{aligned} a_E \cdot b_E &= -a \cdot b \\ a_E^2 &= -a^2 \end{aligned}$$

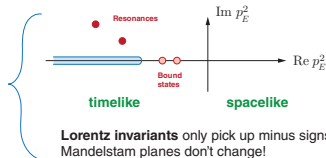
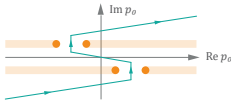
To preserve meaning of slash $\not{a} = a^0 \gamma^0 - a \cdot \gamma$, we must redefine γ matrices:

$$i\gamma_E^\mu = \begin{bmatrix} \gamma \\ i\gamma_0 \end{bmatrix}, \quad \gamma_E^5 = \gamma^5 \Rightarrow \begin{aligned} \not{a}_E &= a_E \cdot \gamma_E = i\not{a} \\ \{\gamma_E^\mu, \gamma_E^\nu\} &= 2\delta^{\mu\nu} \end{aligned} \quad \begin{aligned} (\gamma_E^4)^2 &= 1 \\ \gamma_E^\mu &= (\gamma_E^\mu)^\dagger \end{aligned}$$

⇒ Euclidean action is non-negative, defines probability measure:

$$e^{iS} = \exp \left[i \int d^4x \bar{\psi} (i\not{\partial} - m) \psi \right] = \exp \left[- \int d^4x_E \bar{\psi} (\not{\partial}_E + m) \psi \right] = e^{-S_E}$$

Disclaimer: This does not mean you can simply close your eyes when computing loop integrals... if there are cuts in the way, you must still deform the integration contour (also in Euclidean)!



Minkowski	Euclidean
$a \cdot b$	$-a \cdot b$
a^μ	a^μ
γ^μ	$i\gamma^\mu$
γ_5	γ_5
$\not{a}$	$-i\not{a}$
$g^{\mu\nu}$	$-\delta^{\mu\nu}$
$a^\mu b^\nu$	$a^\mu b^\nu$
$[\gamma^\mu, \gamma^\nu]$	$-[\gamma^\mu, \gamma^\nu]$
$[\gamma^\mu, \not{a}]$	$[\gamma^\mu, \not{a}]$
$[\gamma^\mu, \gamma^\nu, \not{a}]$	$i[\gamma^\mu, \gamma^\nu, \not{a}]$
$[\not{a}, \not{b}]$	$-\not{a}\not{b}$
$\varepsilon^{\mu\nu\rho\alpha} a_\alpha$	$i\varepsilon^{\mu\nu\rho\alpha} a_\alpha$
$\varepsilon^{\mu\nu\alpha\beta} a_\alpha b_\beta$	$-i\varepsilon^{\mu\nu\alpha\beta} a_\alpha b_\beta$
$\varepsilon^{\mu\alpha\beta\gamma} a_\alpha b_\beta c_\gamma$	$i\varepsilon^{\mu\alpha\beta\gamma} a_\alpha b_\beta c_\gamma$
$\varepsilon^{\mu\nu\alpha\beta} a_\alpha \gamma_\beta$	$-\varepsilon^{\mu\nu\alpha\beta} a_\alpha \gamma_\beta$

Euclidean conventions

From now on, I will use Euclidean conventions:

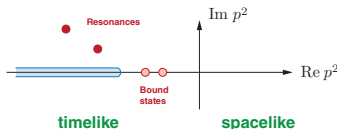
- No distinction between upper & lower indices
- $p^2 > 0$ means spacelike, $p^2 < 0$ timelike
- For an onshell momentum in the **rest frame**, we write

$$P = \begin{bmatrix} 0 \\ 0 \\ 0 \\ iM \end{bmatrix} \Rightarrow P^2 = -M^2$$

- For a general four momentum (e.g. **loop momenta**)

$$p = \sqrt{p^2} \begin{bmatrix} \sqrt{1-z^2} \sqrt{1-y^2} \sin \psi \\ \sqrt{1-z^2} \sqrt{1-y^2} \cos \psi \\ \sqrt{1-z^2} y \\ z \end{bmatrix}$$

- Easy to code!

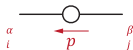


When people say
“We need to go to Minkowski space”,
what they usually mean is:
*We need to go to the **timelike region**!*

1. Example: $2 \rightarrow 2$ scattering
2. Euclidean metric
- 3. QCD's n-point functions**
4. Tensor bases
5. Kinematics

Correlation functions in QCD

• Quark propagator



$$S(p) = \frac{1}{A(p^2)} \frac{-i\not{p} + M(p^2)}{p^2 + M(p^2)^2} \delta_{ij}$$

$$S(p)^{-1} = A(p^2)(i\not{p} + M(p^2)) \delta_{ij}$$

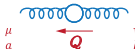
quark
mass
function

color

1 momentum,
2 Dirac indices $\Rightarrow 1, \not{p}$

$$S_{\alpha\beta}(p) = \sum_{i=1}^2 f_i(p^2) \tau_i(p)_{\alpha\beta}$$

• Gluon propagator



$$D^{\mu\nu}(Q) = \frac{1}{Q^2} \left(Z(Q^2) T_Q^{\mu\nu} + \xi L_Q^{\mu\nu} \right) \delta_{ab}$$

gluon dressing

$$T_Q^{\mu\nu} = \delta^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}$$

$$L_Q^{\mu\nu} = \frac{Q^\mu Q^\nu}{Q^2}$$

transverse &
longitudinal
projectors

1 momentum,
2 Lorentz indices $\Rightarrow \delta^{\mu\nu}, Q^\mu Q^\nu$

$$D^{\mu\nu}(Q) = \sum_{i=1}^2 f_i(Q^2) \tau_i(Q)^{\mu\nu}$$

• Ghost propagator



$$D_G(Q) = -\frac{G(Q^2)}{Q^2} \delta_{ab}$$

if (+) here, then also (+)
in ghost-gluon vertex

Correlation functions in QCD

• Ghost-gluon vertex



2 momenta,
1 Lorentz index $\Rightarrow p^\mu, Q^\mu$

$$\Gamma_{gh}^\mu(p, Q) = -ig f_{abc} \left[(1 + A(p^2, p \cdot Q, Q^2)) p^\mu + B(p^2, p \cdot Q, Q^2) Q^\mu \right]$$

• Quark-gluon vertex



2 momenta,
1 Lorentz index,
2 Dirac indices

$$\Gamma_q^\mu(k, Q) = ig t_a \sum_{i=1}^{12} f_i(k^2, k \cdot Q, Q^2) \tau_i^\mu(k, Q)$$

• Three-gluon vertex



2 momenta,
3 Lorentz indices

$$\Gamma_{3g}^{\mu\nu\rho}(p_1, p_2, p_3) = ig f_{abc} \sum_{i=1}^{14} f_i(p_1^2, p_2^2, p_3^2) \tau_i^{\mu\nu\rho}(p_1, p_2, p_3)$$

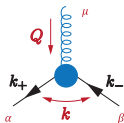
• Four-gluon vertex



3 momenta,
4 Lorentz indices,
5 color tensors

$$\Gamma_{4g}^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) = -g^2 \sum_{i=1}^{136} \sum_{j=1}^5 f_{ij}(\dots) \tau_i^{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4) \tau_{abcd}^{(j)}$$

Fermion-vector vertex



Let's consider a **general** fermion-vector vertex:

$$\Gamma^\mu(k, Q) = i \sum_{i=1}^{12} f_i(k^2, k \cdot Q, Q^2) \tau_i^\mu(k, Q)$$

quark-gluon vertex,
quark-photon vertex,
nucleon-photon vertex, ...

2 independent momenta: k_+, k_- or (simpler in practice) k, Q

$$k_\pm = k \pm \frac{Q}{2} \quad \Leftrightarrow \quad k = \frac{k_+ + k_-}{2}, \quad Q = k_+ - k_-$$

12 linearly independent tensors:

$$\left. \begin{array}{cccc} \gamma^\mu & \gamma^\mu \not{k} & \gamma^\mu \not{Q} & \gamma^\mu \not{k} \not{Q} \\ k^\mu & k^\mu \not{k} & k^\mu \not{Q} & k^\mu \not{k} \not{Q} \\ Q^\mu & Q^\mu \not{k} & Q^\mu \not{Q} & Q^\mu \not{k} \not{Q} \end{array} \right\}$$

This is a complete basis.
But is it also the *best* choice?

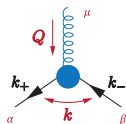
The vertex has a **charge-conjugation symmetry**:

$$\Gamma^\mu(k, Q) \stackrel{!}{=} -C \Gamma^\mu(-k, Q)^T C^T,$$

where $C = \gamma^4 \gamma^2$ is the charge-conjugation matrix and satisfies

$$C^T = C^\dagger = C^{-1} = -C, \quad C \gamma_5^T C^T = \gamma_5, \quad C \gamma_\mu^T C^T = -\gamma_\mu.$$

Fermion-vector vertex (2)



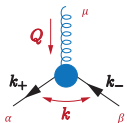
γ^μ	$\gamma^\mu \not{k}$	$\gamma^\mu \not{Q}$	$\gamma^\mu \not{k} \not{Q}$
$\not{k}^\mu$	$\not{k}^\mu \not{k}$	$\not{k}^\mu \not{Q}$	$\not{k}^\mu \not{k} \not{Q}$
Q^μ	$Q^\mu \not{k}$	$Q^\mu \not{Q}$	$Q^\mu \not{k} \not{Q}$

$$\Gamma^\mu(k, Q) \stackrel{!}{=} -C \Gamma^\mu(-k, Q)^T C^T$$

$$C^T = C^\dagger = C^{-1} = -C,$$

$$C \gamma_5^T C^T = \gamma_5, \quad C \gamma_\mu^T C^T = -\gamma_\mu$$

Fermion-vector vertex (2)



γ^μ	$\gamma^\mu \not{k}$	$\gamma^\mu \not{Q}$	$\gamma^\mu \not{k} \not{Q}$
k^μ	$k^\mu \not{k}$	$k^\mu \not{Q}$	$k^\mu \not{k} \not{Q}$
Q^μ	$Q^\mu \not{k}$	$Q^\mu \not{Q}$	$Q^\mu \not{k} \not{Q}$

$$\Gamma^\mu(k, Q) \stackrel{!}{=} -C \Gamma^\mu(-k, Q)^T C^T$$

$$C^T = C^\dagger = C^{-1} = -C,$$

$$C \gamma_5^T C^T = \gamma_5, \quad C \gamma_\mu^T C^T = -\gamma_\mu$$

But not all tensors satisfy this constraint:

$$\gamma_\mu \rightarrow -C \gamma_\mu^T C^T = \gamma_\mu \quad \checkmark$$

$$k^\mu \rightarrow -C(-k^\mu)C^T = k^\mu \quad \checkmark$$

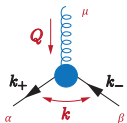
$$Q^\mu \rightarrow -C Q^\mu C^T = -Q^\mu \quad \times$$

$$\gamma_\mu \not{Q} \rightarrow -C \not{Q}^T \gamma_\mu^T C^T = \not{Q} C \gamma_\mu^T C^T = -\not{Q} \gamma_\mu = \gamma_\mu \not{Q} - 2Q^\mu \quad \times$$

$$C^T C = 1$$

leads to symmetry relations
between dressing functions

Fermion-vector vertex (2)



γ^μ	$\gamma^\mu \not{k}$	$\gamma^\mu \not{Q}$	$\gamma^\mu \not{k} \not{Q}$
k^μ	$k^\mu \not{k}$	$k^\mu \not{Q}$	$k^\mu \not{k} \not{Q}$
Q^μ	$Q^\mu \not{k}$	$Q^\mu \not{Q}$	$Q^\mu \not{k} \not{Q}$

$$\Gamma^\mu(k, Q) \stackrel{!}{=} -C \Gamma^\mu(-k, Q)^T C^T$$

$$C^T = C^\dagger = C^{-1} = -C,$$

$$C \gamma_5^T C^T = \gamma_5, \quad C \gamma_\mu^T C^T = -\gamma_\mu$$

But not all tensors satisfy this constraint:

$$\gamma_\mu \rightarrow -C \gamma_\mu^T C^T = \gamma_\mu \quad \checkmark$$

$$k^\mu \rightarrow -C(-k^\mu)C^T = k^\mu \quad \checkmark$$

$$Q^\mu \rightarrow -C Q^\mu C^T = -Q^\mu \quad \times$$

$$\gamma_\mu \not{Q} \rightarrow -C \not{Q}^T \gamma_\mu^T C^T = \not{Q} C \gamma_\mu^T C^T = -\not{Q} \gamma_\mu = \gamma_\mu \not{Q} - 2Q^\mu \quad \times$$

$$C^T C = 1$$

leads to symmetry relations
between dressing functions

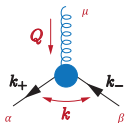
Simpler: implement symmetry directly for the basis elements

$$k \cdot Q \not{Q} \rightarrow +k \cdot Q C \not{Q} C^T = k \cdot Q \not{Q} \quad \checkmark$$

$$[\gamma_\mu, \not{Q}] \rightarrow -C [\not{Q}^T, \gamma_\mu^T] C^T = -[C \not{Q}^T C^T, C \gamma_\mu^T C^T] = -[\not{Q}, \gamma_\mu] = [\gamma_\mu, \not{Q}] \quad \checkmark$$

In short: use **commutators** & attach factors $k \cdot Q$ where necessary

Fermion-vector vertex (3)



Abbreviating $\omega = k \cdot Q$, we arrive at the new basis

γ^μ	$i\omega[\gamma^\mu, \not{k}]$	$i[\gamma^\mu, \not{Q}]$	$[\gamma^\mu, \not{k}, \not{Q}]$	← Triple commutator is antisymmetric combination of A, B, C :
ik^μ	$k^\mu \not{k}$	$\omega k^\mu \not{Q}$	$ik^\mu [\not{k}, \not{Q}]$	$[A, B, C] = [A, B]C + [B, C]A + [C, A]B$
$i\omega \not{Q}^\mu$	$\omega \not{Q}^\mu \not{k}$	$\not{Q}^\mu \not{Q}$	$i\omega \not{Q}^\mu [\not{k}, \not{Q}]$	← i factors make all dressing functions real

Now all tensors satisfy the constraint $\Gamma^\mu(k, Q) \stackrel{!}{=} -C \Gamma^\mu(-k, Q)^T C^T$ individually, so their dressing functions must be **even** in ω , can only depend on ω^2 :

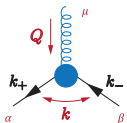
$$\Gamma^\mu(k, Q) = i \sum_{i=1}^{12} f_i(k^2, \omega^2, Q^2) \tau_i^\mu(k, Q)$$

This also simplifies their momentum dependence:



Mathematically speaking, we have arranged the tensors and dressing functions into **singlets** of the **permutation group** S_2 !

Fermion-vector vertex (4)



Amplitudes with gluon or photon legs are subject to **gauge invariance**, e.g. **Ward-Takahashi identity** (WTI) for quark-photon vertex:

$$Q^\mu \Gamma^\mu(k, Q) = S^{-1}(k_+) - S^{-1}(k_-) \quad \rightarrow \text{fixes longitudinal part of vertex}$$

We could split the vertex into transverse and longitudinal part:

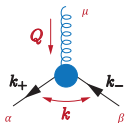
$$\begin{aligned} \Gamma^\mu(k, Q) &= T_Q^{\mu\nu} \Gamma^\nu(k, Q) + L_Q^{\mu\nu} \Gamma^\nu(k, Q) \\ &= \Gamma_\perp^\mu(k, Q) + \frac{Q^\mu}{Q^2} [S^{-1}(k_+) - S^{-1}(k_-)] \end{aligned}$$

But this has a **kinematic singularity** at $Q^2 = 0$, violates **Ward identity**:

$$\Gamma^\mu(k, 0) = \frac{dS(k)^{-1}}{dk^\mu}$$

- ⇒ Constructing tensor bases with T + L projectors is **dangerous** if we are interested in kinematic limits...
- ⇒ Must work out transverse part of vertex without introducing kinematic singularities

Fermion-vector vertex (5)



Simpler example: **scalar vertex**

$$\Gamma^\mu(k, Q) = c_1 k^\mu + c_2 \omega Q^\mu \quad \rightarrow \text{has correct permutation symmetry}$$

Ward-Takahashi identity:

$$Q^\mu \Gamma^\mu(k, Q) = D^{-1}(k_+) - D^{-1}(k_-)$$

Let's first work out the transverse part:

$$Q^\mu \Gamma^\mu(k, Q) = \omega (c_1 + c_2 Q^2) \stackrel{!}{=} 0 \quad \Rightarrow \quad c_1 = -c_2 Q^2$$

$$\begin{aligned} \Rightarrow \Gamma_\perp^\mu(k, Q) &= -c_2 (Q^2 k^\mu - \omega Q^\mu) \\ &= -c_2 \underbrace{(Q^2 \delta^{\mu\nu} - Q^\mu Q^\nu)}_{=: t_{QQ}^{\mu\nu}} k^\nu \quad \rightarrow \text{No } 1/Q^2 \text{ terms here!} \end{aligned}$$

Careful: don't use

$$c_2 = -\frac{c_1}{Q^2},$$

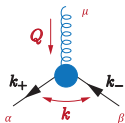
which would introduce kinematic singularity!

Full vertex $\rightarrow$ we solved for c_1 , so we must put back k^μ :

$$\Gamma^\mu(k, Q) = g_1 k^\mu + f_1 t_{QQ}^{\mu\nu} k^\nu$$

“**minimal basis**”: no kinematic singularities, tensors are ordered with powers of momenta

Fermion-vector vertex (5)



Now we can work out the Ward-Takahashi identity:

$$Q^\mu \Gamma^\mu(k, Q) = D^{-1}(k_+) - D^{-1}(k_-) = g_1 \omega$$

Define

$$\Delta = \frac{D^{-1}(k_+) - D^{-1}(k_-)}{k_+^2 - k_-^2} \quad k_\pm^2 = k^2 + \frac{Q^2}{4} \pm k \cdot Q \quad \Rightarrow \quad k_+^2 - k_-^2 = 2k \cdot Q = 2\omega$$

$$\Rightarrow g_1 = 2\Delta$$

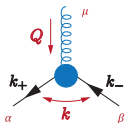
Full vertex $\rightarrow$ we solved for c_1 , so we must put back k^μ :

$$\Gamma^\mu(k, Q) = g_1 k^\mu + f_1 t_{QQ}^{\mu\nu} k^\nu$$



“minimal basis”: no kinematic singularities, tensors are ordered with powers of momenta

Fermion-vector vertex (6)



Final result for **scalar vertex**:

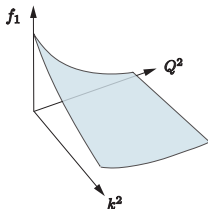
$$\Gamma^\mu(k, Q) = 2\Delta k^\mu + f_1 t_{QQ}^{\mu\nu} k^\nu$$

$$\Delta = \frac{D^{-1}(k_+) - D^{-1}(k_-)}{k_+^2 - k_-^2}$$

Satisfies WTI **and** Ward identity:

$$\Gamma^\mu(k, 0) = 2\Delta k^\mu = \frac{dD^{-1}(k)}{dk^2} 2k^\mu = \frac{dD^{-1}(k)}{dk^\mu} \quad \checkmark$$

In this way, we eliminated all kinematic effects from tensor basis:
Dressing functions are free of kinematic constraints at $k=0$ or $Q=0$
(kinematically independent), no kinematic singularities or zeros:
their only singularities are physical poles and cuts!



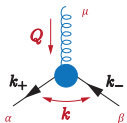
Also: tensors with higher momentum powers are suppressed at low momenta
 $\Rightarrow$ ordering principle!

$$2\Delta k^\mu + f_1 t_{QQ}^{\mu\nu} k^\nu$$

important

less important

Fermion-vector vertex (7)



The same can be done for **fermion-vector vertex**:

$$\Gamma^\mu(k, Q) = i \left[\underbrace{\sum_{j=1}^4 g_j G_j^\mu}_{\text{Gauge part: constrained by STI}} + \underbrace{\sum_{j=1}^8 f_j T_j^\mu}_{\text{Transverse part, carries dynamics}} \right]$$

Gauge part:
constrained by STI

$$\begin{aligned} &\gamma^\mu \\ &k^\mu \not{k} \\ &i k^\mu \\ &\omega \frac{i}{2} [\gamma^\mu, \not{k}] \end{aligned}$$

Transverse part,
carries dynamics

$$\begin{aligned} &t_{QQ}^{\mu\nu} \gamma^\nu \\ &\omega t_{QQ}^{\mu\nu} \frac{i}{2} [\gamma^\nu, \not{k}] \\ &\frac{i}{2} [\gamma^\mu, \not{Q}] \\ &\frac{1}{6} [\gamma^\mu, \not{k}, \not{Q}] \end{aligned}$$

$$\begin{aligned} &t_{QQ}^{\mu\nu} i k^\nu \\ &t_{QQ}^{\mu\nu} k^\nu \not{k} \\ &\omega t_{Qk}^{\mu\nu} \gamma^\nu \\ &t_{Qk}^{\mu\nu} \frac{i}{2} [\gamma^\nu, \not{k}] \end{aligned}$$

$$\begin{aligned} t_{ab}^{\mu\nu} &= a \cdot b \delta^{\mu\nu} - b^\mu a^\nu \\ &\Rightarrow a^\mu t_{ab}^{\mu\nu} = 0 \\ &\quad t_{ab}^{\mu\nu} b^\nu = 0 \end{aligned}$$

For quark-photon vertex: **Ball-Chiu vertex**

Ball, Chiu, PRD 22 (1980)

$$g_1 = \Sigma_A \quad g_2 = 2\Delta_A \quad g_3 = -2\Delta_B \quad g_4 = 0$$

$$\begin{cases} \Sigma_A = \frac{A(k_+) + A(k_-)}{2} \\ \Delta_A = \frac{A(k_+) - A(k_-)}{k_+^2 - k_-^2} \\ B(k^2) = A(k^2)M(k^2) \end{cases}$$

Everything else is **dynamics**! Quark-photon vertex can be calculated from BSE, f_j must have physical vector-meson poles at timelike momenta

Maris, Tandy, PRC 61 (2000)