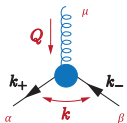


Fermion-vector vertex (7)



The same can be done for **fermion-vector vertex**:

$$\Gamma^\mu(k, Q) = i \left[\underbrace{\sum_{j=1}^4 g_j G_j^\mu}_{\text{Gauge part: constrained by STI}} + \underbrace{\sum_{j=1}^8 f_j T_j^\mu}_{\text{Transverse part, carries dynamics}} \right]$$

Gauge part:
constrained by STI

$$\begin{array}{c} \gamma^\mu \\ k^\mu \not{k} \\ i k^\mu \\ \omega \frac{i}{2} [\gamma^\mu, \not{k}] \end{array}$$

lowest
momentum
powers

Transverse part,
carries dynamics

$$\begin{array}{c} t_{QQ}^{\mu\nu} \gamma^\nu \\ \omega t_{QQ}^{\mu\nu} \frac{i}{2} [\gamma^\nu, \not{k}] \\ \frac{i}{2} [\gamma^\mu, \not{Q}] \\ \frac{1}{6} [\gamma^\mu, \not{k}, \not{Q}] \end{array}$$

$$\begin{array}{c} t_{QQ}^{\mu\nu} i k^\nu \\ t_{QQ}^{\mu\nu} k^\nu \not{k} \\ \omega t_{Qk}^{\mu\nu} \gamma^\nu \\ t_{Qk}^{\mu\nu} \frac{i}{2} [\gamma^\nu, \not{k}] \end{array}$$

$$\begin{aligned} t_{ab}^{\mu\nu} &= a \cdot b \delta^{\mu\nu} - b^\mu a^\nu \\ &\Rightarrow a^\mu t_{ab}^{\mu\nu} = 0 \\ &\quad t_{ab}^{\mu\nu} b^\nu = 0 \end{aligned}$$

For quark-photon vertex: **Ball-Chiu vertex**

Ball, Chiu, PRD 22 (1980)

$$g_1 = \Sigma_A \quad g_2 = 2\Delta_A \quad g_3 = -2\Delta_B \quad g_4 = 0$$

$$\left\{ \begin{array}{l} \Sigma_A = \frac{A(k_+) + A(k_-)}{2} \\ \Delta_A = \frac{A(k_+) - A(k_-)}{k_+^2 - k_-^2} \\ B(k^2) = A(k^2) M(k^2) \end{array} \right.$$

Everything else is **dynamics**! Quark-photon vertex can be calculated from BSE, f_j must have physical vector-meson poles at timelike momenta

Maris, Tandy, PRC 61 (2000)

Fermion-vector vertex (8)

We have exploited all symmetries of the system:

- Permutation-group symmetries (here: charge conjugation)
- Gauge invariance

Procedure is general:

gluon self-energy,
hadronic vacuum
polarization



$$1 + 1$$

fermion-vector:
quark-gluon vertex,
quark-photon vertex



$$4 + 8$$

fermion-2-vector:
nucleon Compton
amplitude



$$14 + 18$$

3-gluon vertex



$$10 + 4$$

 Gauge + Transverse

Bardeen, Tung, Phys. Rev. 173 (1968)
Tarrach, Nuovo Cim. A28 (1975)
Ball, Chiu, PRD 22 (1980)
Drechsel et al., PRC 55 (1997)
GE, Fischer, PRD 87 (2013)
GE, Ramalho, PRD 98 (2018)

Everything else is **dynamics**!

1. Example: $2 \rightarrow 2$ scattering
2. Euclidean metric
3. QCD's n-point functions
- 4. Tensor bases**
5. Kinematics

Tensor bases

What is the **number of tensors** for a general n -point function?

[GE, Fischer, PRD 87 \(2013\)](#)

[GE, Fischer, Heupel, PRD 92 \(2015\)](#)

[Wallbott, GE, Fischer, PRD 100 \(2019\)](#)

$$\Gamma(p, q, k, l, \dots) \rightarrow p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bullet \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ \bullet \\ \bullet \end{bmatrix}, \quad k = \begin{bmatrix} 0 \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}, \quad l = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}, \quad \dots$$

The diagram illustrates the nesting of tensor bases for $n=3, 4, 5$. It shows three levels of complexity, each represented by a blue symbol and a bracket indicating its inclusion in the previous level:

- $n=3$: A blue dot with three lines.
- $n=4$: A blue dot with four lines.
- $n=5$: A blue star with five lines.

Tensor bases

What is the **number of tensors** for a general n-point function?

GE, Fischer, PRD 87 (2013)
GE, Fischer, Heupel, PRD 92 (2015)
Wallbott, GE, Fischer, PRD 100 (2019)

$$\Gamma(p, q, k, l, \dots) \rightarrow p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bullet \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ \bullet \\ \bullet \end{bmatrix}, \quad k = \begin{bmatrix} 0 \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}, \quad l = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}, \quad \dots$$

These are all Lorentz-covariant definitions, in principle we never need to go into a frame!

Construct orthonormal **unit vectors**:

$$\mathbf{n}_4 = \hat{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{n}_3 = \widehat{q_{\perp 4}} = \text{Hat}[q - (q \cdot n_4) n_4] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{n}_2 = \widehat{k_{\perp 4, \perp 3}} = \text{Hat}[k - (k \cdot n_4) n_4 - (k \cdot n_3) n_3] = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{n}_1 = \widehat{l_{\perp 4, \perp 3, \perp 2}} = \text{Hat}[\dots] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

... but then it stops: we are limited by **four dimensions**.

⇒ For arbitrary n-point functions, still at most **4 momenta** as building blocks!

Tensor bases

Three-point functions:

two vectors n_3, n_4



$$\Gamma(p, q) \rightarrow p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bullet \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ \bullet \\ \bullet \end{bmatrix}$$

- Fermion-scalar vertex:



$$\{1, \not{n}_3, \not{n}_4, \not{n}_3 \not{n}_4\} \Rightarrow 4 \text{ tensors}$$

Same for fermion-pseudoscalar vertex (e.g., pion Bethe-Salpeter amplitude):

$$\{1, \not{n}_3, \not{n}_4, \not{n}_3 \not{n}_4\} \gamma_5$$

- Fermion-vector vertex:



$$\{\gamma^\mu, n_3^\mu, n_4^\mu\} \times \{1, \not{n}_3, \not{n}_4, \not{n}_3 \not{n}_4\} \Rightarrow 12 \text{ tensors}$$

- Three-gluon vertex:



$$\underbrace{n_i^\mu n_j^\nu n_k^\rho}_{2^3=8}, \quad \underbrace{\delta^{\mu\nu} n_i^\rho}_2, \quad \underbrace{\delta^{\nu\rho} n_i^\mu}_2, \quad \underbrace{\delta^{\rho\mu} n_i^\nu}_2 \Rightarrow 14 \text{ tensors}$$

Tensor bases

Four-point functions:

three vectors n_2, n_3, n_4



But with 3 vectors we can also do this:

$$v^\mu = \varepsilon^{\mu\alpha\beta\gamma} n_1^\alpha n_2^\beta n_3^\gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

this has opposite parity
 $\Rightarrow$ we will need even numbers of v 's

$$\left\{ \begin{array}{l} \delta^{\mu\nu} = v^\mu v^\nu + \sum_{i=1}^3 n_i^\mu n_i^\nu \\ \gamma^\mu = v^\mu \not{v} + \sum_{i=1}^3 n_i^\mu \not{n}_i \\ \not{v} = \gamma_5 \not{n}_2 \not{n}_3 \not{n}_4 \end{array} \right.$$

These relations hold in simple frame, but **Lorentz-covariant**
 $\Rightarrow$ they are true in any frame!

$\Rightarrow$ we no longer need $\delta^{\mu\nu}$ and γ^μ
 in the basis construction,
 the n_i^μ and v^μ are enough!

Tensor bases

Four-point functions:

three vectors n_2, n_3, n_4



But with 3 vectors we can also do this:

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$$\begin{cases} \delta^{\mu\nu} = v^\mu v^\nu + \sum_{i=1}^3 n_i^\mu n_i^\nu \\ \gamma^\mu = v^\mu \not{v} + \sum_{i=1}^3 n_i^\mu \not{n}_i \\ \not{v} = \gamma_5 \not{v} \not{n}_2 \not{n}_3 \not{n}_4 \end{cases}$$

These relations hold in simple frame, but **Lorentz-covariant**
 $\Rightarrow$ they are true in any frame!

$\Rightarrow$ we no longer need $\delta^{\mu\nu}$ and γ^μ
 in the basis construction,
 the n_i^μ and v^μ are enough!

- Fermion-2-scalar vertex:



$$\Omega = \{ \underbrace{1}_1, \underbrace{\not{n}_i}_3, \underbrace{\not{n}_i \not{n}_j}_3, \underbrace{\not{n}_i \not{n}_j \not{n}_k}_1 \}, \quad i < j < k \Rightarrow \text{8 tensors}$$

$\swarrow = \gamma_5 \not{v}$

- Fermion 4-point function:



$$\underbrace{\Omega_{\alpha\beta} \Omega_{\gamma\delta}}_{8 \times 8 = 64}, \quad \underbrace{(\gamma_5 \Omega)_{\alpha\beta} (\gamma_5 \Omega)_{\gamma\delta}}_{8 \times 8 = 64} \Rightarrow \text{128 tensors}$$

For quarks
 there are also
 2 color tensors:
 256 in total

Tensor bases

Four-point functions:

three vectors n_2, n_3, n_4



But with 3 vectors we can also do this:

$$v^\mu = \epsilon^{\mu\alpha\beta\gamma} n_1^\alpha n_2^\beta n_3^\gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

this has opposite parity
 $\Rightarrow$ we will need even numbers of v 's

$$\begin{cases} \delta^{\mu\nu} = v^\mu v^\nu + \sum_{i=1}^3 n_i^\mu n_i^\nu \\ \gamma^\mu = v^\mu \not{v} + \sum_{i=1}^3 n_i^\mu \not{n}_i \\ \not{v} = \gamma_5 \not{n}_2 \not{n}_3 \not{n}_4 \end{cases}$$

These relations hold in simple frame, but **Lorentz-covariant**
 $\Rightarrow$ they are true in any frame!

$\Rightarrow$ we no longer need $\delta^{\mu\nu}$ and γ^μ
 in the basis construction,
 the n_i^μ and v^μ are enough!

- Fermion-2-vector vertex:



$$\underbrace{\{v^\mu v^\nu\}}_1, \underbrace{\{n_i^\mu n_j^\nu\}}_9 \times \underbrace{\Omega}_8, \quad \underbrace{\{v^\mu n_i^\nu\}}_3, \underbrace{\{n_i^\mu v^\nu\}}_3 \times \underbrace{\gamma_5 \Omega}_8 \quad \Rightarrow \quad \mathbf{128 \text{ tensors}}$$

- 4-gluon vertex:



$$\underbrace{v^\mu v^\nu v^\rho v^\sigma}_1, \quad \underbrace{v^\mu v^\nu n_i^\rho n_j^\sigma}_{3^2=9, \text{ 6 permutations}}, \quad \underbrace{n_i^\mu n_j^\nu n_k^\rho n_l^\sigma}_{3^4=81} \quad \Rightarrow \quad \mathbf{136 \text{ tensors}} \quad (\times \text{ color})$$

Tensor bases

> Four-point functions: four vectors n_1, n_2, n_3, n_4

$$\Rightarrow \left\{ \begin{array}{l} \delta^{\mu\nu} = \sum_{i=1}^4 n_i^\mu n_i^\nu \\ \gamma^\mu = \sum_{i=1}^4 n_i^\mu \not{n}_i \\ \not{n}_1 \not{n}_2 \not{n}_3 \not{n}_4 = -\varepsilon \gamma_5 \end{array} \right. \quad \varepsilon = \varepsilon^{\mu\nu\rho\sigma} n_1^\mu n_2^\nu n_3^\rho n_4^\sigma \quad \Rightarrow \gamma_5 \text{ becomes redundant too}$$

$$\Rightarrow \Omega = \{ \underbrace{1}_1, \underbrace{\not{n}_i}_4, \underbrace{\not{n}_i \not{n}_j}_6, \underbrace{\not{n}_i \not{n}_j \not{n}_k}_4, \underbrace{\not{n}_i \not{n}_j \not{n}_k \not{n}_l}_1 \}, \quad i < j < k < l \quad \Rightarrow \text{16 tensors}$$

- Fermion 6-point function:



$$\underbrace{\Omega_{\alpha\beta} \Omega_{\gamma\delta} \Omega_{\lambda\tau}}_{16^3} \Rightarrow 4096 \text{ tensors} \quad (\times \text{ color})$$

1. Example: $2 \rightarrow 2$ scattering
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- 5. Kinematics**

Lorentz invariants

$$G_{\alpha\beta\ldots}^{\mu\nu\ldots}(p_1, \ldots, p_n) = \sum_{i=1}^N f_i(p_1^2, p_2^2, \ldots) \tau_i(p_1, \ldots, p_n)_{\alpha\beta\ldots}^{\mu\nu\ldots}$$


What is the **number of Lorentz invariants** for a general n-point function?

Lorentz invariants

$$G_{\alpha\beta\dots}^{\mu\nu\dots}(p_1, \dots, p_n) = \sum_{i=1}^N f_i(p_1^2, p_2^2, \dots) \tau_i(p_1, \dots, p_n)_{\alpha\beta\dots}$$

What is the **number of Lorentz invariants** for a general n-point function?
It's the number of entries in the vectors:

$$\Gamma(p, q, k, l, \dots) \rightarrow p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}, \quad k = \begin{bmatrix} 0 \\ \vdots \end{bmatrix}, \quad l = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}, \quad \dots$$

n	# Lorentz invariants	Examples
2	1	
3	3	
4	6	
5	10	
6	14	
7	18	
$n \geq 4$	$4n - 10$	

apart from possible
onshell constraints:
 $p^2 = -m^2$, etc.

Lorentz invariants

$$G_{\alpha\beta\ldots}^{\mu\nu\ldots}(p_1, \ldots, p_n) = \sum_{i=1}^N f_i(p_1^2, p_2^2, \ldots) \tau_i(p_1, \ldots, p_n)_{\alpha\beta\ldots}^{\mu\nu\ldots}$$

What is the **number of Lorentz invariants** for a general n-point function?
It's the number of entries in the vectors:

$$\Gamma(p, q, k, l, \ldots) \rightarrow p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \end{bmatrix}, \quad k = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad l = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad \ldots$$

n	# Lorentz invariants	Examples
2	1	
3	3	
4	6	
5	10	
6	14	
7	18	
$n \geq 4$	$4n - 10$	

apart from possible
onshell constraints:
 $p^2 = -m^2$, etc.

$$\begin{array}{l} p_1^2, \quad p_2^2, \quad p_3^2, \quad p_4^2, \quad p_5^2, \quad p_6^2 \\ p_1 \cdot p_2, \quad p_1 \cdot p_3, \quad p_1 \cdot p_4, \quad p_1 \cdot p_5, \quad p_1 \cdot p_6 \\ p_2 \cdot p_3, \quad p_2 \cdot p_4, \quad p_2 \cdot p_5, \quad p_2 \cdot p_6 \\ p_3 \cdot p_4, \quad p_3 \cdot p_5, \quad p_3 \cdot p_6 \\ p_4 \cdot p_5, \quad p_4 \cdot p_6 \\ p_5 \cdot p_6 \end{array}$$

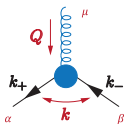
... 6
... 5
... 4
... 3
... 2
... 1
= 21, not 18...
even the
Lorentz-invariants
become redundant!

Lorentz invariants

$$G_{\alpha\beta\ldots}^{\mu\nu\ldots}(p_1, \ldots, p_n) = \sum_{i=1}^N f_i(p_1^2, p_2^2, \ldots) \tau_i(p_1, \ldots, p_n)_{\alpha\beta\ldots}^{\mu\nu\ldots}$$

What is the most **convenient** set of Lorentz invariants?

E.g. fermion vector vertex: kinematics invariant under $k_+ \leftrightarrow -k_-$



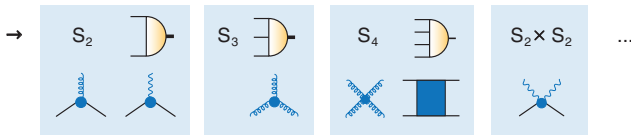
$$k = \frac{k_+ + k_-}{2} \rightarrow \text{antisymmetric}$$

$$Q = k_+ - k_- \rightarrow \text{symmetric}$$

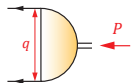
$\Rightarrow k^2, Q^2$ are symmetric, $k \cdot Q$ is antisymmetric

$\Rightarrow$ These variables are **multiplets** under permutation group S_2

Use permutation-group symmetries!



Example: diquark amplitude



$$\Gamma(p, P) = \underbrace{\sum_{i=1}^4 f_i(q^2, q \cdot P, P^2) \tau_i(q, P)}_{\substack{\mathcal{A} \\ S}} \otimes \underbrace{\text{Flavor}}_{\substack{\mathcal{A} : ud - du \\ S : \begin{matrix} uu \\ ud + du \\ dd \end{matrix}}} \otimes \underbrace{\text{Color}}_{\mathcal{A} : \varepsilon_{ABC}} \stackrel{!}{=} \mathcal{A}$$

Pauli principle

Scalar diquarks: $\tau_i = \{1, \dots\} \gamma_5 C \quad \dots \mathcal{A}$

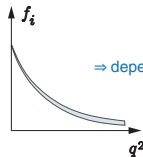
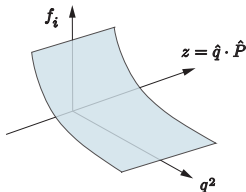
Axialvector diquarks: $\tau_i^\mu = \{\gamma^\mu, \dots\} C \quad \dots \mathcal{S}$

$$(\gamma_5 C)^T = C^T \gamma_5^T = -\gamma_5 C$$

$$(\gamma_\mu C)^T = C^T \gamma_\mu^T = +\gamma_\mu C$$

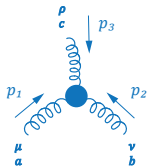
$\Rightarrow$ Scalar diquarks have **isospin I = 0**, axialvector diquarks **isospin I = 1**

can construct remaining tensors to have definite symmetry (commutators, factors $q \cdot P$), such that f_i are singlets



$\Rightarrow$ depend mainly on q^2 !

Three-gluon vertex



$$\Gamma_{3g}^{\mu\nu\rho}(p_1, p_2, p_3) = ig \underbrace{\sum_{i=1}^{14} f_i(p_1^2, p_2^2, p_3^2)}_{\mathcal{A}} \underbrace{\tau_i^{\mu\nu\rho}(p_1, p_2, p_3)}_{\leftarrow} \underbrace{f_{abc}}_{\mathcal{A}} \stackrel{!}{=} \mathcal{S}$$

Bose symmetry

How can we make the symmetry manifest?

- (1) Arrange kinematic variables in multiplets of permutation group S_3
- (2) Arrange tensors in multiplets of permutation group S_3 : then all tensors & dressing functions have definite symmetry
- (3) Make tensors fully antisymmetric $\Rightarrow$ then all dressing functions are fully symmetric

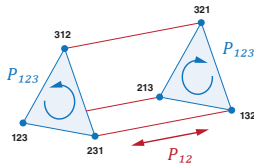
This requires some algebra... let's focus on (1)

GE, Williams, Alkofer, Vujanovic, PRD 89 (2014)

Permutation group S_3

Permutation group S_3 consists of 6 group elements, can be reconstructed from **transposition** P_{12} ($1 \leftrightarrow 2$) and **cyclic permutation** P_{123} ($1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1$):

$$\begin{aligned} 1 \psi_{123} &= \psi_{123} & P_{12} \psi_{123} &= \psi_{213} \\ P_{123} \psi_{123} &= \psi_{231} & P_{12} P_{123} \psi_{123} &= \psi_{132} \\ P_{123}^2 \psi_{123} &= \psi_{312} & P_{12} P_{123}^2 \psi_{123} &= \psi_{321} \end{aligned}$$



Cayley graph

Find combinations transforming under **irreps** of S_3 :

- Singlet** $\mathcal{S}$ is invariant under any permutation:

$$\begin{aligned} P_{12} \mathcal{S} &= \mathcal{S} \\ P_{123} \mathcal{S} &= \mathcal{S} \end{aligned} \quad \mathcal{S} = \psi_{123} + \psi_{231} + \psi_{312} + \psi_{213} + \psi_{132} + \psi_{321}$$

- Antisingleton** $\mathcal{A}$ is antisymmetric under any transposition $i \leftrightarrow j$:

$$\begin{aligned} P_{12} \mathcal{A} &= -\mathcal{A} \\ P_{123} \mathcal{A} &= \mathcal{A} \end{aligned} \quad \mathcal{A} = \psi_{123} + \psi_{231} + \psi_{312} - \psi_{213} - \psi_{132} - \psi_{321}$$

- Doublets** $\mathcal{D}_1, \mathcal{D}_2$ form 2-dim subspace:

$$\begin{aligned} P_{12} \mathcal{D}_j &= \mathbf{M}_{12}^T \mathcal{D}_j & \mathbf{M}_{12} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ P_{123} \mathcal{D}_j &= \mathbf{M}_{123}^T \mathcal{D}_j & \mathbf{M}_{123} &= \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathcal{D}_1 &= \begin{bmatrix} (\psi_{231} - \psi_{132}) - (\psi_{312} - \psi_{321}) \\ \frac{1}{\sqrt{3}} ((\psi_{231} + \psi_{132}) + (\psi_{312} + \psi_{321}) - 2(\psi_{123} + \psi_{213})) \end{bmatrix} \\ \mathcal{D}_2 &= \begin{bmatrix} -\frac{1}{\sqrt{3}} ((\psi_{231} - \psi_{132}) + (\psi_{312} - \psi_{321}) - 2(\psi_{123} - \psi_{213})) \\ (\psi_{231} + \psi_{132}) - (\psi_{312} + \psi_{321}) \end{bmatrix} \end{aligned}$$

Permutation group S_3

Let's abbreviate $x_1 = p_1^2$, $x_2 = p_2^2$, $x_3 = p_3^2$, start with $\psi_{123} = x_3$:

$$\begin{array}{ll} \psi_{123} = x_3 & \psi_{213} = x_3 \\ \psi_{231} = x_1 & \psi_{132} = x_2 \\ \psi_{312} = x_2 & \psi_{321} = x_1 \end{array} \quad \Rightarrow \text{only } \mathcal{S} \text{ and } \mathcal{D}_1 \text{ survive, while } \mathcal{A} = 0 \text{ and } \mathcal{D}_2 = 0:$$

three variables!

$$\Rightarrow \text{Singlet: } \mathcal{S}_0 = \frac{x_1 + x_2 + x_3}{6}$$

$$\text{Doublet: } \mathcal{D} = \begin{bmatrix} a \\ s \end{bmatrix}, \quad a = \sqrt{3} \frac{x_1 - x_2}{x_1 + x_2 + x_3}, \quad s = \frac{x_1 + x_2 - 2x_3}{x_1 + x_2 + x_3}$$

$$\mathcal{S} = \psi_{123} + \psi_{231} + \psi_{312} + \psi_{213} + \psi_{132} + \psi_{321}$$

$$\mathcal{A} = \psi_{123} + \psi_{231} + \psi_{312} - \psi_{213} - \psi_{132} - \psi_{321}$$

$$\mathcal{D}_1 = \begin{bmatrix} (\psi_{231} - \psi_{132}) - (\psi_{312} - \psi_{321}) \\ \frac{1}{\sqrt{3}} ((\psi_{231} + \psi_{132}) + (\psi_{312} + \psi_{321}) - 2(\psi_{123} + \psi_{213})) \end{bmatrix}$$

$$\mathcal{D}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}} ((\psi_{231} - \psi_{132}) + (\psi_{312} - \psi_{321}) - 2(\psi_{123} - \psi_{213})) \\ (\psi_{231} + \psi_{132}) - (\psi_{312} + \psi_{321}) \end{bmatrix}$$

Three-gluon vertex

Let's abbreviate $x_1 = p_1^2$, $x_2 = p_2^2$, $x_3 = p_3^2$, start with $\psi_{123} = x_3$:

$$\psi_{123} = x_3$$

$$\psi_{213} = x_3$$

$$\psi_{231} = x_1$$

$$\psi_{132} = x_2$$

$$\psi_{312} = x_2$$

$$\psi_{321} = x_1$$

$\Rightarrow$ only $\mathcal{S}$ and $\mathcal{D}_1$ survive, whereas $\mathcal{A} = 0$ and $\mathcal{D}_2 = 0$:
three variables!

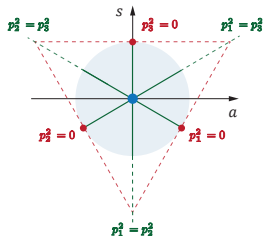
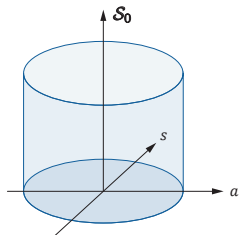
$$\Rightarrow \text{Singlet: } \mathcal{S}_0 = \frac{x_1 + x_2 + x_3}{6}$$

$$\text{Doublet: } \mathcal{D} = \begin{bmatrix} a \\ s \end{bmatrix}, \quad a = \sqrt{3} \frac{x_1 - x_2}{x_1 + x_2 + x_3}, \quad s = \frac{x_1 + x_2 - 2x_3}{x_1 + x_2 + x_3}$$

For spacelike momenta
(like in loop integrations),
the kinematic phase space
is a **cylinder**:

$$\mathcal{S}_0 > 0$$

$$a^2 + s^2 < 1$$



Three-gluon vertex

Singlet $\mathcal{S}_0$, Doublet $\mathcal{D} = \begin{bmatrix} a \\ s \end{bmatrix}$, $a = r \sin \varphi$
 $s = -r \cos \varphi$

If tensor basis is arranged in antisinglets,
 dressing functions must be fully symmetric
 $\Rightarrow$ can only depend on **symmetric variables**:

$$\mathcal{S}_0 > 0$$

$$\mathcal{S}_1 = a^2 + s^2 = r^2 < 1$$

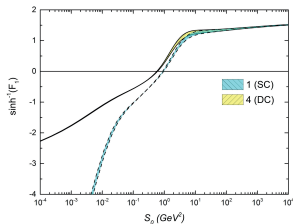
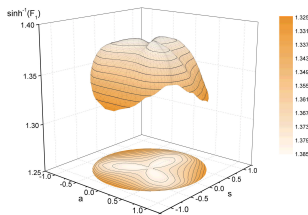
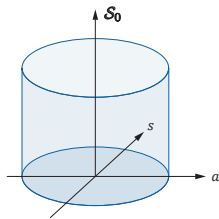
$$\mathcal{S}_2 = 3a^2s - s^3 = r^3 \cos 3\varphi \in [-1, 1]$$

In practice, angular dependence is very weak.
 Here: leading dressing function of three-gluon vertex
 GE, Williams, Alkofer, Vujanovic, PRD 89 (2014)

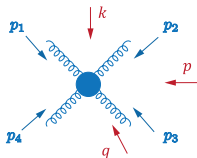
“Planar degeneracy”

Ferreira, Papavassiliou, Particles 6 (2023),

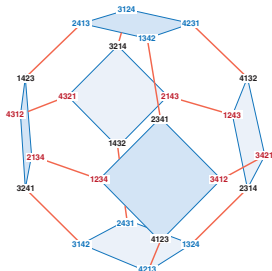
Aguilar, Ferreira, Papavassiliou, Santos, EPJ C 83 (2023)



Four-point functions?



Permutation group S_4 has 24 permutations:

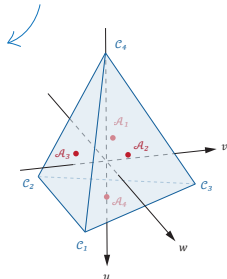
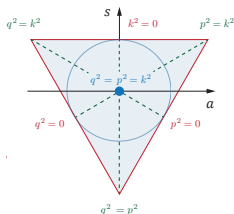


6 Lorentz invariants – which ones?

- $p_1^2, p_2^2, p_3^2, p_4^2, p_1 \cdot p_2, p_3 \cdot p_4$? ... not efficient
- $p^2, q^2, k^2, p \cdot q, q \cdot k, k \cdot p$? ... better, but still no definite symmetry

6 variables can be arranged into singlet $\mathcal{S}_0$, doublet $\mathcal{D}$ and triplet $\mathcal{T}$:
GE, Fischer, Heupel, PRD 92 (2015)

If all external legs are onshell, the doublet becomes the **Mandelstam plane** from earlier



Dynamical calculations: Huber, PRD 101 (2020),
Aguilar, Ferreira, Papavassiliou, Santos, EPJ C 84 (2024)

Summary

$$G_{\alpha\beta\dots}^{\mu\nu\dots}(p_1,\dots p_n) = \sum_{i=1}^N f_i(p_1^2, p_2^2, \dots) \tau_i(p_1,\dots p_n)_{\alpha\beta\dots}^{\mu\nu\dots}$$



Physics



Don't be afraid of Dirac and Lorentz indices!

- Symmetries help
- They make your life easier
- They make your code faster
- They make you smarter
- They are **fun** to play with

1. Introduction
2. Spectrum & symmetries
3. Quark models
4. QFT toolbox
5. Functional methods
6. n-point functions & tensor bases
- 7. Multiquark states**
8. Light-front wave functions

Lecture notes “QCD and hadron physics,”
<https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei>

GE, “*Hadron physics with functional methods*,”
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

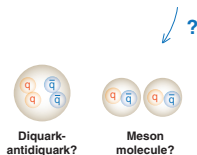
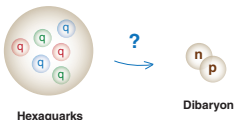
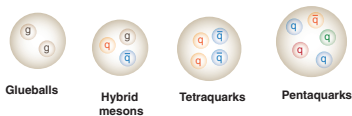
Burkert, GE, Klempt, “*The impact of γN and $\gamma^* N$ interactions on our understanding of nucleon excitations*,” arXiv:2506.16482

GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
“*Baryons as relativistic three-quark bound states*”,
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

Barabanov, Bedolla, Brooks, Cates, Chen et al.,
“*Diquark correlations in hadron physics: Origin, impact and evidence*”, Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

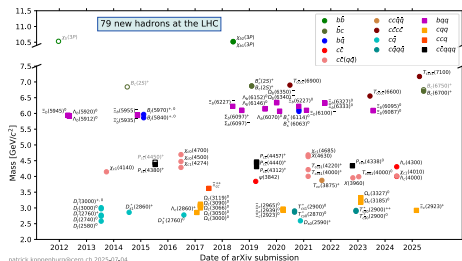
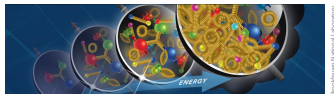
Motivation

• Understanding exotic hadrons



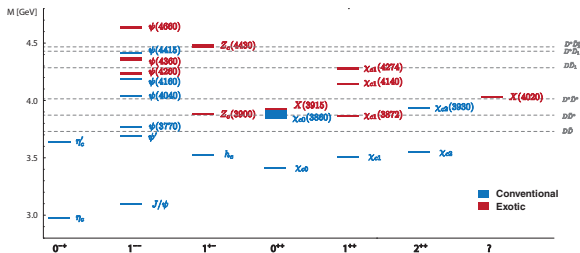
• Quark-gluon structure of hadrons & nuclei:

Hadron tomography at [EIC](#), [JLab](#), [AMBER](#), ...



<https://www.nikhef.nl/~pkoppenb/particles.html>
<https://qwg.ph.nat.tum.de/exoticshub>

Exotic mesons



Reviews:

Chen, Chen, Liu, Zhu,
Phys. Rept. 639 (2016), 1601.02092

Lebed, Mitchell, Swanson
PPNP 93 (2017), 1610.04528

Esposito, Pilloni, Polosa,
Phys. Rept. 668 (2017), 1611.07920

Guo, Hanhart, Meißner et al.,
Rev. Mod. Phys. 90 (2018), 1705.00141

Ali, Lange, Stone,
PPNP 97 (2017), 1706.00610

Olsen, Skwarnicki, Zieminska,
Rev. Mod. Phys. 90 (2019), 1708.04012

Liu, Chen, Chen, Liu, Zhu,
PPNP 107 (2019), 1903.11976

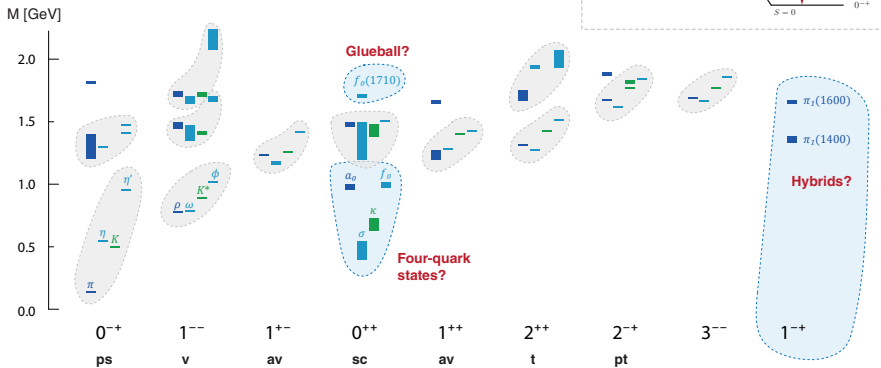
Brambilla, Eidelman, Hanhart et al.,
Phys. Rept. 873 (2020)

...

- Several tetraquark candidates in **charmonium spectrum**: $X(3872)$, $X(3915)$, $Z_c(3900)$,
- Z states cannot be $c\bar{c}$ since they carry charge
- Recent additions: all-charm $X(6900)$, open-charm T_{cc}^+ , ...
- Oldest tetraquark candidates: **light scalar mesons**

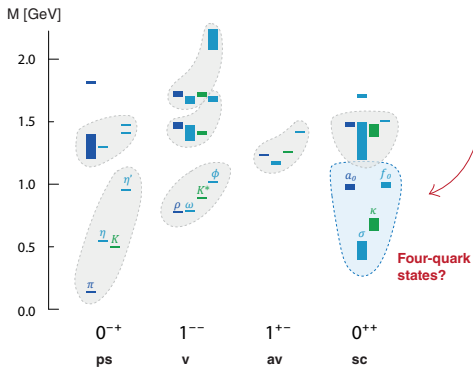
Light exotic mesons

Light meson spectrum
(PDG 2020)



Light exotic mesons

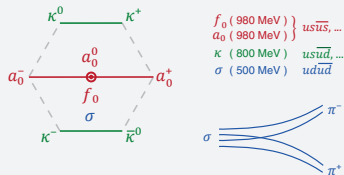
Light meson spectrum (PDG 2020)



• Diquark-antidiquark?

Explains mass ordering & decay widths

Jaffe 1977, Close, Tornqvist 2002,
Maiani, Polosa, Riquer 2004



• Meson molecules?

Weinstein, Isgur 1982, 1990; Close, Isgur, Kumano 1993

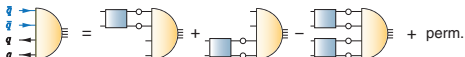
• Non- $q\bar{q}$ nature supported by various approaches

Pelaez, Phys. Rept. 658 (2016)

Four-quark states

- Light scalar mesons (σ , κ , a_0 , f_0) as **four-quark states**:

GE, Fischer, Heupel, PLB 753 (2016)



$$\Gamma(p, q, k, P) = \sum_i f_i(p^2, q^2, k^2, \{\omega_j\}, \{\eta_j\}) \tau_i(p, q, k, P) \otimes \text{Color} \otimes \text{Flavor}$$

9 Lorentz invariants:

$$p^2, \quad q^2, \quad k^2, \quad P^2 = -M^2$$

$$\omega_1 = q \cdot k \quad \eta_1 = p \cdot P$$

$$\omega_2 = p \cdot k \quad \eta_2 = q \cdot P$$

$$\omega_3 = p \cdot q \quad \eta_3 = k \cdot P$$

256 Dirac-Lorentz tensors

2 Color tensors:

$$3 \otimes \bar{3}, \quad 6 \otimes \bar{6} \quad \text{or}$$

$$1 \otimes 1, \quad 8 \otimes 8$$

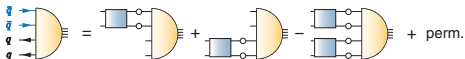
(Fierz-equivalent)

		dim K	memory
$K \psi_i = \lambda_i \psi_i$	Mesons	10^3	20 MB
	Baryons	10^8	10^7 GB
	Tetraquarks	10^{13}	10^{18} GB

Four-quark states

- Light scalar mesons (σ , κ , a_0 , f_0) as **four-quark states**:

GE, Fischer, Heupel, PLB 753 (2016)



$$\Gamma(p, q, k, P) = \sum_i f_i(p^2, q^2, k^2, \{\omega_j\}, \{\eta_j\}) \tau_i(p, q, k, P) \otimes \text{Color} \otimes \text{Flavor}$$

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$$\omega_2 = p \cdot k \quad \eta_2 = q \cdot P$$

$$\omega_3 = p \cdot q \quad \eta_3 = k \cdot P$$

**256 Dirac-
Lorentz tensors**

**2 Color
tensors:**

$$3 \otimes \bar{3}, \quad 6 \otimes \bar{6} \quad \text{or}$$

$$1 \otimes 1, \quad 8 \otimes 8$$

(Fierz-equivalent)

- Group momentum variables into multiplets of **permutation group S4**:
can switch off groups of variables without destroying symmetries

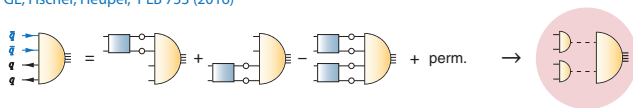
GE, Fischer, Heupel, PRD 92 (2015)

$$f_i(\mathcal{S}_0, \nabla, \triangle, \circ)$$

Four-quark states

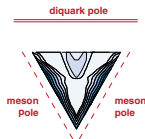
- Light scalar mesons (σ , κ , a_0 , f_0) as **four-quark states**:

GE, Fischer, Heupel, PLB 753 (2016)

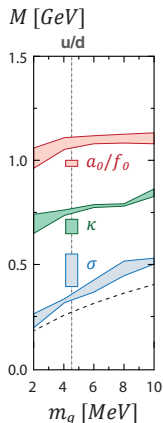


- BSE dynamically generates **meson poles** in BS amplitude:

$$\begin{aligned}
 f_i(\mathcal{S}_0, \nabla, \triangle, \circ) &\rightarrow 1500 \text{ MeV} \\
 f_i(\mathcal{S}_0, \nabla, \triangle, \circ) &\rightarrow 1500 \text{ MeV} \\
 f_i(\mathcal{S}_0, \nabla, \triangle, \circ) &\rightarrow 1200 \text{ MeV} \\
 f_i(\mathcal{S}_0, \nabla, \triangle, \circ) &\rightarrow \mathbf{350 \text{ MeV} !}
 \end{aligned}$$



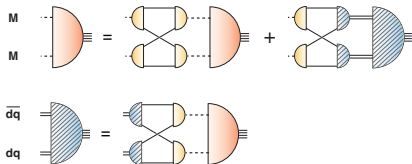
- “Light scalar mesons” look like **meson molecules**, diquark-antidiquark components almost negligible. Lightness is inherited from pseudoscalar Goldstone bosons!



Four-quark states

Two-body formulation: **meson-meson / diquark-antidiquark**, follows from four-quark eq. (analogue of quark-diquark for baryons)

[Heupel, GE, Fischer, PLB 718 \(2012\)](#)

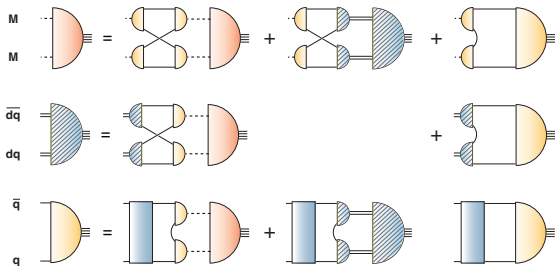


- Interaction by **quark exchange**
- System 'wants' to be **meson-meson-like** (no diagonal $dq\text{-}\overline{dq}$ term)
- Similar results as in 4-quark approach: $m_\sigma \sim 400$ MeV, etc.

Four-quark states

Two-body formulation: **meson-meson / diquark-antidiquark**, follows from four-quark eq. (analogue of quark-diquark for baryons)

Heupel, GE, Fischer, PLB 718 (2012)



Include **mixing with $q\bar{q}$** :
 $\pi\pi$ still dominant

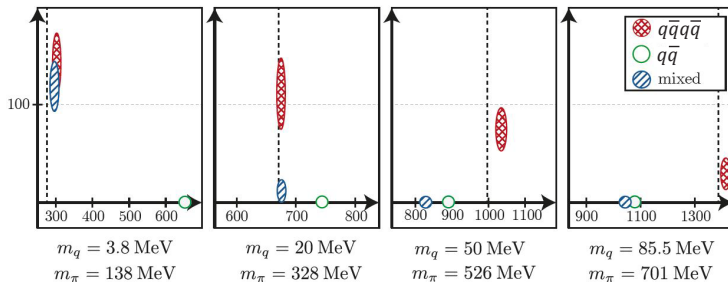
Santowsky, GE, Fischer, Wallbott,
 Williams, PRD 102 (2020)

[MeV]	ground state mass	first excitation
$\pi\pi$	416 ± 26	970 ± 130
$\pi\pi + 0^+0^+$	416 ± 26	970 ± 130
$q\bar{q}$	667 ± 2	1036 ± 8
$\pi\pi + q\bar{q}$	472 ± 22	1080 ± 280
$\pi\pi + 0^+0^+ + q\bar{q}$	456 ± 24	1110 ± 110

Four-quark states

Four-quark vs. $q\bar{q}$ dominance

Santowsky, Fischer, PRD 105 (2022)



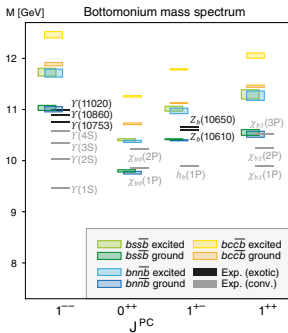
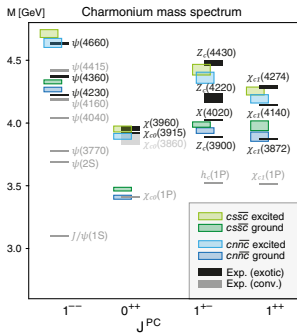
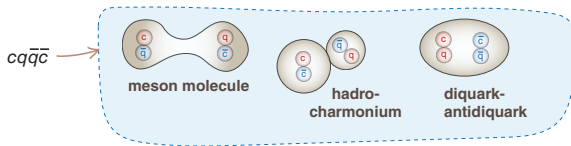
Light quarks:
 $\pi\pi$ dominant

Strange quarks:
 $q\bar{q}$ dominant

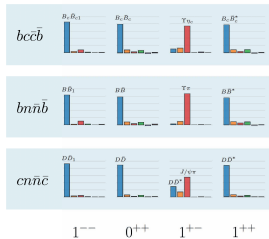
Four-quark states

Hidden charm, hidden bottom

Hoffer, GE, Fischer,
PRD 109 (2024)



Wave-function components:



Four-quark states

Open-flavor states

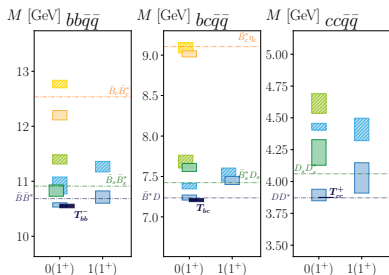
Hoffer, GE, Fischer,
PRD 111 (2025)



meson molecule



diquark-antidiquark

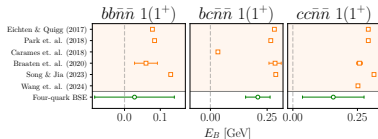
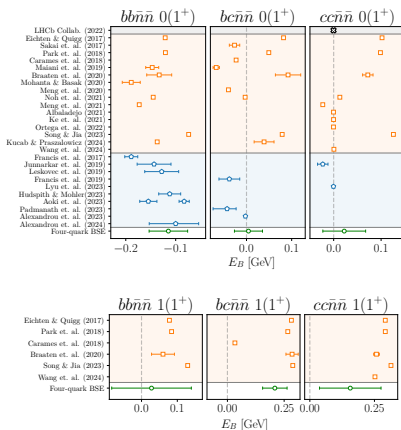


$bb\bar{c}\bar{c}$ { excited
ground }
 $bb\bar{s}\bar{s}$ { excited
ground }
 $bb\bar{n}\bar{n}$ { excited
ground }

$bc\bar{c}\bar{c}$ { excited
ground }
 $bc\bar{s}\bar{s}$ { excited
ground }
 $bc\bar{n}\bar{n}$ { excited
ground }

$cc\bar{s}\bar{s}$ { excited
ground }
 $cc\bar{n}\bar{n}$ { excited
ground }

Binding energies (ground states):



Four-quark states

Open-flavor states

Hoffer, GE, Fischer,
PRD 111 (2025)

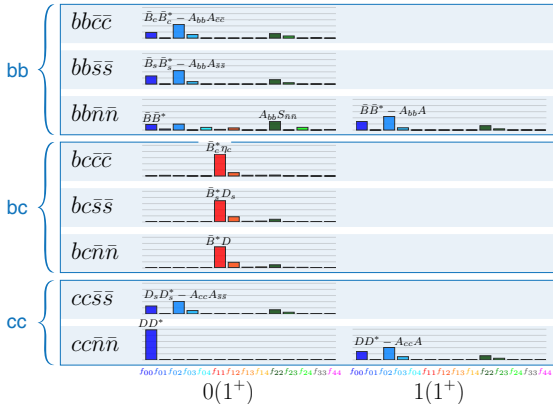


meson molecule



diquark-antidiquark

Wave-function components:



$I(J^P)$	Physical components					
	$1 \otimes 1$		$\bar{3} \otimes 3$	$8 \otimes 8$		$\bar{6} \otimes 6$
	f_0	f_1	f_2	f_3	f_4	f_5
$0(1^+) bb\bar{n}\bar{n}$	BB^*	$B^* B^*$	$A_{bb} S$	BB^*	$B^* B^*$	$S_{bb} A$
$bc\bar{n}\bar{n}$	BD^*	$B^* D$	$A_{bc} S$	BD^*	$B^* D$	$S_{bc} A$
$cc\bar{n}\bar{n}$	DD^*	$D^* D^*$	$A_{cc} S$	DD^*	$D^* D^*$	$S_{cc} A$
$bb\bar{s}\bar{s}$	$B_s B_s^*$	—	$A_{bb} A_{ss}$	$B_s B_s^*$	—	—
$bc\bar{s}\bar{s}$	$B_s D_s^*$	$B_s^* D_s$	$S_{bc} A_{ss}$	$B_s D_s^*$	$B_s^* D_s$	$A_{bc} S_{ss}$
$cc\bar{s}\bar{s}$	$D_s D_s^*$	—	$A_{cc} A_{ss}$	$D_s D_s^*$	—	—
$1(1^+) bb\bar{q}\bar{q}$	BB^*	—	$A_{bb} A$	BB^*	—	—
$cc\bar{q}\bar{q}$	DD^*	—	$A_{cc} A$	DD^*	—	—

→ T_{bb} is complicated mixture of molecule and diquark-antidiquark

Can **quantify** meson-meson and diquark-antidiquark contributions

→ T_{cc} sits on top of threshold, pure molecule

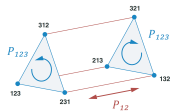
Five-body systems

- **Five-body equation**

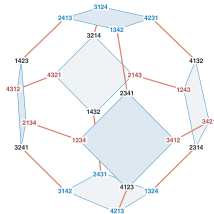
GE, Torres, PRD 111 (2025),
GE, Peña, Torres, PLB 866 (2025)

$$\Gamma = \sum_{\text{10 permutations}} \Gamma - \sum_{\text{15 permutations}} \Gamma$$

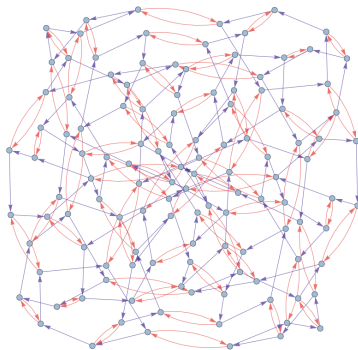
- Multiplets of permutation group S_5 ?



Cayley graph of S_3



Cayley graph of S_4



Cayley graph of S_5

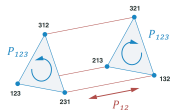
Five-body systems

- Five-body equation**

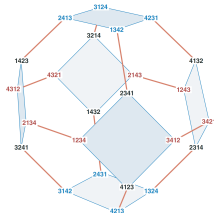
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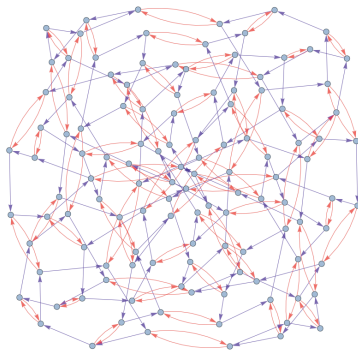
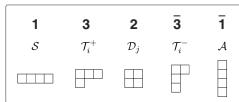
- Multiplets of permutation group S_5 ?**



Cayley graph of S_3



Cayley graph of S_4



Cayley graph of S_5

Five-body systems

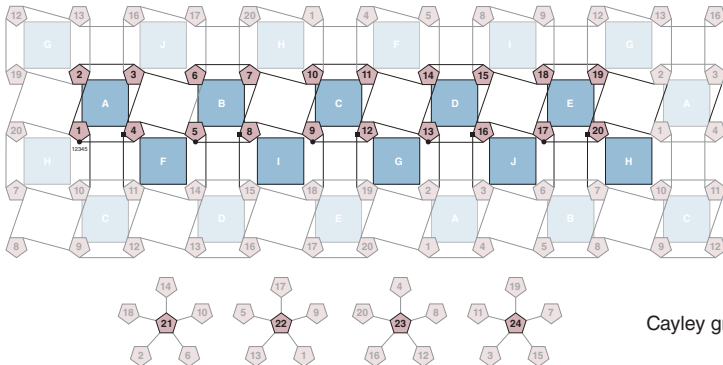
- Five-body equation**

GE, Torres, PRD 111 (2025),
GE, Peña, Torres, PLB 866 (2025)

$$\Gamma = \sum_{\text{10 permutations}} \text{Diagram 1} - \sum_{\text{15 permutations}} \text{Diagram 2}$$

The diagram shows the five-body equation for the scattering amplitude Γ . It is expressed as the difference between two sums of diagrams. The first sum, labeled "10 permutations", involves a diagram with a green semi-circular block Γ and a blue block with four external lines. The second sum, labeled "15 permutations", involves a similar diagram but with a different internal structure for the blue block.

- Multiplets of permutation group S_5 ?**

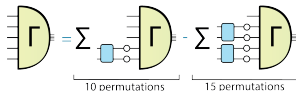


Cayley graph of S_5

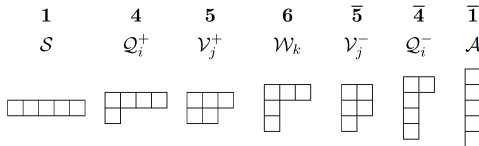
Five-body systems

- Five-body equation**

GE, Torres, PRD 111 (2025),
GE, Peña, Torres, PLB 866 (2025)



- Multiplets of permutation group S_5 :**



- E.g., 32 linearly independent color structures for **five-gluon vertex**:**

$$\delta_{ab} d_{cde} \rightarrow \mathbf{1}, \mathbf{4}, \mathbf{5}$$

$$\delta_{ab} f_{cde} \rightarrow \bar{\mathbf{4}}, \mathbf{6} \quad \rightarrow$$

$$f_{abr} f_{cds} f_{ers} \rightarrow \mathbf{6}$$

$$f_{abr} f_{cds} d_{ers} \rightarrow \bar{\mathbf{5}}, \bar{\mathbf{1}}$$

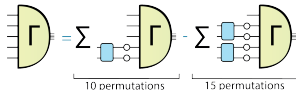
$$Q^- = \begin{bmatrix} \psi_1 - \psi_2 \\ \frac{1}{\sqrt{3}}(\psi_1 + \psi_2 - 2\psi_3) \\ \frac{1}{\sqrt{6}}(\psi_1 + \psi_2 + \psi_3 - 3\psi_4) \\ \frac{1}{\sqrt{10}}(\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_5) \end{bmatrix},$$

$$W = \begin{bmatrix} 2(f_{45} - \frac{1}{5}(\psi_4 - \psi_5)) \\ \frac{1}{\sqrt{2}}(3f_{35} - f_{45} + \frac{1}{5}(3\psi_3 + \psi_4 - 4\psi_5)) \\ \sqrt{\frac{3}{10}}(4f_{34} - f_{35} - f_{45} - \psi_3 + \psi_4) \\ \sqrt{\frac{3}{2}}(f_{15} + f_{25} + \frac{1}{5}(\psi_1 - \psi_2)) \\ \frac{1}{\sqrt{10}}(-2f_{12} + f_{13} + 3f_{14} + f_{23} + 3f_{24}) \\ \frac{2}{\sqrt{5}}(f_{12} + f_{13} + f_{23}) \end{bmatrix}$$

Five-body systems

• Five-body equation

GE, Torres, PRD 111 (2025),
GE, Peña, Torres, PLB 866 (2025)



$$\Gamma(p, q, k, l, P) = \sum_i f_i(p^2, q^2, k^2, l^2, P^2, \{\omega_i\}, \{\eta_i\}) \tau_i(p, q, k, l, P)$$

**15 Lorentz invariants,
group into multiplets of S_5 :
1 + 1 + 4 + 4 + 5**

$$\begin{aligned} \omega_1 &= q \cdot p, & \eta_1 &= q \cdot P, \\ \omega_2 &= q \cdot k, & \eta_2 &= p \cdot P, \\ \omega_3 &= q \cdot l, & \eta_3 &= k \cdot P, \\ \omega_4 &= p \cdot k, & \eta_4 &= l \cdot P, \\ \omega_5 &= p \cdot l, & \eta_5 &= k \cdot l, \\ \omega_6 &= k \cdot l, \end{aligned}$$

+ dimensional constraint: only 14 are independent

$$P = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bullet \end{bmatrix}, q = \begin{bmatrix} 0 \\ 0 \\ \bullet \\ \bullet \end{bmatrix}, p = \begin{bmatrix} 0 \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}, k = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}, l = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}$$

unfortunately,
complicated
constraint
equation...

$$\begin{aligned} & \left[(\omega_{12} \omega_{34} + \omega_{13} \omega_{24} + \omega_{14} \omega_{23})^2 - q_1^2 q_2^2 q_3^2 \right] P^2 \\ & + \sum_i \eta_i^2 [q_i^2 q_2^2 q_3^2 - q_1^2 \omega_{12}^2 - q_2^2 \omega_{13}^2 - q_3^2 \omega_{14}^2] \\ & + 2 \sum_i (\eta_i^2 - q_i^2 P^2) \omega_{ijk} \omega_{jil} \omega_{kl} \\ & + 2 \sum_{i < j} \eta_i \eta_j [2 q_{ik}^2 (\omega_{il} \omega_{jl} - q_l^2 \omega_{ij}) + q_l^2 \omega_{ik} \omega_{jk} \\ & \quad + \omega_{il} (\omega_{ij} \omega_{kl} - \omega_{ik} \omega_{jl} - \omega_{il} \omega_{jk})] \\ & + P^2 \sum_{i < j} \omega_{ij}^2 (q_i^2 q_j^2 - \omega_{ij}^2) = 0, \end{aligned}$$

• can be generalized
to arbitrary n-body
equations:

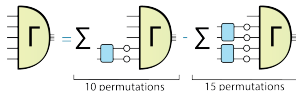
n	S_0	P^2	η_i	$p_1^2 \dots p_n^2$	$\mathcal{M}$	Total	Indep.
2	1	1	1	—	—	3	3
3	1	1	2	2	—	6	6
4	1	1	3	3	2	10	10
5	1	1	4	4	5	15	14
6	1	1	5	5	9	21	18
n	1	1	$n-1$	$n-1$	$n(n-3)/2$	$n(n+1)/2$	$4n-6$

two-/three-body poles
distributed over 4 + 5

Five-body systems

- **Five-body equation**

GE, Torres, PRD 111 (2025),
GE, Peña, Torres, PLB 866 (2025)



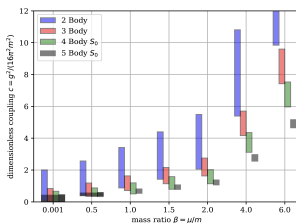
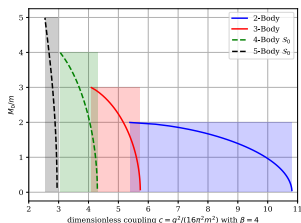
- Approximation: absorb **two-/three-body poles** explicitly in wave function

$$\Gamma(q, p, k, l, P) \approx f(S_0) \sum_{aa'} \mathcal{P}_{aa'} , \quad \mathcal{P}_{(12)(345)} = \frac{1}{(p_1 + p_2)^2 + M_M^2} \frac{1}{(p_3 + p_4 + p_5)^2 + M_B^2}$$

⇒ needs solutions of two-/three-body equations

- Scalar (massive Wick-Cutkosky) model:

Parameters are coupling strength c , exchange particle mass β

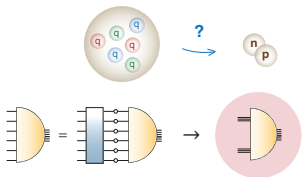


Regions in c, β where system is **Borromean**:
5-body eq. has solution
but 2- and 3-body eqs.
do not

Six-body systems

- 6-body equations: Transition from quarks & gluons to **light nuclei**

GE, Hoffer, in preparation

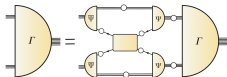


- Wave function depends on 21 variables ($1 + 1 + 5 + 5 + 9$), 3 constraint equations

n	S_0	P^2	η_i	$p_1^2 \dots p_n^2$	$\mathcal{M}$	Total	Indep.
2	1	1	1	—	—	3	3
3	1	1	2	2	—	6	6
4	1	1	3	3	2	10	10
5	1	1	4	4	5	15	14
6	1	1	5	5	9	21	18
n	1	1	$n-1$	$n-1$	$n(n-3)/2$	$n(n+1)/2$	$4n-6$

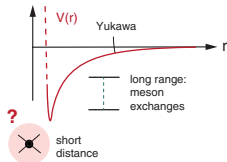
- Relativistic structure of the **deuteron**

Arriaga, GE, Nunes, Peña, Stadler, in preparation



- Microscopic origin of **short-range nuclear force?**

GE, Mirlt, in preparation



1. Introduction
2. Spectrum & symmetries
3. Quark models
4. QFT toolbox
5. Functional methods
6. n-point functions & tensor bases
7. Multiquark states
- 8. Light-front wave functions**

Lecture notes “QCD and hadron physics,”
<https://particle.uni-graz.at/en/quarks-hadrons-and-nuclei>

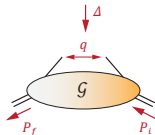
GE, “*Hadron physics with functional methods*,”
arXiv: 2503.10397 (for Encyclopedia of Particle Physics)

Burkert, GE, Klempt, “*The impact of γN and $\gamma^* N$ interactions on our understanding of nucleon excitations*,” arXiv:2506.16482

GE, Sanchis-Alepuz, Williams, Alkofer, Fischer,
“*Baryons as relativistic three-quark bound states*”,
Prog. Part. Nucl. Phys. 91 (2016), arXiv: 1606.09602

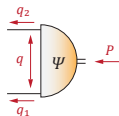
Barabanov, Bedolla, Brooks, Cates, Chen et al.,
“*Diquark correlations in hadron physics: Origin, impact and evidence*”, Prog. Part. Nucl. Phys. 116 (2021),
arXiv: 2008.07630

Parton distributions



Hadron-to-hadron correlator

$$\mathcal{G}(z, P, \Delta) = \langle P_f | \mathcal{T} \Phi(z) \mathcal{O} \Phi(0) | P_i \rangle$$



Bethe-Salpeter WF:

vacuum-to-hadron correlator

$$\Psi(z, P) = \langle 0 | \mathcal{T} \Phi(z) \Phi(0) | P \rangle$$

	$\mathcal{G}(q, P, \Delta = 0)$	$\mathcal{G}(q, P, \Delta)$	$\Psi(q, P)$
$\int dq^-$	TMD	GTMD	LFWF
$\int d^2 \mathbf{q}_\perp \int dq^-$	PDF	GPD	PDA

Diehl, Phys. Rept. 388 (2003)

Belitsky, Radyushkin,
Phys. Rept. 418 (2005)

Lorcé, Pasquini, Vanderhaeghen,
JHEP 05 (2011)

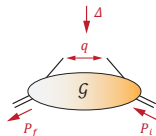
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Light-front wave functions:

coefficients of Fock expansion
in light-front quantization

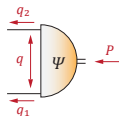
Brodsky, Pauli, Pinsky, Phys. Rept. 301 (1998)

Parton distributions



Hadron-to-hadron correlator

$$\mathcal{G}(z, P, \Delta) = \langle P_f | T \Phi(z) \mathcal{O} \Phi(0) | P_i \rangle$$



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Diehl, Phys. Rept. 388 (2003)
Belitsky, Radyushkin,
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Lorcé, Pasquini, Vanderhaeghen,
JHEP 05 (2011)
...

Light-front wave functions:
coefficients of Fock expansion
in light-front quantization

Brodsky, Pauli, Pinsky, Phys. Rept. 301 (1998)

Progress in lattice QCD:
Quasi-PDFs, pseudo-PDFs, ...

Ji, PRL 110 (2013),
Radyushkin, PLB 767 (2017),
Lin et al., PPNP 100 (2018)
Constantinou et al., PPNP 121 (2021)

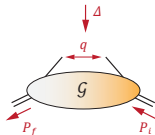
Nakanishi representation

Nakanishi 1963, 1969, 1988
Kusaka, Williams, PRD 51 (1995)
Sauli, Adam, PRD 67 (2003)
Karmanov, Carbonell, EPJ A 27 (2006)
Frederico, Salmé, Viviani, PRD 85 (2012)

Many continuum studies of
LFWFs, PDFs, GPDs, TMDs, ...

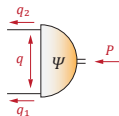
Tiburzi, Miller, PRD 65 (2002)
Nguyen, Bashir, Roberts, Tandy, PRC 83 (2011)
Chang et al., PRL 110 (2013)
Frederico, Salmé, Viviani, PRD 89 (2014)
Mezrag et al., PLB 741 (2015)
de Paula et al., PRD 94 (2016)
Mezrag, Segovia, Chang, Roberts, PLB 783 (2018)
Bednar, Cloet, Tandy, PRL 124 (2020)
Ding et al., PRD 101 (2020)
Serna et al., EPJ C 80 (2020)
Freese, Cloet, PRC 103 (2021)
Zhang et al., PLB 815 (2021)
Ydrefors, Frederico, PRD 104 (2021)
...

Parton distributions



Hadron-to-hadron correlator

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$\int dq^-$	TMD	GTMD	LFWF
$\int d^2 q_\perp \int dq^-$	PDF	GPD	PDA

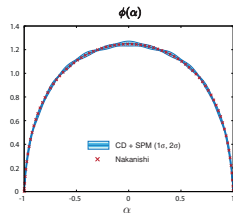
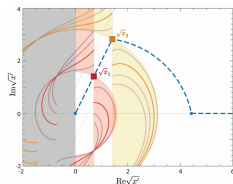
Diehl, Phys. Rept. 388 (2003)
 Belitsky, Radyushkin,
 Phys. Rept. 418 (2005)
 Lorcé, Pasquini, Vanderhaeghen,
 JHEP 05 (2011)

...

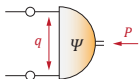
Novel method to compute
light-front wave functions
 via contour deformations

Editors' Suggestion:

GE, Ferreira, Stadler, PRD 105 (2022)

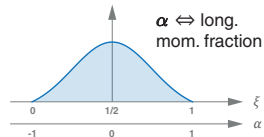


Light-front wave function

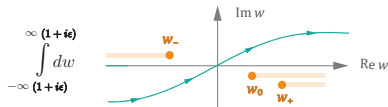
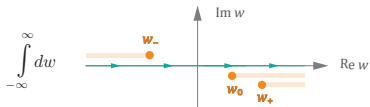


LFWF = BSWF integrated over q^-

$$\psi(\alpha, \mathbf{k}_\perp) = \mathcal{N} P^+ \int_{-\infty}^{\infty} \frac{dq^-}{2\pi} \Psi(q, P) \Big|_{q^+ = \frac{\alpha}{2} P^+, \mathbf{q}_\perp = \mathbf{k}_\perp}$$

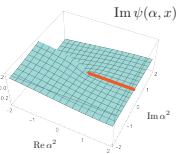
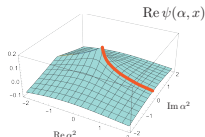
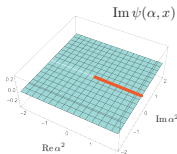
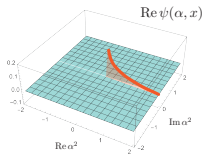


How is this related to an analytic function? Set $x = \frac{k_\perp^2}{m^2}$, $w = \frac{M}{2m} q^-$, $t = -\frac{M^2}{4m^2}$

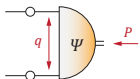


$\Rightarrow$ support only for $-1 < \alpha < 1$,
not an analytic function

$\Rightarrow$ analytic function in α^2 for any $x, t \in \mathbb{C}$,
for $-1 < \alpha < 1$ result is the same

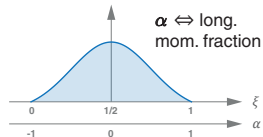


Light-front wave function

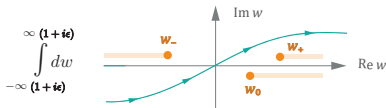
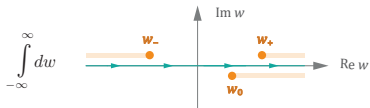


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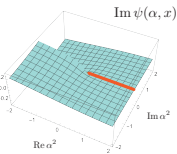
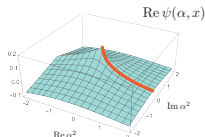
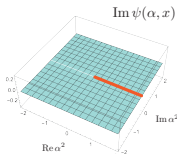
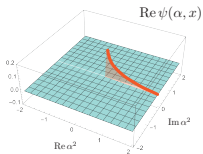


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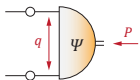


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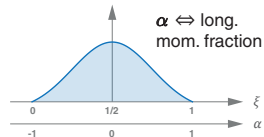


Light-front wave function

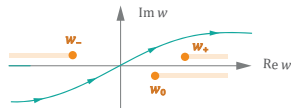


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How is this related to an **analytic function**? Set $x = \frac{k_\perp^2}{m^2}$, $w = \frac{M}{2m} q^-$, $t = -\frac{M^2}{4m^2}$

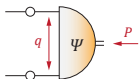


$$\psi(\alpha, x, t) = \frac{\mathcal{N} m^2}{i\pi} 2\sqrt{x t} \int_{-\infty}^{\infty} d\omega \Psi(x, \omega, t, \alpha)$$

$$\phi(\alpha) = \frac{m^2}{(4\pi)^2 f} \int_0^\infty dx \psi(\alpha, x, t)$$

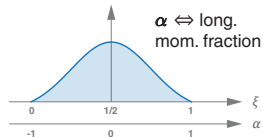
- Can be integrated **numerically** (check for monopole: same result)
- BSWF is **Lorentz-invariant**, can be calculated in any frame (also rest frame)
- But now the **branch cuts** in complex x plane will look different, need to stay inside $\mathcal{R}_\epsilon$

Light-front wave function



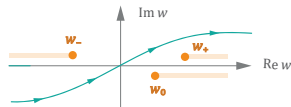
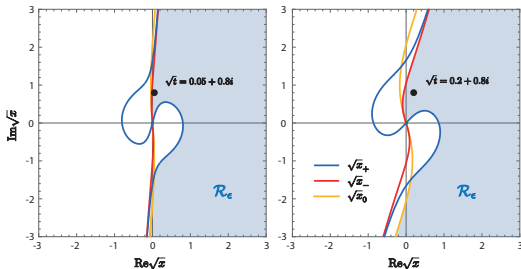
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How is this related to an **analytic function**? Set $x = \frac{k_\perp^2}{m^2}$, $w = \frac{M}{2m} q^-$, $t = -\frac{M^2}{4m^2}$

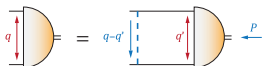
After integration $\Rightarrow$ **branch cuts** in complex $\sqrt{x}$ plane



- Propagator poles
- Pole in BS amplitude

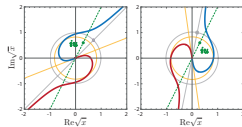
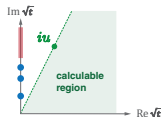
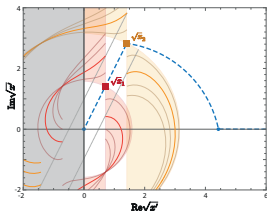
- Correct result for LFWF inside $\mathcal{R}_\epsilon$
- Physical region $0 < M < 2m$ is imaginary axis: $\sqrt{t} = \frac{iM}{2m}$
Not possible $\Rightarrow$ complex $\sqrt{t}$

Light-front wave function

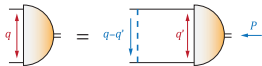


$$\Gamma(x, \omega, t, \alpha) = \int_0^\infty dx' \int_{-1}^1 d\omega' \int_{-1}^1 dy K(x, x', \Omega) G_0(x', \omega', t, \alpha) \Gamma(x', \omega', t, \alpha)$$

- BSE must be solved for $\sqrt{x} \in \mathcal{R}_e$
- BSE is **integral equation** $\Rightarrow$ must be solved along path $\sqrt{x}$ that coincides with **integration path** $\sqrt{x'}$, must lie inside $\mathcal{R}_e \Rightarrow$ need contour deformations
- **Kernel** has pole $\Rightarrow$ after integrating over y and ω' , becomes branch cut in $\sqrt{x'}$, automatically avoided if $\text{Re}\sqrt{x'}$ and $|\sqrt{x'}|$ increase along integration path
- **Propagators** have poles $\Rightarrow$ branch cuts from before, automatically avoided if path within $\mathcal{R}_e$
- **BS amplitude** may dynamically generate singularities $\Rightarrow$ avoided as long as $\arg \sqrt{t} < \arg(iu)$

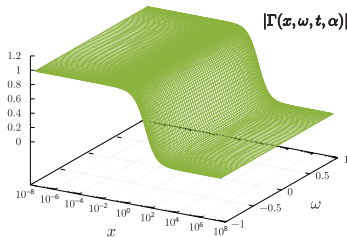


Light-front wave function



$$\Gamma(x, \omega, t, \alpha) = \int_0^\infty dx' \int_{-1}^1 d\omega' \int_{-1}^1 dy K(x, x', \Omega) G_0(x', \omega', t, \alpha) \Gamma(x', \omega', t, \alpha)$$

$$\Omega = \omega\omega' + y\sqrt{1-\omega^2}\sqrt{1-\omega'^2}$$



$$\alpha = 0.6, \\ \sqrt{i} = 0.2 + 0.2i$$

- BS amplitude falls off like $1/x$
- only weak dependence on ω and α
 $\Rightarrow \alpha$ dependence in LFWF comes from propagators:

$$\psi(\alpha, x, t) \propto \int_{-\infty}^{\infty} d\omega G_0(x, \omega, t, \alpha) \Gamma(x, \omega, t, \alpha)$$

- LFWF needs Γ for $\omega \in (-\infty, \infty)$, but
 BSE solution only known for $\omega \in [-1, 1]$

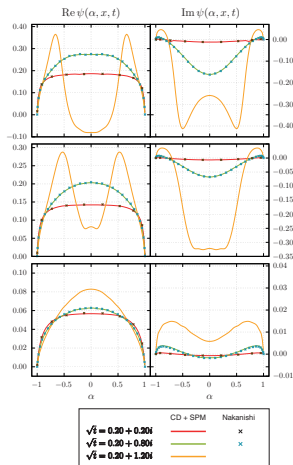
Use **Schlessinger point method (SPM)**
 for analytic continuation

Schlessinger, Phys. Rev. 167 (1968), Tripolt, Haritan, Wambach,
 Moiseyev, PLB 774 (2017), Binosi, Tripolt, PLB 801 (2020), ...

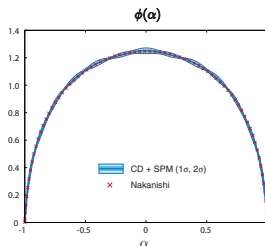
$$f(\omega) = \frac{c_1}{1 + \frac{c_2(\omega - \omega_1)}{1 + \frac{c_3(\omega - \omega_2)}{1 + \frac{c_4(\omega - \omega_3)}{\dots}}}}$$

Light-front wave function

Light-front wave function
(IR, mid-momentum, UV)



Parton distribution amplitude



On the physical axis:

$$\sqrt{t} = 0.5i$$

use Schlessinger
method in t

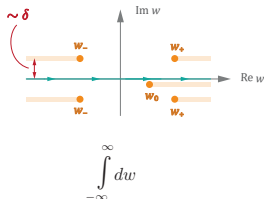
GE, Ferreira, Stadler,
PRD 105 (2022)

- Results agree with Nakanishi method { Frederico, Salmè, Viviani, PRD 85 (2012)
- LFWF vanishes at endpoints $\alpha = \pm 1$
- No expansion in moments involved, plain numerical result
- Also works above threshold (unphysical, no poles on 1st sheet, no resonances either)

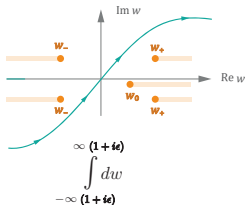
Complex conjugate singularities

Straightforward to implement in scalar model,
contour deformations work in same way

$$D(q^2) = \frac{1}{2} \left(\frac{1}{q^2 + m^2(1 + i\delta)} + \frac{1}{q^2 + m^2(1 - i\delta)} \right) = \frac{q^2 + m^2}{(q^2 + m^2)^2 + m^4 \delta^2}$$

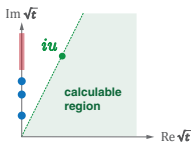


does not give correct
limit for $\delta \rightarrow 0$

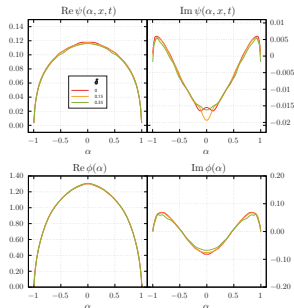


correct limit for $\delta \rightarrow 0$,
proper analytic continuation

- One does not even need to know pole positions!
 $\Rightarrow$ works for general singularities in n-point functions!



LFWF and PDA not sensitive to δ :



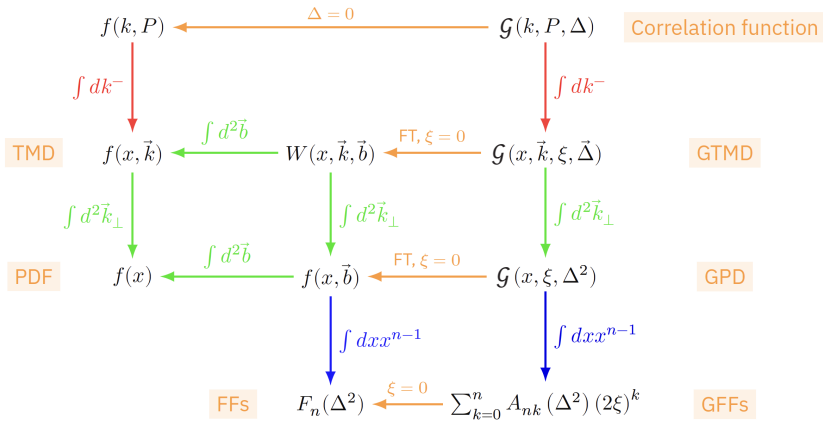
GE, Ferreira, Stadler, PRD 105 (2022)

PDFs, TMDs, GPDs?

All generated from hadron-to-hadron correlator

Diehl 2003, Lorcé, Pasquini, Vanderhaeghen 2011,

Meißner, Goeke, Metz, Schlegel 2008, Meißner, Metz, Schlegel 2009



Eduardo Ferreira, ACHT 2025

Thank you!

Backup slides

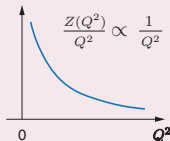
Gluon mass generation

Gluon propagator:

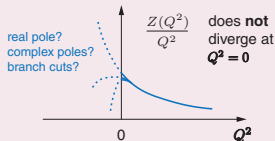


$$D^{\mu\nu}(Q) = \underbrace{\frac{Z(Q^2)}{Q^2}}_{\text{transverse dressing}} \left(\delta^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) + \xi \underbrace{\frac{L(Q^2)}{Q^2}}_{\text{longitudinal dressing} = 1} \frac{Q^\mu Q^\nu}{Q^2}$$

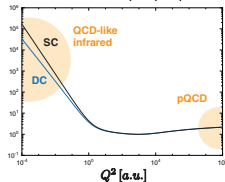
- **Perturbation theory:**
Massless gluon pole



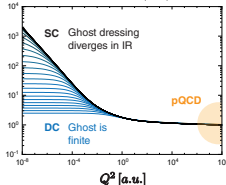
- **Nonperturbative calculations:**
Massless pole disappears!



Gluon $1/Z(Q^2)$



Ghost $G(Q^2)$



Family of “**decoupling**” solutions, also seen in lattice QCD

Cucchieri, Maas, Mendes, PRD 77 (2008)
Aguilar, Binosi, Papavassiliou, PRD 78 (2008)
Boucaud et al., JHEP 06 (2008)
Bogolubsky et al., PLB 676 (2009)
Fischer, Maas, Pawłowski, Ann. Phys. 324 (2009)
Duarte, Oliveira, Silva, PRD 94 (2016)
Aguilar et al., EPJ C 80 (2020)

Endpoint is “**scaling**” solution, confinement manifest

Lerche, Smekal, PRD 65 (2002)
Fischer, Alkofer, PLB 536 (2002)
Alkofer, Fischer, Llanes-Estrada, MPLA 23 (2008)

All solutions show **gluon mass gap**

$$\lim_{r \rightarrow \infty} \int \frac{d^3 Q}{(2\pi)^3} \frac{Z(Q^2)}{Q^2} e^{i \mathbf{x} \cdot \mathbf{Q}} \propto e^{-m_{\text{gap}} r}$$

Schwinger mechanism:
Gluon correlation functions develop massless longitudinal poles

Aguilar, Ibanez, Mathieu, Papavassiliou, PRD 85 (2012)
Aguilar, Binosi, Papavassiliou, PRD 95 (2017)
GE, Pawłowski, Silva, PRD 104 (2021)
Aguilar, Ferreira, Papavassiliou, PRD 105 (2022)

Gluon mass generation

$$(D^{-1})^{\mu\nu}(Q) = (D_0^{-1})^{\mu\nu}(Q) + \Pi^{\mu\nu}(Q)$$

$\text{gluon}^{-1} = \text{gluon}_0^{-1} + \dots$

Expand self-energy in overcomplete basis:

$$\Delta_T(Q^2) (Q^2 \delta^{\mu\nu} - Q^\mu Q^\nu) + \Delta_0(Q^2) \delta^{\mu\nu} + \Delta_L(Q^2) Q^\mu Q^\nu$$

contains
quadratic
divergences,
need to be
subtracted

must contain
longitudinal
massless poles

$$\Delta_L = -\frac{\Delta_0}{Q^2}$$

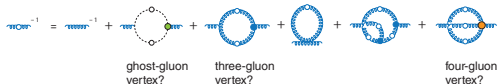
Mass generation
must come from here!

$$Z(Q^2)^{-1} = Z_A + \Delta_T + \frac{\Delta_0}{Q^2}$$

$$L(Q^2)^{-1} = 1 + \xi \left[\Delta_L + \frac{\Delta_0}{Q^2} \right]$$

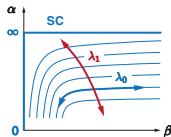
$$\stackrel{!}{=} 0$$

⇒ Gauge consistency needs **longitudinal massless poles** in vertices



- YM solutions depend on 2 parameters: coupling α , mass parameter $\beta \Leftrightarrow \Delta_0$

GE, Pawłowski, Silva, PRD 104 (2021)



$\lambda_0(\alpha, \beta)$ only rescales solutions,
 $\lambda_1(\alpha, \beta)$ distinguishes SC & DC

- SC solution: ghost-gluon vertex has long. massless pole

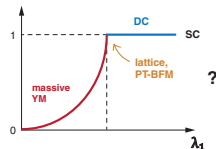
GE, Pawłowski, Silva, PRD 104 (2021)

- DC solution: three-gluon vertex has long. massless pole

Aguilar, Ibanez, Mathieu, Papavassiliou, PRD 85 (2012),
Ferreira, Papavassiliou, Particles 6 (2023)

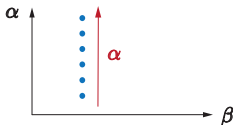
- SC & DC physically equivalent?
Test in hadron observables!

BSE eigenvalue
(=1 means: long. massless pole)



Family of solutions

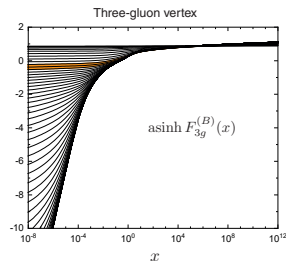
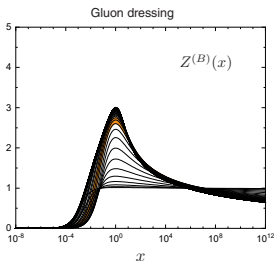
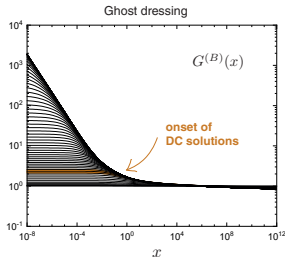
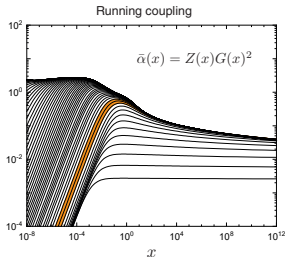
Fixed β , vary $\alpha \in (0, \infty)$:



- Running coupling $\bar{\alpha}(x)$ is RG-invariant, sets scale (so far x is arbitrary)
- SC = envelope of DC solutions for $\alpha \rightarrow \infty$

$$\begin{aligned} Z &\sim (Q^2)^{2\kappa} \\ G &\sim (Q^2)^{-\kappa} \\ F_{3g} &\sim (Q^2)^{-3\kappa} \\ \bar{\alpha} &\sim \text{const.} \end{aligned}$$

- Onset of DC solutions $\leftrightarrow$ lattice solutions

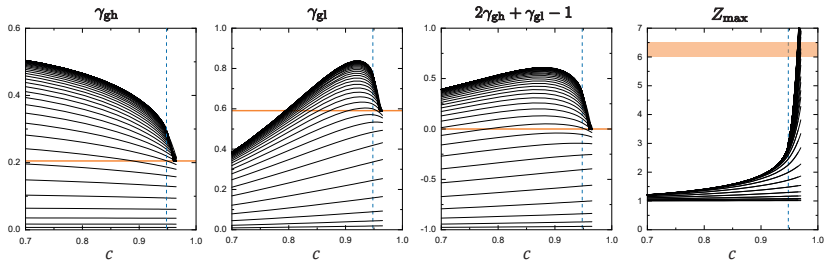


Truncation error

- Set $Z_{3g} \rightarrow c Z_{3g}$... quantifies deviation from STI (without truncation: $c = 1$), same effect from “over-renormalizing” 3-gluon vertex
- YM system only converges up to $c_{\max} < 1$
- Anomalous dimensions reproduced for

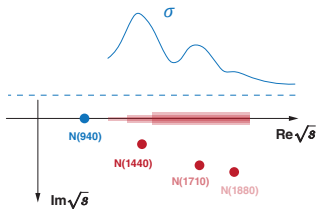
1	$c \sim 0.4$
2	$c \sim 0.9$
3	$c \sim 0.96$

 } $\Rightarrow$ identifies “physical point” for each truncation
 GE, Pawłowski, Silva, PRD 104 (2021)



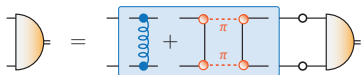
Resonances

- Most hadrons are **resonances** and decay
 $\Leftrightarrow$ poles in complex momentum plane



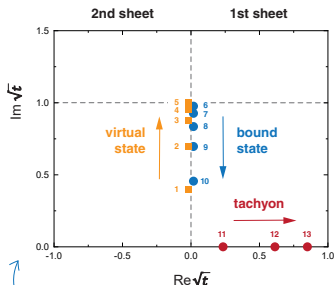
- BSE kernel must include decay channels:
 ρ meson becomes resonance

Williams, PLB 798 (2019), Miramontes, Sanchis-Alepuz, EPJA 55 (2019),
 Santowsky, GE, Fischer, Wallbott, PRD 102 (2020),
 Miramontes, Sanchis-Alepuz, Alkofer, PRD 103 (2021)



- Contour deformations** as tool
 to go beyond thresholds

GE, Duarte, Peña, Stadler, PRD 100 (2019)

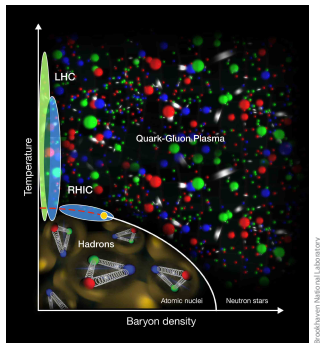


Scattering equation for 4-point function

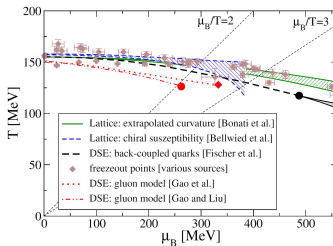
$$\text{Diagram} = \text{Diagram} + \text{Diagram} + \text{Diagram} + \text{Diagram}$$

The equation shows the scattering equation for a 4-point function, represented by a diagram with four external lines and a central loop. The diagram is equal to the sum of four terms, each representing a different contribution to the scattering process.

QCD phase diagram



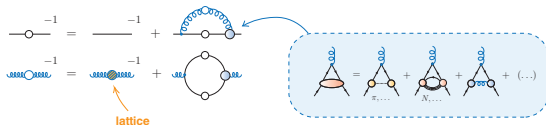
Search for **critical endpoint (CEP)** from DSEs & lattice:



Fischer, Prog. Part.
Nucl. Phys. 105 (2019)

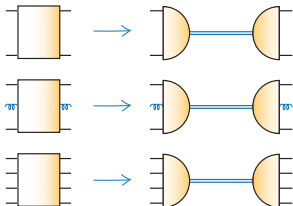
Location of CEP sensitive to **baryons**?

GE, Fischer, Welzbacher, PRD 93 (2016)

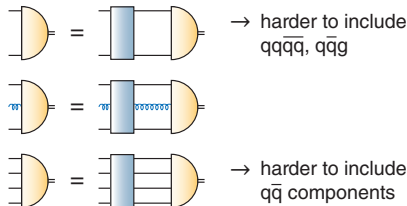


Some thoughts

- Same spectral representation for **all** correlation functions that produce $q\bar{q}$:



- Without** truncations, BSEs for qq , $qq\bar{q}\bar{q}$, $q\bar{q}g$, ... should produce **same spectrum**



With truncations, $q\bar{q}$ ($qq\bar{q}\bar{q}$, $q\bar{q}g$, ...) BSE should give more reliable spectrum for $q\bar{q}$ ($qq\bar{q}\bar{q}$, $q\bar{q}g$, ...) dominated states

Exotic quantum numbers not excluded in principle (need gluon-rich kernel in $q\bar{q}$ BSE)

Some thoughts

- Why don't we see exotic quantum numbers in **lattice calculations** with $q\bar{q}$ operators?

$$\langle q_\alpha \bar{q}_\beta q_\gamma \bar{q}_\delta \rangle =$$



gauge-dep.



gauge-dep.

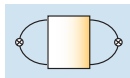


gauge-dep.

gauge-inv.



$$\langle (\bar{q} \Gamma q)(\bar{q} \Gamma q) \rangle =$$



gauge-inv.

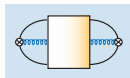


gauge-inv.



gauge-inv.

= 0 for exotic quantum nrs (?)



≠ 0 for exotic quantum nrs

Could test with **gauge-fixed** lattice calculations: same spectrum?

