# Lattice Regularised QCD and Baryon DAs

JunHua SCNU

On behalf of Lattice Parton Collaboration (LPC)

International Workshop School on Hadron Structure & Strong Interactions2025

10/16 @ Nanjing University

# OUT LINE

01 **Motivation**
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02 **LaMET Framework**
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03 **Precision Calculation of Meson DA**
LCDA in LaMET; Precision check for meson LCDA;

04 **Advances on Baryon LCDA**
From light meson to  light baryon & heavy meson

05 **Review on Advances of LCDA**
Precision in future

# Motivation About Meson & Baryon

- **Meson — ubiquitous messengers**

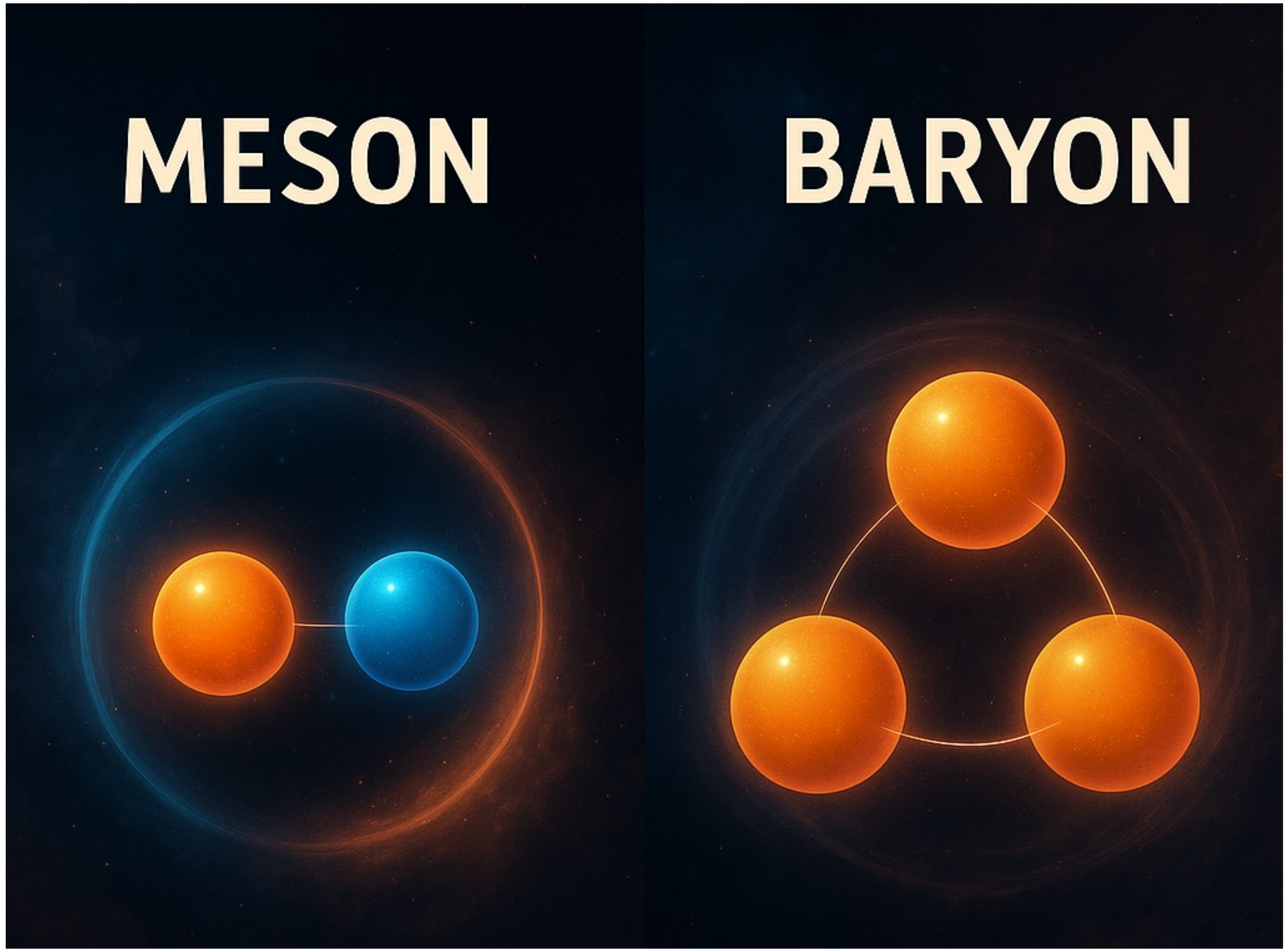
  **Light Meson $SU(3)_{flavor}$ $8 \otimes 1$**

  **Heavy Meson $SU(4)_{flavor}$ …**

- **Baryon — cornerstone particles**

  **Octet baryons**

  **Decuplet baryons**



# Motivation About Meson & Baryon

## Meson and Baryon in Flavor Physics

- 1956, Parity violation in weak interaction
- 1964, Observation of CP violation in **Kaon meson**
- 1973, Kobayashi-Maskawa mechanism
- 2004, Observation of direct CPV in **B meson**
- 2019, Observation of direct CPV in **D meson**
- 2025, Observation of direct CPV in **Baryon $\Lambda_b^0 \to p$**

physicsworld
TOP 10
BREAKTHROUGH
2019

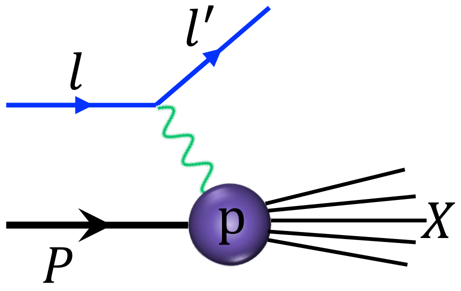
*LHCb, Nature 643 8074 (2025)*； Theoretical: *J.J.Han, et.al. PRL 134 221801(2025)*

# Motivation About LCDA

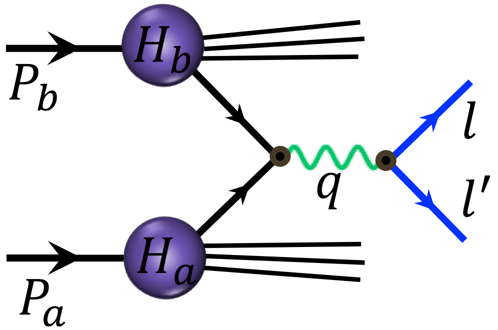
- One try to probe internal structure of nucleons → **Inclusive beam-target collision**

## Defined PDFs

DIS



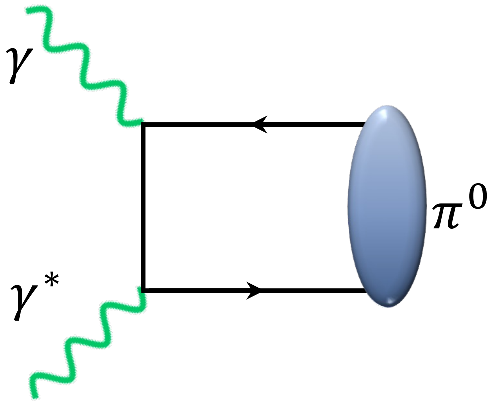

Drell-Yan



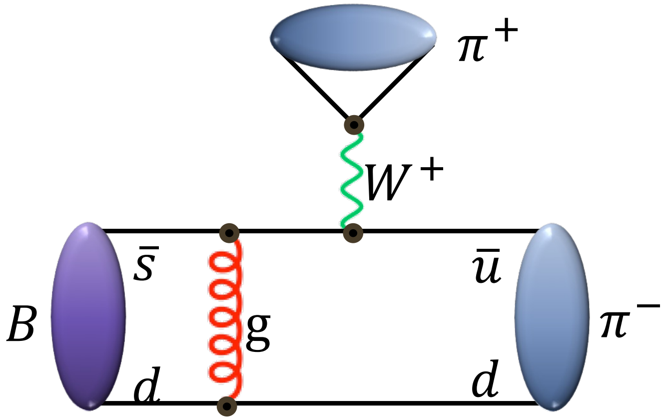

- One try to obtain richer QCD dynamical information → **Exclusive scattering**

## Defined LCDAs

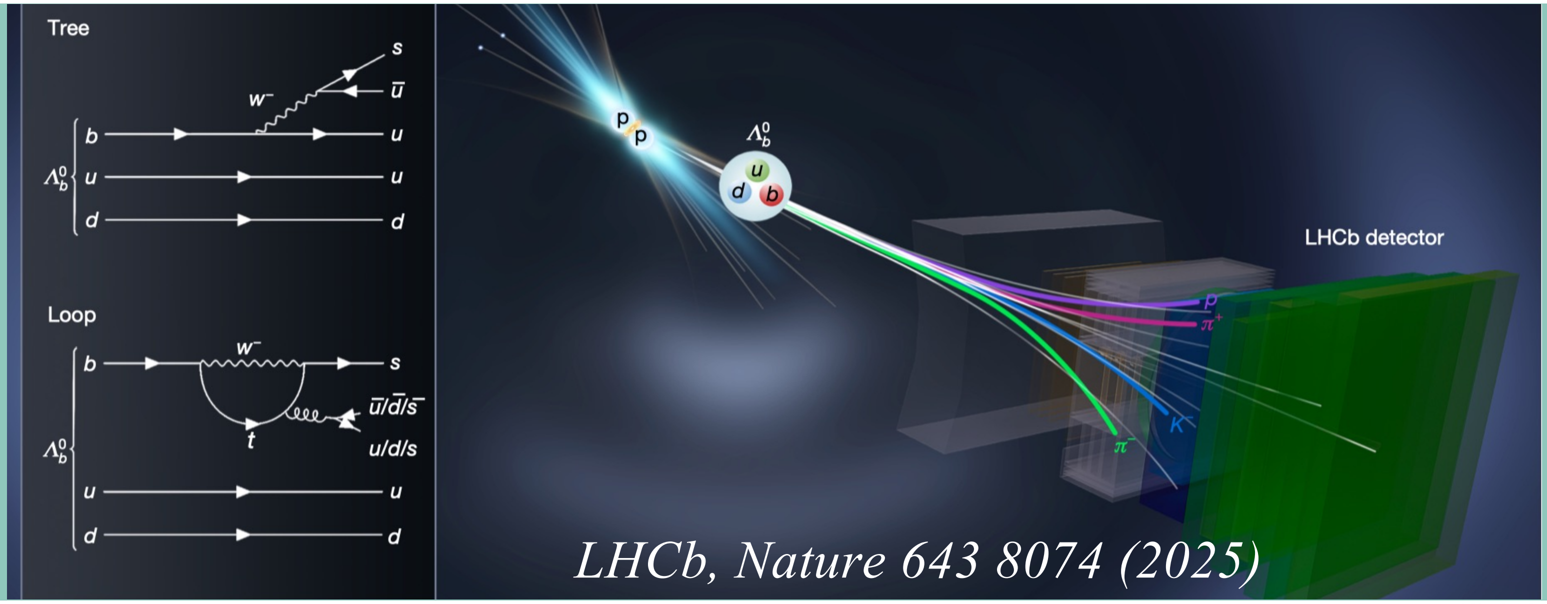
$\gamma\gamma^* \to \pi$



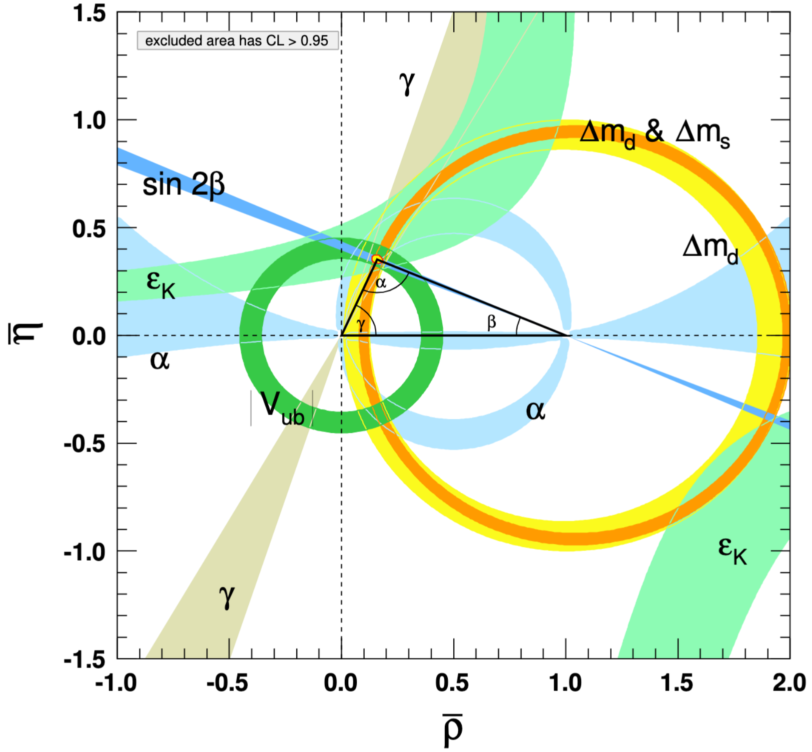

$B \to \pi^+\pi^-$



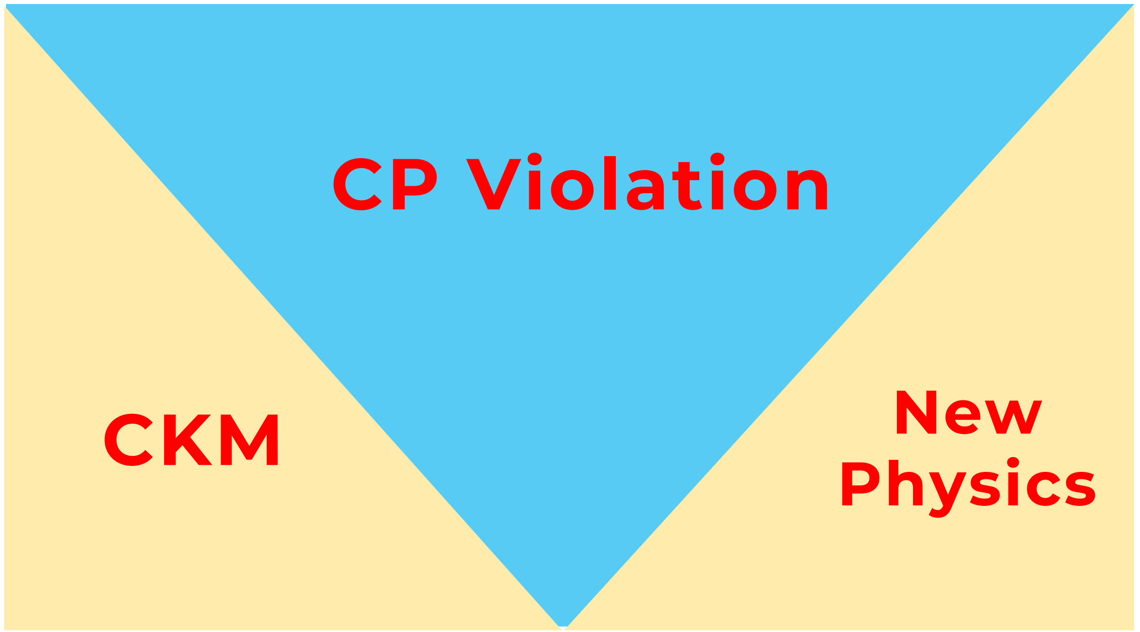
# Motivation Exclusive Target



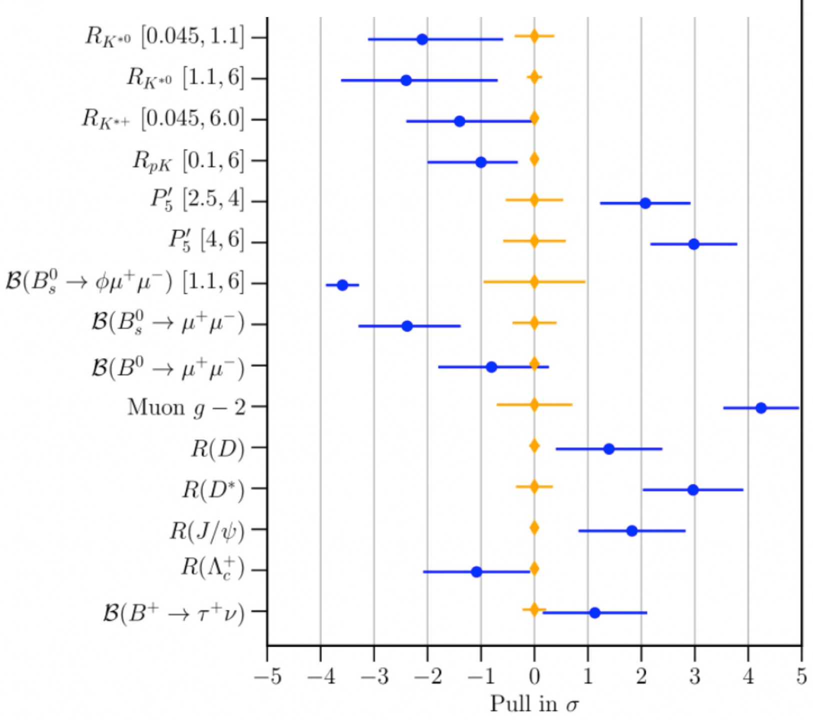

*LHCb, Nature 643 8074 (2025)*




*PDG2024, PRD 110, 030001*




*Snowmass 2021, 2207.01851*



# Motivation About LCDA

## Light Meson LCDAs: (1977 - now)

- Asymptotic form

  Chernyak, Zhitnitsky, 1977; Lepage, Brodsky, 1979; ……
- QCD Sum rules

  Chernyak, Zhitnitsky, 1982; Braun, Filyanov, 1989; ……
- Dyson-Schwinger Equation

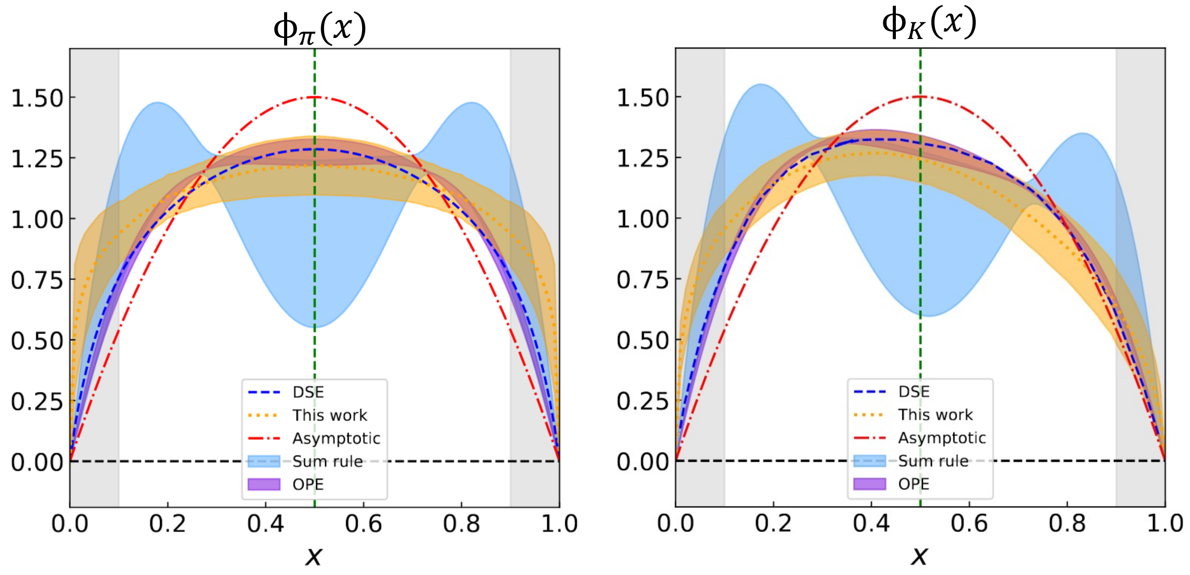
  Chang, Cloet, et.al, 2013; Gao, Chang, et.al, 2014; ……
- Global Fits

  Cheng, et.al, 2020; Hua, Li, Lu, Wang, Xing, 2021; ……
- Models

  Arriola, Broniowski, 2002; Zhong, Zhu, et.al, 2021; ……
- **Lattice QCD with OPE**

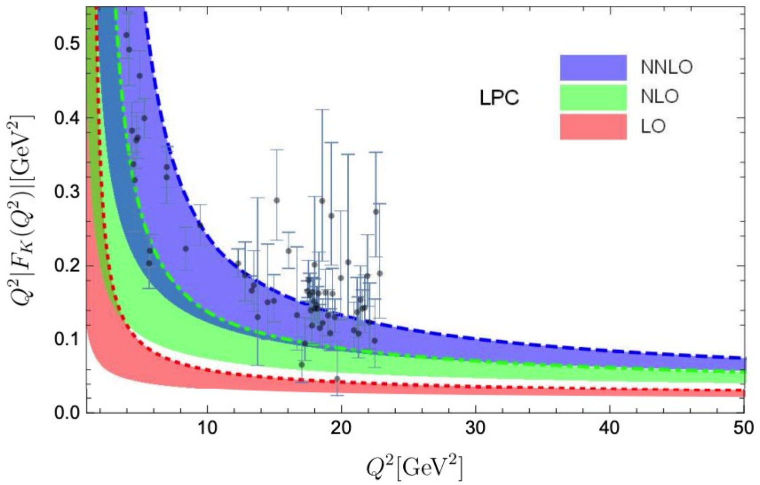
  Braun, et al., 2016; RQCD collaboration, 2019, 2020; ……
- **Lattice QCD with LaMET**

  LP3, 2019; LPC, 2021, 2022; ……




**Application:**

*L.B.Chen et.al*
*PRL 132 201901(2024),*
*PLB 870 139886(2025)*
*PRL 135 131903 (2025)*

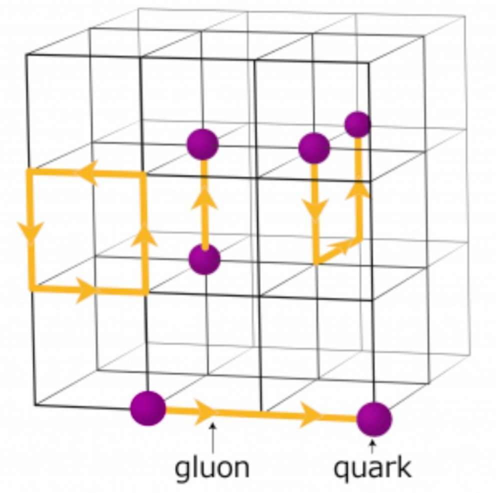


# Motivation Lattice QCD

**LQCD is formulated as a Feynman path integral on a 4D Euclidean grid.**

**Simulations provide a stochastic computation follows QCD Lagrangian:**

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}D_{\mu} - m)\psi - \frac{1}{4}G^{a}_{\mu\nu}G^{a,\mu\nu}$$




- Gluon fields on links of a hypercube;
- Quark fields on sites: approaches to fermion discretization – Wilson, Staggered, Overlap;

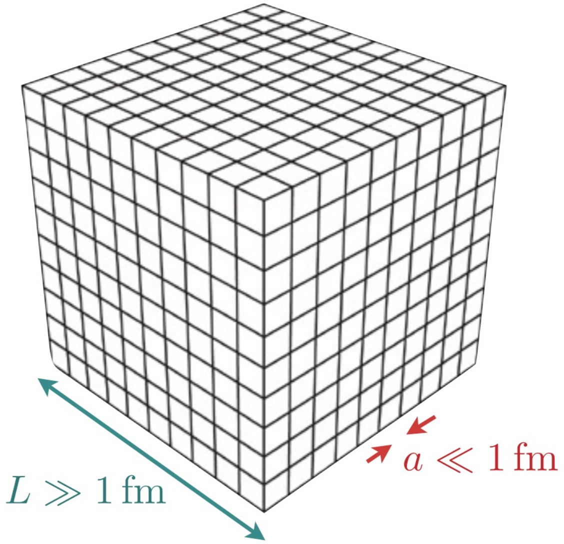
☞ **Discrete: lattice spacing $a$ → UV regulator; box length $L$ → IR regulator;**

☞ **Derivatives: discretization errors ($a \to 0$); $\mathcal{O}(a)$ improved actions; ……**

☞ **Finite volume ($M_{\pi}L \to \infty$): FV errors exponentially small for $M_{\pi}L > 4$;**

☞ **Chiral extrapolation ($M_{\pi} \to 135\text{MeV}$);**

☞ **Numerical importance sampling of path integral: statistical errors.**



# ◉ Motivation Lattice QCD ◉

## LQCD Observables:

- Building blocks: ensembles of gauge configurations; quark propagators
- Hadron & interactions put in as external probes: N-point correlation function

## LQCD Methodology:

☞ Generate gauge configurations;

☞ Calculate quark propagators on the gauge configurations;

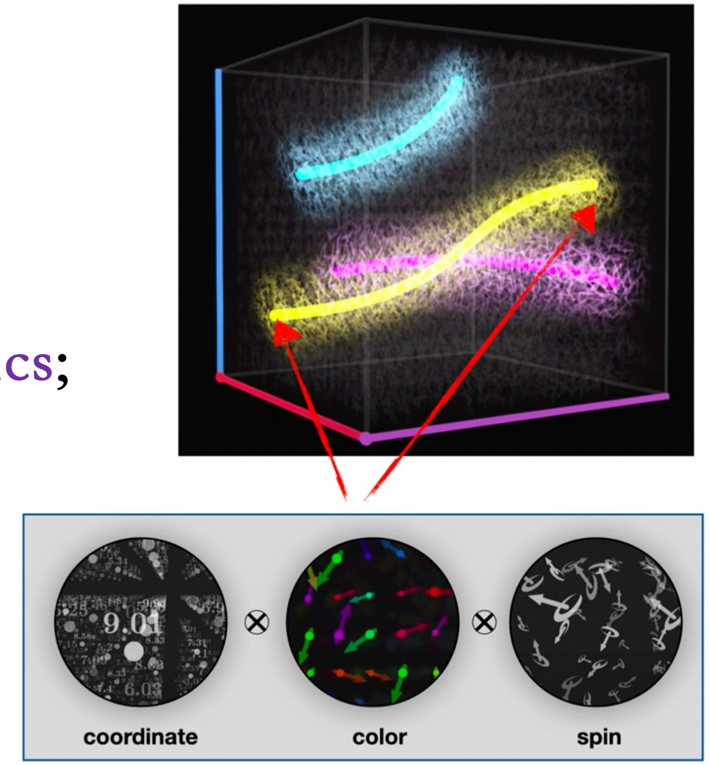
☞ Formulate operators that best probe the physics:

- Low energy effective operators encapsulating SM/BSM physics;

☞ Construct hadronic correlation functions by the building blocks;

☞ Extract hadron ground states by reduction formula;

☞ Evaluate the hadronic matric elements.



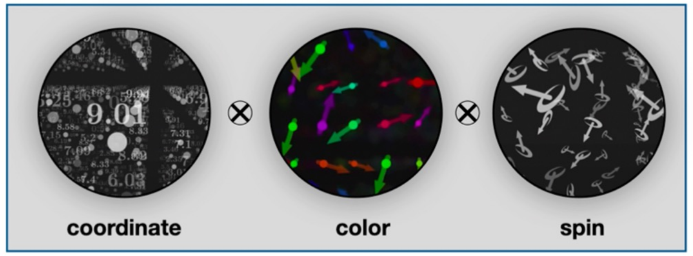
# Motivation Lattice QCD

| Lattice v.s. Continuum | |
|---|---|
| **We simulate:** | **We want:** |
| 😀 At finite lattice spacing $a$ | 🤔 $a \to 0$ |
| 😀 In finite volume $L^3$ | 🤔 $L \to \infty$ |
| 😀 Euclidean space | 🤔 Minkowski space ⇨ **Lost the real time information!** |
| 😀 Lattice regularization | 🤔 Some continuum scheme |
| 😀 Some bare input quark masses: $am_l, \ am_s, \ am_c, \ am_b$ In general, $m_\pi^{\mathrm{lat}} \neq m_\pi^{\mathrm{phy}}$ | 🤔 $m_q^{\mathrm{lat}} = m_q^{\mathrm{phy}}$ |

⇨ **Need to control all limits:** particularly simultaneously control FV and discretization

⇨ **Universality:** different input parameters **must** give converge results.

# Motivation Lattice QCD




Top 500

| | Countries | Count | System Share (%) | Rmax (GFlops) |
|---|---|---|---|---|
| 1 | China | 162 | 32.4 | 514,491,614 |
| 2 | United States | 127 | 25.4 | 2,122,791,370 |
| 3 | Germany | 34 | 6.8 | 219,253,860 |
| 4 | Japan | 31 | 6.2 | 624,251,300 |
| 5 | France | 24 | 4.8 | 174,854,530 |
| 6 | United Kingdom | 15 | 3 | 64,078,644 |
| 7 | Canada | 10 | 2 | 41,208,360 |
| 8 | South Korea | 8 | 1.6 | 88,682,560 |
| 9 | Netherlands | 8 | 1.6 | 33,959,120 |
| 10 | Brazil | 8 | 1.6 | 46,729,150 |

Millions of dimensions or even more !

eg: $32^3 * 64 * 2^2 * 3^2$

Super computers



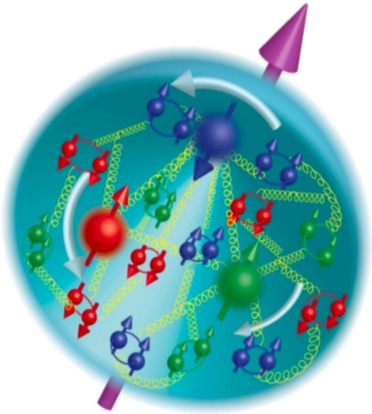
# Motivation Lattice QCD

FLAG Flavour Lattice Averaging Group http://flag.unibe.ch/2024/

Contents



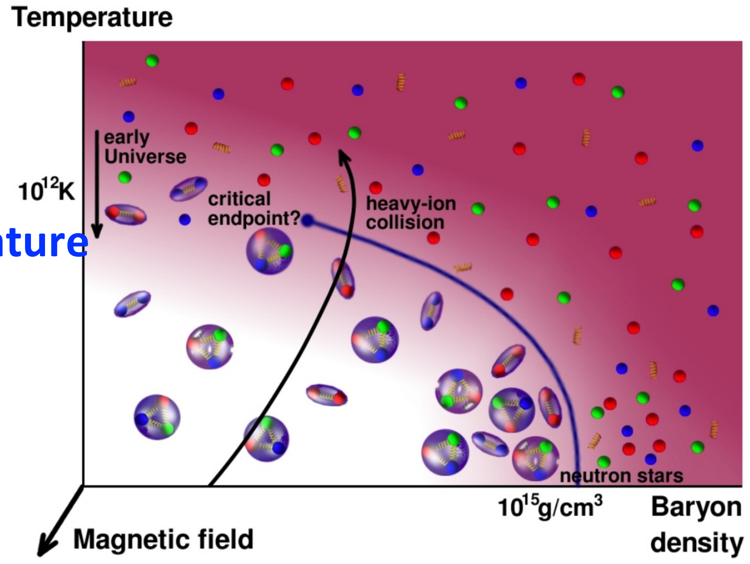
➤ **Hadron Structure and Distributions：**



- **Proton mass, spin, and internal composition；**
- **Parton distribution functions (PDFs)；**
- **Light-cone distribution amplitudes (LCDAs)**
- **Three-dimensional structure:TMDs**

➤ **Lattice QCD under Extreme Conditions:**

- **Quark and gluon condensates**
- **equation of state**
- **QCD phase diagram at finite temperature and density**



02

# LaMET Framework

Moments VS LaMET; Renormalization

# Moments or LaMET

**Light-like correlators cannot be simulated on Euclidean lattice directly**

$\Rightarrow$?

# Moments or LaMET

**Light-like correlators cannot be simulated on Euclidean lattice directly**

**⇒ Local correlators can !**

- OPE moments : **OPE moments ⇒ Gegenbauer moments**

$$\xi \equiv 2x - 1, \qquad \langle \xi^n \rangle_K(\mu) \equiv \int_0^1 dx\, (2x-1)^n\, \phi_K(x,\mu)$$

- OPE for light-cone operator (non-local → local):

$$\mathcal{O}(z^-) \equiv \bar{q}(0)\,\gamma^+\gamma_5\, W(0,z^-)\, s(z^-), \quad \longrightarrow \quad \mathcal{O}(z^-) = \sum_{n=0}^{\infty} \frac{(iz^-)^n}{n!}\, \bar{q}(0)\,\gamma^+\gamma_5\, (i\overleftrightarrow{D}^+)^n s(0) + \text{higher-twist},$$

Expansion at short z

- Moments of the kaon DA can relate to matrix elements of local operators:

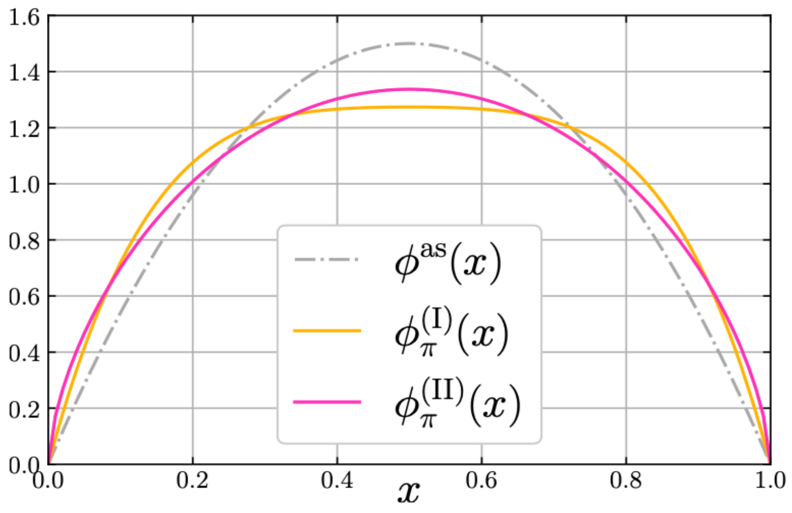
$$\langle 0|\bar{q}(0)\gamma^+\gamma_5(i\overleftrightarrow{D}^+)^n s(0)|K(P)\rangle = i f_K P^{+(n+1)} \langle \xi^n \rangle_K(\mu)$$

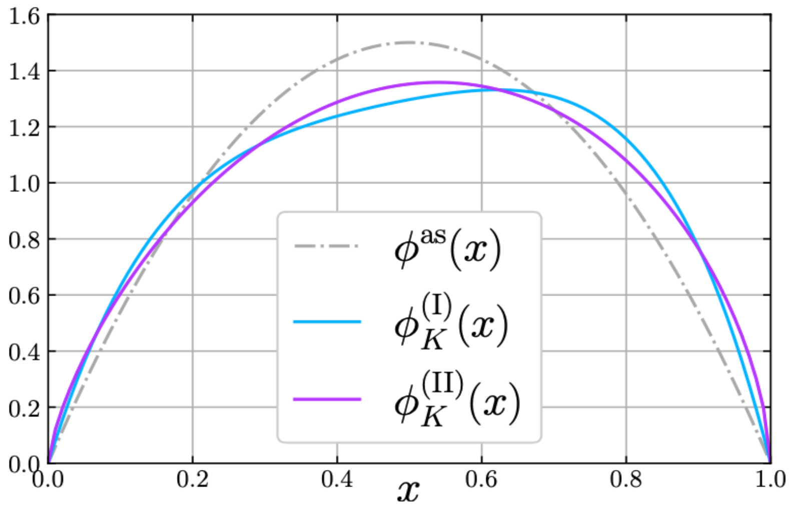
# Moments or LaMET

The nonlocal operator can be defined as a generating function for **renormalized local operators**

**OPE moments ⇒ Gegenbauer moments**

*[RQCD collaboration, 2019]*




$a_2^\pi = 0.101^{+24}_{-24}$ $a_4^\pi = 0.002^{+71}_{-71}$

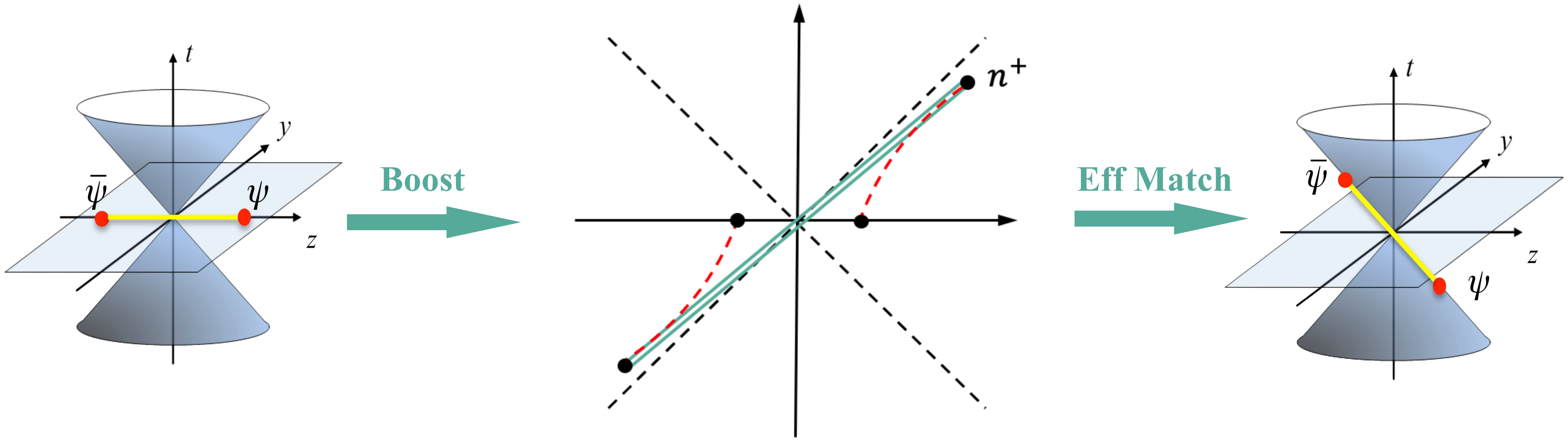
$a_1^K = 0.0533^{+34}_{-35}$ $a_2^K = 0.090^{+19}_{-20}$ ← $\phi_\pi(x) = 6x(1-x)\sum_{n=1,2,\cdots} a_{2n-2}^\pi C_{2n-2}^{(3/2)}(2x-1)$

- 😀 Precise at low order moments
- 🤔 Operator mixing
- Convergence problem
- Computational complexity

# Moments or LaMET

**Large momentum effective theory (LaMET): the light-cone non-local operator correlated to Euclidean non-local operator with a large momentum boost**

*Ji, PRL 110 262002 (2013)*




**LaMET expansion:** $\phi(y,\mu_R) = \int dx C(x,y,\mu_R,\mu)\tilde{\phi}(x,\mu) + \mathcal{O}\left(\frac{1}{(yP^z)^2}, \frac{1}{[(1-y)P^z]^2}\right)$

Perturbative calculated Matching kernel

03

# Precision Calculation of Meson DA

LCDA in LaMET; Precision check for meson LCDA

## ➢ Pion LCDA:

✓ **Calculate the bare quasi-LF correlation**

$$\tilde{h}\left(z,a,P_z\right)=\langle 0|\bar{\psi}_1(0)n_z\cdot\gamma\gamma_5 U(0,z)\psi_2(z)|\pi(P)\rangle$$

✓ **Non-perturbative renormalization**

$$\tilde{h}\left(z,a,P_z\right)=Z(z,a)\tilde{h}_R\left(z,a,P_z\right)$$

✓ **Take appropriate limits and do Fourier transform**

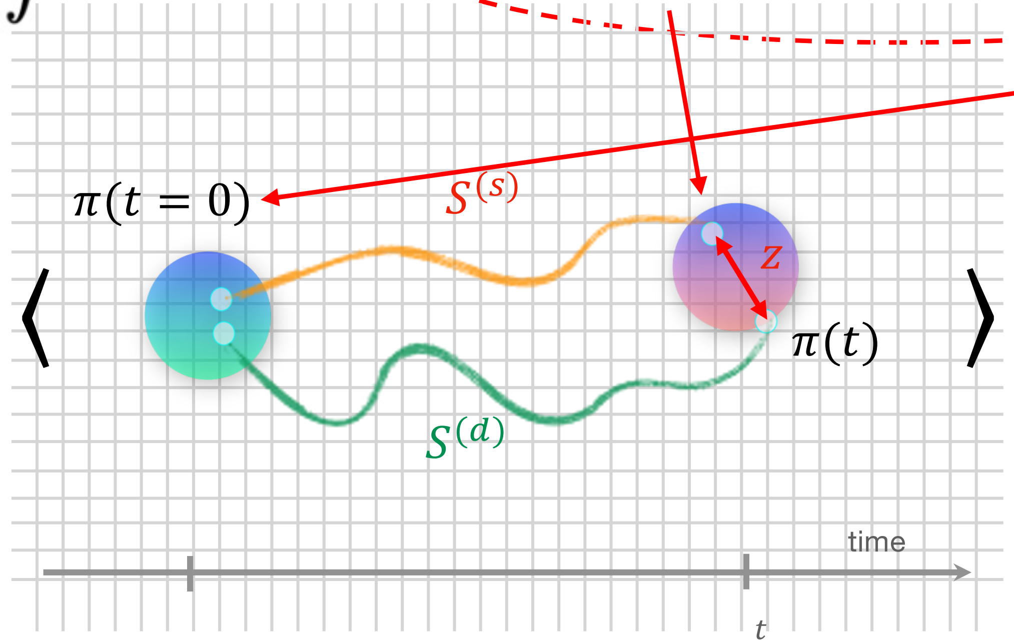
$$if_\pi\tilde{\phi}_\pi\left(x,P_z\right)=\int\frac{dz}{2\pi}e^{-ixzP_z}\tilde{h}_R\left(z,a\to 0,P_z\right)$$

✓ **Matching to light clone**

$$\phi(y,\mu_R)=\int dx C(x,y,\mu_R,\mu)\tilde{\phi}(x,\mu)+\mathcal{O}\left(\frac{1}{(yP^z)^2},\frac{1}{[(1-y)P^z]^2}\right)$$

➢ **Simulate on the lattice the quasi-LF correlation:**

$$C_2^m(z,\vec{P},t) = \int d^3y e^{-i\vec{P}\cdot\vec{y}} \langle 0|\bar{\psi}_1(\vec{y},t)\Gamma_1 U(\vec{y},\vec{y}-z\hat{z})\psi_2(\vec{y}-z\hat{z},t)\bar{\psi}_2(0,0)\Gamma_2\psi_1(0,0)|0\rangle$$




$$\Gamma_{1,\pi} = \gamma_z\gamma_5$$
$$\Gamma_{1,K^*,L} = \gamma_t$$
$$\Gamma_{1,K^*,T} = \sigma(z,y)$$

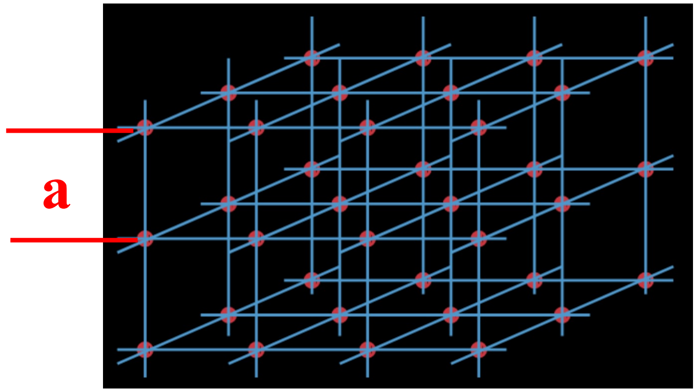
➢ **The ground state quasi-DA extracted by reduction:**

$$\frac{C_2^m(z,\vec{P},t)}{C_2^m(z=0,\vec{P},t)}$$

- **Scheme Conversion**

$a * N_s$ *IR cut* → finite volume effects

$\frac{1}{a}$ *UV cut* → certain scheme




Renorm ⟹ $\overline{\text{MS}}$

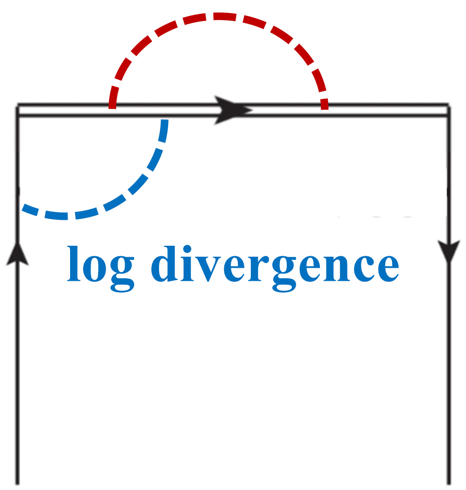
**Lattice bare results** ⟹ Renormalization (Eliminating the lattice divergences) ⟹ **A well-defined intermediate scheme** ⟹ Matching ⟹ **Physical results**

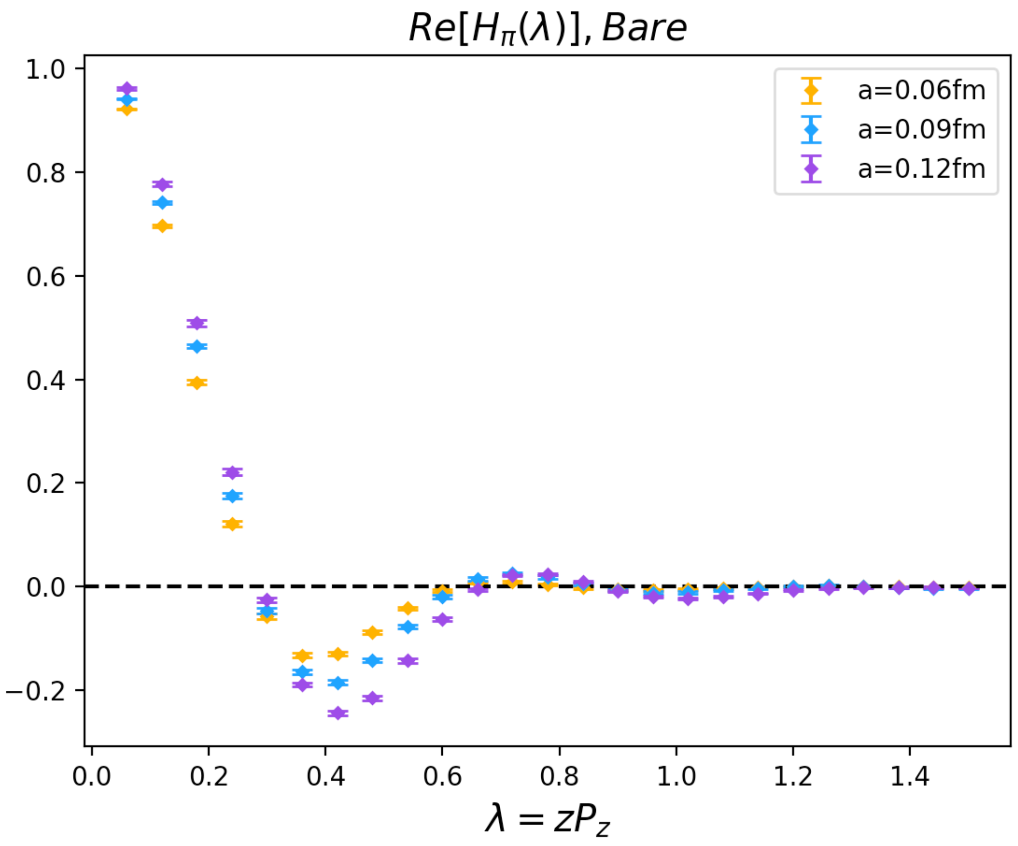
**RI-MOM, Ratio, Self, Hybrid, …**

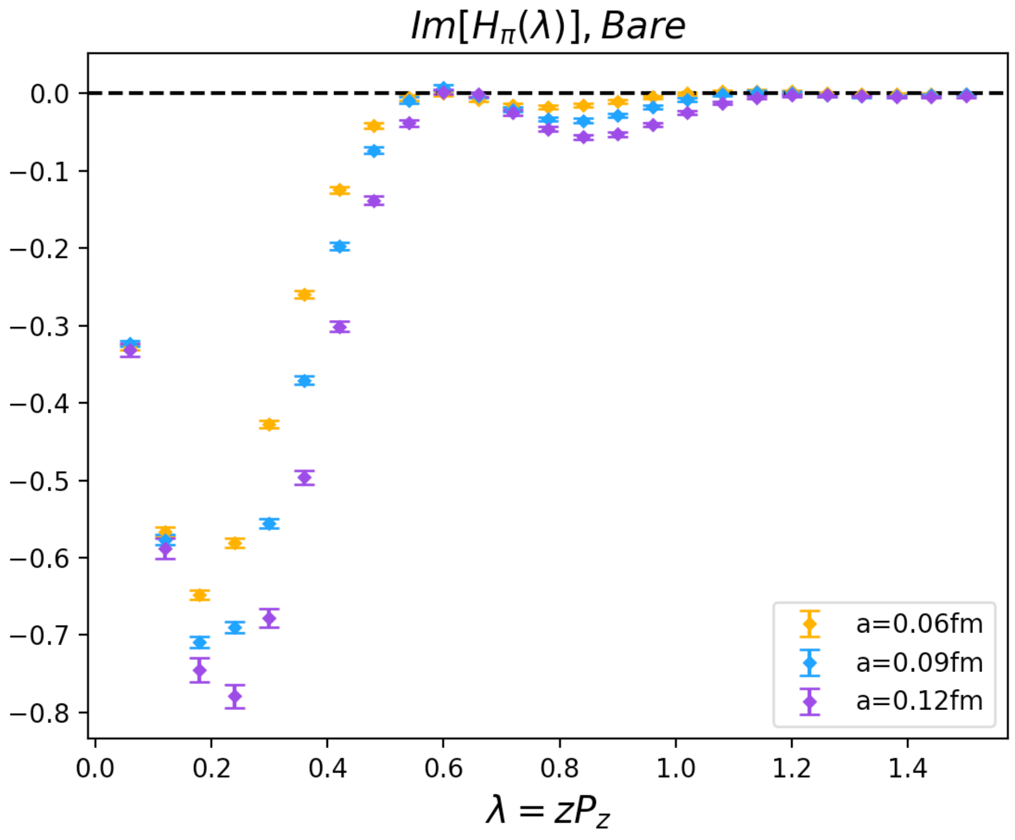
- 1) Renormalization requires the ability to eliminate the divergence caused by the lattice effect.
- 2) The converted scheme is well-defined for effective matching or running …

## ➢ The bare(ground state) quasi-DA:

$$\tilde{\psi}(z,\vec{P}) = \frac{C_2^m(z,\vec{P},t)}{C_2^m(z=0,\vec{P},t)}$$




$P_z = 2.15GeV$

*Figure from(LPC) PRL 129,132001 (2022)*
*Three lattice spacings on MILC ensembles*

- **RI/MOM RI/SMOM Scheme**

  *Martinelli et.al, NPB (1994);*

  *Alexandrou et.al, NPB (2017); Stewart, Zhao, PRD (2018)*

  *He et.al, PRD (2022); Bi et.al, PRD (2023)*

- **Ratio Scheme**

  *Radyushkin et.al, PRD (2017)*

- **Self Renormalization and Hybrid Scheme**

  *Ji et.al, NPB (2021); Huo et.al, NPB (2021)*

| Pros | Cons |
| --- | --- |
| Strict subtraction<br>Well established<br>Widely applied<br>… | Not suitable for non-local<br>(e.g. Gauge dependence,<br>IR behavior large-z …) |
| Easy to use<br>Well subtraction at short-distance | Extra IR behavior at large-z |

—— The best match for the LaMET formalism

# Self renorm

*Ji et.al, NPB (2021); Huo et.al, NPB (2021)*

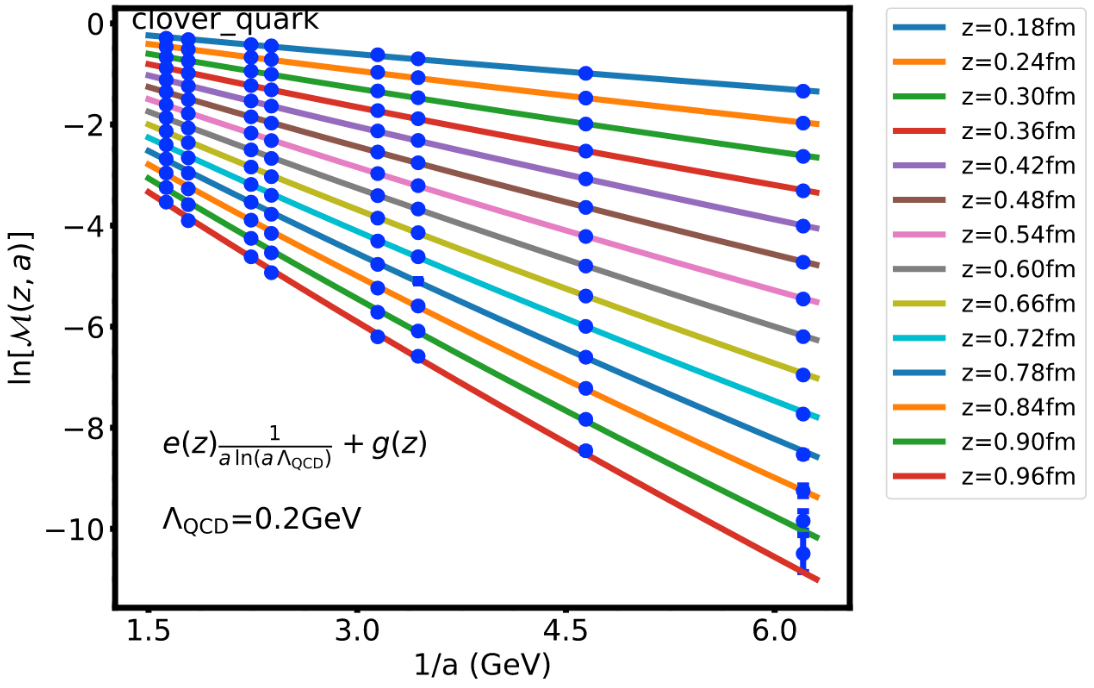
Linear divergence → Multi-a, extract linear-div → Match with perturbative $\overline{MS}$

$$Z(z,a)_R = \exp\left[\frac{kz}{a\ln[a\Lambda_{QCD}]} + m_0 z + f(z)a + \frac{3C_F}{b_0}\ln\left[\frac{\ln[1/(a\Lambda_{QCD})]}{\ln[\mu/\Lambda_{QCD}]}\right] + \ln\left[1+\frac{d}{\ln(a\Lambda_{QCD})}\right]\right]$$

- Extract from multi lattice spacings of zero momentum matrix
- Extract from matching with perturbative $\overline{MS}$ quasi

**Hybrid scheme**

- Short distance: Ratio scheme
- Middle region: Self renormalization
- Large distance: extrapolation

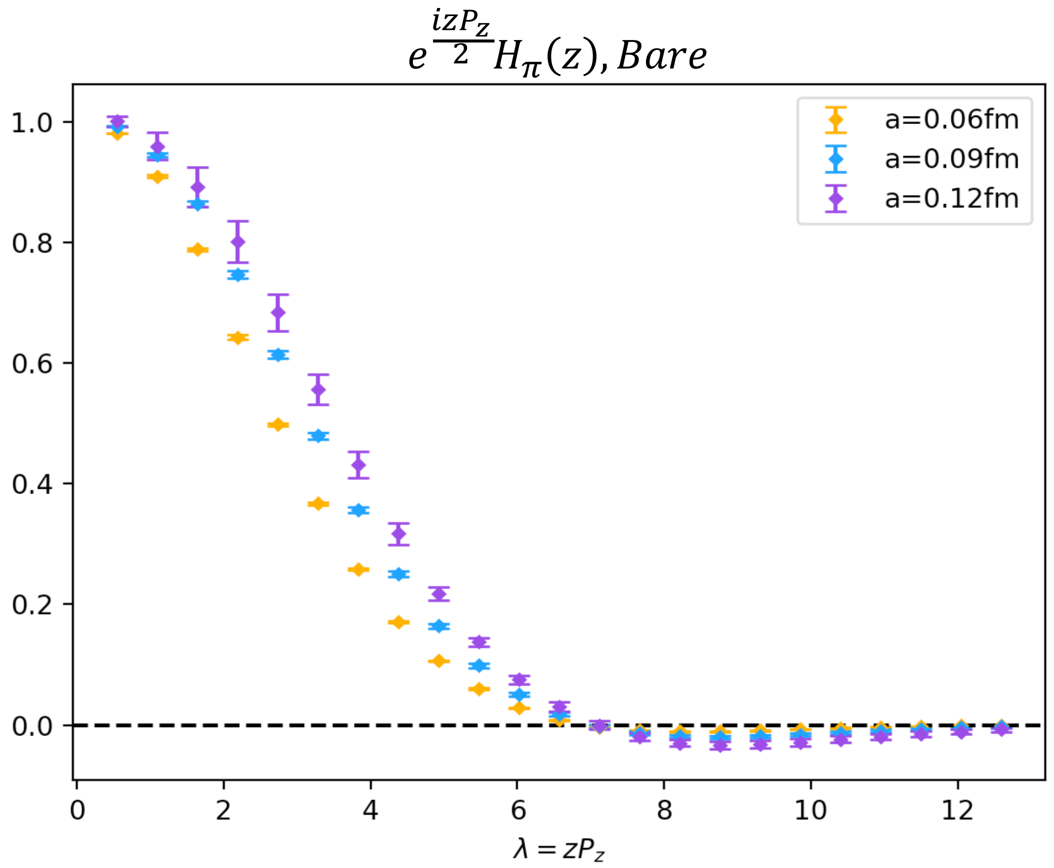


| Meson LCDA in LaMET | Quasi -LF correlation | Non-perturbative renorm | Fourier Transform | Effective match |
|---|---|---|---|---|

# Self renorm

*Ji et.al, NPB (2021); Huo et.al, NPB (2021)*

Linear divergence → Multi-a, extract linear-div → Match with perturbative $\overline{MS}$




Ratio scheme | self scheme

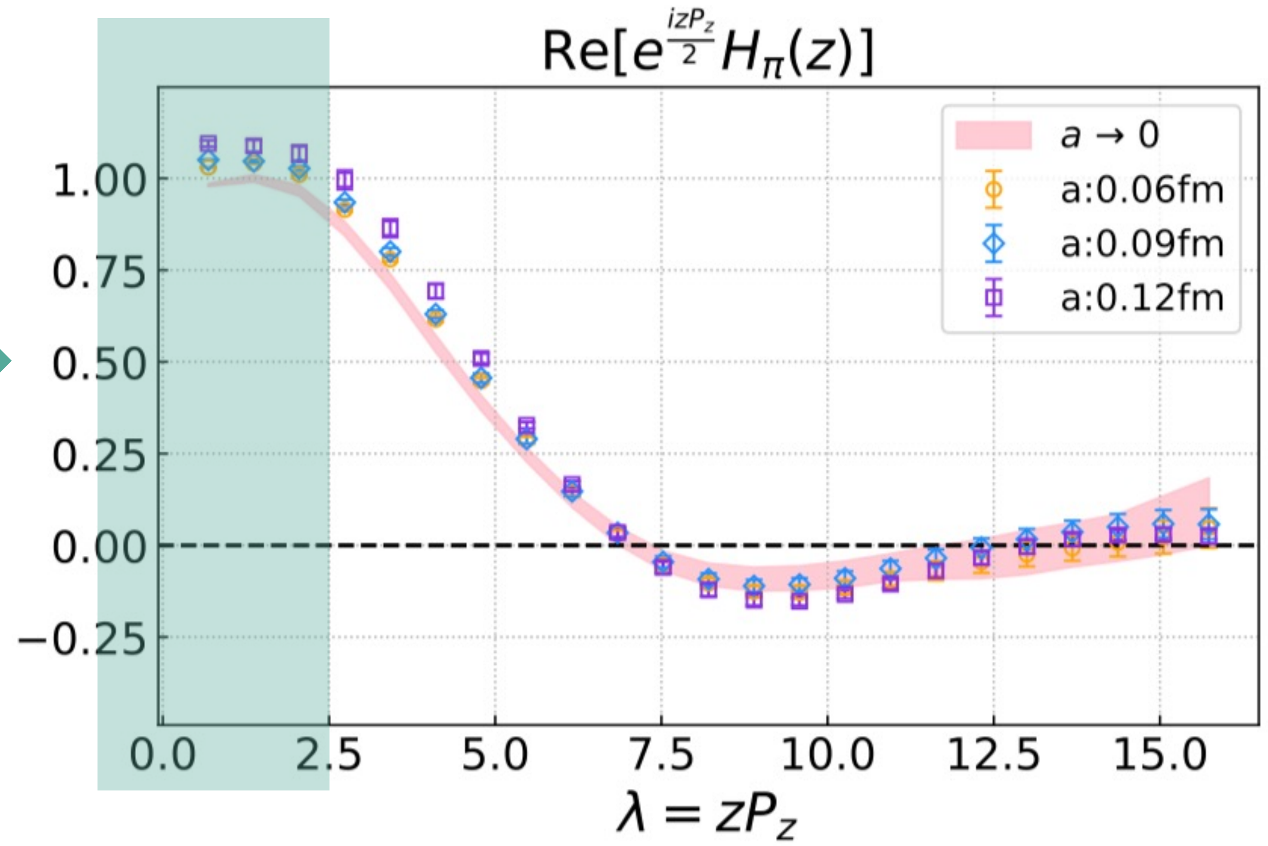
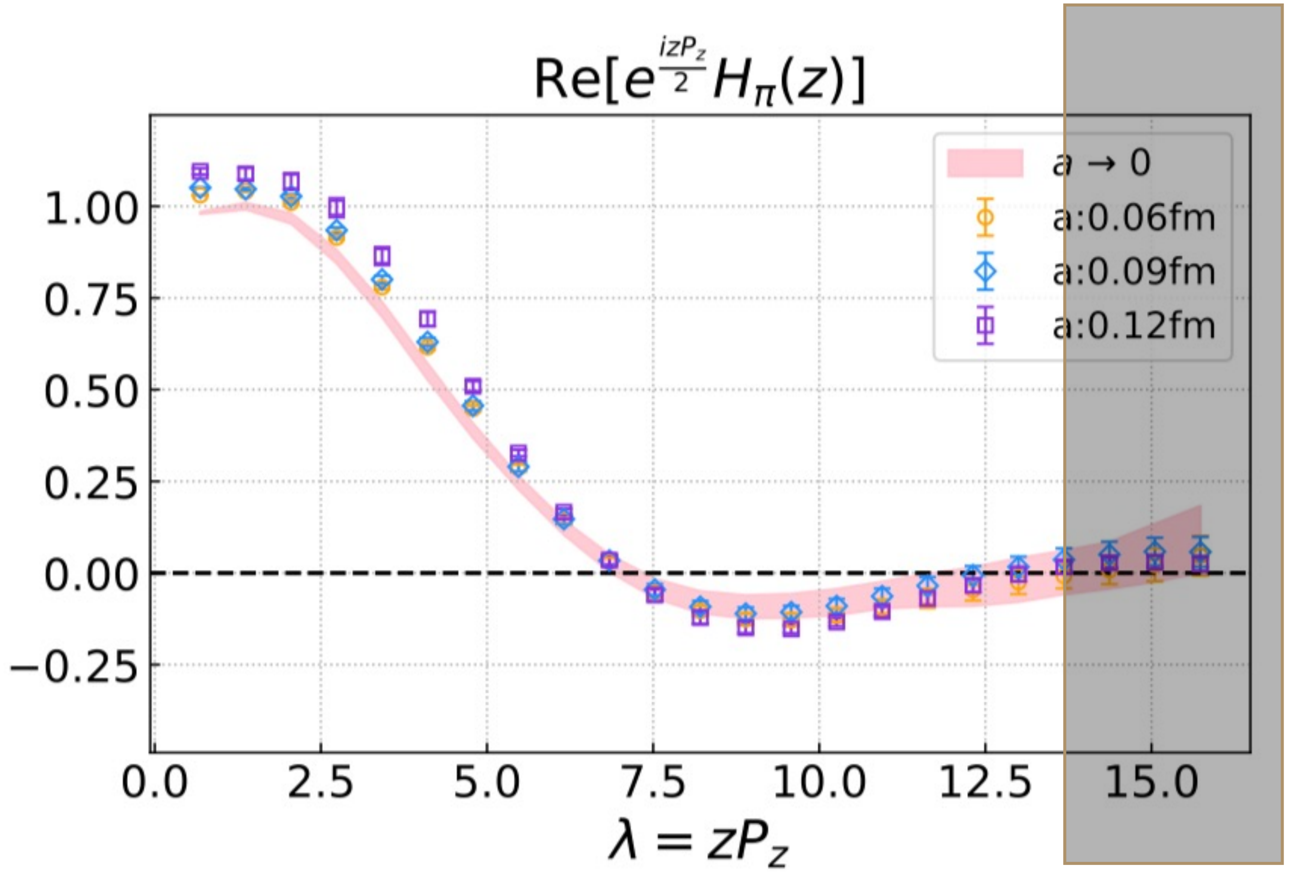
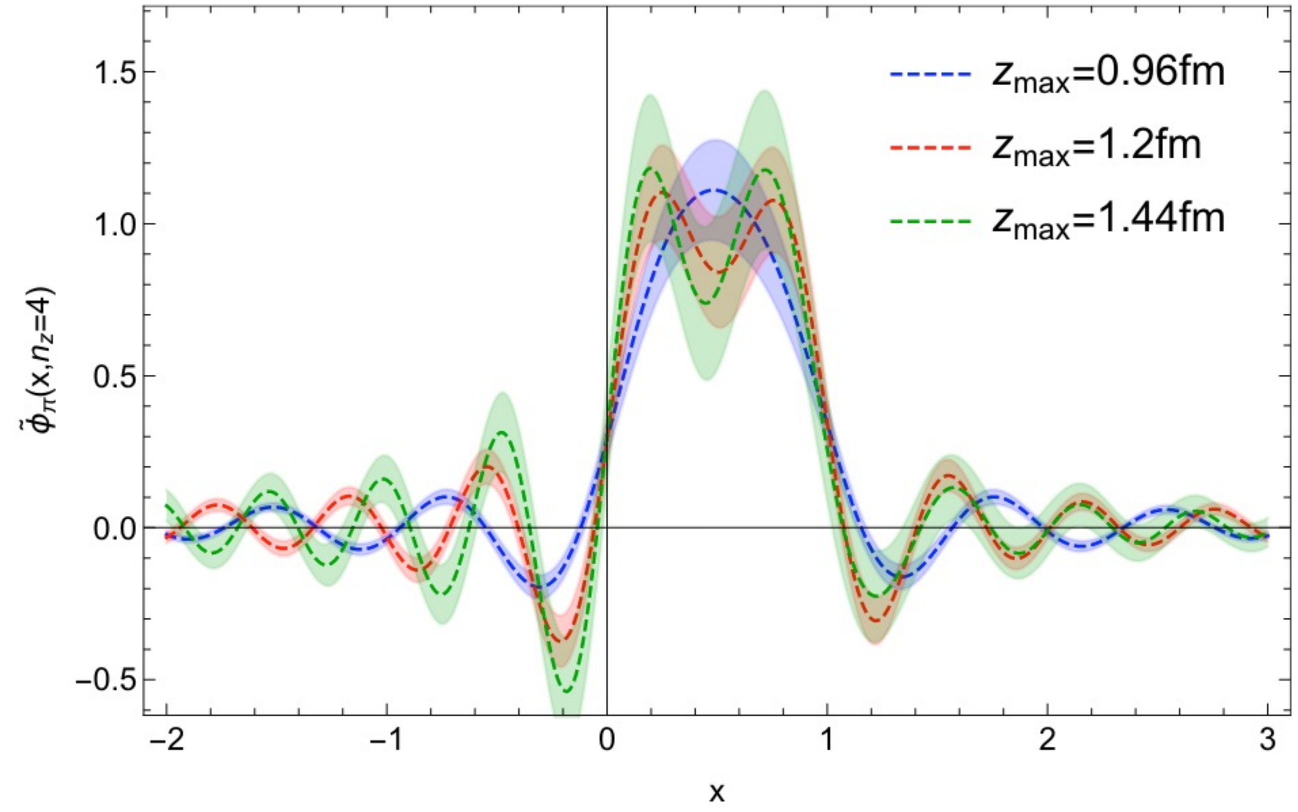



*Figure from(LPC) PRL 129,132001 (2022)*
*Three lattice spacings on MILC ensembles*

## Limited inverse discrete Fourier transform:

$$g(\lambda) = \int dx e^{-i\lambda x} f(x) \to f(x) = \frac{1}{2\pi} \sum_{\lambda_{\min}}^{\lambda_{max}} \Delta\lambda e^{i\lambda x} g(\lambda)$$



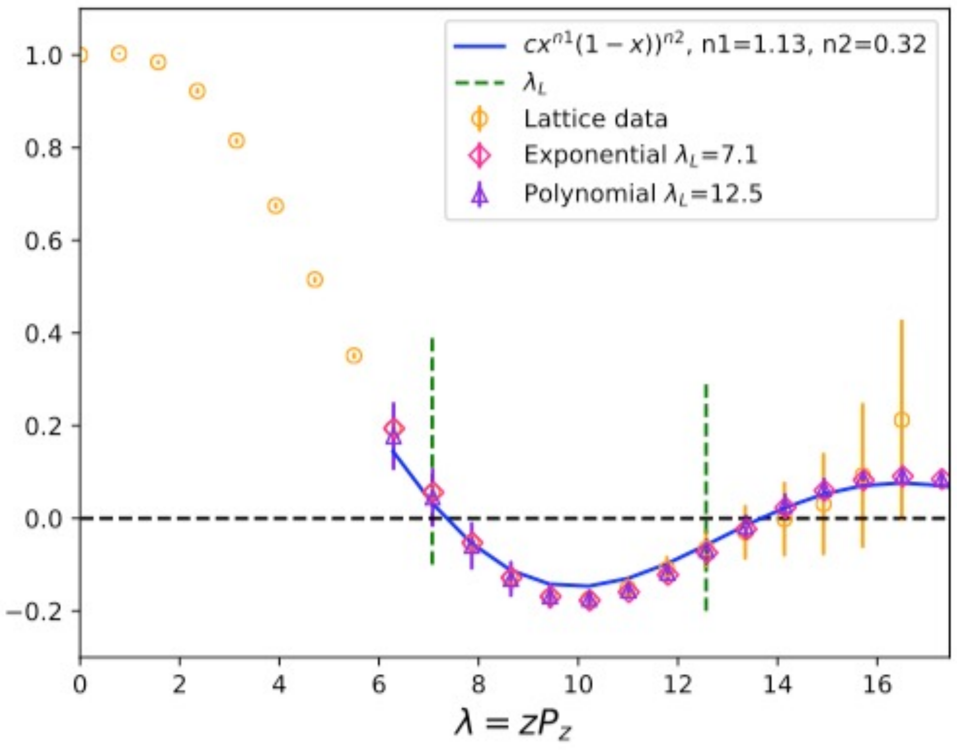



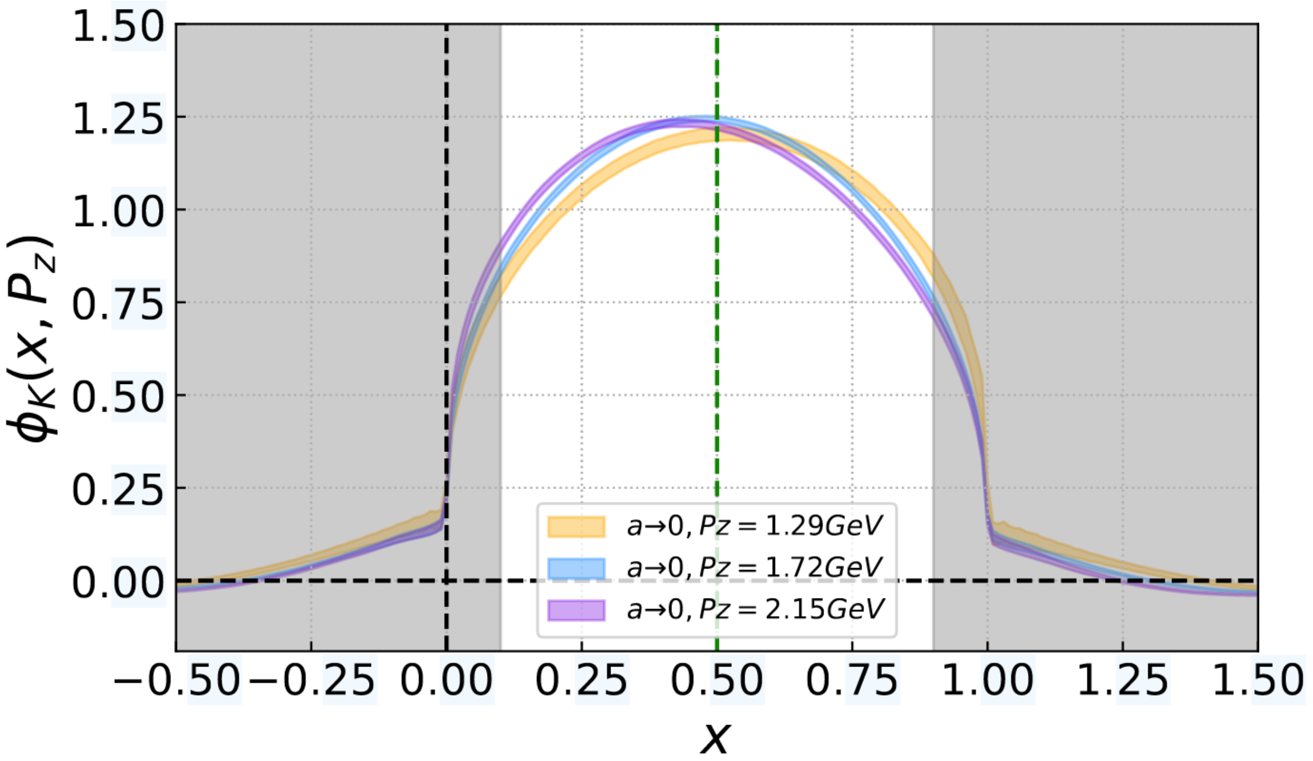

*R. Zhang et al, PRD 20'*

## Limited inverse discrete Fourier transform:

$$g(\lambda) = \int dx e^{-i\lambda x} f(x) \to f(x) = \frac{1}{2\pi} \sum_{\lambda_{\min}}^{\lambda_{max}} \Delta\lambda e^{i\lambda x} g(\lambda)$$

- Start from asymptotic behavior at x~0,1: $\psi(x) \sim x^a (1-x)^b$
- Extrapolation form: $\tilde{H}(z, P_z) = \left[\frac{c_1}{(-i\lambda)^a} + e^{i\lambda} \frac{c_2}{(i\lambda)^b}\right] e^{-\frac{\lambda}{\lambda_0}},$



Meson LCDA in LaMET | Quasi -LF correlation | Non-perturbative renorm | Fourier Transform | Effective match

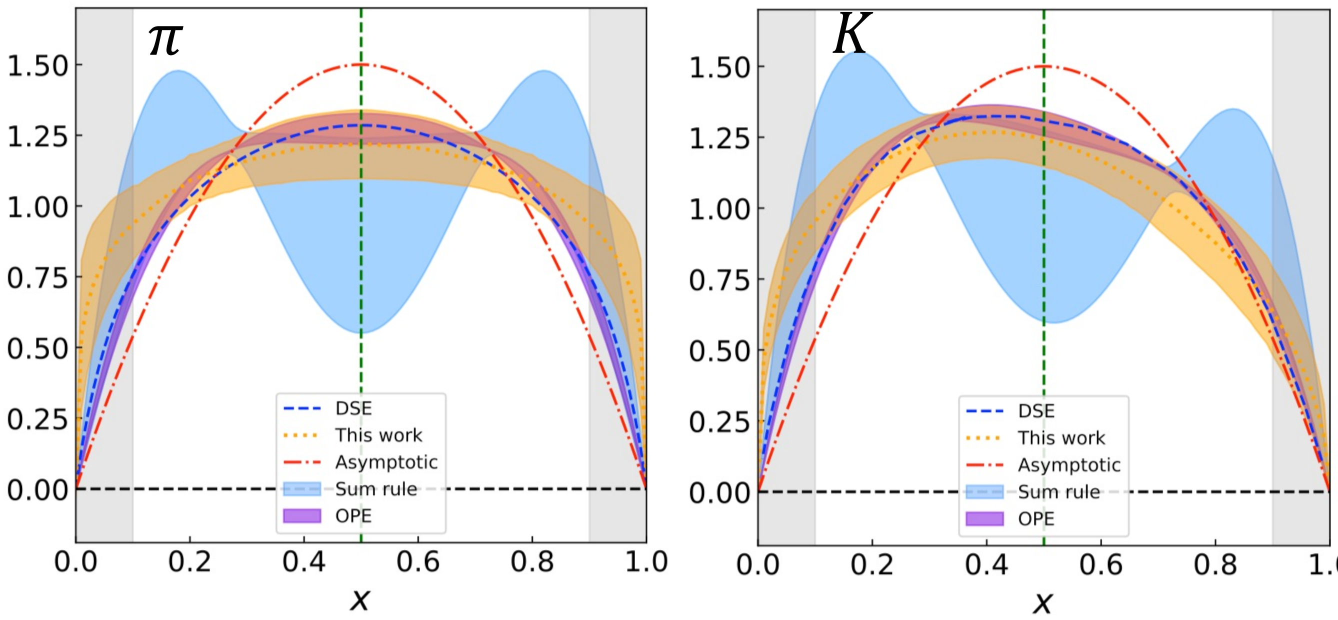
**LaMET expansion:**

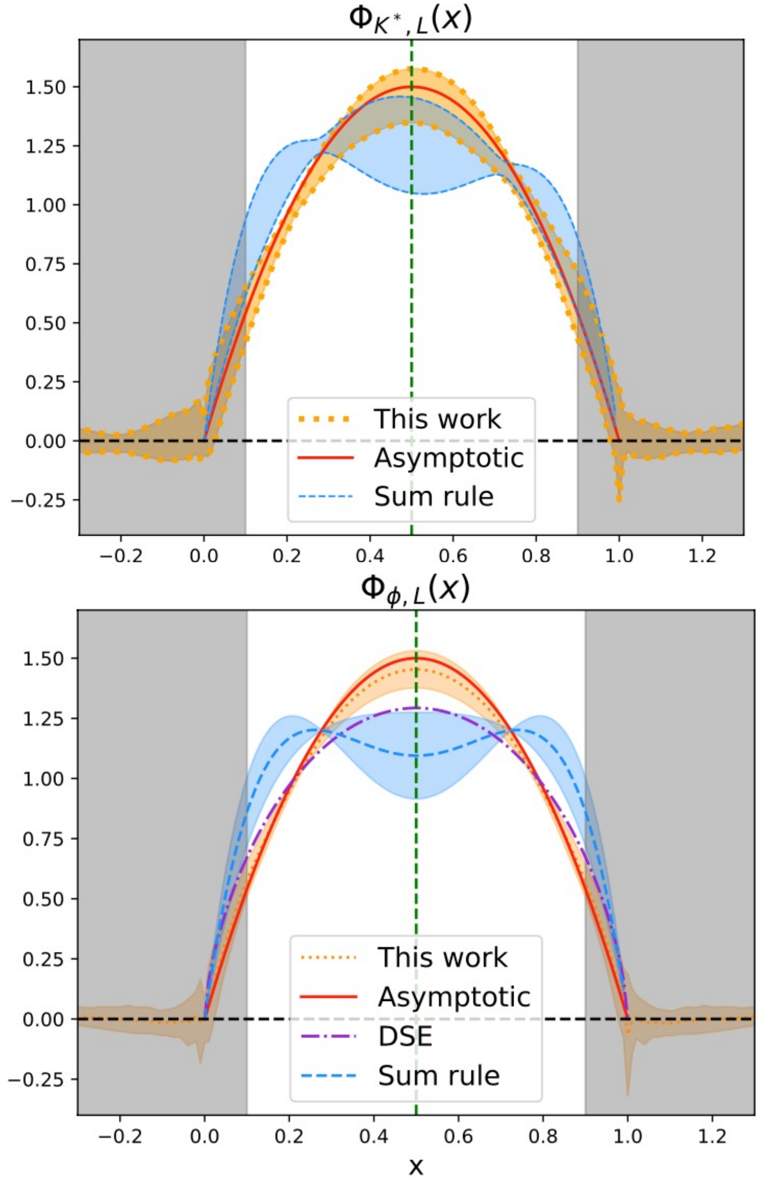
$$\phi\left(y,\mu_R\right)=\int dx\, C\left(x,y,\mu_R,\mu\right)\tilde{\phi}(x,\mu)+\mathcal{O}\left(\frac{1}{(yP^z)^2},\frac{1}{[(1-y)P^z]^2}\right)$$

*Perturbative Matching kernel*

*(LPC) PRL 127,062002 (2021) — vector meson*

*(LPC) PRL 129,132001 (2022) — pseudoscalar meson*



04

# Advances on Baryon LCDA

From light meson to  light baryon & heavy meson

# Light Baryon LCDAs: (1983 - now)

- Asymptotic form

  Chernyak, Zhitnitsky, 1983 ; ……
- QCD Sum rules

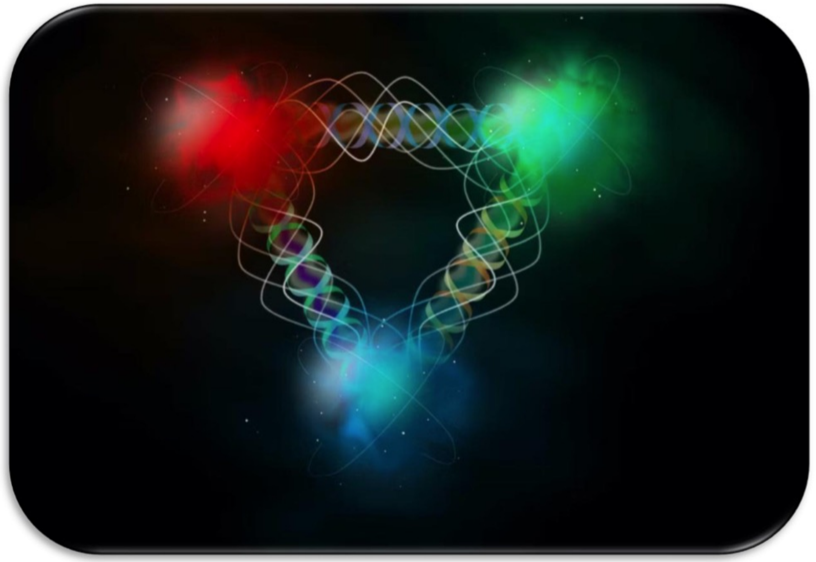
  King, Sachrajda,1987; Stefanis, Bergmann, 1993; Braun, et.al, 2000,2006; ……
- Models parametrization

  Bell, Feldmann, Wang, Matthew 2013; ……
- Lattice QCD with OPE

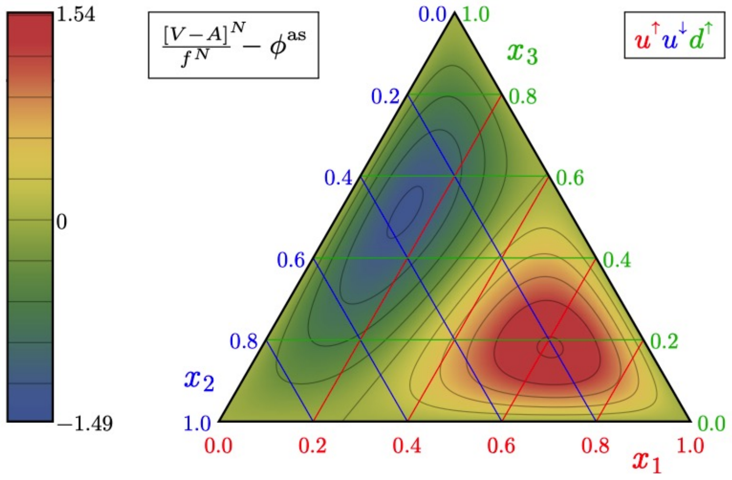
  QCDSF, 2008, 2009; RQCD, 2016, 2019(2025) ……
- Lattice QCD with LaMET

  LPC, 2024,2025; ……

  *PRD 111, 034510, arXiv:2508.08971*



**More is Different**



# Operator Product Expansion (OPE) VS Large Momentum Effective theory(LaMET)

Limited for only few moments:

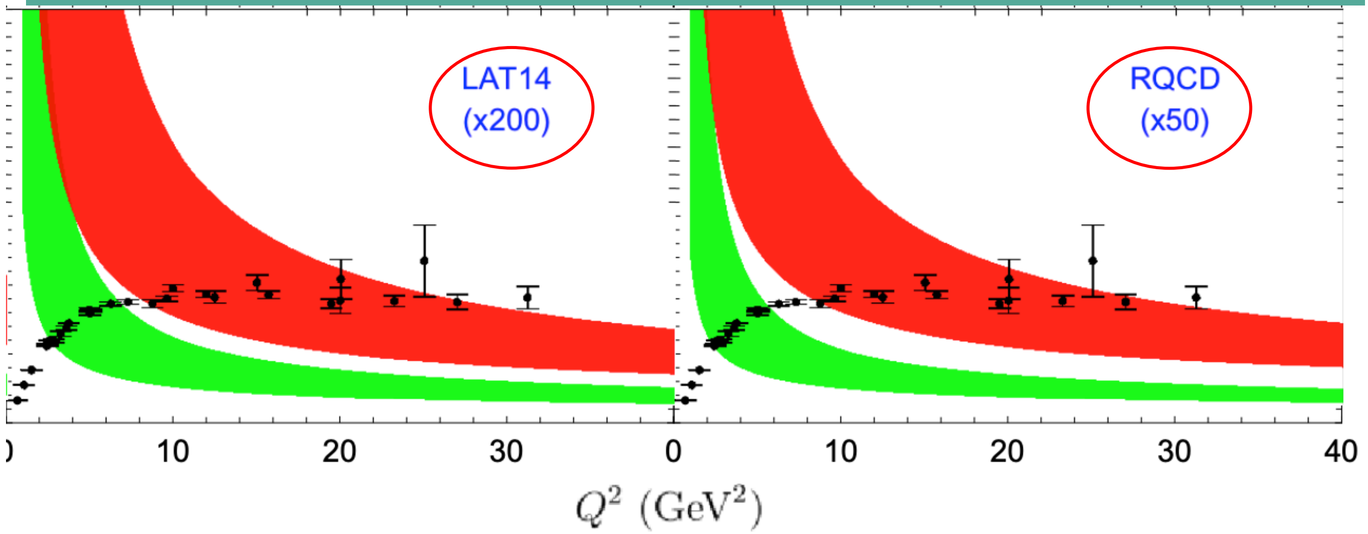
- First two moments for light mesons
- First moments for light baryons
- Inverse problem from moments to LCDA

High twist/moments are more important in Baryon

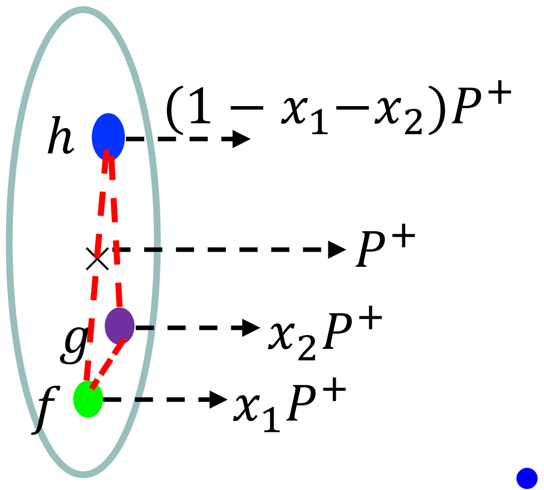
*J.J.Han, et.al. EPJC 82 8 (2022), 686PRL 134 221801(2025)*

*L.B.Chen, et.al. PRL 135 131903 (2025)*

**Nucleon Dirac form factors:**




Soft contribution dominates in baryon decay , leading twist far from enough




- **Definition of baryon LCDA:**

$$\phi(x_1,x_2,\mu)=\int\frac{P^+\mathrm{d}\xi_1^-}{2\pi}\frac{P^+\mathrm{d}\xi_2^-}{2\pi}\,\mathrm{e}^{\mathrm{i}(x_1\xi_1^-+x_2\xi_2^-)P^+}\,\Phi(\xi_1,\xi_2,\mu)$$

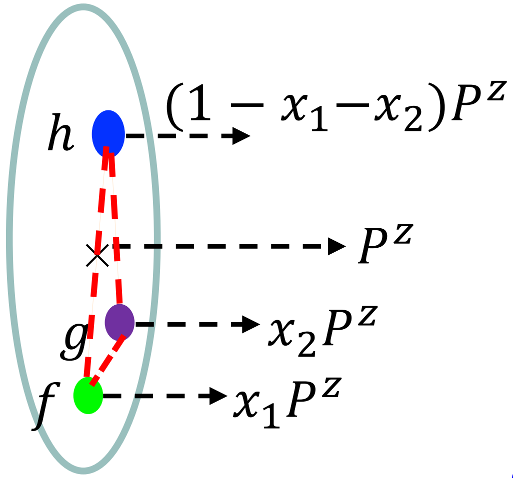
- **Leading twist for octet baryon LCDA**

$$-f_V\Phi_V(\xi_i n\cdot P)P^+\gamma_5 u_B=\langle 0|f^T(\xi_1 n)(C\not{n})g(\xi_2 n)\;h(0)|B\rangle$$

*PRD 111, 034510 (2025)* —— Λ: $f_V\Phi_A(\xi_i n\cdot P)P^+u_B=\langle 0|f^T(\xi_1 n)(C\gamma_5\not{n})g(\xi_2 n)\;h(0)|B\rangle$

$$2f_T\Phi_T(\xi_i n\cdot P)P^+\gamma_5 u_B=\langle 0|f^T(\xi_1 n)(\mathrm{i}C\sigma_{\mu\nu}n^\nu)g(\xi_2 n)\;\gamma^\mu h(0)|B\rangle$$

*C.Han et.al. JHEP 07 019 (2024);*
*V.L.C & I.R.Z NPB 246 52(1984);*
*G.R.Farrar et.al. NPB 311 585(1989)*




- **Definition of baryon quasi-DA:**

$$\tilde{\phi}(x_1,x_2,\mu)=\int\frac{P^z dz_1}{2\pi}\frac{P^z dz_2}{2\pi}\,e^{-i(x_1z_1+x_2z_2)P^z}\,\tilde{\Phi}(z_1,z_2,\mu)$$

- **Matrix elements of quasi-DA:**

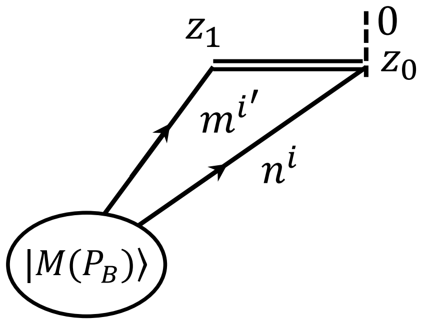
$$-f_V\tilde{\Phi}_V(z_i,P^z)P^\nu\gamma_5u_B=\langle 0|f^T(z_1)(C\gamma^\nu)g(z_2)\ h(0)|B\rangle$$

PRD 111, 034510 (2025) —— Λ: $f_V\tilde{\Phi}_A(z_i,P^z)P^\nu u_B=\langle 0|f^T(z_1)(C\gamma_5\gamma^\nu)g(z_2)\ h(0)|B\rangle$

$$2f_T\tilde{\Phi}_T(z_i,P^z)P^\nu\gamma_5u_B=\langle 0|f^T(z_1)\left(\frac{1}{2}C[\gamma^\nu,\gamma^\mu]\right)g(z_2)\ \gamma^\mu h(0)|B\rangle$$

C.Han et.al. JHEP 07 019 (2024);

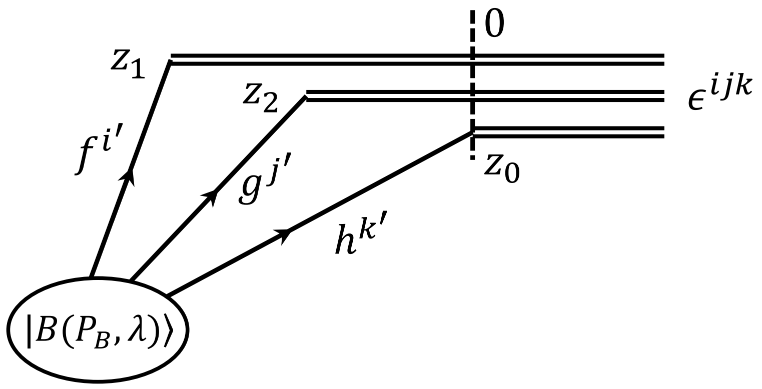
## ➢ Definition of Meson Quasi-DAs:




**1D** → **2D**

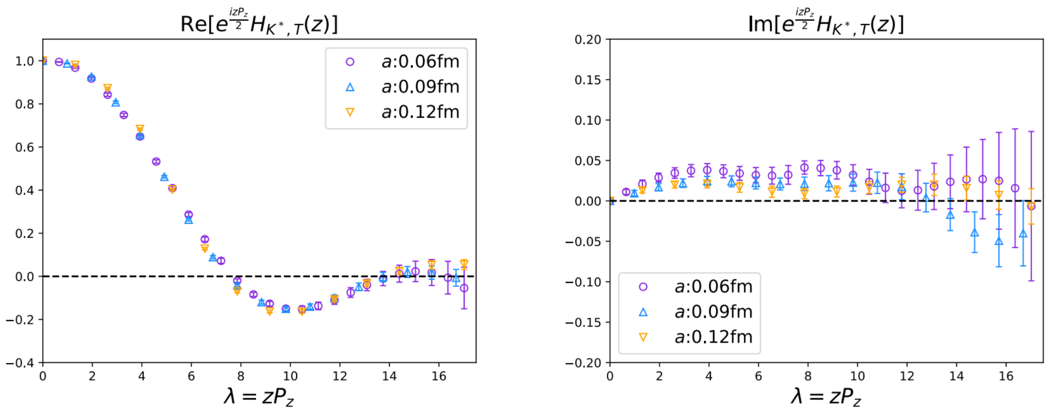
- 1 Direction
- 2 Dimensional

## ➢ Definition of Baryon Quasi-DAs:

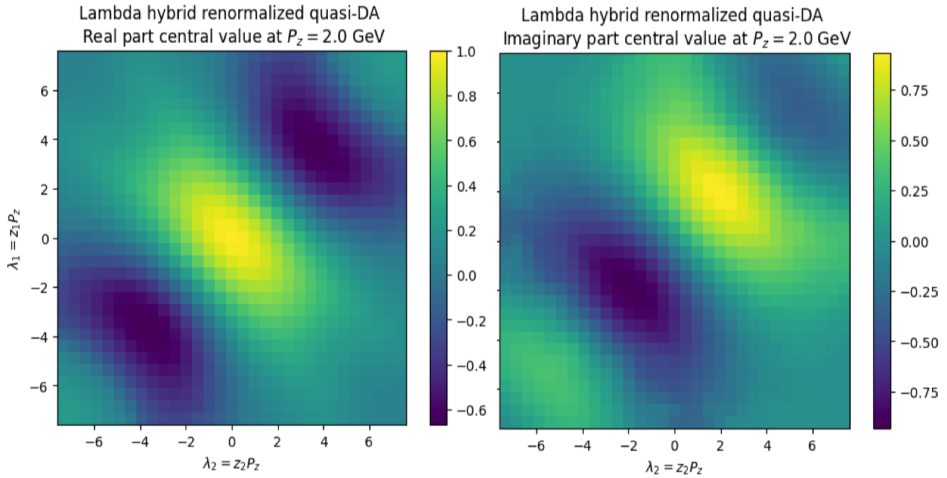



$$\hat{O}(\vec{x},t;z) \quad = m_\alpha^{i'}(\vec{x}+zn_z,t)\tilde{\Gamma}_{\alpha\beta}W^{i'i}(\vec{x}+zn_z,\vec{x},t)n_\beta^i(\vec{x},t)$$

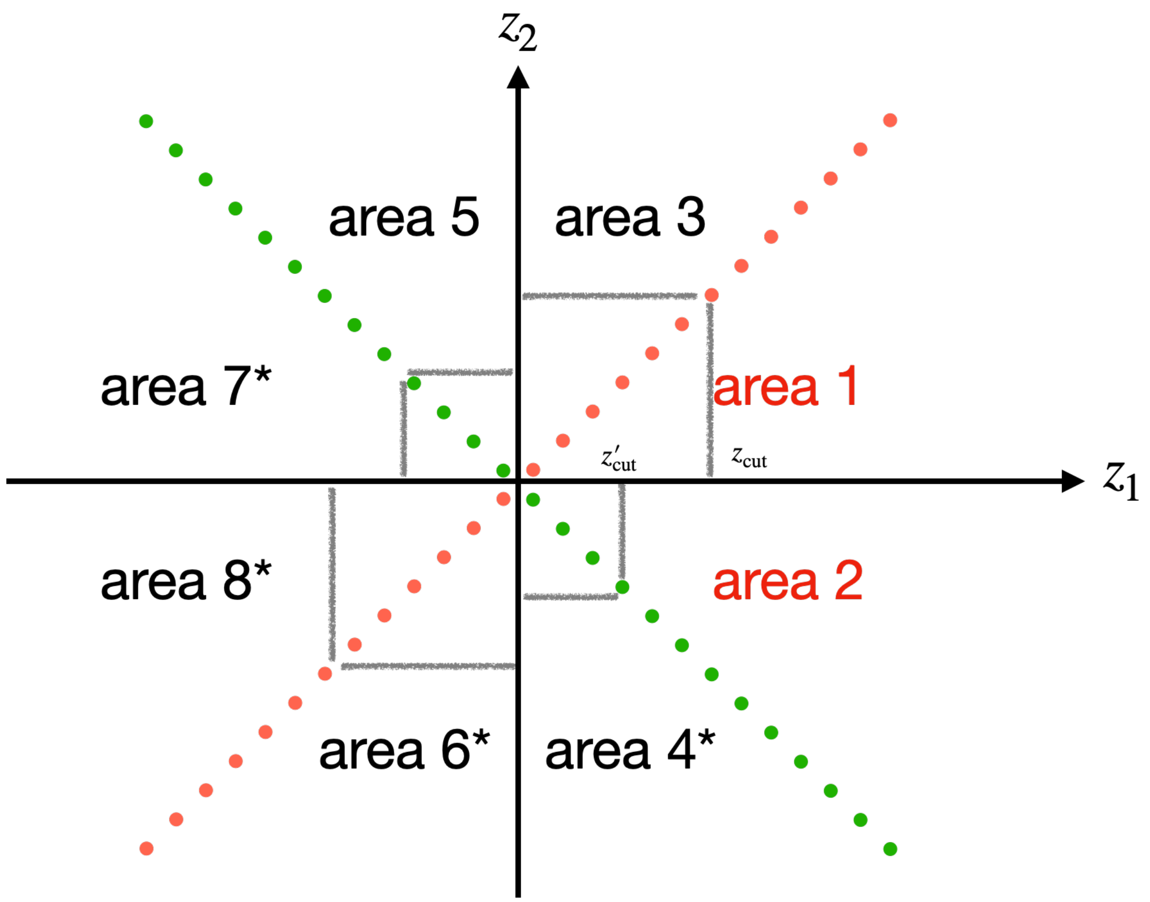
$$\hat{O}_\gamma(\vec{x},t;z_1,z_2) = \epsilon^{ijk}W^{ii'}(\infty,\vec{x}+z_1n_z)f_\alpha^{i'}(\vec{x}+z_1n_z,t)$$
$$\times\tilde{\Gamma}_{\alpha\beta}W^{jj'}(\infty,\vec{x}+z_2n_z,t)g_\beta^{j'}(\vec{x}+z_2n_z)$$
$$\times W^{kk'}(\infty,\vec{x})h_\gamma^{k'}(\vec{x},t)$$

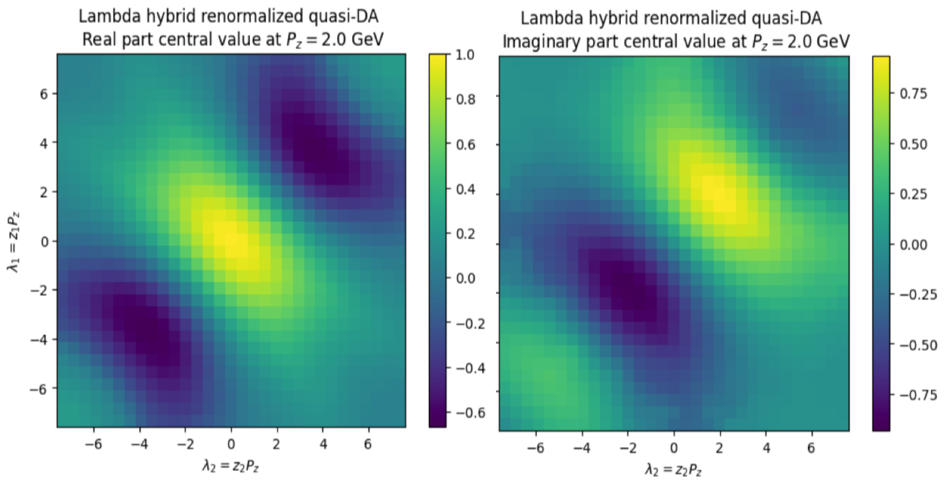



*LPC, PRL 127, 062002 (2021)*




*LPC, arXiv:2508.08971 (2025)*




## ➢ Two symmetries for Quasi-DA:

- iso-spin symmetry for "u, d" quarks:

$$\tilde{\phi}(z_1, z_2) = \tilde{\phi}(z_2, z_1)$$

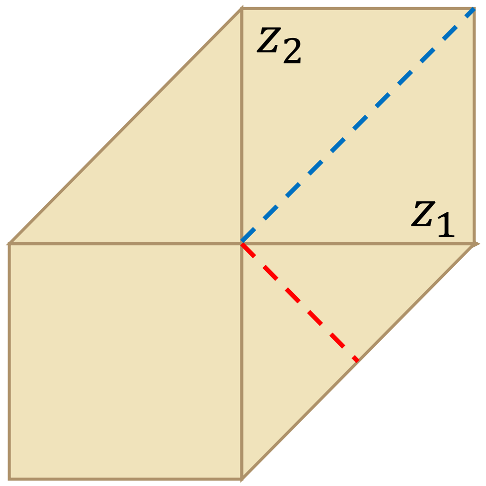
- The constrain by real $\tilde{\phi}(x_1, x_2)$:

$$\tilde{\phi}(z_1, z_2) = \tilde{\phi}^*(-z_1, -z_2)$$

- ☐ Thus for these areas: $1 = 3 = 6^* = 8^*$, $2 = 4^* = 5 = 7^*$

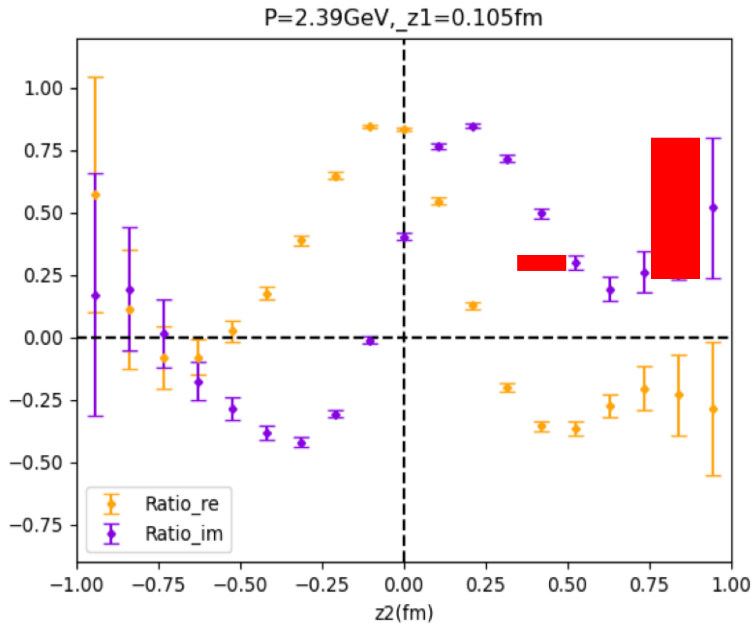
  only area 1,2 are independent

- ☐ $\tilde{\phi}(z_1 = -z_2)_{Im} = 0$




*Rhombus-shaped*

- **From effective Wilson length → Signal to Noise Ratio**



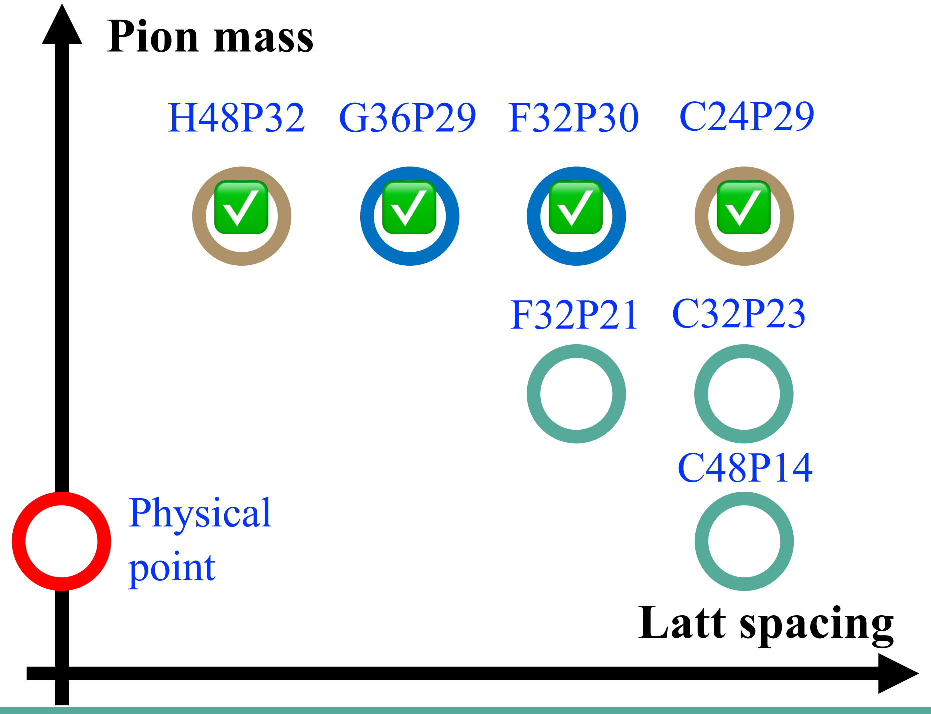

- Signal to Noise Ratio * 8 -> Statistics * 64
- Totally 64*10*2

- It would take about **nine years** for computer performance to increase by a factor of 64

| Scenario | Time per Doubling | Total Time for 6 Doubling | Target Date (from 2025-07-19) | Explanation |
|---|---|---|---|---|
| Optimistic | 18 months (1.5 years) | 6 × 1.5 = 9 years | July 19, 2034 | Based on the classic assumption of Moore's Law (rapid technological progress). |
| Benchmark | 24 months (2 years) | 6 × 2 = 12 years | July 19, 2037 | A common estimate at present (balanced model). |
| Conservative | 36 months (3 years) | 6 × 3 = 18 years | July 19, 2043 | Recent trend (Moore's Law has significantly slowed down). |

# CLQCD Ensembles

- 4 lattice spacings for $a \to 0$ limit & renormalization
- 4 momentums for $P_z \to \infty$ limit
- More pion masses to **physical mass** limit




| Ensemble | Lattice spacing | $\pi$ mass | measurements |
|---|---|---|---|
| C24P29 | 0.1052 fm | 292 MeV | 864 *4 src *9 it0 |
| F32P30 | 0.0775 fm | 300 MeV | 777 *4 src *8 it0 |
| H48P32 | 0.0520 fm | 317 MeV | 550 *6 src *9 it0 |
| G36P29 | 0.0689 fm | 297 MeV | 656 *6 src *8 it0 |
| Other ensembles with different pion masses | | | |

# Kinematically-enhance:

*RZ et.al. PRD 112 L051502(2025)*

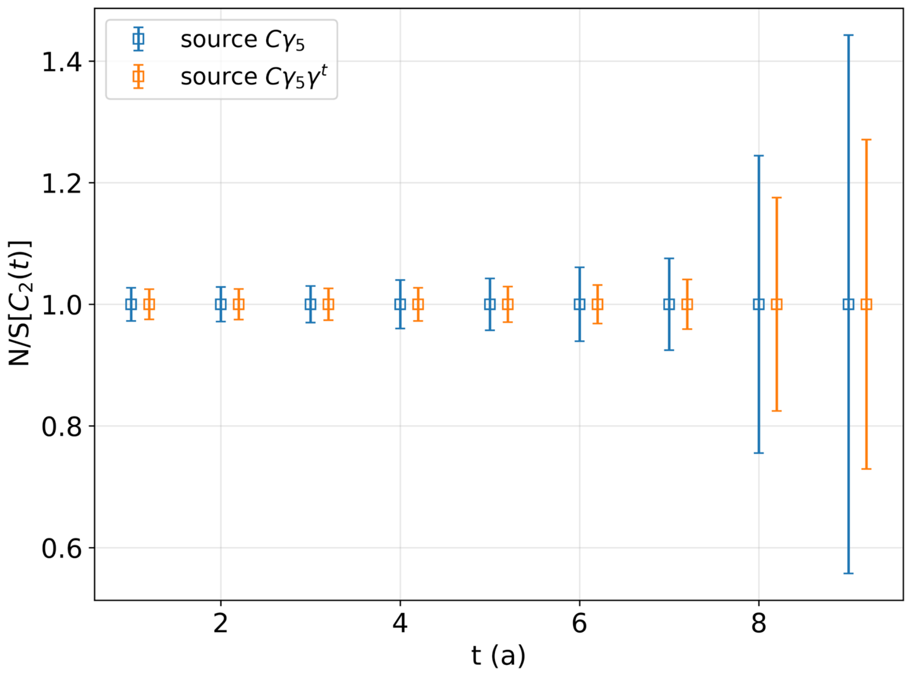
- Standard interpolator: $N_{\gamma_5} = \epsilon^{ijk}(f_i^T\ C\gamma_5\ g_j)h_k$ for a static baryon states

$$\langle 0|N_{\gamma_5}|N(P)\rangle = \lambda u(P)$$

- Enhanced interpolator : $N_{\gamma_5\gamma_\mu} = \epsilon^{ijk}(f_i^T\ C\gamma_5\gamma_\mu\ g_j)h_k$ better overlap with boosted states

$$\langle 0|N_{\gamma_5\gamma_\mu}|N(P)\rangle = \alpha P_\mu u(P) + \beta\gamma_\mu u(P)$$

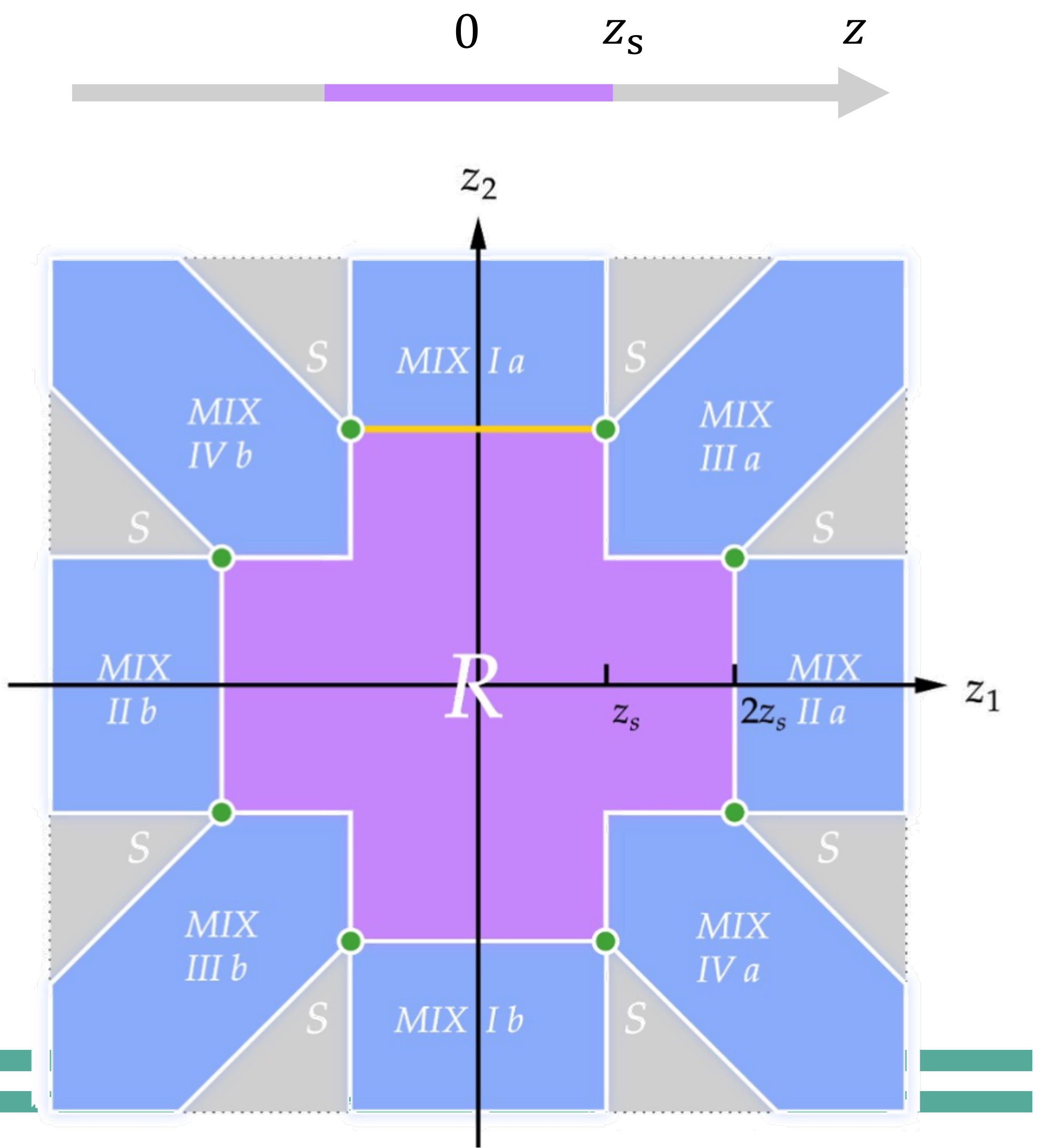
$$\Rightarrow \quad \mathrm{SNR}(C_{2\mathrm{pt}}(t\to\infty)) \propto \frac{P_\mu}{M_0} e^{-E't}$$




- **Dynamic enhance: SNR 200% improve (eff 400%) !**
- **Coding in Pyquda: Computing efficiency 800% !**

*X.Y.Jiang, arXiv: 2411.08461*

# Hybrid scheme: use suitable schemes for different regions




- 1) **Short-distance region**

$$\frac{\hat{M}_{\overline{\mathrm{MS}}}(z_1, z_2, 0, P^z, \mu)}{\hat{M}_{\overline{\mathrm{MS}}}(z_1, z_2, 0, 0, \mu)} \cdot S_{\mathrm{short}}(z_1, z_2)$$

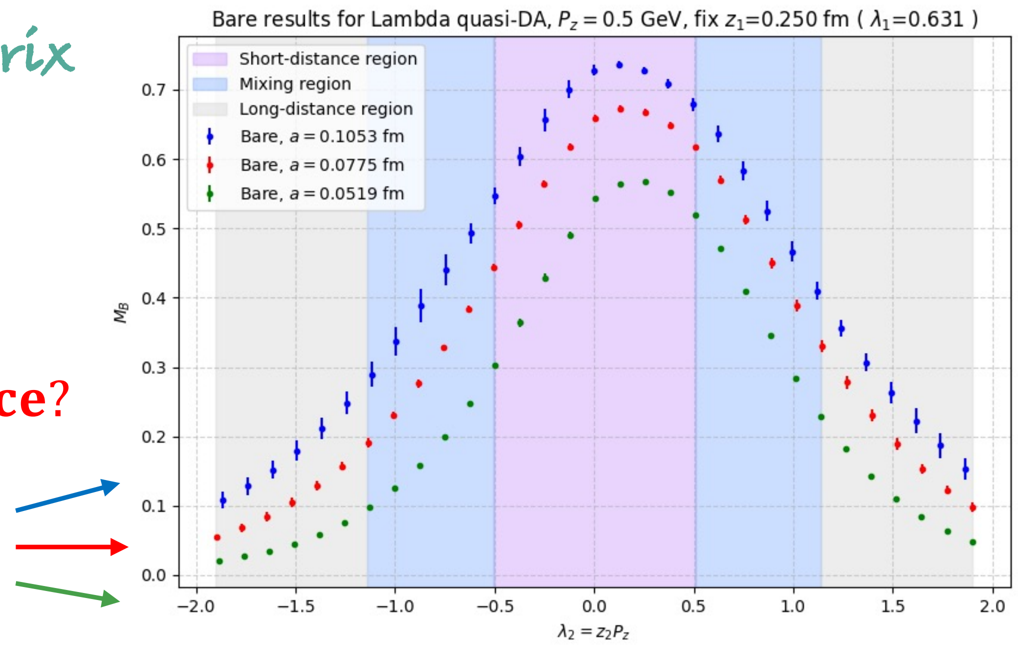
- 2) **Long-distance region**

$$\frac{\hat{M}_{\overline{\mathrm{MS}}}(z_1, z_2, 0, P^z, \mu)}{\hat{M}_{\overline{\mathrm{MS}}}(\mathrm{sign}(z_1)z_s, \mathrm{sign}(z_2)2z_s, 0, 0, \mu)} \cdot S_{\mathrm{long}}(z_1, z_2)$$

- 3) **Mixing regions**

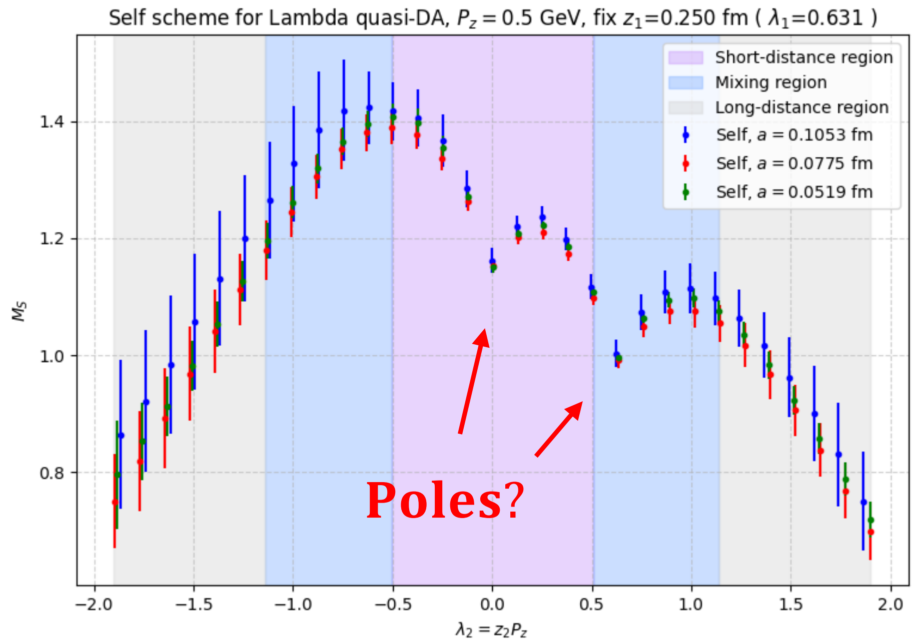
$$\frac{\hat{M}_{\overline{\mathrm{MS}}}(z_1, z_2, 0, P^z, \mu)}{\hat{M}_{\overline{\mathrm{MS}}}(z_1, \mathrm{sign}(z_2)2z_s, 0, 0, \mu)} \cdot \theta(z_s - |z_1|)\theta(|z_2| - 2z_s)$$

Bare Matrix Element

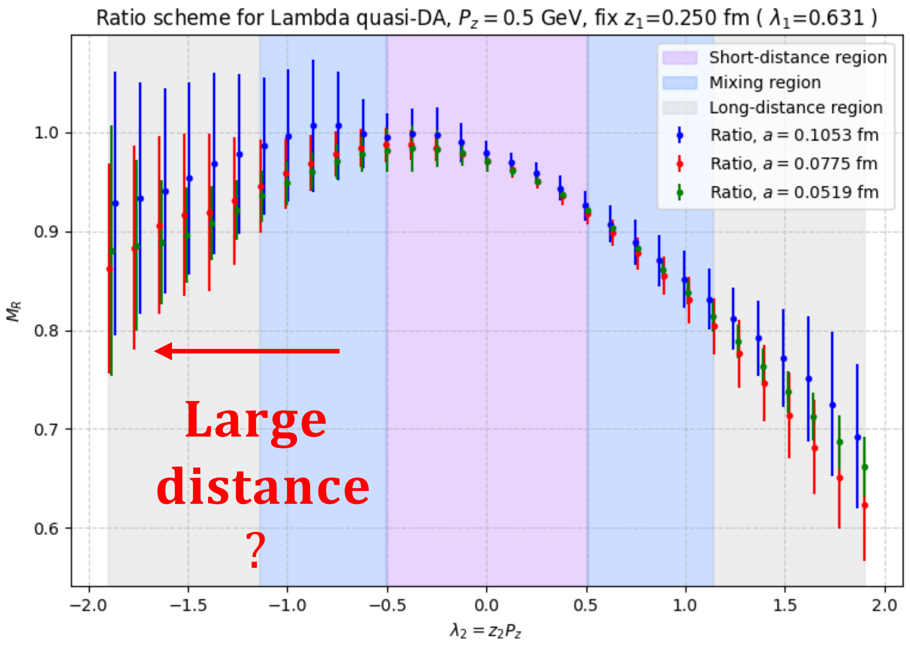



Divergence?

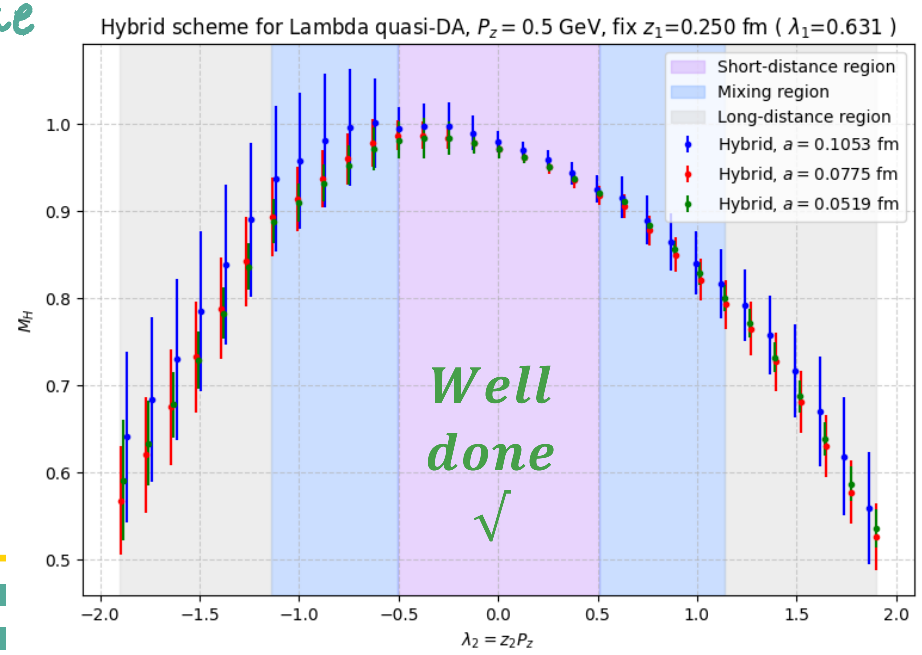
Self scheme




Ratio scheme



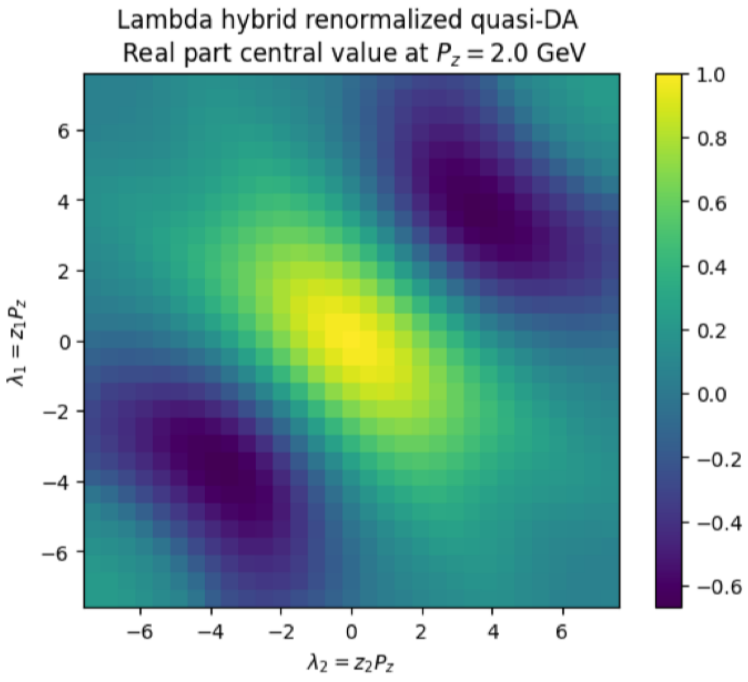

Hybrid scheme



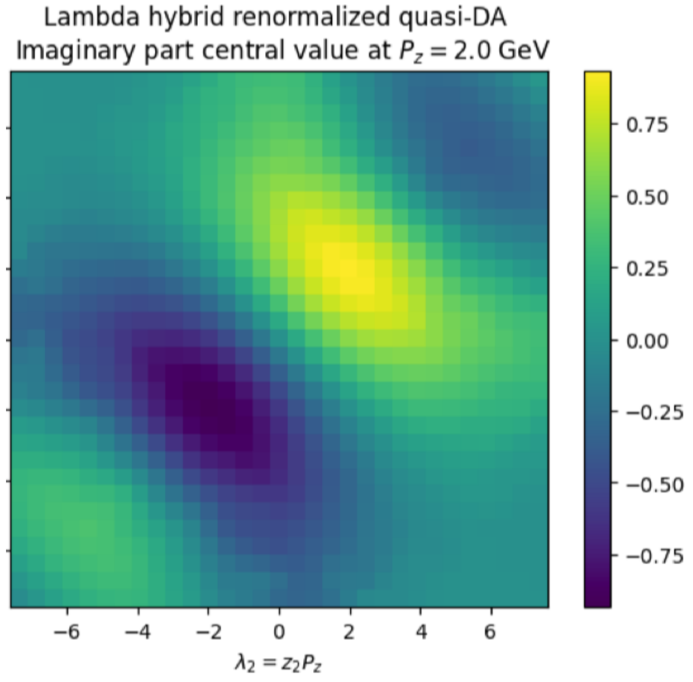
- Hybrid renormalized large momentium baryon quasi-DA ( $P^z = 2.0\text{GeV}$)

## Λ quasi-DA

**Re**

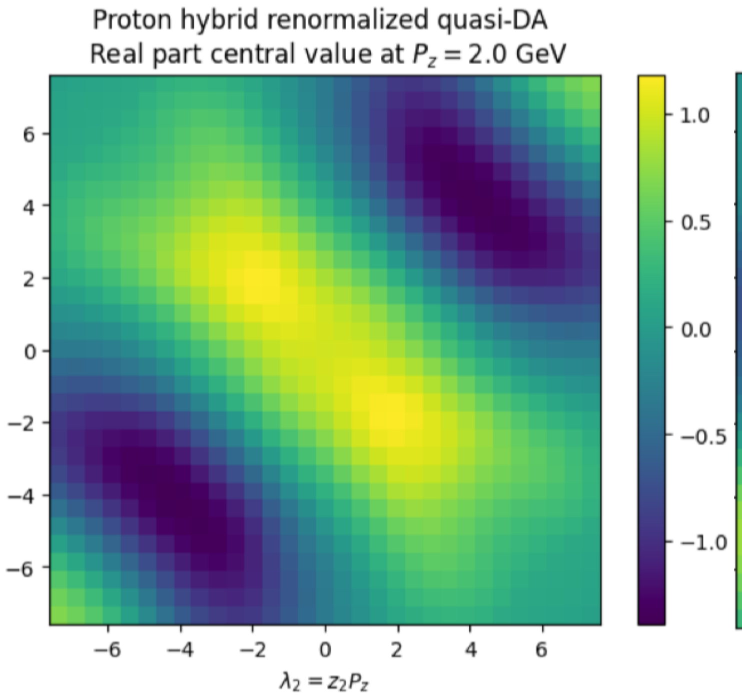



**Im**

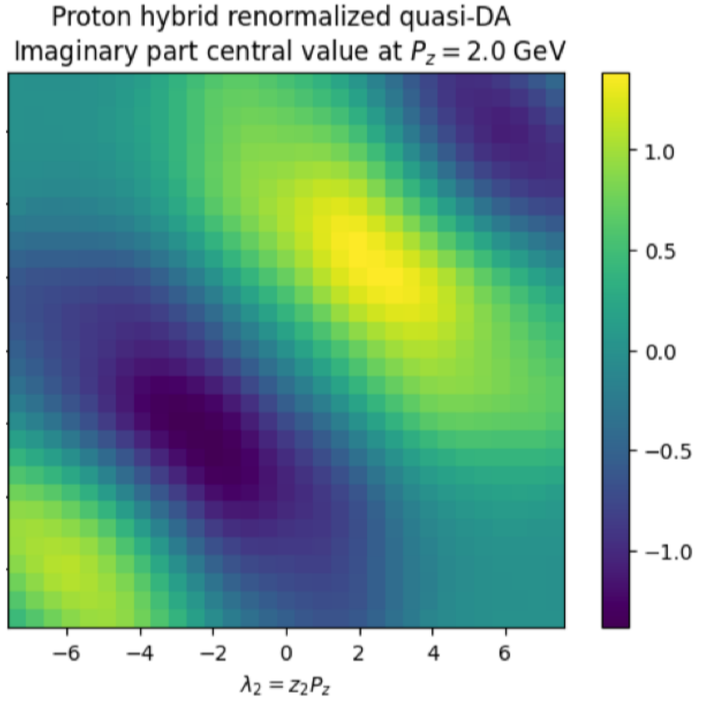



## Proton quasi-DA

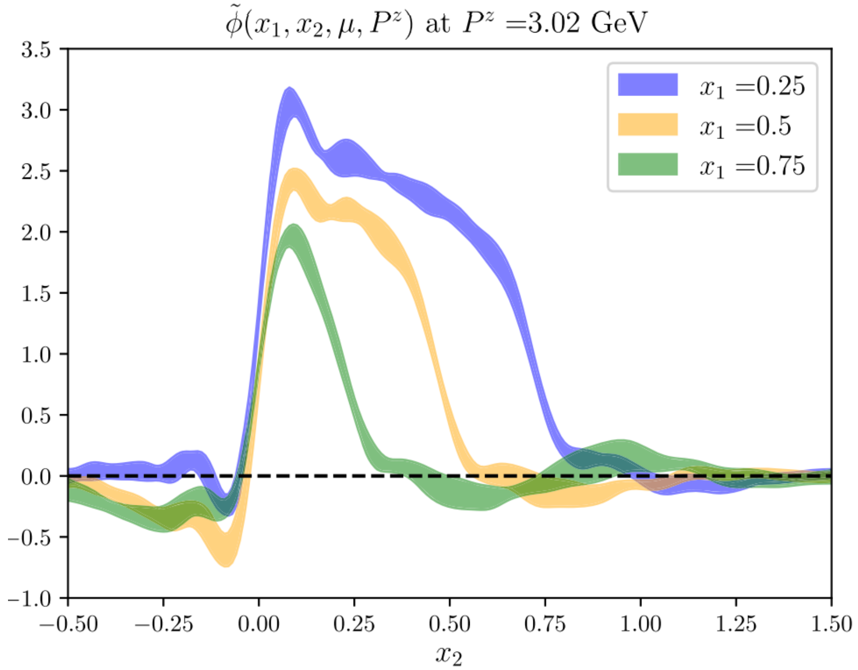
**Re**



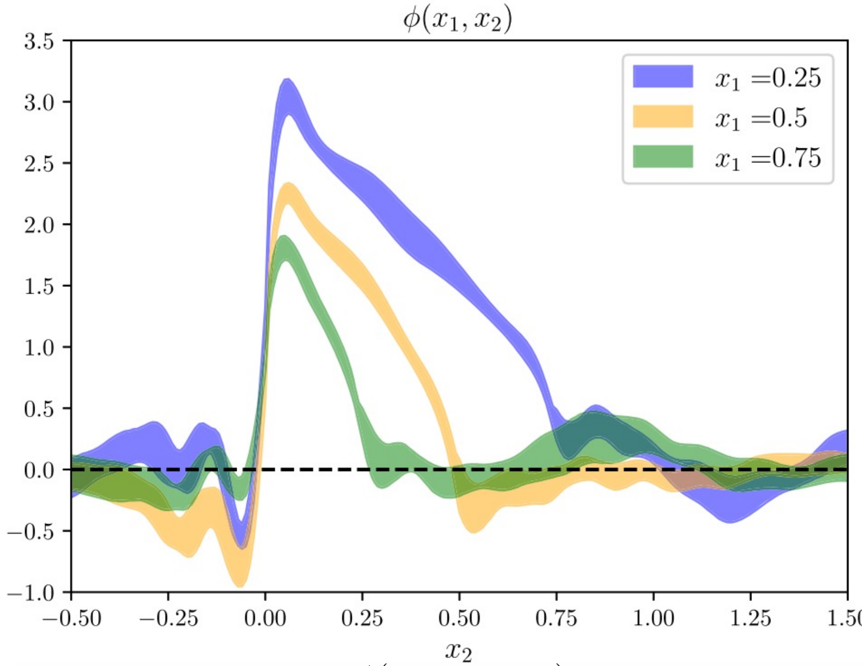

**Im**



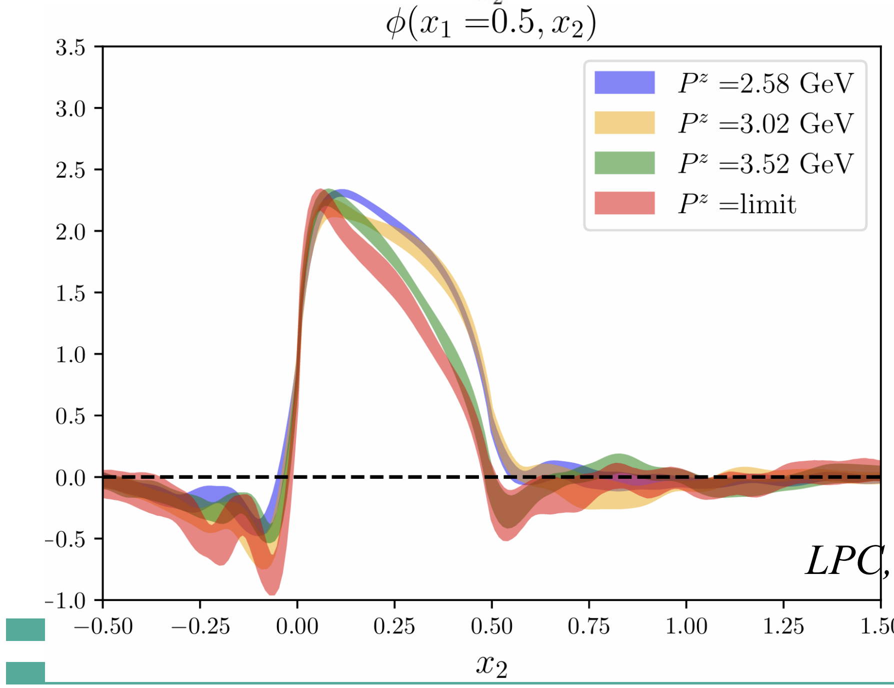

**well-defined** and **smooth** in all regions



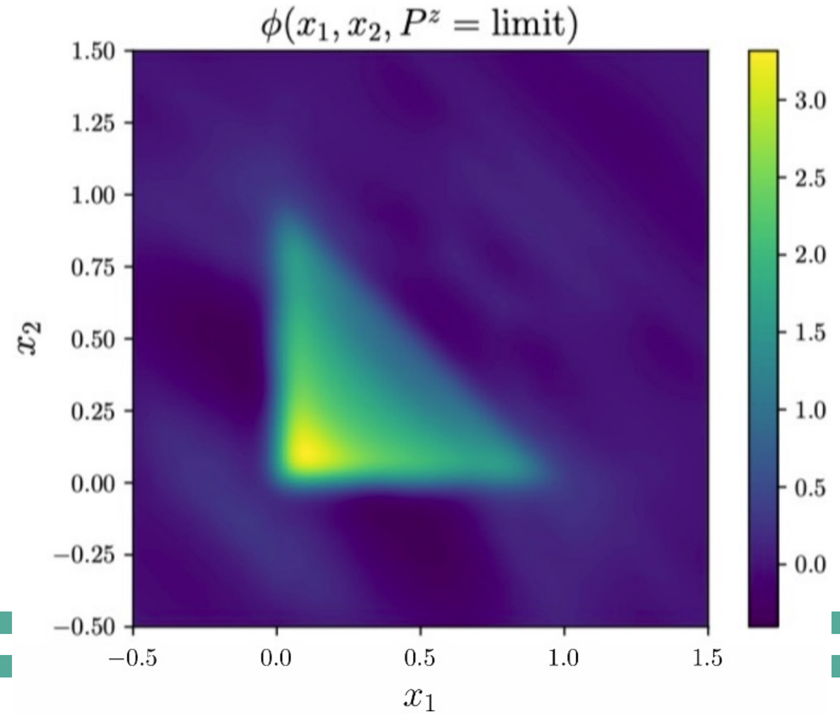

**Matching**




**Large Pz limit**




**LCDA**




*LPC, PRD 111, 034510 (2025)*

05

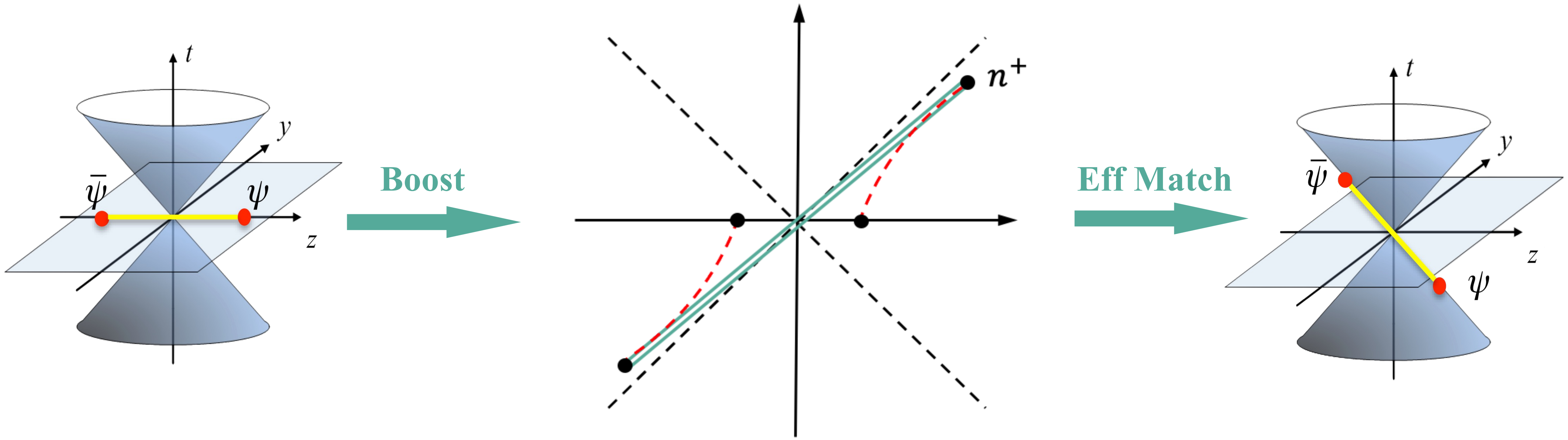
# Review on Advances of LCDA

From light meson to heavy meson & light baryon

# Advances of Framework

**Large momentum effective theory (LaMET): the light-cone non-local operator correlated to Euclidean non-local operator with a large momentum boost**

*Ji, PRL 110 262002 (2013)*




**LaMET factorization:**

$$\phi(y,\mu_R) = \int dx C(x,y,\mu_R,\mu)\tilde{\phi}(x,\mu) + \mathcal{O}\left(\frac{1}{(yP^z)^2}, \frac{1}{[(1-y)P^z]^2}\right)$$

***What's update Version ?***

*(Focus on LCDA related)*

# Advances of Framework

**Advances:** (2021- )

- Hybrid & Self renormalization

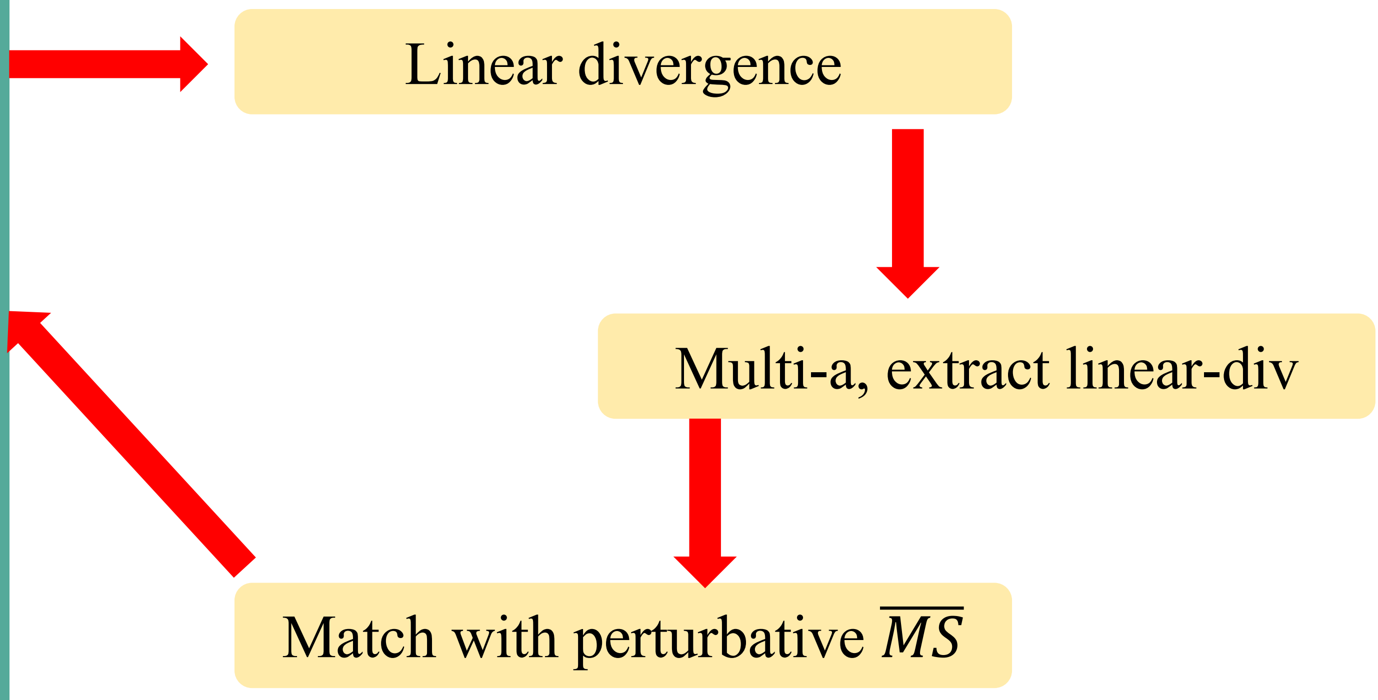
  LPC NPB 964 115311, 969 115443(2021)…
- Renormalon resummation (LRR)

  NPB 993 116282(2023)…
- RGR & Threshold resummation

  NPB 993, JHEP 03 045(2025)…
- Two loop matching

  arXiv:2504.09367 …
- Power correction estimation

  PRD 110 094038(2025), 112 016013(2025)…




*Renormalon in perturbative results*

# Self renorm & Renormalon resum

*JH et.al. NPB 993 116282(2023), XDJ et.al. JHEP 03 045(2025)*

Linear divergence → Multi-a, extract linear-div → Match with perturbative $\overline{MS}$

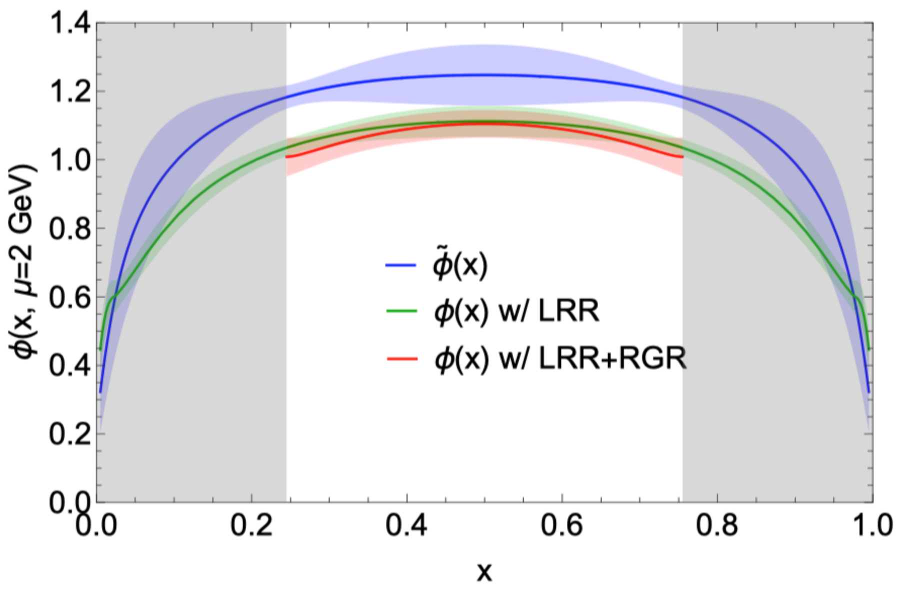
$$Z(z,a)_R = \exp\left[\frac{kz}{a\ln[a\Lambda_{\rm QCD}]} + m_0 z + f(z)a + \frac{3C_F}{b_0}\ln\left[\frac{\ln[1/(a\Lambda_{\rm QCD})]}{\ln[\mu/\Lambda_{\rm QCD}]}\right] + \ln\left[1+\frac{d}{\ln(a\Lambda_{\rm QCD})}\right]\right]$$

- Extract from multi lattice spacings of zero momentum matrix
- Extract from matching with perturbative $\overline{MS}$ quasi

*Renormalon ambiguity occurs*

**A LRR guarantee a more comprehensive self renormalization**

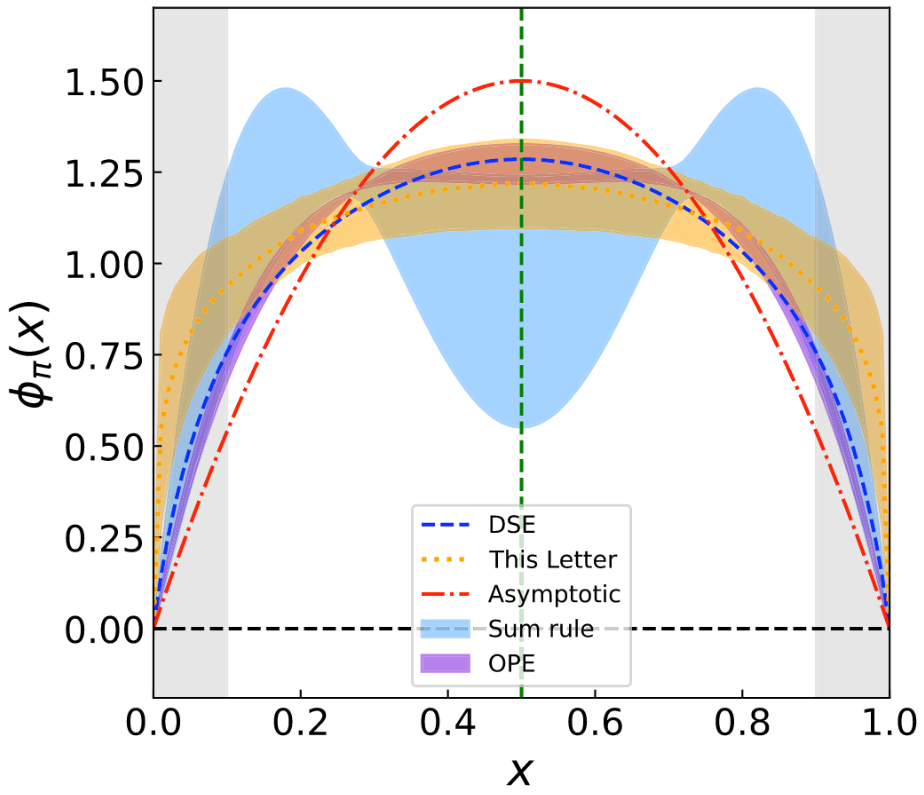
LRR+RGR



# RGR & Threshold resummation

*LPC NPB 969 115443(2021), JH et.al. NPB 993 116282(2023)*

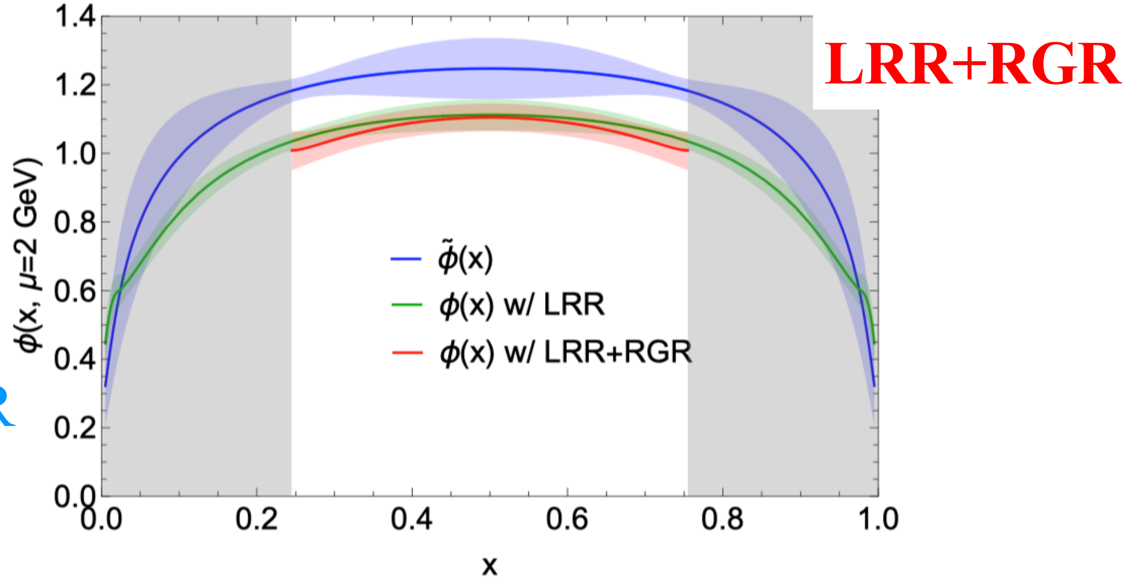
$$\phi(y,\mu_R) = \int dx C(x,y,\mu_R,\mu)\tilde{\phi}(x,\mu) + \mathcal{O}\left(\frac{1}{(yP^z)^2}, \frac{1}{[(1-y)P^z]^2}\right)$$

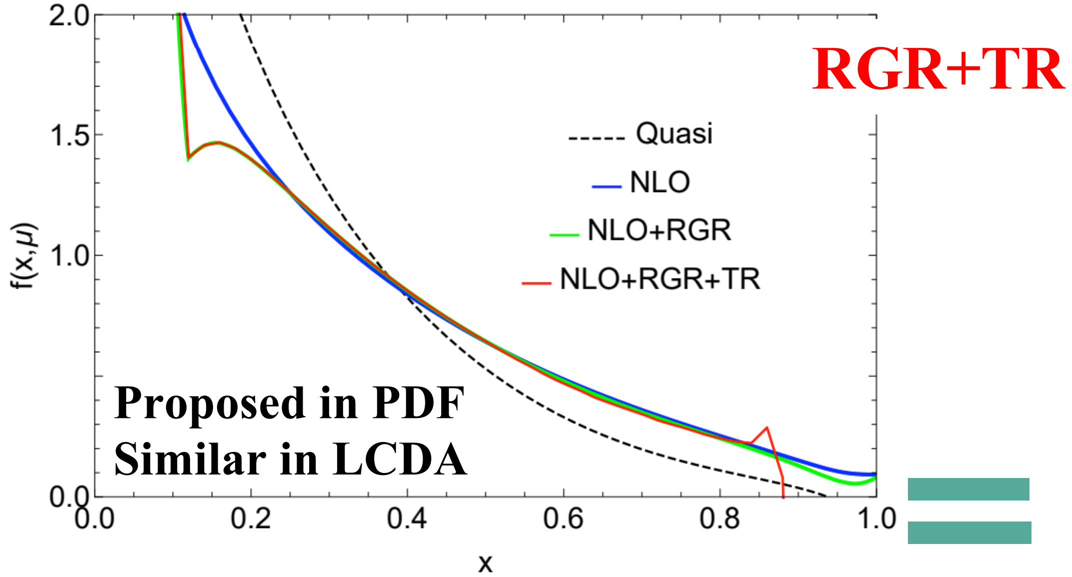



*LPC PRL 129,132001 (2022)*

- Scale of x: RGR
- Sum log in kernel: TR

**Towards small x region control !**



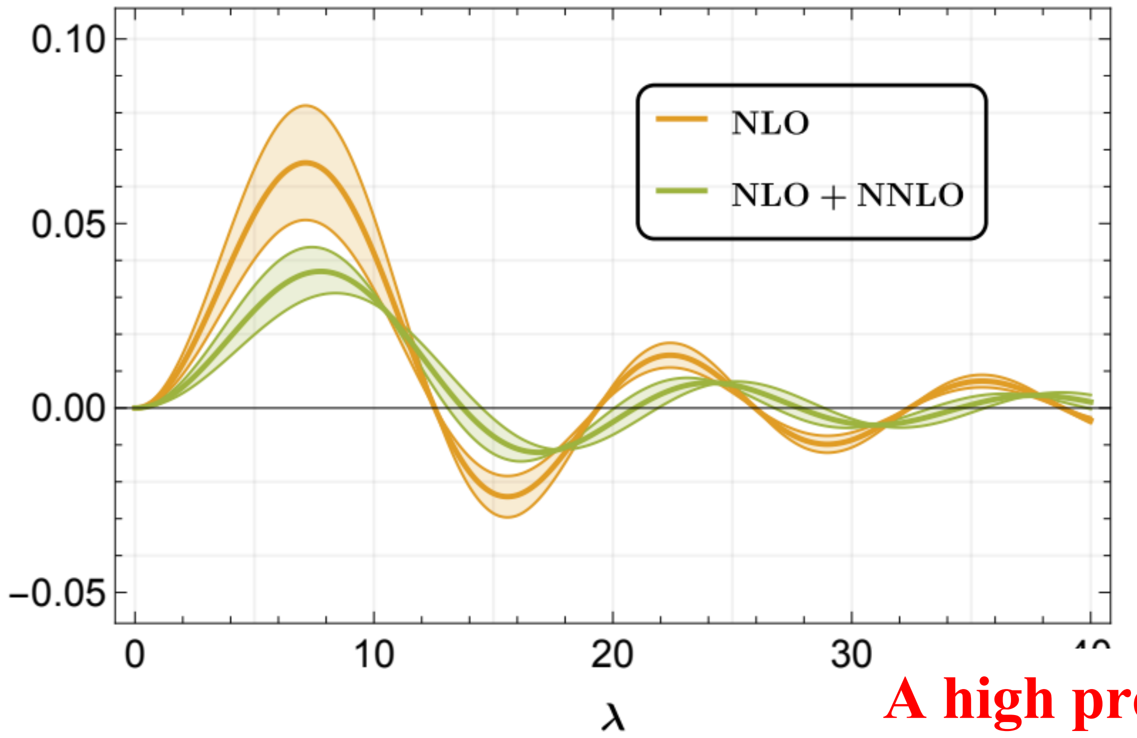
# NNLO corrections:

*YJ, FY & JHZ arXiv:2504.09367*

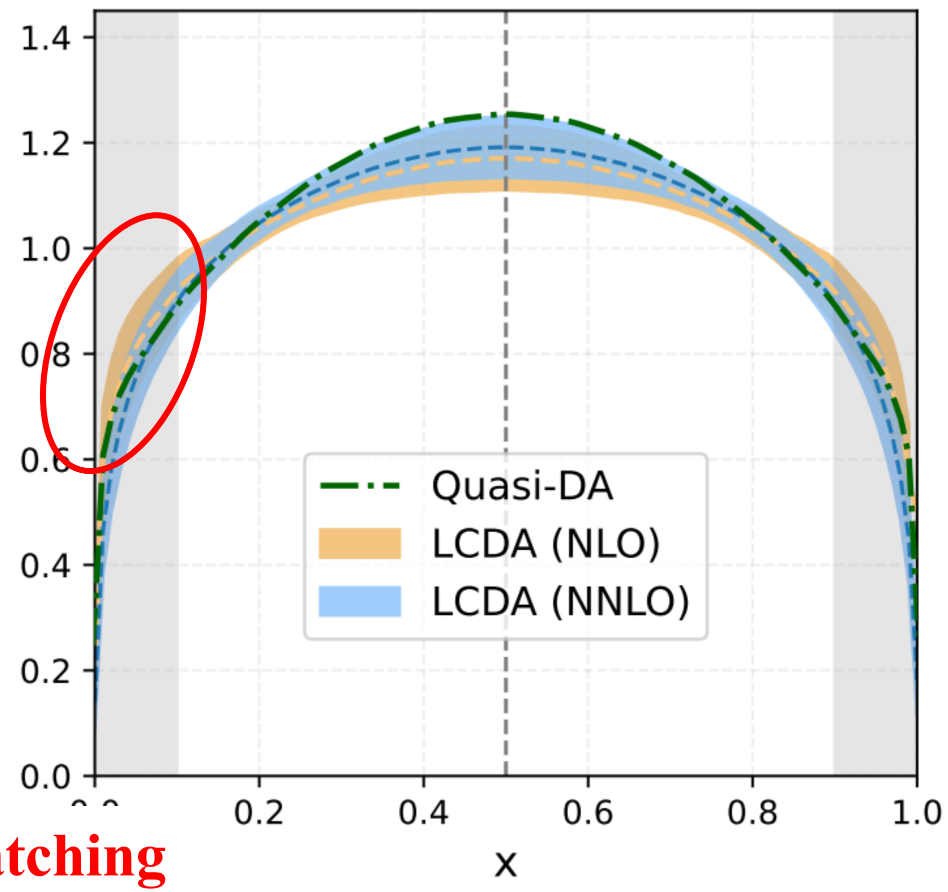
$$\phi_R(x, P^z) = \int dy C\left(x, y, \frac{\mu}{P^z}\right) \tilde{\phi}_R(y, \mu) + \mathcal{O}\left(\frac{m_H^2}{(P^z)^2}, \frac{\Lambda_{\rm QCD}^2}{(P^z)^2}\right)$$

*Matching kernel*

**Coordinate space:** $\mu = 2\ \mathrm{GeV},\ z = 0.2\ \mathrm{fm}$




**Momentum space:**



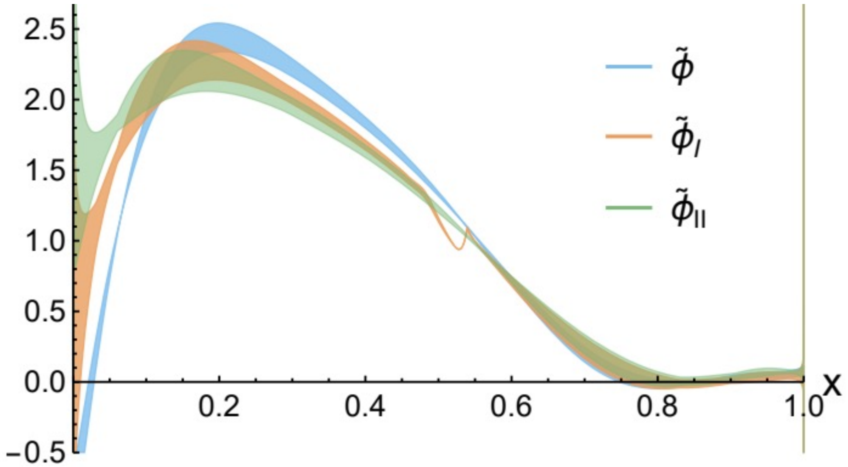

**A high precision matching**

# Power corrections in LaMET:

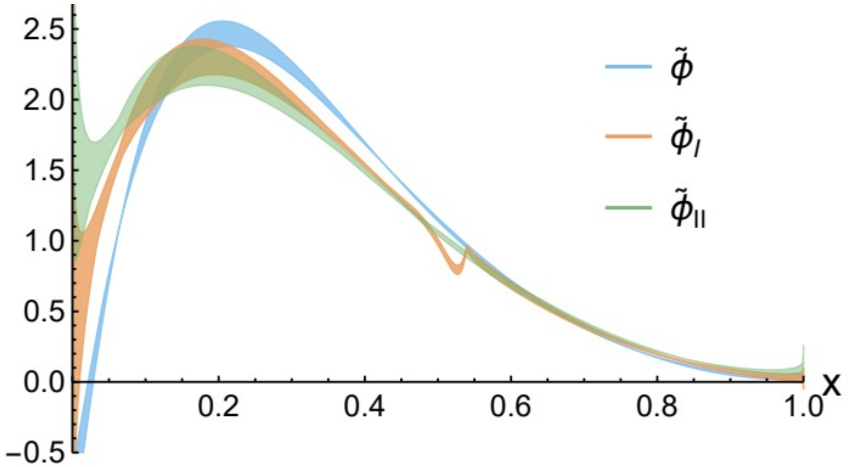
*JLZ et.al. PRD 110 094038(2025), 112 016013(2025)*

$$\phi_R(x, P^z) = \int dy C\left(x, y, \frac{\mu}{P^z}\right)\tilde{\phi}_R(y,\mu) + \mathcal{O}\left(\frac{m_H^2}{(P^z)^2}, \frac{\Lambda_{\rm QCD}^2}{(P^z)^2}\right)$$

- $\frac{m_H^2}{(P^z)^2}$: Match with OPE
- $\frac{\Lambda_{QCD}^2}{(P^z)^2}$: Estimated by renormalons




(e)




(f)

**Towards precision calculation on updated framework. How ?**

# Advances of Precision Calculation

**Advances:** (2021- )

- Kinematically-enhance
  PRD 112 L051502(2025)
- Precision check for meson LCDA
  … JHW et.al.
- Inverse Fourier transform
  arXiv:2505.14619, arXiv:2506.16689…
- CLQCD ensembles & Pyquda
  CLQCD PRD 109(2024) , arXiv: 2411.08461…
- … Gradient flow
  JHEP 06 210 (2024), arXiv:2507.18233…

➢ **Precision**

Signal Improve

# Kinematically-enhance:

*RZ et.al. PRD 112 L051502(2025)*

$$C_2^{\pi}(z,\vec{P},t)=\int d^3y e^{-i\vec{P}\cdot\vec{y}}\langle 0|\underbrace{\bar{\psi}_1(\vec{y},t)\Gamma_1 U(\vec{y},\vec{y}-z\hat{z})\psi_2(\vec{y}-z\hat{z},t)}_{Sink}\underbrace{\bar{\psi}_2(0,0)\Gamma_2\psi_1(0,0)}_{Src}|0\rangle$$

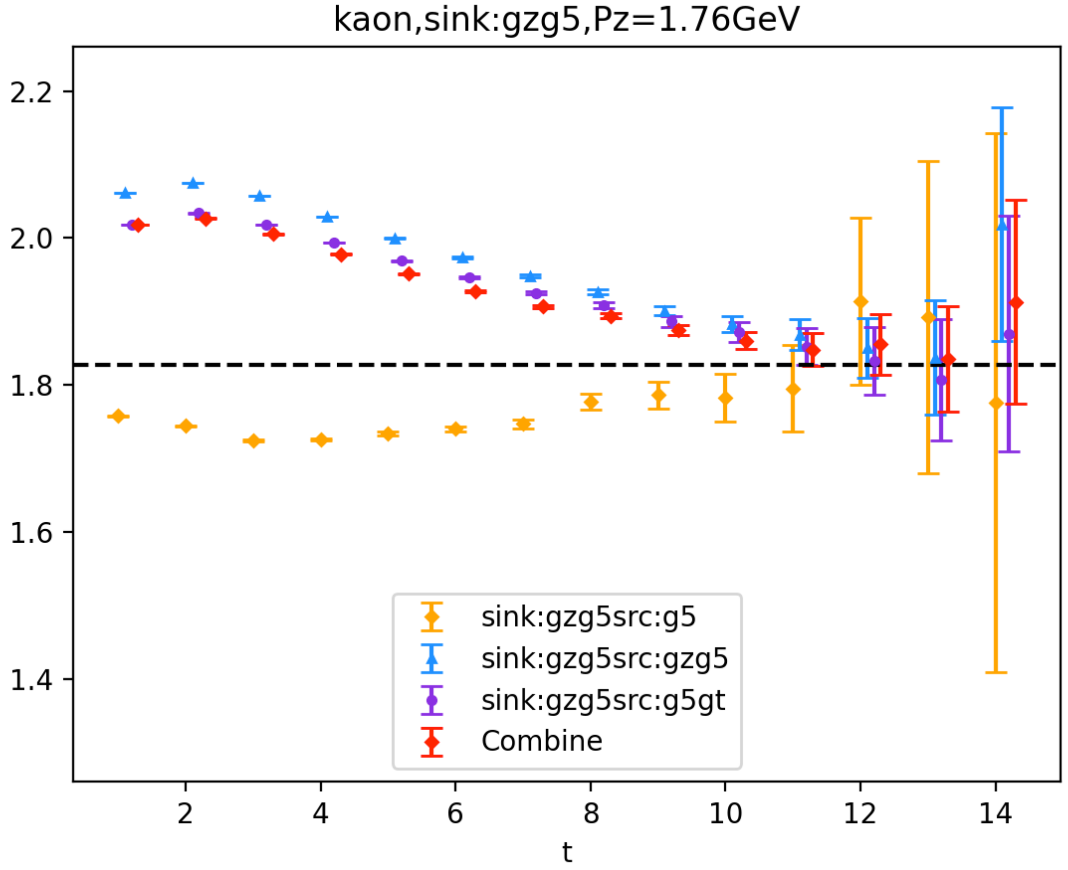
**Decomposition in light-cone limit :**

*PLB 545,345 (2002), EPJC 33, 75(2004)*

$$u_+^{\dagger}\gamma_5 d_+ = \sqrt{2}\bar{u}\gamma_+\gamma_5 d$$

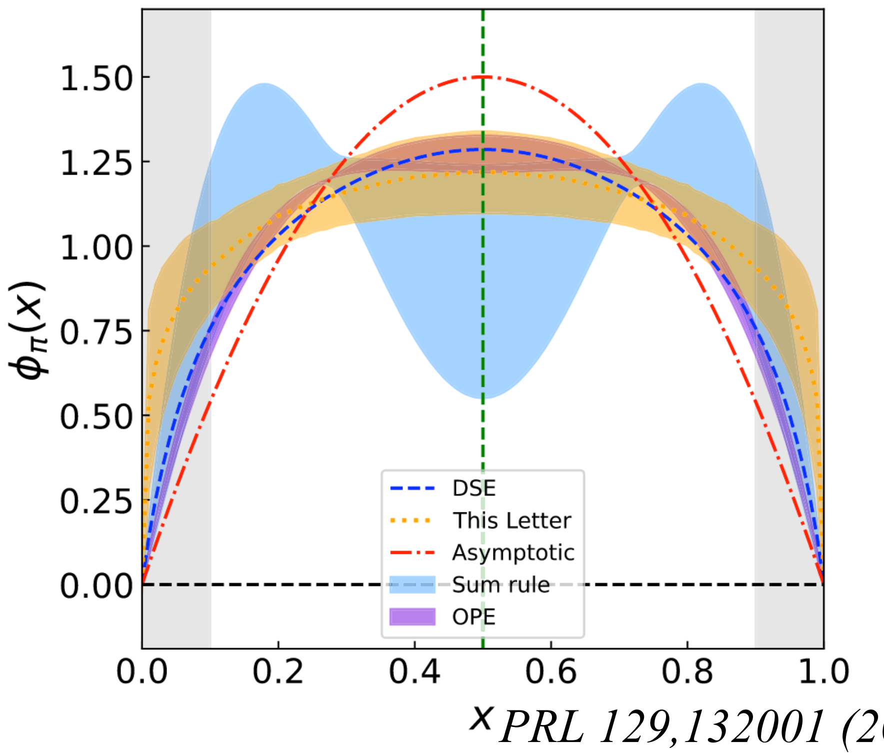
$$(\bar{u}\gamma_5 d) = \left(u_+^{\dagger}\gamma_t\gamma_5 d_- + u_-^{\dagger}\gamma_t\gamma_5 d_+\right)/2$$

Can be further extended in baryon case.

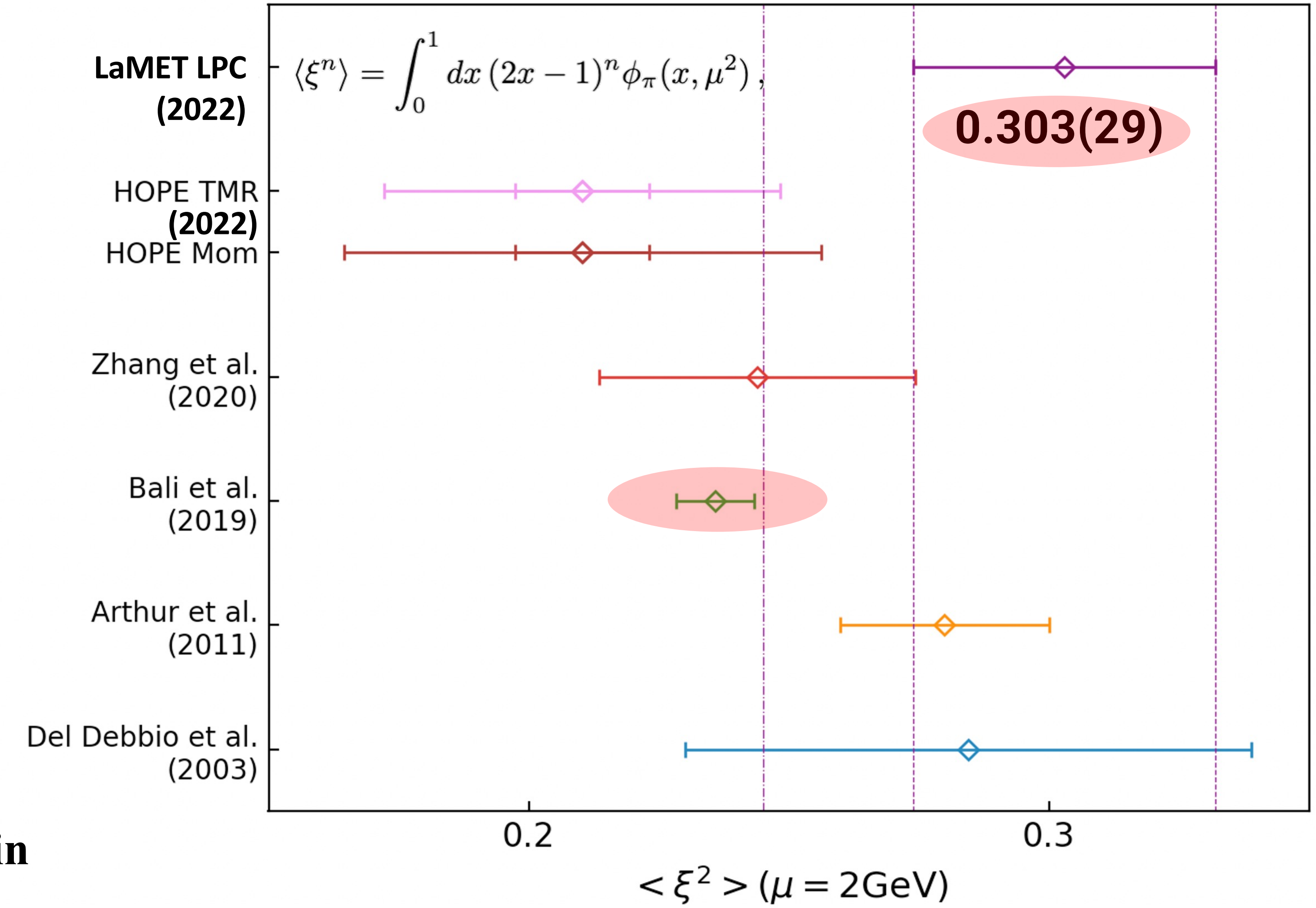



**1 step towards precision calculation**

# Moments tension:




*PRL 129,132001 (2022)*




- **High precision under going in LaMET**
- **Moments tension between LaMET &OPE remain**

- **OPE: Local moments**

- **Small z expansion: Non-local moments**

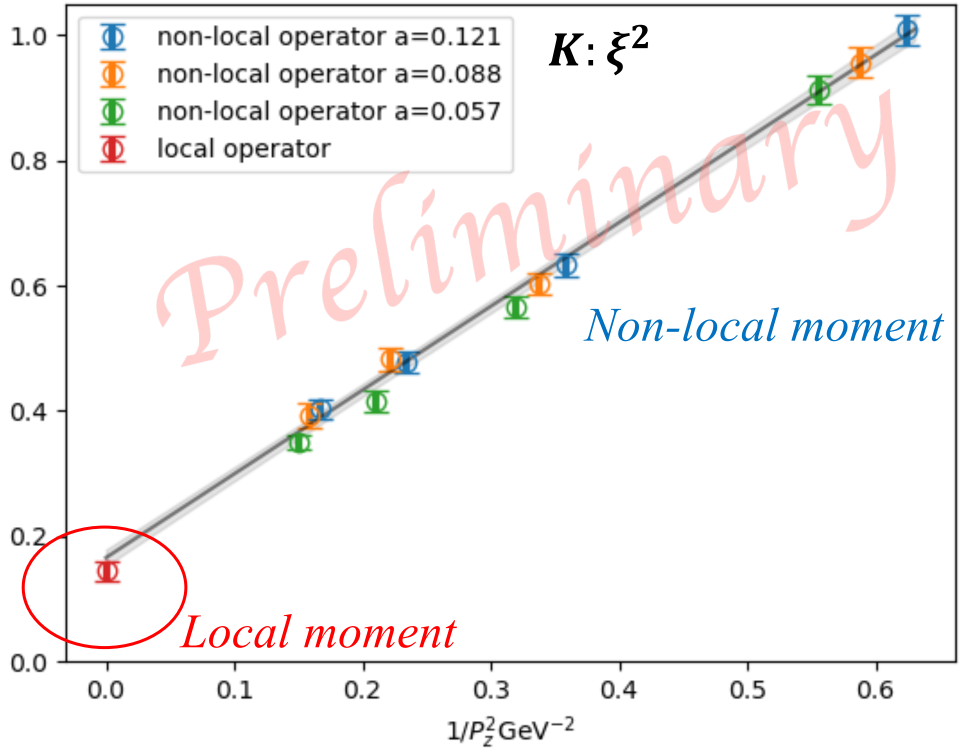
$$\xi \equiv 2x-1, \qquad \langle \xi^n \rangle_K(\mu) \equiv \int_0^1 dx\,(2x-1)^n\,\phi_K(x,\mu)$$

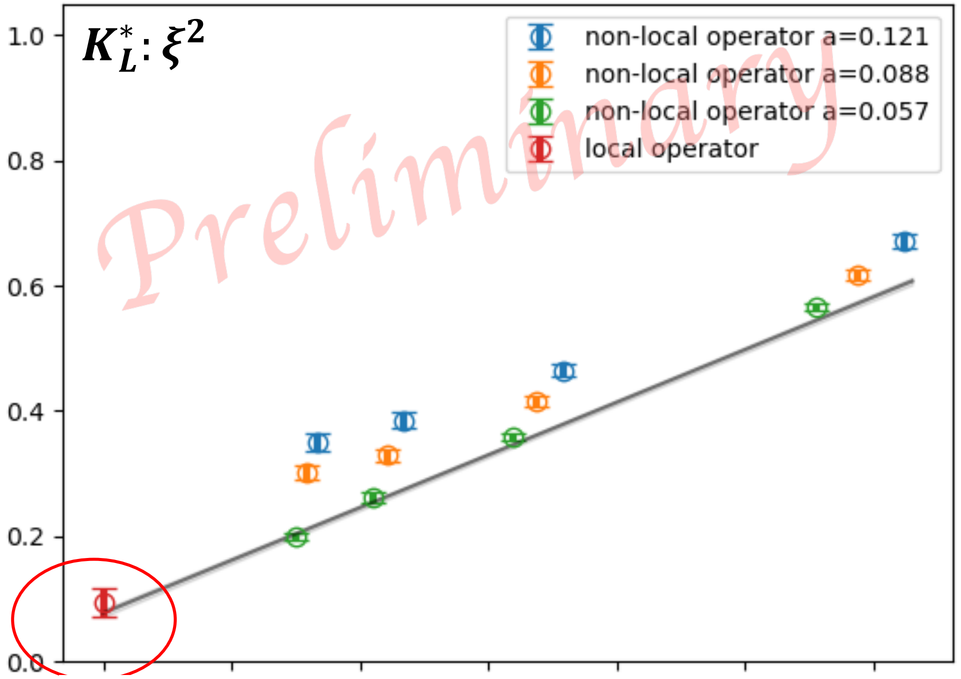
□□□□□□□□ ID □ rld3 □□□□□□

$$\tilde{\Phi}(z,P_z) \sim \int_0^1 dx\, e^{i(2x-1)P_z z}\phi_K(x)$$

$$\tilde{\Phi}(z,P_z) = \int_0^1 dx\left[1 + i(2x-1)P_z z - \frac{1}{2}(2x-1)^2 P_z^2 z^2 + \cdots\right]\phi_K(x)$$

**Consistent moments from local and non-local**




**The first precision check in LaMET, then ?**

# Quasi from coordinate to momentum space: Limited discrete Fourier transform

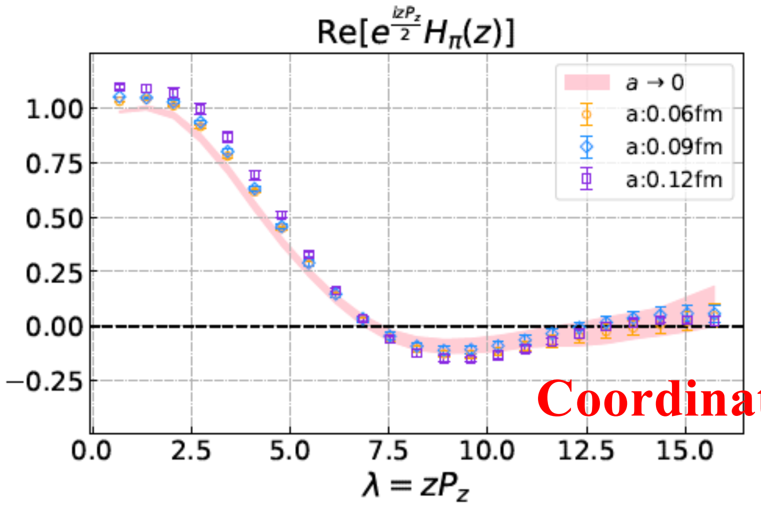
$$g(\lambda) = \int dx e^{-i\lambda x} f(x) \to f(x) = \frac{1}{2\pi} \sum_{\lambda_{\min}}^{\lambda_{max}} \Delta\lambda e^{i\lambda x} g(\lambda)$$

*LPC NPB 964 115311(2021);—hybrid scheme Extrapolation*

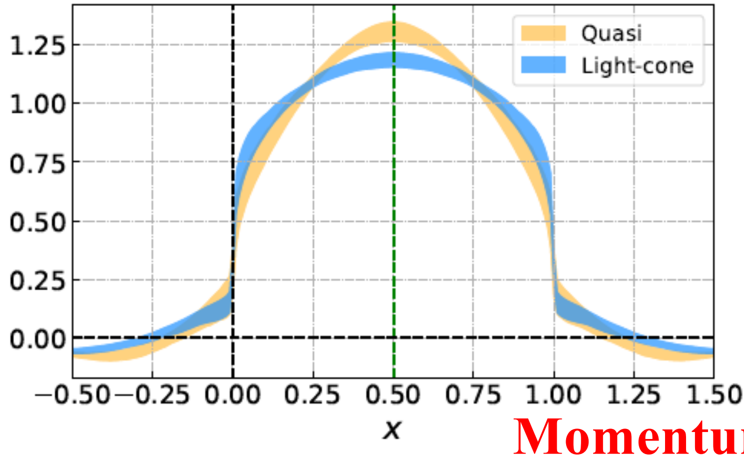
*H.Dutrieux et.al, arXiv: 2504.17706; — inverse problem exist*

*J.W.Chen et.al. arXiv: 2505.14619; — can be circumvented*

*A.S.Xiong et.al. arXiv: 2506.16689; — can be addressed*




Coordinate space:




Momentum space:

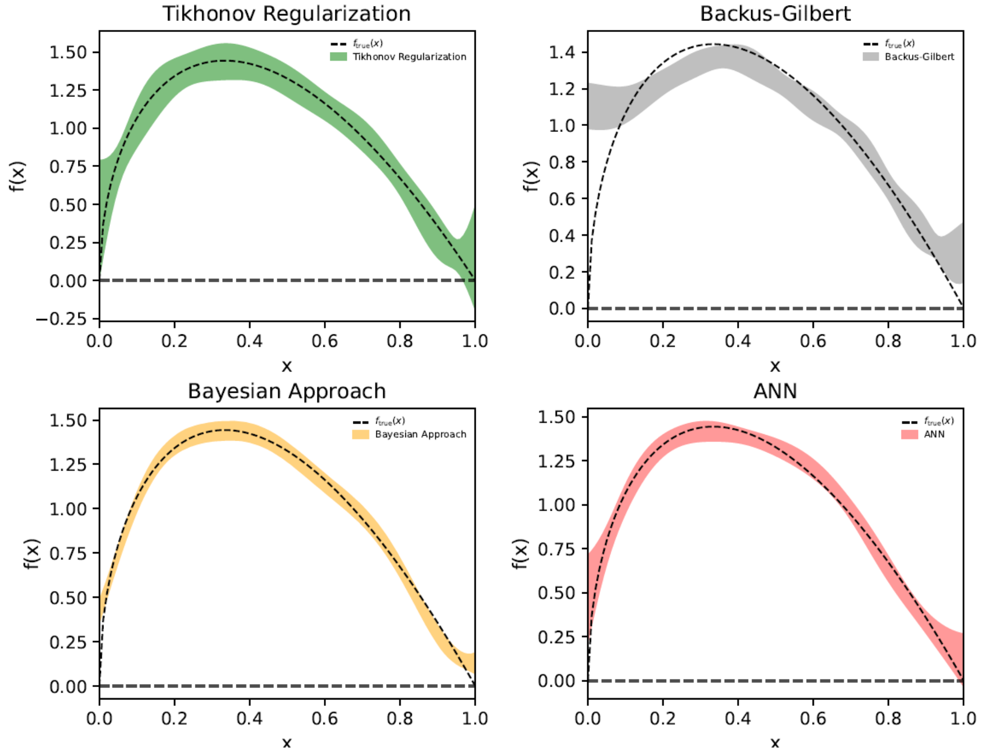
# Limited discrete Fourier transform: Can be circumvented

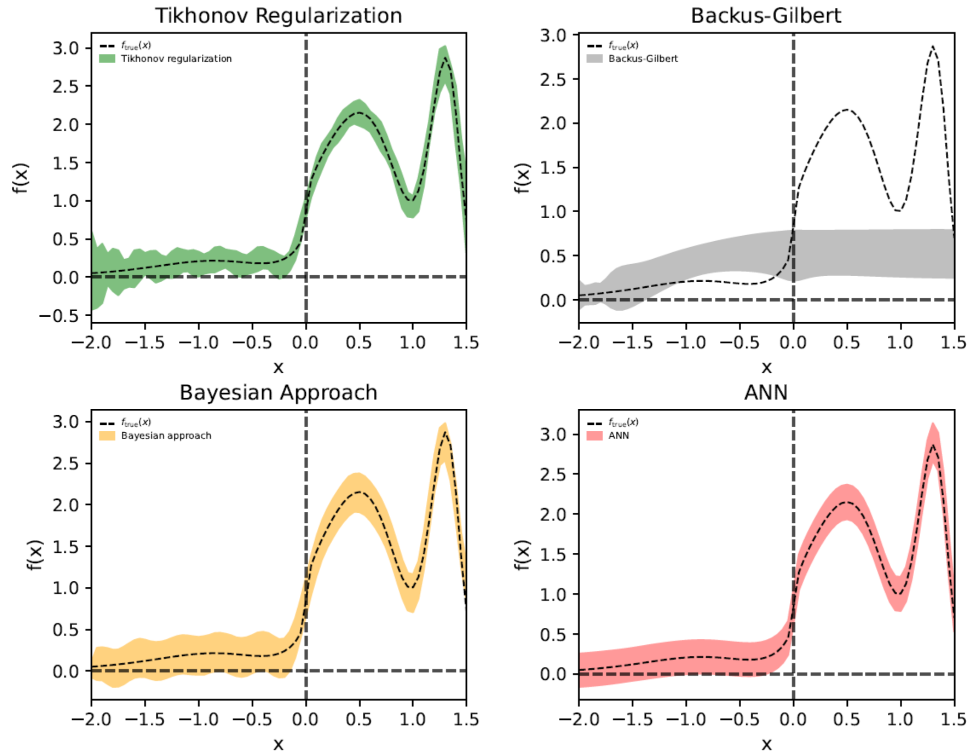
- Tikhonov regularization
- Backus-Gilbert
- Bayesian approach

Regulator method

➢ Artificial neurons network

Inverse method give
- a support plan for extrapolation
- a safer error estimate
- Maybe more …



# B meson LCDA cannot be directly determined

× *OPE*

× *LaMET*

**The most important input in heavy flavor physics**

- Only models for heavy meson LCDAs

  Grozin, Neubert, 1997; Braun, Ivanov, Korchemsky, 2004; Beneke, Braun, Ji, Wei, 2018…
- Only models for heavy meson SF

  Korchemsky, Sterman, 1994; Bauer, Luke, Mannel, 2001; Neubert, 2005; Lee, Ligeti, Stewart, Tackmann, 2006; ……
- A new approach from Lattice ?

  XuJi et.al , PRD 2020, 2022, EPJC 2024…

## B meson LCDA cannot be directly determined

× *OPE*

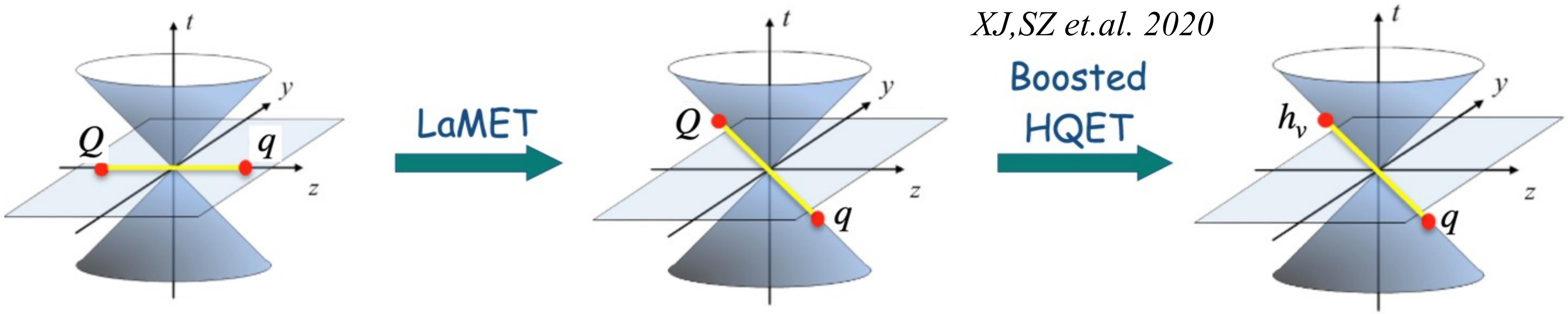
× *LaMET*

**The most important input in heavy flavor physics**

- **QCD LCDA**: quark distribution in **final-state** heavy mesons $B \to \boldsymbol{D}\ell\nu$:
- **HQET LCDA**: quark distribution in **initial-state** heavy mesons $\boldsymbol{B} \to D\ell\nu$

——Cusp divergence → Moments forbidden

**Only solution： Two step matching**




*LPC XYH, QAZ el.al. PRD 111, L111503, PRD 111, 034503, (2025)*

# Summary & Outlook

- We have done:
  - *LPC, PRD 111, 034510*, validated the concept of lattice calculation of baryon LCDA from LaMET
  - *arXiv: 2508.08971*, successfully adopted the Hybrid renormalization on baryon matrix element

- We now proceeding:
  - Continuum & physical mass extrapolation
  - More strategies to enhance lattice simulation
  - Methods for limited FT, inverse & matching schemes

- Following:
  - Please stay tuned our results for all leading twists LCDAs of Proton and Lambda !

# Summary & Outlook

**Framework advances:**

- Renormalon resummation(LRR)
- Renormalization group resummation(RGR)
- Two loop matching

**Simulation and analysis advances:**

- Simulation improvement
- Inverse Fourier Transform
- Pyquda coding

**LCDA**
**Precision Calculation**

**Physical targets advances:**

- Consistent and precision meson LCDA
- Heavy meson LCDA
- Light baryon LCDA

**Future potential:**

- Gradient flow
- Lanczos
- ANN … …

**Thanks for Your Attention !**

# Backup slides

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# Backup slides

We can also take start from the light-cone decomposition. We start from some convention on light-cone

$$\gamma^{\pm} = 1/\sqrt{2}\,(\gamma^0 \pm \gamma^3) \;\; \text{or} \;\; \text{on lattice} = 1/\sqrt{2}\,(\gamma^t \pm i\gamma^z) \tag{4}$$

$$P_{\pm} = (1/2)\,\gamma^{\mp}\,\gamma^{\pm}$$
$$\psi = \psi_+ + \psi_-, \quad \psi_{\pm} \equiv P_{\pm}\,\psi$$

For a quark bilinears constructed with only $\psi_+$ components describe the leading Fock states of mesons:

$$u_+^{\dagger}\,\gamma_5\,d_+ = \sqrt{2}\,\bar{u}\,\gamma_+\,\gamma_5\,d \tag{5}$$

Here, we have used

$$P_+\,\gamma_5\,P_+ = \sqrt{2}\,\gamma^+\,\gamma_5 \tag{6}$$

It will be trival for a leading twist contribution of a pseudoscalar meson should be $\gamma_+\gamma_5$ and $\gamma^t + i\gamma^z$ on lattice