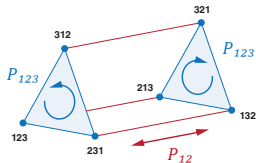


Flavor

Permutation group S_3 consists of 6 group elements, can be reconstructed from **transposition** P_{12} ($1 \leftrightarrow 2$) and **cyclic permutation** P_{123} ($1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1$):

$$\begin{aligned} 1 \psi_{123} &= \psi_{123} & P_{12} \psi_{123} &= \psi_{213} \\ P_{123} \psi_{123} &= \psi_{231} & P_{12} P_{123} \psi_{123} &= \psi_{132} \\ P_{123}^2 \psi_{123} &= \psi_{312} & P_{12} P_{123}^2 \psi_{123} &= \psi_{321} \end{aligned}$$



Cayley graph

Find combinations transforming under **irreps** of S_3 :

- **Singlet** $\mathcal{S}$ is invariant under any permutation:

$$\begin{aligned} P_{12} \mathcal{S} &= \mathcal{S} \\ P_{123} \mathcal{S} &= \mathcal{S} \end{aligned} \quad \mathcal{S} = \psi_{123} + \psi_{231} + \psi_{312} + \psi_{213} + \psi_{132} + \psi_{321}$$

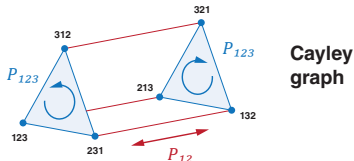
- **Antisingleton** $\mathcal{A}$ is antisymmetric under any transposition $i \leftrightarrow j$:

$$\begin{aligned} P_{12} \mathcal{A} &= -\mathcal{A} \\ P_{123} \mathcal{A} &= \mathcal{A} \end{aligned} \quad \mathcal{A} = \psi_{123} + \psi_{231} + \psi_{312} - \psi_{213} - \psi_{132} - \psi_{321}$$

Flavor

Permutation group S_3 consists of 6 group elements, can be reconstructed from **transposition** P_{12} ($1 \leftrightarrow 2$) and **cyclic permutation** P_{123} ($1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1$):

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- **Antisinglet** $\mathcal{A}$ is antisymmetric under any transposition $i \leftrightarrow j$:

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- **Doublets** $\mathcal{D}_1, \mathcal{D}_2$ form 2-dim subspace:

$$\begin{aligned} P_{12} \mathcal{D}_j &= \mathbf{M}_{12}^T \mathcal{D}_j & \mathbf{M}_{12} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ P_{123} \mathcal{D}_j &= \mathbf{M}_{123}^T \mathcal{D}_j & \mathbf{M}_{123} &= \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathcal{D}_1 &= \begin{bmatrix} (\psi_{231} - \psi_{132}) - (\psi_{312} - \psi_{321}) \\ \frac{1}{\sqrt{3}} ((\psi_{231} + \psi_{132}) + (\psi_{312} + \psi_{321}) - 2(\psi_{123} + \psi_{213})) \end{bmatrix} \\ \mathcal{D}_2 &= \begin{bmatrix} -\frac{1}{\sqrt{3}} ((\psi_{231} - \psi_{132}) + (\psi_{312} - \psi_{321}) - 2(\psi_{123} - \psi_{213})) \\ (\psi_{231} + \psi_{132}) - (\psi_{312} + \psi_{321}) \end{bmatrix} \end{aligned}$$

Flavor

Young diagrams:



$\mathcal{S}$



$\mathcal{D}_1, \mathcal{D}_2$



$\mathcal{A}$

Find combinations transforming under **irreps** of S_3 :

- **Singlet** $\mathcal{S}$ is invariant under any permutation:

$$P_{12} \mathcal{S} = \mathcal{S}$$

$$P_{123} \mathcal{S} = \mathcal{S}$$

$$\mathcal{S} = \psi_{123} + \psi_{231} + \psi_{312} + \psi_{213} + \psi_{132} + \psi_{321}$$

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$$P_{12} \mathcal{A} = -\mathcal{A}$$

$$P_{123} \mathcal{A} = \mathcal{A}$$

$$\mathcal{A} = \psi_{123} + \psi_{231} + \psi_{312} - \psi_{213} - \psi_{132} - \psi_{321}$$

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$$P_{123} \mathcal{D}_j = \mathbf{M}_{123}^T \mathcal{D}_j$$

$$\mathbf{M}_{12} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{M}_{123} = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$\mathcal{D}_1 = \begin{bmatrix} (\psi_{231} - \psi_{132}) - (\psi_{312} - \psi_{321}) \\ \frac{1}{\sqrt{3}} ((\psi_{231} + \psi_{132}) + (\psi_{312} + \psi_{321}) - 2(\psi_{123} + \psi_{213})) \end{bmatrix}$$

$$\mathcal{D}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}} ((\psi_{231} - \psi_{132}) + (\psi_{312} - \psi_{321}) - 2(\psi_{123} - \psi_{213})) \\ (\psi_{231} + \psi_{132}) - (\psi_{312} + \psi_{321}) \end{bmatrix}$$

Flavor

Now combine this with **SU(3)**, e.g. for quark content **uud**:

$$\begin{aligned}\psi_{123} &= u_1 u_2 d_3 = (uud)_{123} \\ \psi_{231} &= u_2 u_3 d_1 = (duu)_{123} \\ \psi_{312} &= u_3 u_1 d_2 = (udu)_{123} \\ \psi_{213} &= u_2 u_1 d_3 = (uud)_{123} \\ \psi_{132} &= u_1 u_3 d_2 = (udu)_{123} \\ \psi_{321} &= u_3 u_2 d_1 = (duu)_{123}\end{aligned}$$

$$\Rightarrow \mathcal{S} = 2(uud + udu + duu) \dots \Delta^+ \\ \mathcal{A} = 0$$

$$\mathcal{D}_1 = 2 \left[\frac{duu - udu}{\sqrt{3}} (duu + udu - 2uud) \right] \dots \text{proton} \\ \mathcal{D}_2 = 0$$

Find combinations transforming under irreps of S_3 :

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$$\begin{aligned}P_{12} \mathcal{S} &= \mathcal{S} \\ P_{123} \mathcal{S} &= \mathcal{S}\end{aligned}$$

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Now combine this with **SU(3)**, e.g. for quark content ***uud***:

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$$\psi_{231} = u_2 u_3 d_1 = (duu)_{123}$$

$$\psi_{312} = u_3 u_1 d_2 = (udu)_{123}$$

$$\psi_{213} = u_2 u_1 d_3 = (uud)_{123}$$

$$\psi_{132} = u_1 u_3 d_2 = (udu)_{123}$$

$$\psi_{321} = u_3 u_2 d_1 = (duu)_{123}$$

$$\Rightarrow \mathcal{S} = 2 (uud + udu + duu) \dots \Delta^+$$

$$\mathcal{A} = 0$$

$$\mathcal{D}_1 = 2 \left[\frac{duu - udu}{\sqrt{3}} (duu + udu - 2uud) \right] \dots \text{proton}$$

$$\mathcal{D}_2 = 0$$

• ***ddu***: exchange $u \leftrightarrow d \Rightarrow \mathcal{S} \dots \Delta^0, \mathcal{D}_1 \dots \text{neutron}$

• ***uuu*** $\Rightarrow \mathcal{S} \dots \Delta^{++}$

• ***ddd*** $\Rightarrow \mathcal{S} \dots \Delta^-$

• ***uds*** $\Rightarrow \mathcal{S} \dots \Sigma^0, \mathcal{D}_1 \dots \Sigma^0, \mathcal{D}_2 \dots \Lambda^0, \mathcal{A} \dots \Lambda^0$

$\Rightarrow$ **SU(3) flavor wave functions:**

	<i>uuu</i>	<i>uud</i>	<i>ddu</i>	<i>ddd</i>	<i>uus</i>	<i>uds</i>	<i>dds</i>	<i>ssu</i>	<i>ssd</i>	<i>sss</i>
$\mathcal{S}$	Δ^{++}	Δ^+	Δ^0	Δ^-	Σ^+	Σ^0	Σ^-	Ξ^0	Ξ^-	Ω^-
$\mathcal{D}_1$		p	n		Σ^+	Σ^0	Σ^-	Ξ^0	Ξ^-	
$\mathcal{D}_2$						Λ^0				
$\mathcal{A}$						Λ^0				

$$\begin{aligned}
 & \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \\
 & \text{Decuplet} \quad \mathbf{10}_S \oplus \\
 & \text{Octet} \quad \mathbf{8}_{M_A} \oplus \mathbf{8}_{M_S} \oplus \\
 & \text{Singlet} \quad \mathbf{1}_A
 \end{aligned}$$

Flavor

		I	I_3	S	
uud	p	$1/2$	$1/2$	0	octet
udd	n	$1/2$	$-1/2$	0	
uus	Σ^+	1	1	-1	
uds	Σ^0	1	0	-1	
dds	Σ^-	1	-1	-1	
uds	Λ^0	0	0	-1	
uss	Ξ^0	$1/2$	$1/2$	-2	
dss	Ξ^-	$1/2$	$-1/2$	-2	
uuu	Δ^{++}	$3/2$	$3/2$	0	decuplet
uud	Δ^+	$3/2$	$1/2$	0	
udd	Δ^0	$3/2$	$-1/2$	0	
ddd	Δ^-	$3/2$	$-3/2$	0	
uus	Σ^+	1	1	-1	
uds	Σ^0	1	0	-1	
dds	Σ^-	1	-1	-1	
uss	Ξ^0	$1/2$	$1/2$	-2	
dss	Ξ^-	$1/2$	$-1/2$	-2	singlet
sss	Ω^-	0	0	-3	
uds	Λ^0	0	0	-1	

$SU(3)_f$ is broken,
states with same I , I_3 , Y can mix

Flavor

Analogous for **SU(4)**: $4 \otimes 4 \otimes 4 = 20_S \oplus 20_{M_A} \oplus 20_{M_S} \oplus 4_A$

	<i>uuu</i>	<i>uud</i>	<i>ddu</i>	<i>ddd</i>	<i>uus</i>	<i>uds</i>	<i>dds</i>	<i>ssu</i>	<i>ssd</i>	<i>sss</i>	<i>uuc</i>	<i>udc</i>	<i>ddc</i>	<i>usc</i>	<i>dsc</i>	<i>ssc</i>	<i>ucc</i>	<i>dcc</i>	<i>scc</i>	<i>ccc</i>
<i>S</i>	Δ^{++}	Δ^+	Δ^0	Δ^-	Σ^+	Σ^0	Σ^-	Ξ^0	Ξ^-	Ω^-	Σ_c^{++}	Σ_c^+	Σ_c^0	Ξ_c^+	Ξ_c^0	Ω_c^0	Ξ_{cc}^{++}	Ξ_{cc}^+	Ω_{cc}^+	Ω_{ccc}^+
<i>D</i> ₁		<i>p</i>	<i>n</i>		Σ^+	Σ^0	Σ^-	Ξ^0	Ξ^-		Σ_c^{++}	Σ_c^+	Σ_c^0	Ξ_c^+	Ξ_c^0	Ω_c^0	Ξ_{cc}^{++}	Ξ_{cc}^+	Ω_{cc}^+	
<i>D</i> ₂						Λ^0						Λ_c^+		Ξ_c^+	Ξ_c^0					
<i>A</i>						Λ^0						Λ_c^+		Ξ_c^+	Ξ_c^0					

single
charm

double
charm

triple
charm

SU(2): $2 \otimes 2 \otimes 2 = 4_S \oplus 2_{M_A} \oplus 2_{M_S}$

	<i>uuu</i>	<i>uud</i>	<i>ddu</i>	<i>ddd</i>
<i>S</i>	Δ^{++}	Δ^+	Δ^0	Δ^-
<i>D</i> ₁		<i>p</i>	<i>n</i>	
<i>D</i> ₂				
<i>A</i>				

If this were **spin** instead of flavor:

Spin 3/2:

$\uparrow\uparrow\uparrow$,
 $\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow$,
 $\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow$,
 $\downarrow\downarrow\downarrow$

Spin 1/2:

$\left[\begin{array}{c} \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow \\ -\frac{1}{\sqrt{3}}(\uparrow\uparrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow) \end{array} \right]$
 $\left[\begin{array}{c} \uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow \\ \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \downarrow\uparrow\downarrow - 2\downarrow\uparrow\downarrow) \end{array} \right]$

Total wave function

$$\Psi_{\text{dynamics}} \otimes \Psi_{\text{flavor}} \otimes \Psi_{\text{color}} \stackrel{!}{=} \mathcal{A}$$

$\{ \mathcal{S}_f, \mathcal{D}_f, \mathcal{A}_f \}$ $\mathcal{A}_c$

suppose $\downarrow$

$$= \Psi_{\text{orbital}} \otimes \Psi_{\text{spin}}$$

$$\{ \mathcal{S}_o, \mathcal{D}_o, \mathcal{A}_o \} \quad \{ \mathcal{S}_s, \mathcal{D}_s \}$$

⇒ All **symmetric** group-theoretical combinations:

orb	spin-flavor	spin	flavor	
$\mathcal{S}_o$	$(\mathcal{D}_s \cdot \mathcal{D}_f)$	$\frac{1}{2}$	8	56-plet
$\mathcal{S}_o$	$(\mathcal{S}_s \mathcal{S}_f)$	$\frac{3}{2}$	10	(2·8 + 4·10)
$\mathcal{D}_o \cdot (\mathcal{D}_s * \mathcal{D}_f)$		$\frac{1}{2}$	8	70-plet (2·8 + 2·10 + 2·1 + 4·8)
$\mathcal{D}_o \cdot (\mathcal{D}_s \mathcal{S}_f)$		$\frac{1}{2}$	10	
$\mathcal{D}_o \cdot (\varepsilon \mathcal{D}_s) \mathcal{A}_f$		$\frac{1}{2}$	1	
$\mathcal{D}_o \cdot (\mathcal{S}_s \mathcal{D}_f)$		$\frac{3}{2}$	8	
$\mathcal{A}_o$	$(\mathcal{D}_s \wedge \mathcal{D}_f)$	$\frac{1}{2}$	8	20-plet
$\mathcal{A}_o$	$(\mathcal{S}_s \mathcal{A}_f)$	$\frac{3}{2}$	1	(2·8 + 4·1)

“SU(6)-symmetric quark model”:

$$6 \otimes 6 \otimes 6 = 56_S \oplus 70_{MA} \oplus 70_{MS} \oplus 20_A$$

Total wave function

$$\Psi_{\text{dynamics}} \otimes \Psi_{\text{flavor}} \otimes \Psi_{\text{color}} \stackrel{!}{=} \mathcal{A}$$

$\{ \mathcal{S}_f, \mathcal{D}_f, \mathcal{A}_f \}$ $\mathcal{A}_c$

suppose $\downarrow$

$$= \Psi_{\text{orbital}} \otimes \Psi_{\text{spin}}$$

$$\{ \mathcal{S}_o, \mathcal{D}_o, \mathcal{A}_o \} \quad \{ \mathcal{S}_s, \mathcal{D}_s \}$$

$\Rightarrow$ All **symmetric** group-theoretical combinations:

- Spin and flavor determined;
orbital part from dynamics;
total angular momentum $J = |L - S| \dots L + S$;
parity $P \Rightarrow J^P$
- Orbital ground states:** $\mathcal{S}_o$, $L = 0$, $J = S$, $P = +$
 $\Rightarrow$ octet with $J^P = \frac{1}{2}^+$, decuplet with $J^P = \frac{3}{2}^+$
- But: assumes that S and L are good quantum numbers (nonrelativistic quark model!)

orb	spin-flavor	spin	flavor	
$\mathcal{S}_o$	$(\mathcal{D}_s \cdot \mathcal{D}_f)$	$\frac{1}{2}$	8	56-plet
$\mathcal{S}_o$	$(\mathcal{S}_s \mathcal{S}_f)$	$\frac{3}{2}$	10	(2·8 + 4·10)
$\mathcal{D}_o \cdot (\mathcal{D}_s * \mathcal{D}_f)$		$\frac{1}{2}$	8	70-plet (2·8 + 2·10 + 2·1 + 4·8)
$\mathcal{D}_o \cdot (\mathcal{D}_s \mathcal{S}_f)$		$\frac{1}{2}$	10	
$\mathcal{D}_o \cdot (\mathcal{E} \mathcal{D}_s) \mathcal{A}_f$		$\frac{1}{2}$	1	
$\mathcal{D}_o \cdot (\mathcal{S}_s \mathcal{D}_f)$		$\frac{3}{2}$	8	
$\mathcal{A}_o$	$(\mathcal{D}_s \wedge \mathcal{D}_f)$	$\frac{1}{2}$	8	20-plet
$\mathcal{A}_o$	$(\mathcal{S}_s \mathcal{A}_f)$	$\frac{3}{2}$	1	(2·8 + 4·1)

“SU(6)-symmetric quark model”:

$$6 \otimes 6 \otimes 6 = 56_S \oplus 70_{MA} \oplus 70_{MS} \oplus 20_A$$

Missing resonances (?)

Set up orbital wave functions in spherical harmonic oscillator basis

Klempt, Richard, Rev. Mod. Phys. 82 (2010), Klempt, Metsch, EPJ A 48 (2012), Crede, Roberts, Rept. Prog. Phys. 76 (2013)

$$\phi_L(\rho, \lambda) = \sum_{n_\rho, l_\rho, n_\lambda, l_\lambda} c_{n_\rho, l_\rho, n_\lambda, l_\lambda}^L [\phi_{n_\rho, l_\rho}(\rho) \otimes \phi_{n_\lambda, l_\lambda}(\lambda)]_L$$

Depend on Jacobi coordinates:

$$\rho = \frac{x_1 - x_2}{\sqrt{2}}, \quad \lambda = \frac{x_1 + x_2 - 2x_3}{\sqrt{6}}$$

‘Band quantum number’ N with same energy:

$$N = 2n + l$$

$$n = n_\rho + n_\lambda \quad n_\alpha > 0 \quad \dots \text{radial excitations}$$

$$l = l_\rho + l_\lambda \quad l_\alpha > 0 \quad \dots \text{orbital excitations}$$

$$P = (-1)^l$$

$$L = |l_\rho - l_\lambda| \dots l_\rho + l_\lambda$$

Combining this with spin & flavor gives *lots* of states... “Missing-resonances problem”

N	L ^P	O	SF	S	F	J ^P
0	0 ⁺	S ₀	56	8	10	8⁺ 10⁺
1	1 ⁻	D ₀	70	8 10 1 8	8 10 1 8	8⁻ 10⁻ 1⁻ 8⁻
2	0 ⁺	S ₀	56	8 10 1 8	8 10 1 8	8⁺ 10⁺ 1⁺ 8⁺
		D ₀	70	8 10 1 8	8 10 1 8	8⁺ 10⁺ 1⁺ 8⁺
	1 ⁺	A ₀	20	8 1	8 1	8⁺ 1⁺
	2 ⁺	S ₀	56	8 10 1 8	8 10 1 8	8⁺ 10⁺ 1⁺ 8⁺
		D ₀	70	8 10 1 8	8 10 1 8	8⁺ 10⁺ 1⁺ 8⁺

Bold: well-established states

Gray: no exp. candidate or 1 star

N	L ^P	O	SF	S	F	J ^P
3	1 ⁻	S ₀	56	8 10 1 8	8 10 1 8	8⁻ 10⁻ 1⁻ 8⁻
		D ₀	70	8 10 1 8	8 10 1 8	8⁻ 10⁻ 1⁻ 8⁻
		D ₀	70	8 10 1 8	8 10 1 8	8⁻ 10⁻ 1⁻ 8⁻
		A ₀	20	8 1	8 1	8⁻ 1⁻
	2 ⁻	D ₀	70	8 10 1 8	8 10 1 8	8⁻ 10⁻ 1⁻ 8⁻
		S ₀	56	8 10 1 8	8 10 1 8	8⁻ 10⁻ 1⁻ 8⁻
		D ₀	70	8 10 1 8	8 10 1 8	8⁻ 10⁻ 1⁻ 8⁻
		A ₀	20	8 1	8 1	8⁻ 1⁻

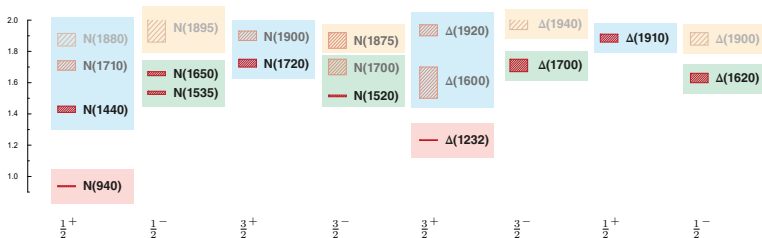
Missing resonances (?)

Prediction of many states, but also problems with **level ordering**, e.g. Roper vs. N(1535)

N	L^P	O	SF	S	F	J^P
0	0^+	S_0	56	8	10	8^+ 10^+
1	1^-	D_0	70	8	10	8^- 10^- 1^- 8^-
2	0^+	S_0	56	8	10	8^+ 10^+
		D_0	70	8	10	8^- 10^- 1^- 8^-

N	L^P	O	SF	S	F	J^P
3	1^-	S_0	56	8	10	8^- 10^-
		D_0	70	8	10	8^- 10^- 1^- 8^-
		D_0	70	8	10	8^- 10^- 1^- 8^-
		A_0	20	8	10	8^- 10^- 1^- 8^-

M [GeV]



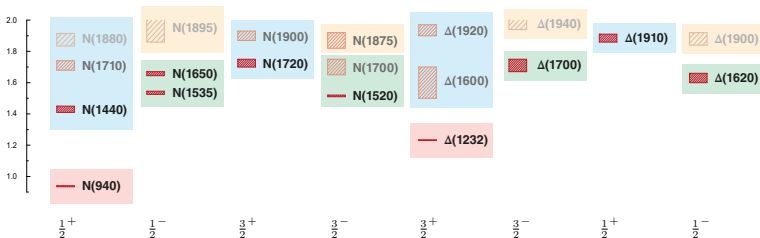
Missing resonances (?)

Prediction of many states, but also problems with **level ordering**, e.g. Roper vs. N(1535)

- We may not have found them yet (weak coupling to $N\pi$)
- **Diquark** clustering reduces # states?
- Or maybe our assumptions so far (non-relativistic quark model, harmonic oscillator) have been too restrictive?



M [GeV]



1. Introduction

2. Spectrum & symmetries

3. Quark models

4. QFT toolbox

5. Functional methods

6. n-point functions & tensor bases

7. Multiquark states

8. Light-front wave functions

Lecture notes “QCD and hadron physics,”
<https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei>

GE, “*Hadron physics with functional methods*,”
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

Burkert, GE, Klempt, “*The impact of γN and $\gamma^* N$ interactions on our understanding of nucleon excitations*,” arXiv:2506.16482

GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
“*Baryons as relativistic three-quark bound states*”,
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

Barabanov, Bedolla, Brooks, Cates, Chen et al.,
“*Diquark correlations in hadron physics: Origin, impact and evidence*”, Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

Quantum Field Theory

$$\mathbf{H} \Psi = E \Psi \quad ?$$

- Relativity
- No “wave function” with probability interpretation
- Particle creation & annihilation, Hilbert space becomes ∞ dim
- Renormalization
- ...



Some issues addressed e.g. in
Light-front Hamiltonian approaches

[Brodsky, Teramond, Dosch, Erlich, Phys. Rept. 584 \(2015\)](#)
[Hiller, Prog. Part. Nucl. Phys. 90 \(2016\)](#)

Typical baryon spectrum calculations in QFT do not start from Hamiltonian, except in certain limits or using approximations (non-relativistic, heavy-quark, ...)

... then how else?

Quantum Field Theory

- **Classical action:**

$$S[\phi] = \int d^4x \mathcal{L}(\phi)$$

- **Relativity & unitarity:**

$\phi_i(x)$... operators acting on states $|p_1 \dots p_n\rangle$: vacuum $|0\rangle$, 1-particle states $|p\rangle$, etc.

States transform under unitary representations of Poincaré group, probability interpretation for S-matrix elements

$$\langle p_1 \dots p_n | q_1 \dots q_n \rangle$$

- **Causality:**

Two measurements at spacelike distance cannot affect each other:
 $[O_1(x), O_2(y)] = 0 \quad \text{for } (x-y)^2 < 0$

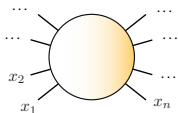
$\Rightarrow$ (anti) commutation relations for bosonic (fermionic) fields

- **Spectral condition**

- **Renormalizability** (for renormalizable theories)

- **Gauge invariance** (for gauge theories)

Interacting QFT: troubles at operator level, but **correlation functions** mathematically well-defined

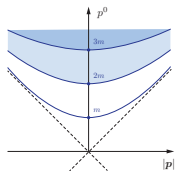


$$G(x_1, \dots, x_n) := \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle$$

- contain full content of QFT
- related to onshell S-matrix elements (LSZ formula)
- can be computed perturbatively (e.g. QED) but also **non-perturbatively**

Quantum Field Theory

Spectral condition



$|0\rangle \dots$ vacuum

$|p\rangle \dots$ **1-particle state** with onshell momentum $p^2 = m^2$, $E_p^2 = \mathbf{p}^2 + m^2$

$|p_1 \dots p_n\rangle \dots$ **multiparticle states** with total momentum p and relative momenta, form continuum $p^2 = m_\lambda^2 > (2m)^2$

Completeness relation

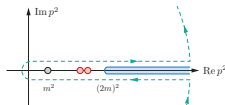
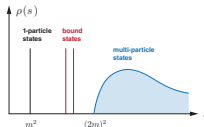
$$\mathbb{1} = \sum_{\lambda} \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2E_p} |\lambda\rangle \langle \lambda| \quad p^2 = m_\lambda^2$$

Inserting this in 2-point function yields **spectral representation**:

$$G(x, y) = \langle 0 | T \phi(x) \phi(y) | 0 \rangle$$

$$G(p^2) = \sum_{\lambda} \frac{iR_{\lambda}}{p^2 - m_{\lambda}^2 + i\epsilon} = \int_0^{\infty} \frac{ds}{2\pi} \frac{i\rho(s)}{p^2 - s + i\epsilon}$$

Spectral
function



$\Rightarrow$ 1-particle states produce **poles** at $p^2 = m^2$

$\Rightarrow$ multiparticle states produce **branch cuts** for $p^2 > (2m)^2$

$\Rightarrow$ no singularities except on positive real axis (analyticity)

(what about
gauge theories...?)

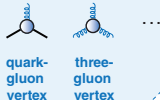
Quantum Field Theory

Correlation functions in **QCD**:

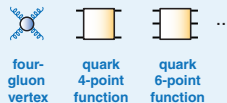
Two-point functions



Three-point functions

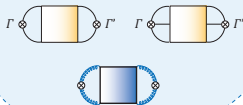


Higher n-point functions



Correlation functions with composite operators

$$\langle 0 | [\bar{\psi} \Gamma \psi](x) [\bar{\psi} \Gamma' \psi](y) | 0 \rangle$$



The information on each correlation function is contained in its (Lorentz-invariant)

redressing functions:

$$G_{\alpha\beta\ldots}^{\mu\nu\ldots}(p_1, \ldots, p_n) = \sum_{i=1}^N f_i(p_1^2, p_2^2, \ldots) \tau_i(p_1, \ldots, p_n)_{\alpha\beta\ldots}^{\mu\nu\ldots}$$

Lorentz, Dirac,
flavor, color

Quantum Field Theory

Hadrons produce poles in QCD's correlation functions:

$$G(x_1, \dots, x_r) = \langle 0 | T \phi(x_1) \dots \phi(x_n) \phi(y_1) \dots \phi(y_r) | 0 \rangle$$

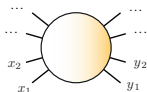
Proof: see e.g. books by
S. Weinberg, *The QFT of Fields* Vol. 1, 1995 (Sec. 10.2)
M. D. Schwartz, *QFT and the SM*, 2014 (Sec. 24.3)



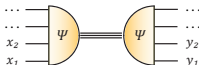
Insert completeness relation with
asymptotic, **onshell** states $|\lambda\rangle$:

$$1 = \sum_{\lambda} \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2E_p} |\lambda\rangle \langle \lambda| \quad p^2 = m_{\lambda}^2, \quad E_p^2 = \mathbf{p}^2 + m_{\lambda}^2$$

$$G(x_1, \dots, x_r) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipz} \left[\frac{i \Psi(\{x_i\}, p) \Psi^{\dagger}(\{y_j\}, p)}{p^2 - m_{\lambda}^2 + i\epsilon} + \text{finite} \right]$$



$$p^2 \longrightarrow m_{\lambda}^2$$



Bethe-Salpeter wave function
encodes information about hadron

$$\Psi(\{x_i\}, p) = \langle 0 | T \phi(x_1) \dots \phi(x_n) | \lambda \rangle$$

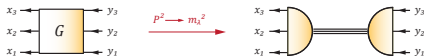
QFT analogue of wave function in QM
(but no probability interpretation)

- also holds for composite operators
- can insert 1 anywhere (and repeat)
- LSZ formula is special case
- poles in scattering amplitudes $\rightarrow$ poles in **experimental cross sections**

Quantum Field Theory

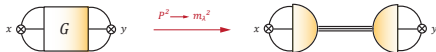
Baryon poles appear in any (elementary or composite) correlation function that is compatible with baryon quantum numbers:

- Elementary **quark 6-point function**:

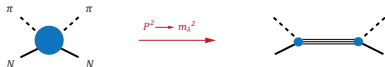


- Gauge-invariant (2-point) **current correlators**:

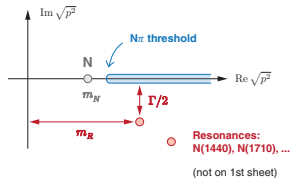
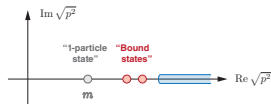
$$G(x-y) = \langle 0 | T \underbrace{[\Gamma_{\alpha\beta\gamma} \psi_\alpha \psi_\beta \psi_\gamma](x)}_{B(x)} \underbrace{[\bar{\Gamma}_{\rho\sigma\tau} \bar{\psi}_\rho \bar{\psi}_\sigma \bar{\psi}_\tau](y)}_{\bar{B}(y)} | 0 \rangle$$



- Hadronic scattering amplitudes**:



Singularity locations are independent of correlation function!



Effective field theories

- **2 → 2 scattering amplitude:**

Invariant matrix element

$$\mathcal{M}(s, z) = \sum_{l=0}^{\infty} (2l+1) f_l(s) P_l(z)$$

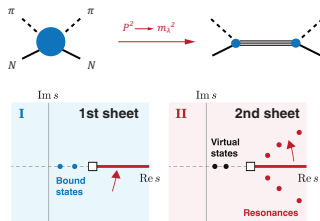
in CM frame:

$\sqrt{s}$... CM energy

z ... cosine of scattering angle

partial waves

Legendre polynomials



- **Unitarity** constrains imaginary parts of partial waves:

$$S^\dagger S = (1 + iT)^\dagger (1 + iT) \stackrel{!}{=} 1 \quad \Rightarrow \quad T - T^\dagger = iT^\dagger T \quad \Rightarrow$$

$$\text{Im } f_l(s) = \tau(s) |f_l(s)|^2$$

$$\tau(s) = \frac{1}{16\pi} \sqrt{\frac{s - 4m^2}{s}}$$

- Can be expressed through **phase shift**:

$$f_l(s) = R_l(s) e^{i\delta_l(s)} \Rightarrow R_l = \frac{1}{\tau} \sin \delta_l \Rightarrow f_l = \frac{1}{\tau} e^{i\delta_l} \sin \delta_l = \frac{1}{\tau} \frac{1}{\cot \delta_l - i} \Rightarrow \text{Re } \frac{1}{f_l} = \tau \cot \delta_l,$$

- Alternatively, through **K matrix**:

$$\text{Re } \frac{1}{f_l} = \tau \cot \delta_l = \frac{1}{K_l} \Rightarrow f_l^{-1} = K_l^{-1} - i\tau \Leftrightarrow f_l = K_l + K_l i\tau f_l$$

$$\text{Im } \frac{1}{f_l} = -\tau$$

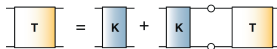
Effective field theories

- Bethe-Salpeter equation (BSE):**

$$T = K + K G_0 T \Leftrightarrow T^{-1} = K^{-1} - G_0$$

T matrix:
connected part
of 4-point CF

2-particle
irreducible
kernel



free hadron
propagators

symbolic notation:
each loop stands
for 4-momentum integral

- BSE automatically satisfies **unitarity!**

for partial
waves

$$T_+ - T_- = T_+ (T_-^{-1} - T_+^{-1}) T_- = T_+ (G_{0+} - G_{0-}) T_- \Rightarrow \text{Im } f_l(s) = \tau(s) |f_l(s)|^2$$

$\sim 2 \text{ Im } T$ T matrix slightly above and below the cut difference of propagators $\sim \delta$ function

- BSE returns K-matrix equation:

$$G_0 = G_1 + G_2 \Rightarrow T^{-1} = \underbrace{K^{-1} - G_1 - G_2}_{=: K_1^{-1}} \Rightarrow T = K_1 + K_1 G_2 T$$

if principle-value
integral

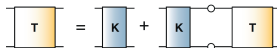
δ function

$$\Rightarrow f_l = K_l + K_l i\tau f_l$$

Effective field theories

What is the kernel K?

- Particles in loops are hadrons
⇒ must be interactions between hadrons



- Derive from effective Lagrangians ⇒ **EFTs**
- Prime example is **chiral perturbation theory**: interactions between nucleons, pions, ...

Weinberg, *Physica A* 96 (1979), Gasser, Leutwyler, *Annals Phys.* 158 (1984), *Nucl. Phys. B* 250 (1985)



Include all possible terms compatible with QCD symmetries (e.g., **chiral symmetry**)

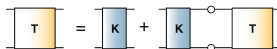
- systematic expansion in powers of derivatives & pion masses
- non-renormalizable, ∞ many terms with ∞ many free parameters, but higher loops suppressed by derivative couplings ⇒ **low-energy EFT**, introduce new **low-energy constants** (LECs) at each order
- perturbative calculations of $M_N(m_\pi^2)$, $\pi\pi$ scattering, $N\pi$ scattering, ...

Reviews: Meißner, *Rept. Prog. Phys.* 56 (1993),
Ecker, *Prog. Part. Nucl. Phys.* 35 (1995),
Pich, *Rept. Prog. Phys.* 58 (1995),

Bernard, Kaiser, Meißner, *Int. J. Mod. Phys. E* 4 (1995),
Scherer, *Adv. Nucl. Phys.* 27 (2003),
Bernard, Meißner, *Ann. Rev. Nucl. Part. Sci.* 57 (2007), ...

Effective field theories

But ChPT is perturbative $\Rightarrow$ no **bound states & resonances** $\Rightarrow$ **use BSE!** (“Unitarized ChPT”)



- BSE is **nonperturbative!** Like geometric series:

$$f(x) = 1 + x + x^2 + x^3 + \dots \Rightarrow f(x) = 1/(1-x) \text{ only if } |x| < 1$$

$$\text{But } f(x) = 1 + xf(x) = 1 + x + x^2 f(x) = 1 + x + x^2 + x^3 f(x) = \dots \Leftrightarrow f(x)^{-1} = 1 - x \text{ for any } x$$

- BSEs with EFT kernels $\rightarrow$ **dynamical coupled channel equations:** $N\pi$, ΛK , ΣK , $N\gamma$, etc. ANL-Osaka, Jülich-Bonn-Washington, MAID, SAID, JLab, KSU, Giessen, Bonn-Gatchina, ...

Mai, Meißner, Urbach, Phys. Rept. 1001 (2023), Guo et al., Rev. Mod. Phys. 90 (2018), Oller, Prog. Part. Nucl. Phys. 110 (2020), Döring, Haidenbauer, Mai, Sato, 2505.02745

- **Dynamically generated resonances / hadronic molecules** $\leftrightarrow$ hadrons made of other hadrons. Many candidates: deuteron, light scalar mesons, heavy exotic mesons, $\Lambda(1405)$, Roper, ...

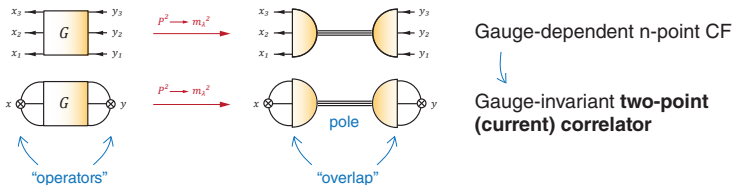
Sometimes argued: if a state (e.g., baryon) is dynamically generated, it cannot be qqq state

- ▶ But: a “state” is a pole in the scattering matrix, the same pole appears in qqq , $qqqq\bar{q}$, ... (unless forbidden by symmetry) $\rightarrow$ dynamically generated states **must** have qqq component
- ▶ A BSE in EFT does not know about quarks, “all it can do” is generate resonances dynamically... need QCD calculations to decide!

Lattice QCD

- Calculate correlation functions directly from path integral:

$$G(x-y) = \langle 0 | T \underbrace{[\Gamma_{\alpha\beta\gamma} \psi_\alpha \psi_\beta \psi_\gamma]}_{B(x)}(x) \underbrace{[\bar{\Gamma}_{\rho\sigma\tau} \bar{\psi}_\rho \bar{\psi}_\sigma \bar{\psi}_\tau]}_{\bar{B}(y)}(y) | 0 \rangle = \int \mathcal{D}[\psi, \bar{\psi}, A] e^{-S[\psi, \bar{\psi}, A]} B(x) \bar{B}(y)$$

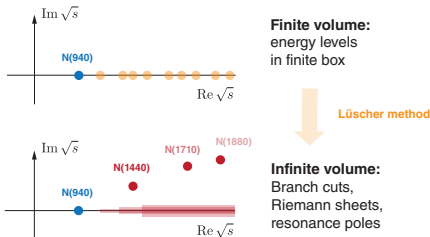


- Nonperturbative, ab-initio.** Only need to supply current-quark masses & scale.
- Generate gauge-field ensembles, “measure” correlation functions on these ensembles, extract physical observables. Challenges: $a > 0$, $L < \infty$, physical pion masses (noisy)
- Extract **stable hadron masses** from correlator at large Euclidean times:

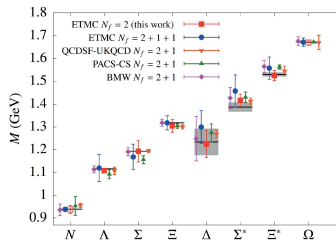
$$G(P) \xrightarrow{P^2 \rightarrow -m_\lambda^2} \frac{R_\lambda}{P^2 + m_\lambda^2} = \frac{R_\lambda}{P_4^2 + E_\lambda^2}, \quad G(\tau, P) \xrightarrow{P^2 \rightarrow -m_\lambda^2} R_\lambda \int \frac{dP_4}{2\pi} \frac{e^{iP_4\tau}}{P_4^2 + E_\lambda^2} = \frac{R_\lambda}{2E_\lambda} e^{-E_\lambda\tau}, \quad E_\lambda^2 = P^2 + m_\lambda^2$$

Lattice QCD

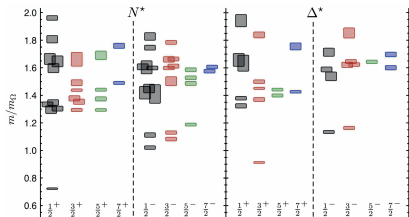
- Stable hadrons (N , Λ , Σ , Ξ , Ω) well established →
- Unstable hadrons** above thresholds more difficult due to finite volume:



- Step 1:** calculate energy levels in finite box, including qqq and multihadron operators (i.e., look for poles in qqq , $qqqq\bar{q}$, ... correlation matrix)
- Many calculations over past 15 years →



Alexandrou, Kallidonis, PRD 96 (2017)



Edwards, Dudek, Richards, Wallace, PRD 84 (2011)

Lattice QCD

• Step 2: apply Lüscher method

Lüscher, Commun. Math. Phys. 105 (1986), Nucl. Phys. B 354 (1991). **Reviews:** Briceño, Dudek, Young, Rev. Mod. Phys. 90 (2018), Hansen, Sharpe, Ann. Rev. Nucl. Part. Sci. 69 (2019), Mai, Döring, Rusetsky, EPJ ST 230 (2021)

Can be formulated with BSE: lattice meets EFT

$$\mathbf{T}^{-1} = \mathbf{K}^{-1} - \mathbf{G}_0,$$

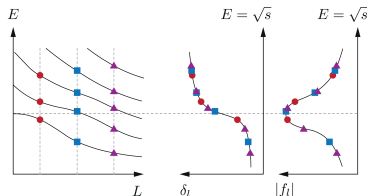
↑
infinite volume:
physical poles

$$\mathbf{T}_L^{-1} = \mathbf{K}_L^{-1} - \mathbf{G}_{0L}$$

↑
finite volume: poles $\Leftrightarrow$
finite-volume energy levels

$$\mathbf{T}^{-1} - \mathbf{T}_L^{-1} = -(\mathbf{G}_0 - \mathbf{G}_{0L}) =: -\mathbf{F} \quad \text{known function}$$

$$\det(\mathbf{T} + \mathbf{F}^{-1}) = 0 \quad \text{at} \quad E = E_i(L) \quad \text{Lüscher condition}$$

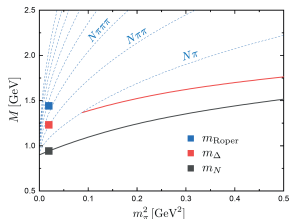


Each $E_i(L)$ maps to a value of $\mathbf{T}$

- Much progress in meson sector. Also generalizations to states above **3-particle thresholds** (3-body BSE!)
- **Excited baryons:** first applications underway, but at low pion masses many thresholds open up $\rightarrow$ difficult

e.g. Roper: no consensus yet, FV calculations differ, $N\pi\pi$ likely important, dynamically generated state?

Lang, Leskovec, Padmanath, Prelovsek, PRD 95 (2017), Liu, IJMPE 26 (2017), Owa, Leinweber, Thomas, PRD 111 (2025)



1. Introduction
2. Spectrum & symmetries
3. Quark models
4. QFT toolbox
- 5. Functional methods**
6. n-point functions & tensor bases
7. Multiquark states
8. Light-front wave functions

Lecture notes “QCD and hadron physics,”
<https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei>

GE, “*Hadron physics with functional methods*,”
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

Burkert, GE, Klempt, “*The impact of γ N and γ^*N interactions on our understanding of nucleon excitations*,” arXiv:2506.16482

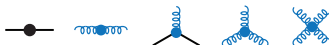
GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
“*Baryons as relativistic three-quark bound states*”,
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

Barabanov, Bedolla, Brooks, Cates, Chen et al.,
“*Diquark correlations in hadron physics: Origin, impact and evidence*”, Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

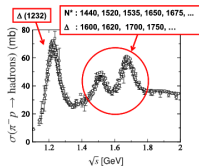
Theory tools

Functional methods

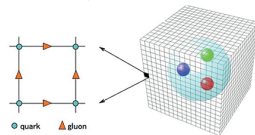
(Dyson-Schwinger & Bethe-Salpeter eqs, FRG, ...)



Amplitude analyses



Lattice QCD



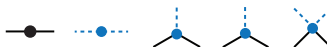
Phenomenological models

(Quark models, ...)



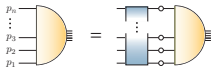
Effective theories

(Chiral perturbation theory, ...)



Functional methods

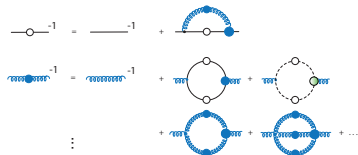
- Hadronic **bound-state equations**
(Bethe-Salpeter & Faddeev eqs)



“QFT analogue of Schrödinger eq.”

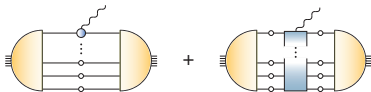
- hadron masses & “wave functions”
- **spectroscopy calculations**

- Ingredients: **QCD’s n-point functions**,
Satisfy quantum eqs. of motion (DSEs)



- Dynamical mass generation,
gluon mass gap, confinement, ...

- **Structure calculations:** form factors, PDFs, GPDs, TMDs,
two-photon processes, ...

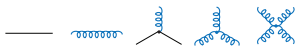


Derivation of DSEs

- **Classical action** $S[\phi]$ $\phi \in \{\psi, \bar{\psi}, A, \dots\}$

Classical eqs. of motion:
(Klein-Gordon, Dirac, Maxwell)

$$\frac{\delta S[\phi]}{\delta \phi(x)} = 0$$



- **QFT: n-point correlation functions**

$$G(x_1, \dots, x_n) = \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle = \frac{\int \mathcal{D}\phi e^{iS[\phi]} \phi(x_1) \dots \phi(x_n)}{\int \mathcal{D}\phi e^{iS[\phi]}}$$



Lattice: calculate them directly from path integral

Functional methods: calculate them *from each other* by relations derived from path integral

- **How? Define partition function:** $Z[J] = \int \mathcal{D}\phi e^{i(S[\phi] - \int d^4x \phi(x) J(x))}$

$$\left. \begin{aligned} \frac{i\delta}{\delta J(x_1)} \left[-i \int d^4x \phi(x) J(x) \right] &= \phi(x_1) \\ \frac{i\delta}{\delta J(x_1)} Z[J] &= \int \mathcal{D}\phi e^{i(\dots)} \phi(x_1), \\ \frac{i\delta}{\delta J(x_1)} \frac{i\delta}{\delta J(x_2)} Z[J] &= \int \mathcal{D}\phi e^{i(\dots)} \phi(x_1) \phi(x_2) \\ f \left[\frac{i\delta}{\delta J} \right] Z[J] &= \int \mathcal{D}\phi e^{i(\dots)} f[\phi] \end{aligned} \right\}$$

CFs are **functional derivatives** of $Z[J]$:

$$G(x_1 \dots x_n) = \frac{i\delta}{\delta J(x_1)} \dots \frac{i\delta}{\delta J(x_n)} \Big|_{J=0} \frac{Z[J]}{Z[0]}$$

From $Z[J]$ one can get any CF; from all CFs one can reconstruct $Z[J]$
⇒ **contain full QFT**

Useful: define CFs also for $J \neq 0$:

$$\langle f[\phi] \rangle_J := \frac{\int \mathcal{D}\phi e^{i(S[\phi] - \int d^4x \phi(x) J(x))} f[\phi]}{\int \mathcal{D}\phi e^{i(S[\phi] - \int d^4x \phi(x) J(x))}} = \frac{1}{Z[J]} f \left[\frac{i\delta}{\delta J} \right] Z[J]$$

Derivation of DSEs

- $Z[J]$ is invariant under shift $\phi'(x) = \phi(x) + \varepsilon(x)$ (= relabeling of fields under integral):

$$\begin{aligned}
 Z[J] &= \int \mathcal{D}\phi' e^{i(S[\phi'] - \int d^4x \phi'(x) J(x))} \\
 &= \int \mathcal{D}\phi e^{i(S[\phi] - \int d^4x \phi(x) J(x)) + i \int d^4x \varepsilon(x) \left(\frac{\delta S[\phi]}{\delta \phi(x)} - J(x) \right)} \\
 &= Z[J] \left\langle e^{i \int d^4x \varepsilon(x) \left(\frac{\delta S[\phi]}{\delta \phi(x)} - J(x) \right)} \right\rangle_J
 \end{aligned}
 \quad \Rightarrow \quad
 \begin{aligned}
 &\left\langle \frac{\delta S[\phi]}{\delta \phi(x)} \right\rangle_J = J(x) \\
 &\Rightarrow \left\langle \frac{\delta S[\phi]}{\delta \phi(x)} \right\rangle = 0
 \end{aligned}$$

DSEs = { classical eqs. of motion }
= quantum eqs. of motion

- To make this more useful, write:

$$\frac{1}{Z[J]} \frac{\delta S}{\delta \phi} \left[\frac{i\delta}{\delta J} \right] Z[J] = J(x) \quad \dots \text{"master DSE"}$$

functional derivatives of $Z[J]$ generates CFs $\Rightarrow$ **DSEs are relations between CFs**
(similar: classical symmetry relations $\rightarrow$ WTIs, STIs)

Useful: define CFs also for $J \neq 0$:

$$\langle f[\phi] \rangle_J := \frac{\int \mathcal{D}\phi e^{i(S[\phi] - \int d^4x \phi(x) J(x))} f[\phi]}{\int \mathcal{D}\phi e^{i(S[\phi] - \int d^4x \phi(x) J(x))}} = \frac{1}{Z[J]} f \left[\frac{i\delta}{\delta J} \right] Z[J]$$

1PI correlation functions

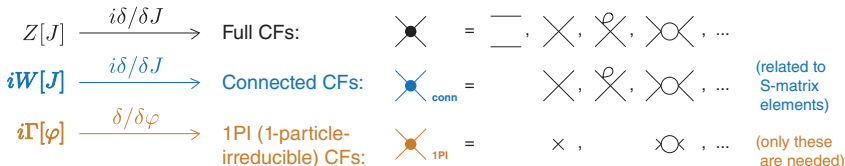
- Define **quantum effective action**:

$$Z[J] = \int \mathcal{D}\phi e^{i(S[\phi] - \int d^4x \phi(x) J(x))} = e^{iW[J]} = e^{i(\Gamma[\varphi] - \int d^4x \varphi(x) J(x))}$$

↑
**Schwinger
functional**

↑
**Effective
action**

(each of these
encodes full QFT)



- $\varphi(x)$ is the **quantum averaged field**:

$$\varphi(x) = -\frac{\delta W[J]}{\delta J(x)} = \frac{1}{Z[J]} \frac{i\delta}{\delta J(x)} Z[J] = \langle \phi(x) \rangle_J \xrightarrow{J=0} 0, \quad J(x) = \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)}$$

- Then the CF for $J \neq 0$ turns into

$$\langle f[\phi] \rangle_J = \frac{\int \mathcal{D}\phi e^{i(S[\phi] - \int d^4x \phi(x) J(x))} f[\phi]}{\int \mathcal{D}\phi e^{i(S[\phi] - \int d^4x \phi(x) J(x))}} = \frac{1}{Z[J]} f\left[\frac{i\delta}{\delta J}\right] Z[J] = \text{after some calculations :)} \quad f\left[\varphi(x) + \int d^4y \overset{\text{full propagator}}{D_{xy}[\varphi]} \frac{\delta}{\delta \varphi(y)}\right] 1$$

1PI correlation functions

- Can be used to relate **full** and **1PI** CFs:

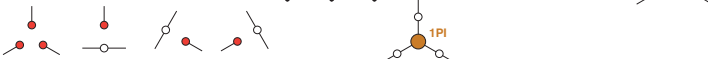
$$\langle f[\phi] \rangle_J = \int \mathcal{D}\varphi \left[\varphi(x) + \int d^4y \overset{\text{full propagator}}{D_{xy}[\varphi]} \frac{\delta}{\delta\varphi(y)} \right] 1$$

$$\langle \phi(x) \rangle_J = \left(\varphi(x) + \int d^4y D_{xy}[\varphi] \frac{\delta}{\delta\varphi(y)} \right) 1 = \varphi(x) \xrightarrow{\varphi=0} 0$$

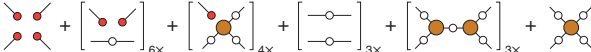
$$\langle \phi(x) \phi(y) \rangle_J = \left(\varphi(x) + \int d^4a D_{xa} \frac{\delta}{\delta\varphi_a} \right) \left(\varphi(y) + \int d^4b D_{yb} \frac{\delta}{\delta\varphi_b} \right) 1 = \varphi(x) \varphi(y) + D_{xy} \xrightarrow{\varphi=0} \text{diagram}$$




$$\begin{aligned} \langle \phi(x) \phi(y) \phi(z) \rangle_J &= \left(\varphi(x) + \int d^4a D_{xa} \frac{\delta}{\delta\varphi_a} \right) (\varphi(y) \varphi(z) + D_{yz}) \\ &= \varphi_x \varphi_y \varphi_z + D_{xy} \varphi_z + D_{xz} \varphi_y + D_{yz} \varphi_x + \int d^4a \int d^4b \int d^4c D_{xa} D_{yb} D_{zc} U_{abc} \xrightarrow{\varphi=0} \text{diagram} \end{aligned}$$



$$\langle \phi(x) \phi(y) \phi(z) \phi(w) \rangle_J = \text{diagram} + \left[\text{diagram} \right]_{6x} + \left[\text{diagram} \right]_{4x} + \left[\text{diagram} \right]_{3x} + \left[\text{diagram} \right]_{3x} + \text{diagram}$$



$$\text{General: } \langle \phi^n \rangle = \left(\varphi + D \frac{\delta}{\delta\varphi} \right) \langle \phi^{n-1} \rangle, \quad \frac{\delta}{\delta\varphi} \underbrace{(-D^{-1}, U, V, \dots)}_{\text{1PI CFs}} = (U, V, \dots)$$

1PI correlation functions

- But can also be read like this:

$$\langle f[\phi] \rangle_J = f \left[\varphi(x) + \int d^4y \overset{\text{full propagator}}{D_{xy}[\varphi]} \frac{\delta}{\delta \varphi(y)} \right] 1$$

classical relation between fields

$$f[\phi] \longrightarrow \langle f[\phi] \rangle_J = f \left(\varphi_x + \int d^4y D_{xy} \frac{\delta}{\delta \varphi_y} \right) = \underbrace{f(\varphi)}_{\text{classical}} + \underbrace{(\dots)}_{\text{quantum}}$$



- Dyson-Schwinger equations:**

“connect the dots” in all possible ways!

$$J(x) = \left\langle \frac{\delta S}{\delta \phi(x)} \right\rangle_J \quad \dots \text{master DSE}$$

↓

$$\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = \frac{\delta S}{\delta \phi} \left[\varphi(x) + \int d^4y D_{xy}[\varphi] \frac{\delta}{\delta \varphi(y)} \right] = \underbrace{\frac{\delta S[\varphi]}{\delta \varphi(x)}}_{\text{classical}} + \underbrace{(\dots)}_{\text{quantum}}$$

↓ functional derivatives

$$\left. \frac{\delta^n \Gamma}{\delta \varphi^n} \right|_{\varphi=0} = \dots \quad \text{DSEs for 1PI CFs}$$

- Graphical rules:**

$$\begin{aligned} \frac{\delta}{\delta \varphi(x)} \text{ (grey semi-circle) } &= \text{ (grey semi-circle) } - x \\ \frac{\delta}{\delta \varphi(x)} \text{ (white circle) } &= \text{ (white circle) } - x \\ \frac{\delta}{\delta \varphi(x)} \text{ (white circle with line) } &= \text{ (white circle with line) } - x \end{aligned}$$

Derivation of DSEs: ϕ^4 theory

- Classical action:

$$S = \int d^4x \left[\frac{1}{2} (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2) - \frac{g}{4!} \phi^4 \right] = \int d^4x \left[-\frac{1}{2} \phi (\square + m^2) \phi - \frac{g}{4!} \phi^4 \right] = \frac{i}{2} \text{---} \overset{-1}{\text{---}} \text{---} - \frac{i}{4!} \text{---} \text{---} \text{---} \text{---}$$

$$\frac{\delta S}{\delta \varphi} = -(\square + m^2) \phi - \frac{g}{3!} \phi^3 = i \text{---} \overset{-1}{\text{---}} \text{---} - \frac{i}{3!} \text{---} \text{---} \text{---} \text{---} \quad (= 0 \Rightarrow \text{classical eqs. of motion})$$

- Quantum eqs. of motion = master DSE: (connect dots in all possible ways)

$$\frac{\delta \Gamma}{\delta \varphi} = i \text{---} \overset{-1}{\text{---}} \text{---} - \frac{i}{3!} \text{---} \text{---} \text{---} \text{---} - \frac{i}{2} \text{---} \text{---} \text{---} \text{---} \text{---} - \frac{i}{3!} \text{---} \text{---} \text{---} \text{---} \text{---}$$

- Another functional derivative gives DSE for full inverse propagator:

$$-i \frac{\delta^2 \Gamma}{\delta \varphi^2} \Big|_{\varphi=0} = \text{---} \overset{-1}{\text{---}} \text{---} = \text{---} \overset{-1}{\text{---}} \text{---} - \frac{1}{2} \text{---} \text{---} \text{---} \text{---} \text{---} - \frac{1}{6} \text{---} \text{---} \text{---} \text{---} \text{---} - \frac{1}{2} \text{---} \text{---} \text{---} \text{---} \text{---}$$

tadpole sunset squint

depends on full
3- and 4-point vertices
→ satisfy own DSEs

- DSE = **nonperturbative integral equation**;
if we knew exact 3- and 4-point vertices, would get exact propagator.
If coupling g is small $\Rightarrow$ recovers perturbation theory

$$D^{-1} = D_0^{-1} - \Sigma \quad \Leftrightarrow \quad D = D_0 + D_0 \Sigma D = D_0 + D_0 \Sigma D_0 + D_0 \Sigma D_0 \Sigma D_0 + \dots$$