Theory and phenomenology of Generalised Partons Distributions

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Summary of Lecture 1

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- We observe that from a wide range of order of magnitude, light scattering allows one to probe the structure of matter usually through a Fourier Transform.

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- We saw that when a parton is hit by a hard photon, it needs to quickly (perturbatively) release this energy in order not to break the photon
  $\Rightarrow$ DVCS factorises between a hard part, computed in pQCD and GPDs (non-perturbative)
- GPDs are generalisation of the EM Form Factor measured in elastic scattering and of PDFs measured in inclusive processes (DIS).
- Finally, we demonstrated that the Fourier Transform of GPDs yield the 2+1D probability density to find a quark or a gluon with fixed momentum fraction at a given $b_\perp$ position in a hadron.

Unpolarised nucleon GPDs

$$\frac{1}{2}\int \frac{e^{ixP^+z^-}}{2\pi}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}$$
$$=\frac{1}{2P^+}\left[H^q(x,\xi,t)\bar{u}\gamma^+u+E^q(x,\xi,t)\bar{u}\frac{i\sigma^{+\alpha}\Delta_\alpha}{2M}u\right].$$

Polarised Nucleon GPDs

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$$=\frac{1}{2P^+}\left[\tilde{H}^q(x,\xi,t)\bar{u}\gamma^+\gamma_5u+\tilde{E}^q(x,\xi,t)\bar{u}\frac{\gamma_5\Delta^+}{2M}u\right].$$

- The number of GPDs depends on the hadron spin
- GPDs E and $\tilde{E}$ do not reduce to PDFs when $\Delta \to 0$

Chiral-Even Nucleon GPDs

Unpolarised nucleon GPDs

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Place of GPDs in the Hadron physics context

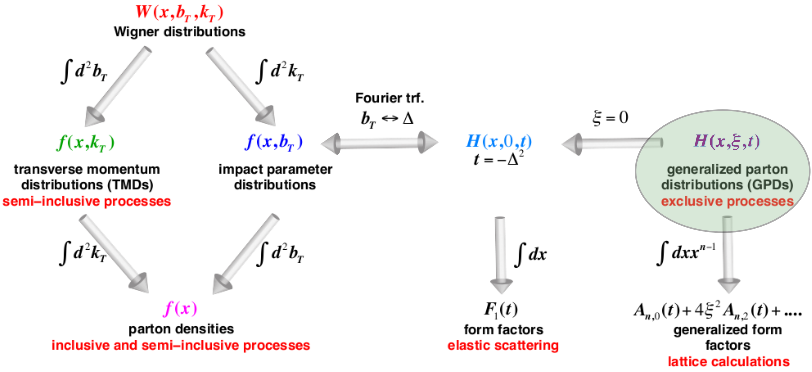

figure from A. Accardi *et al.*, Eur.Phys.J.A 52 (2016) 9, 268

Connection with the Energy-Momentum Tensor

Hadron EMT in QCD

In QCD, the energy momentum tensor of the nucleon is a correlator of the EMT operator, evaluated between two nucleon states:

$$\langle p', s' | T^{\{\mu\nu\}}_{q,g} | p, s\rangle = \bar{u}\left[P^{\{\mu}\gamma^{\nu\}} A_{q,g}(t;\mu) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu}\Delta^2}{M} C_{q,g}(t;\mu) \right.$$
$$\left. + M g^{\mu\nu} \bar{C}_{q,g}(t;\mu) + \frac{P^{\{\mu} i\sigma^{\nu\}\Delta}}{2M} B_{q,g}(t;\mu)\right] u$$

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- The total EMT is scale independent as it defines a conserved current
- Different definitions exist for the EMT, we stick to the one above
- 4 form factors are needed to parameterise the (symmetric) EMT correlator in the spin-1/2 case
- Constraints exist on some of these form factors:

$$A(0) = 1, \quad B(0) = 0, \quad \bar{C}(t) = 0$$

- Note that there is **no** constraint on C.

Connection to GPDs I

The quark sector of the EMT is given as:

$$T_q^{\mu\nu} = \bar{q}\gamma^{\{\mu} i \overleftrightarrow{D}^{\nu\}} q \quad \text{such that} \quad \overleftrightarrow{D}^{\mu} = \frac{1}{2}\left(\overrightarrow{D} - \overleftarrow{D}\right)$$

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Working in the lightcone gauge where $D = \partial$ one can readily see that:

$$\frac{1}{2}\int_{-1}^{1} \mathrm{d}x x \int \frac{e^{ixP^+z^-}}{2\pi} \langle P + \frac{\Delta}{2} | \bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2}) | P - \frac{\Delta}{2} \rangle \mathrm{d}z^- |_{z^+=0, z=0}$$

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Consequently, EMT Form Factors A, B and C are connected to GPDs H and E through:

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Connection to GPDs II

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In principle, from GPDs extracted from experimental data, we would be able to get experimental information on these Form Factors.

Ji sum rule

The quark and gluon contributions to the angular momentum J are

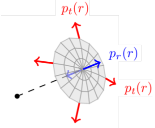
$$\begin{aligned} 2J^q =& A^q(0) + B^q(0) \\ =& \int \mathrm{d}x x \left(H^q(x,\xi,0) + E^q(x,\xi,0) \right) \\ 2J^g =& A^g(0) + B^g(0) \\ =& \int \mathrm{d}x \left(H^g(x,\xi,0) + E^g(x,\xi,0) \right) \end{aligned}$$

X.D. Ji, Phys.Rev.Lett. 78 (1997) 610-613

Pressure in Relativistic hydrodynamics

- In relativistic hydrodynamics $\rightarrow$ pressure for a anisotropic fluid enters the description of the EMT θ:

$$\theta^{\mu\nu}(r) = (\varepsilon + p_t)\frac{P^\mu P^\nu}{M^2} - p_t\eta^{\mu\nu} + (p_r - p_t)\frac{z^\mu z^\nu}{r^2}$$



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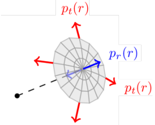

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- On can define isotropic pressure p and pressure anisotropy s:

$$p(\mathrm{r}) = \frac{p_r(\mathrm{r}) + 2p_t(\mathrm{r})}{3}$$
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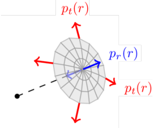

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Question

Can we obtain an analoguous definition within hadron physics?

And from them, extract pressure and shear forces following:

$$\varepsilon_a(r) = M \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\Delta\cdot r} \left\{ A_a(t) + \bar{C}_a(t) + \frac{t}{4M^2}\left[B_a(t) - 4C_a(t)\right] \right\},$$

$$p_{r,a}(r) = M \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\Delta\cdot r} \left\{ -\bar{C}_a(t) - \frac{4}{r^2}\frac{t^{-1/2}}{M^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{3/2}\, C_a(t)\right) \right\},$$

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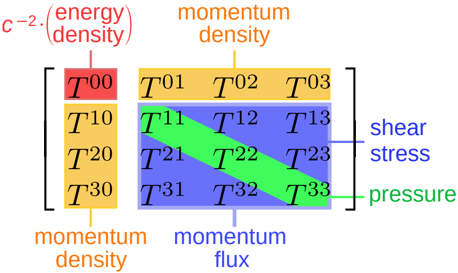
$$s_a(r) = M \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{-i\Delta\cdot r} \left\{ -\frac{4}{r^2}\frac{t^{-1/2}}{M^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t^{5/2}\, C_a(t)\right) \right\},$$

C. Lorcé *et al.*, Eur.Phys.J.C 79 (2019) 1, 89



How energy, momentum, pressure are shared between quarks and gluons

Caveat: renormalization scheme and scale dependence

C. Lorcé *et al.*, PLB 776 (2018) 38-47,
M. Polyakov and P. Schweitzer,
IJMPA 33 (2018) 26, 1830025
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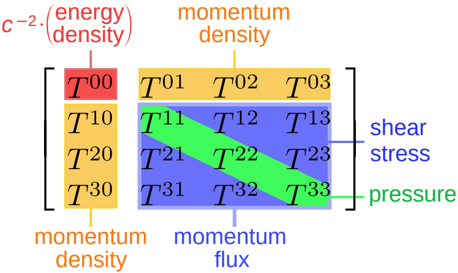
Interpretation of GPDs II

Connection to the Energy-Momentum Tensor



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$$\int_{-1}^{1} \mathrm{d}x\, x\, E_q(x,\xi,t;\mu) = B_q(t;\mu) - 4\xi^2 C_q(t;\mu)$$

- Ji sum rule (nucleon)
- Fluid mechanics analogy

X. Ji, PRL 78, 610-613 (1997)
M.V. Polyakov PLB 555, 57-62 (2003)

Lorentz covariance and its consequences

We can generalise what we obtained on the EFF for higher moments:

$$\frac{1}{2}\int \mathrm{d}x\, x^m \int \frac{e^{ixP^+z^-}}{2\pi} \langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}$$

$$= \int \frac{\mathrm{d}x}{2(iP^+)^m}\frac{\mathrm{d}^m}{(\mathrm{d}z^-)^m}\left[\frac{e^{ixP^+z^-}}{2\pi}\right]\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|^{\mathbf{z}=0}_{z^+=0}$$

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$$= \frac{i^m}{2(P^+)^{m+1}} \langle P+\frac{\Delta}{2}|\frac{\mathrm{d}^m}{(\mathrm{d}z^-)^m}\left[\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})\right]|P-\frac{\Delta}{2}\rangle|_{z=0}$$

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$$
\begin{aligned}
&\frac{1}{2}\int \mathrm{d}x\, x^m \int \frac{e^{ixP^+z^-}}{2\pi}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0} \\
&= \int \frac{\mathrm{d}x}{2(iP^+)^m}\frac{\mathrm{d}^m}{(\mathrm{d}z^-)^m}\left[\frac{e^{ixP^+z^-}}{2\pi}\right]\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|^{\mathbf{z}=0}_{z^+=0} \\
&= \frac{i^m}{2(P^+)^{m+1}}\langle P+\frac{\Delta}{2}|\frac{\mathrm{d}^m}{(\mathrm{d}z^-)^m}\left[\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})\right]|P-\frac{\Delta}{2}\rangle|_{z=0} \\
&= \frac{1}{2(P^+)^{m+1}}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(0)\gamma^+\left(i\overleftrightarrow{\partial}^+\right)^m\psi^q(0)|P-\frac{\Delta}{2}\rangle
\end{aligned}
$$

We can generalise what we obtained on the EFF for higher moments:

$$\begin{aligned}
&\frac{1}{2}\int \mathrm{d}x\, x^m \int \frac{e^{ixP^+z^-}}{2\pi}\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0,\mathbf{z}=0}\\
&=\int \frac{\mathrm{d}x}{2(iP^+)^m}\frac{\mathrm{d}^m}{(\mathrm{d}z^-)^m}\left[\frac{e^{ixP^+z^-}}{2\pi}\right]\langle P+\frac{\Delta}{2}|\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})|P-\frac{\Delta}{2}\rangle \mathrm{d}z^-|_{z^+=0}^{\mathbf{z}=0}\\
&=\frac{i^m}{2(P^+)^{m+1}}\langle P+\frac{\Delta}{2}|\frac{\mathrm{d}^m}{(\mathrm{d}z^-)^m}\left[\bar{\psi}^q(-\frac{z}{2})\gamma^+\psi^q(\frac{z}{2})\right]|P-\frac{\Delta}{2}\rangle|_{z=0}\\
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\end{aligned}$$

- we recover local operators as in DIS $\mathcal{O}^{\mu\mu_1\cdots\mu_m}=\mathbf{S}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}^{\mu_1}\ldots\overleftrightarrow{\partial}^{\mu_m}\psi$
- ... but evaluated between off-diagonal states

Mellin Moments of GPDs I

Polynomiality property

$$\mathcal{M}_m = \frac{1}{2(P^+)^{m+1}} \langle P + \frac{\Delta}{2} | \bar{\psi}^q(0) \gamma^+ \left(i \overleftrightarrow{\partial}^+ \right)^m \psi^q(0) | P - \frac{\Delta}{2} \rangle$$

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$$= \frac{1}{(P^+)^m} \frac{\bar{u}(p')\gamma^+ u(p)}{2P^+} \sum_{\substack{i=0 \\ \text{even}}}^{m} A_{i,m}(t) \Delta^{\mu_1} \ldots \Delta^{\mu_i} P^{\mu_{i+1}} \ldots P^{\mu_m} n_{\mu_1} \ldots n_{\mu_m}$$

$$+ \frac{1}{(P^+)^m} \frac{\bar{u}(p') i\sigma^{+\alpha} \Delta_\alpha u(p)}{4MP^+} \sum_{\substack{i=0 \\ \text{even}}}^{m} B_{i,m}(t) \Delta^{\mu_1} \ldots \Delta^{\mu_i} P^{\mu_{i+1}} \ldots P^{\mu_m} n_{\mu_1} \ldots n_{\mu_m} n_{\mu_1} \ldots n_{\mu_m}$$

$$+ \mathrm{mod}(m, 2) \frac{\Delta^+}{2MP^+} \bar{u}(p') u(p) (\frac{\Delta^+}{P^+})^m C_{m+1}(t)$$

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$$= \frac{\bar{u}(p')\gamma^+ u(p)}{2P^+} \sum_{\substack{i=0 \\ \text{even}}}^{m} A_{i,m}(t)(-2\xi)^i + \frac{\bar{u}(p') i\sigma^{+\alpha}\Delta_\alpha u(p)}{4MP^+} \sum_{\substack{i=0 \\ \text{even}}}^{m} B_{i,m}(t)(-2\xi)^i$$

$$+ \mathrm{mod}(m,2) \frac{\bar{u}(p') u(p)}{2M} (-2\xi)^{m+1} C_{m+1}(t)$$

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$$+ \mathrm{mod}(m,2) \frac{\bar{u}(p') u(p)}{2M} (-2\xi)^{m+1} C_{m+1}(t)$$

Using the Gordon Identity the last structure can be reabsorbed:

$$\bar{u}(p')\gamma^\mu u(p) = \frac{P^\mu}{M} \bar{u}(p') u(p) + \bar{u}(p') \frac{i\sigma^{\mu\nu}\Delta_\nu}{2M} u(p)$$

Mellin Moments of GPDs II

We deduce that the GPDs Mellin moments are:

$$\int_{-1}^{1} \mathrm{d}x \; x^m H^q(x,\xi,t;\mu) = \sum_{j=0}^{\left[\frac{m}{2}\right]} (2\xi)^{2j} A^q_{2j,m}(t;\mu) + mod(m,2)(2\xi)^{m+1} C^q_{m+1}(t;\mu)$$

$$\int_{-1}^{1} \mathrm{d}x \; x^m E^q(x,\xi,t;\mu) = \sum_{j=0}^{\left[\frac{m}{2}\right]} (2\xi)^{2j} B^q_{2j,m}(t;\mu) - mod(m,2)(2\xi)^{m+1} C^q_{m+1}(t;\mu)$$

X. Ji, J.Phys.G 24 (1998) 1181-1205
A. Radyushkin, Phys.Lett.B 449 (1999) 81-88

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Mellin Moments of GPDs are even polynomials in ξ of a given degree !

- $A_{0,m}(0)$ are the moments of the PDF
- $A_{0,0}(t)$ is the Dirac Form Factor
- $B_{0,0}(t)$ is the Pauli Form Factor
- $C_{m+1}(t)$ are the Mellin moment of a new object: the D-term

Introducing the D-term

- We want to define a function D so that for odd m:

$$\int_{-1}^{1} \mathrm{d}y \, y^m D(y,t) = (-2)^{m+1} C_{m+1}(t)$$

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- What is the connection between y, x and ξ (we stick to $\xi > 0$)?

$$\sum_{\substack{i=0\\ \text{even}}}^{m} A_{i,m}(t)(-2\xi)^i = \int_{-1}^{1} \mathrm{d}x\, x^m H(x,\xi,t) - \xi^{m+1} \int_{-1}^{1} \mathrm{d}y\, y^m D(y,t)$$

$$= \int_{-1}^{1} \mathrm{d}x\, x^m \left[H(x,\xi,t) - \Theta(-\xi \leq x \leq \xi) D\left(\frac{x}{\xi}, t\right)\right]$$

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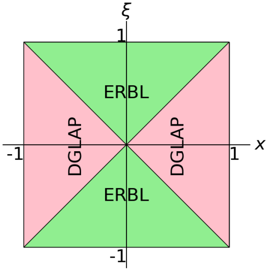

- D-term is a function of 2 variables only ! (like the PDF)
- It lives *only* in the so-called ERBL region
- It triggers singular behaviours ($\xi \to 0$ and $x \to \xi$)

Consequence of Polynomiality

$$\sum_{\substack{i=0 \\ \text{even}}}^{m} A_{i,m}(t)(-2\xi)^i = \int_{-1}^{1} \mathrm{d}x\, x^m \left[H(x,\xi,t) - \Theta(-\xi \le x \le \xi) D\left(\frac{x}{\xi}, t\right)\right]$$

- After introducing the D-term, we obtained a new polynomiality relation with the *same* power on the left and right-hand side.

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- After introducing the D-term, we obtained a new polynomiality relation with the *same* power on the left and right-hand side.
- This has an important consequence: in mathematics, this relation is called the Lugwig-Helgason condition

O. Teryaev, PLB510 125-132 (2001)
N. Chouika *et al.*, EPJC 77 906 (2017)

Consequence of Polynomiality

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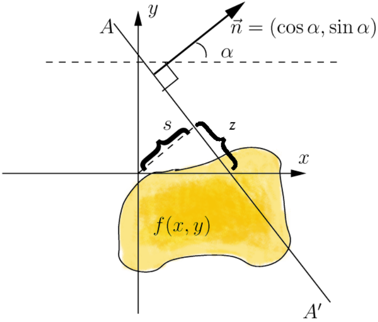
- It implies that $H - D$ is the Radon transform of a third function, called a Double Distribution F.

A brief introduction of the Radon Transform

Radon transform : integral of a function over a line $L \in \mathbb{R}^2$



source: wikipedia

- Definition

$$\mathcal{R}[f][\theta, s] = \int_{-\infty}^{\infty} \mathrm{d}z f(x(z), y(z))$$

$$x(z) = z\sin(\theta) + s\cos(\theta)$$

$$y(z) = -z cos(\theta) + s\sin(\theta)$$

- Connected to 2D Fourier transform through Fourier Slice Theorem.

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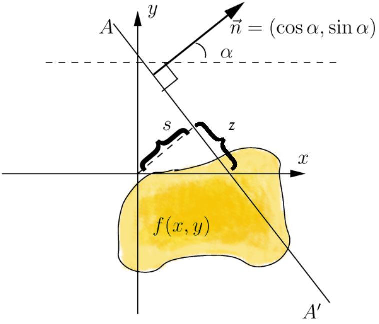

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The Radon transform is a key ingredient of Computed tomography (medical X-ray imaging)

Radon Dictionnary for GPDs

GPD variable are (x, ξ) instead of (s, θ). A way to build the dictionary is to took again at the polynomiality condition :

- For GPDs:

$$\int \mathrm{d}x x^m H(x, \xi) = \sum_i A_{i,m} (2\xi)^i$$

- For canonical Radon transform:

$$\int \mathrm{d}s s^m G(s, \theta) = \sum_i g_{i,m} \cos^{m-i} \theta \sin^i \theta = \cos^m \theta \sum_i g_{i,m} \tan^i \theta$$

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We thus deduce

$$x = \frac{s}{\cos\theta}, \quad \xi = \tan\theta$$

Radon transform and Double Distributions

- The connection between GPDs and DDs is given through:

$$H(x,\xi,t) - \Theta(-\xi \leq x \leq \xi) D\left(\frac{x}{\xi}, t\right) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi) F(\beta,\alpha,t)$$

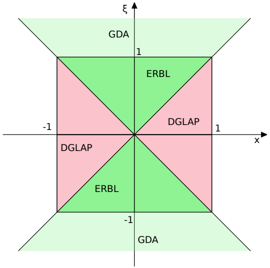
A. Radysuhkin, PRD 56 (1997) 5524-5557
D. Müller *et al.*, Fortsch. Phy. 42 101 (1994)

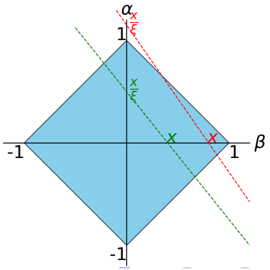
- The D-term can be reabsorbed as:

$$H(x,\xi,t) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi) \left[F(\beta,\alpha,t) + \xi\delta(\beta) D(\alpha,t)\right]$$

M. Polyakov and C. Weiss, PRD60 114017 (1999)

- The properties of the DD guarantee the one of the GPD





Polynomiality revisited with DD

- Polynomiality of GPDs Mellin moments is equivalent to the existence of the DDs.

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- In fact, generalised form factors $A_{i,m}(t)$ can be reinterpreted in terms of DDs:

$$\int \mathrm{d}x\, x^m H(x,\xi,t) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\, (\beta+\alpha\xi)^m F(\beta,\alpha,t) + \xi^{m+1}\int_{-1}^{1} \mathrm{d}\alpha \alpha^m D(\alpha,t)$$

$$= \sum_i^m \xi^i \underbrace{\binom{m}{i}\int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\, \alpha^i \beta^{m-i} F(\beta,\alpha,t)}_{=(-2)^i A_{i,m}(t)} + + \xi^{m+1} \underbrace{\int_{-1}^{1} \mathrm{d}\alpha \alpha^m D(\alpha,t)}_{=(-2)^{m+1} C_{m+1}(t)}$$

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- A direct consequence is the link between the DD and the PDF:

$$q(x) = \int_{-1+|x|}^{1-|x|} \mathrm{d}\alpha F(x,\alpha,0)$$

Model of Double Distributions

- Many GPDs models rely on DD in order to fulfil the polynomiality condition.
- The most common way is to use the Radyushkin DD Ansatz:

$$F(\beta,\alpha,t) = q(\beta,t)\times\pi_N(\beta,\alpha)$$

$$\pi_N(\beta,\alpha) = \frac{\Gamma\left(N+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma(N+1)}\frac{\left((1-|\beta|)^2-\alpha^2\right)^N}{(1-|\beta|)^{2N+1}}$$

$$1 = \int_{-1+|\beta|}^{1-|\beta|} \mathrm{d}\alpha\, \pi_N(\beta,\alpha)$$

Musatov, I.V. and Radyushkin, A.V., PRD61 074027 (2000)

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- This was used for many model, both on the nucleon and the pion several reasons:
  - Simple to implement
  - Gives results driven by the PDF (much better known)
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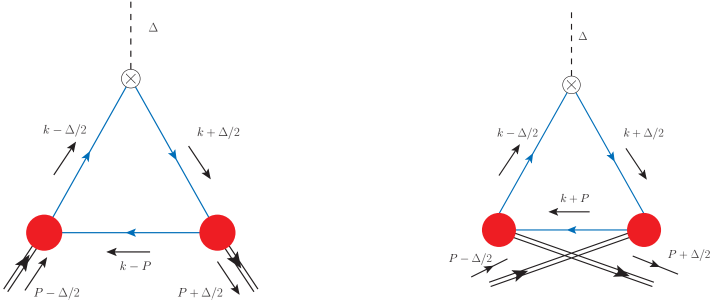
  - Simple to implement
  - Gives results driven by the PDF (much better known)
  - It allows to fulfil easily the GPDs sum rules (connection to EFF)
- However, this functional form has been shown not to be a very flexible fitting parametrisation

C. Mezrag *et al.*,PRD 88 (2013) 1, 014001

Covariant computations and DD

- DDs naturally appear in explicitly covariant computations



- Inserting local operators, one recovers polynomials in ξ and therefore DDs.

B.C. Tiburzi and G. A. Miller, PRD 67 (2003) 113004
C. Mezrag *et al.*, arXiv:1406.7425 and FBS 57 (2016) 9, 729-772

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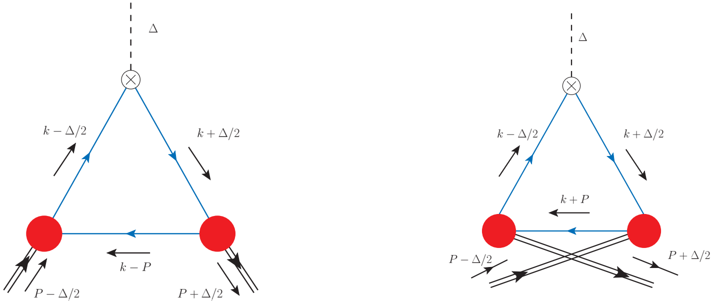

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- However these computations suffer from other issue, for instance regarding the so-called positivity property.

The lightfront wave functions (LFWFs) formalism

Hadrons seen as Fock States

- Lightfront quantization allows to expand hadrons on a Fock basis:

$$|P,\pi\rangle \propto \sum_\beta \Phi_\beta^{q\bar{q}}|q\bar{q}\rangle + \sum_\beta \Phi_\beta^{q\bar{q},q\bar{q}}|q\bar{q},q\bar{q}\rangle + \dots$$

$$|P,N\rangle \propto \sum_\beta \Phi_\beta^{qqq}|qqq\rangle + \sum_\beta \Phi_\beta^{qqq,q\bar{q}}|qqq,q\bar{q}\rangle + \dots$$

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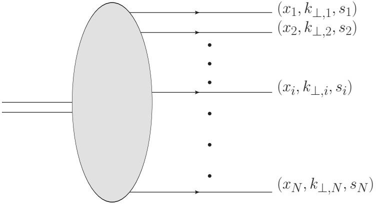
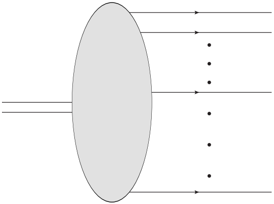
- Non-perturbative physics is contained in the N-particles Lightfront-Wave Functions (LFWF) Φ^N

see for instance S. Brodsky *et al.*, Phys.Rept.S 301 (1998) 299-486



- Momentum information for each parton:
  - Momentum fraction along the lightcone x_i carried by each partons such that $\sum_i^N x_i = 1$ with $0 \leq x_i \leq 1$.
  - Momentum in the transverse plane $k_{\perp,i}$ for each parton
- other quantum number such as parton spin projection

LFWFs



- Momentum information for each parton:
  - Momentum fraction along the lightcone x_i carried by each partons such that $\sum_i^N x_i = 1$ with $0 \leq x_i \leq 1$.
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The pion has two independent two-body LFWFs:

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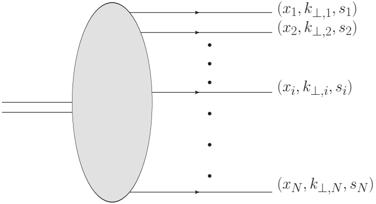

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Overlap of LFWFs and GPDs

- Starting from the matrix element:

$$\langle \pi, P + \frac{\Delta}{2} | \bar{\psi}\left(-\frac{z}{2}\right)\gamma^{+}\psi\left(\frac{z}{2}\right)|\pi, P - \frac{\Delta}{2}\rangle$$

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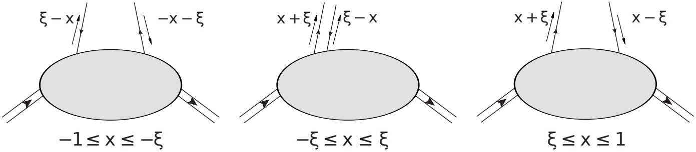
M. Diehl *et al.*, Nucl.Phys. B596 (2001) 33-65

- Two different partonic interpretations:



GPD partonic interpretation

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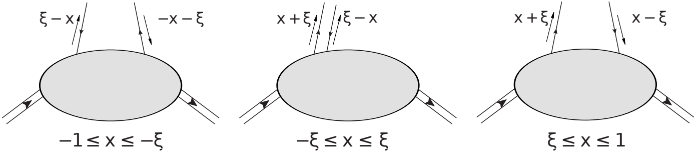
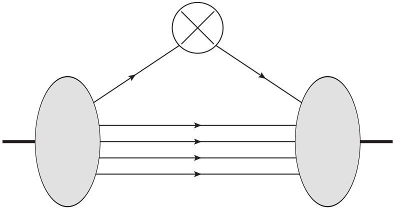
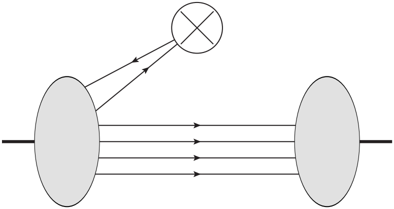


- This has a impact on the way the LFWFs overlap:

DGLAP: $|x| > |\xi|$

ERBL: $|x| < |\xi|$





- Same N LFWFs
- No ambiguity

- N and $N+2$ partons LFWFs
- Ambiguity

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  The PDFs depend only on square modulus of LFWFs.
- Note that we recover formally a expression of a norm:

$$\langle \pi, P|\bar{\psi}\left(-\frac{z}{2}\right)\gamma^+\psi\left(\frac{z}{2}\right)|\pi, P\rangle \sim \sum_N^\infty |\phi^N|^2$$

The positivity property

- Beyond the forward limit, in the DGLAP region, the overlap of LFWFs keeps an interesting structure:

$$\langle \pi, P + \frac{\Delta}{2} | \bar{\psi}\left(-\frac{z}{2}\right) \gamma^+ \psi\left(\frac{z}{2}\right) | \pi, P - \frac{\Delta}{2} \rangle \sim \sum_N^{\infty} (\phi^N_{out})^* \times \phi^N_{in}$$

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$$|H(x,\xi,t)_{x \geq \xi \geq 0}| \leq \sqrt{q\left(\frac{x-\xi}{1-\xi}\right) q\left(\frac{x+\xi}{1+\xi}\right)}$$

A. Radysuhkin, Phys. Rev. **D59**, 014030 (1999)
B. Pire *et al.*, Eur. Phys. J. **C8**, 103 (1999)
M. Diehl *et al.*, Nucl. Phys. **B596**, 33 (2001)
P.V. Pobilitsa, Phys. Rev. **D65**, 114015 (2002)

- Same type of inequality for gluon GPDs.

Nucleon LFWFs classification

- In the nucleon case, the procedure applies with three quarks at leading Fock state:

$$\langle 0|\epsilon^{ijk} u^i_\alpha(z_1) u^j_\beta(z_2) d^k_\gamma(z_3)|P,\uparrow\rangle$$

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- The LFWFs carry different amount of OAM projections:

states	$\langle\downarrow\downarrow\downarrow\|P,\uparrow\rangle$	$\langle\downarrow\downarrow\uparrow\|P,\uparrow\rangle$	$\langle\uparrow\downarrow\uparrow\|P,\uparrow\rangle$	$\langle\uparrow\uparrow\uparrow\|P,\uparrow\rangle$
OAM	2	1	0	-1
LFWFs	ψ^6	ψ^3, ψ^4	ψ^1, ψ^2	ψ^5

Overlap representation for the nucleon

- Starting from the matrix element:

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This can be made more constraining:

$$(1-\xi^2)\left(H^q-\frac{\xi^2}{1-\xi^2}E^q\right)^2+\frac{t_0-t}{4M^2}(E^q)^2\leq q\left(\frac{x-\xi}{1-\xi}\right)q\left(\frac{x+\xi}{1+\xi}\right)$$

Impact parameter space positivity constraints

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The solution is provided by the impact parameter space representation:

$$(1-\xi^2)\left|\hat{H}_\pi(x,\xi,\mathrm{b}_\perp)\right| \leq \sqrt{\hat{H}_\pi\left(x,0,\frac{\mathrm{b}_\perp}{1+\xi}\right)\hat{H}_\pi\left(x,0,\frac{\mathrm{b}_\perp}{1-\xi}\right)}$$

This inequality is more constraining, but I do not know examples of it being used for realistic phenomenology in the nucleon case.

Polynomiality vs. Positivity

Polynomiality

- Properties of Mellin moments (local operators)
- Comes from Lorentz Covariance and discrete symmetries
- Delicate cancellations between DGLAP and ERBL region
- Equivalent to the existence of underlying Double Distributions

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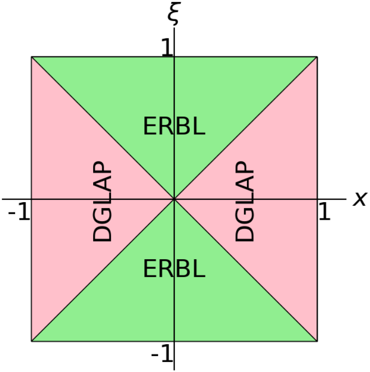
Is there a way to fulfil both?

Delicate cancellation at work



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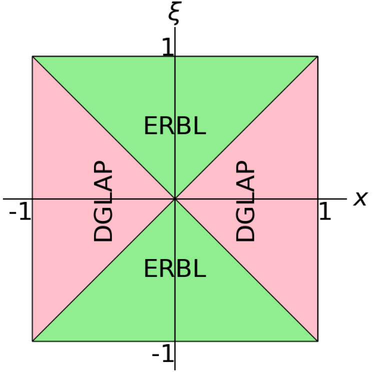

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n	$\int_{+\xi}^{+1} \mathrm{d}x\, x^n H_{\mathrm{DGLAP}}(x,\xi)$	$\int_{-\xi}^{+\xi} \mathrm{d}x\, x^n H_{\mathrm{ERBL}}(x,\xi)$	$\int_{-1}^{+1} \mathrm{d}x\, x^n H(x,\xi)$
0	$\frac{(-1+\xi)^2(1+4\xi)}{(1+\xi)^2}$	$-\frac{4(-2+\xi)\xi^2}{(1+\xi)^2}$	1
1	$\frac{(-1+\xi)^2\left(1+4\xi+10\xi^2\right)}{3(1+\xi)^2}$	$-\frac{4\xi^3(-5+2\xi)}{3(1+\xi)^2}$	$\frac{1}{3}\left(1+2\xi^2\right)$
2	$\frac{(-1+\xi)^2\left(1+4\xi+10\xi^2+20\xi^3\right)}{7(1+\xi)^2}$	$-\frac{4\xi^4(-8+5\xi)}{7(1+\xi)^2}$	$\frac{1}{7}(1+2\xi^2)$
3	$\frac{(-1+\xi)^2\left(1+4\xi+10\xi^2+20\xi^3+35\xi^4\right)}{14(1+\xi)^2}$	$-\frac{4\xi^5(-7+4\xi)}{7(1+\xi)^2}$	$\frac{1}{14}\left(1+2\xi^2+3\xi^4\right)$
4	$\frac{5(-1+\xi)^2\left(1+4\xi+10\xi^2+20\xi^3+35\xi^4+56\xi^5\right)}{126(1+\xi)^2}$	$-\frac{20\xi^6(-10+7\xi)}{63(1+\xi)^2}$	$\frac{5}{126}\left(1+2\xi^2+3\xi^4\right)$

Pragmatic solution: DD-based fit

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  - Specific form better than others

    P.V. Pobilitsa, Phys. Rev. **D65**, 114015 (2002)
  - possibility to reject parameters combinations outside the positivity range

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"Try and test" way to fulfil positivity in DD space

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P.V. Pobilitsa, Phys. Rev. **D65**, 114015 (2002)

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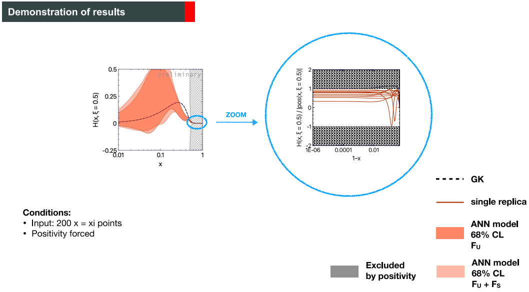
"Try and test" way to fulfil positivity in DD space

- It has been tested on pseudo-data and it really helps constraining GPDs



slide from P. Sznajder *et al.*,

SPIN 2021

Systematic Way: The covariant extension

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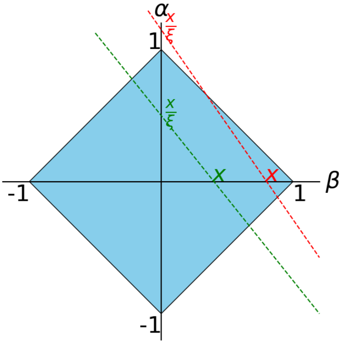

- two types of lines: DGLAP and ERBL lines
- All point of the support are crossed by infinitely many DGLAP lines
- But the line $\beta = 0$!
- when getting close to $\beta = 0$ the slope of DGLAP lines $\rightarrow \infty$

Anwer

- Mathematical Answer : Yes ! We can uniquely extract the DD but not the D-term.

N. Chouika *et al.*, EPJC78, 478 (2018)

- There is a condition : the line $\xi = 0$ should be in the domain probed

P. Dall'Olio *et al.*, PRD 109 (2024) 9, 096013

Anwer

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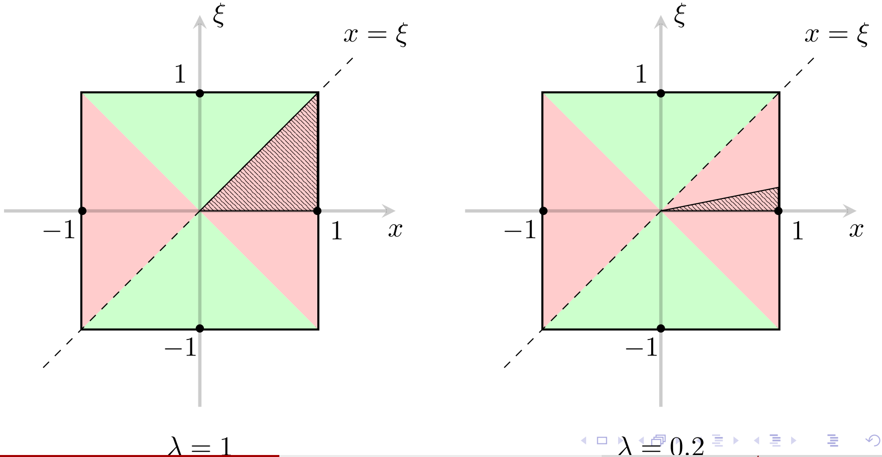

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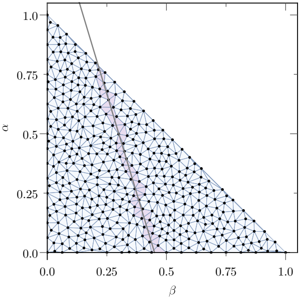

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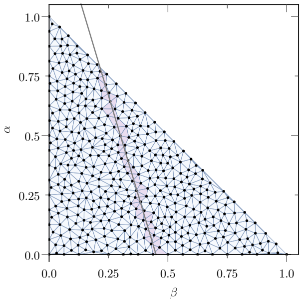

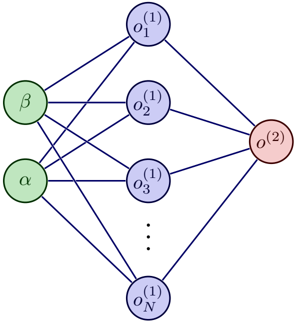

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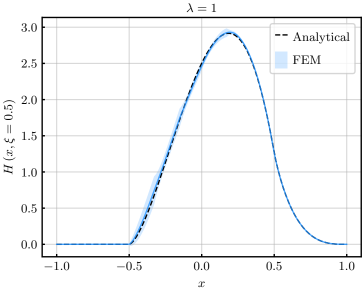

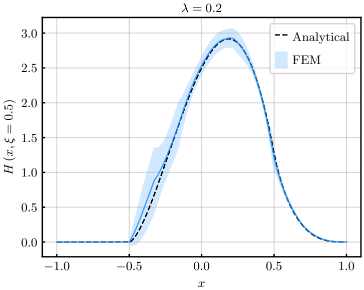

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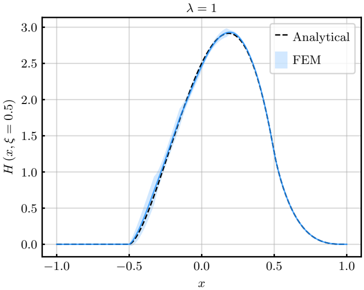

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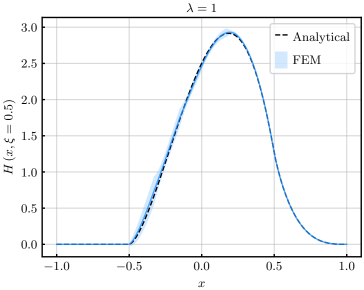

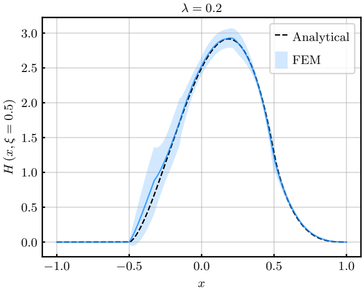

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- This also might be because the target function is quite regular

Modelling GPD: a challenge

Summary so far

- GPDs obeys multiple theoretical constraint
  - Polynomiality coming from Lorentz covariance
  - Positivity derived from LF Hilbert space
- Modelling them so that they fulfil these properties is *difficult*

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Dura Physicae Lex sed Physicae Lex

- Requires deep understanding of the physics at stake
- Strongly help constraining experimental extraction

Questions ?