# Pion: An Ageless Messenger of QCD

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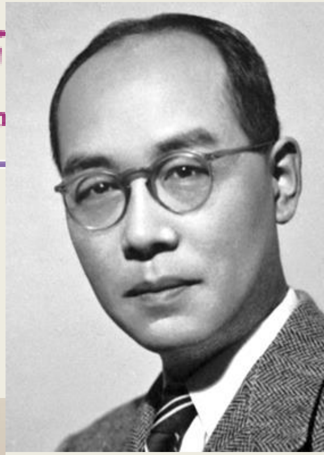
**Nankai University**

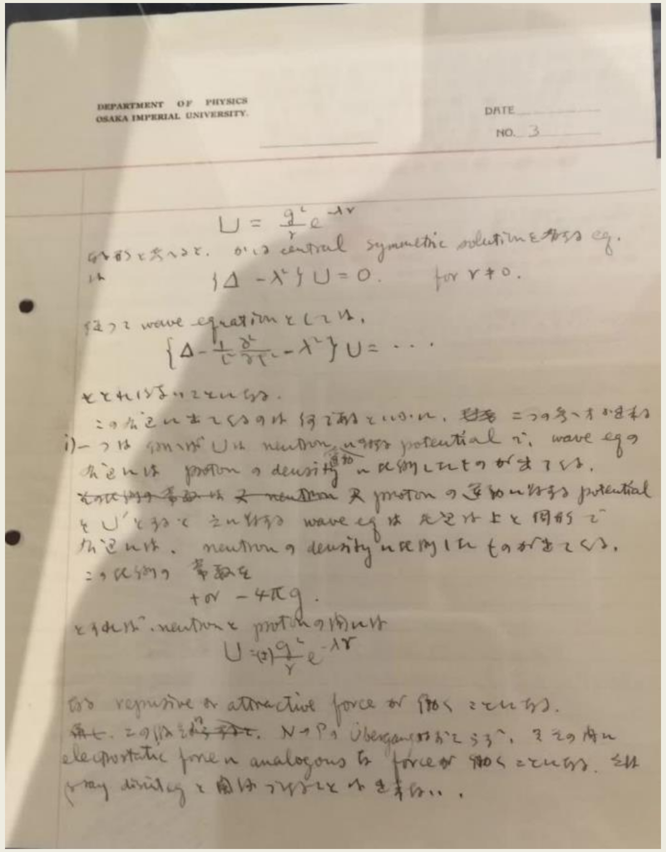
2025 International Workshop and School on Hadron Structure and Strong Interactions (IWSHSSI)

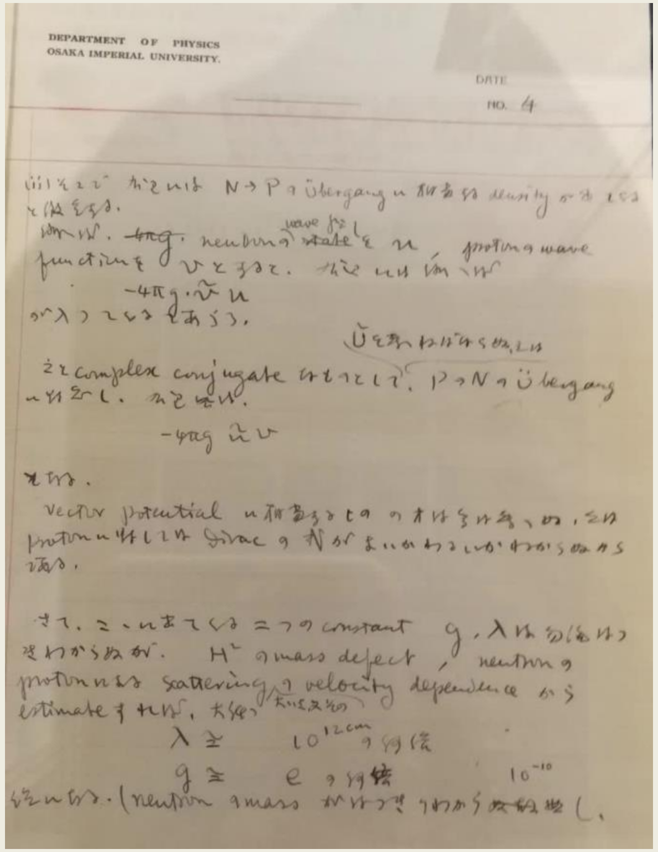
南京大学, 10/17/2025

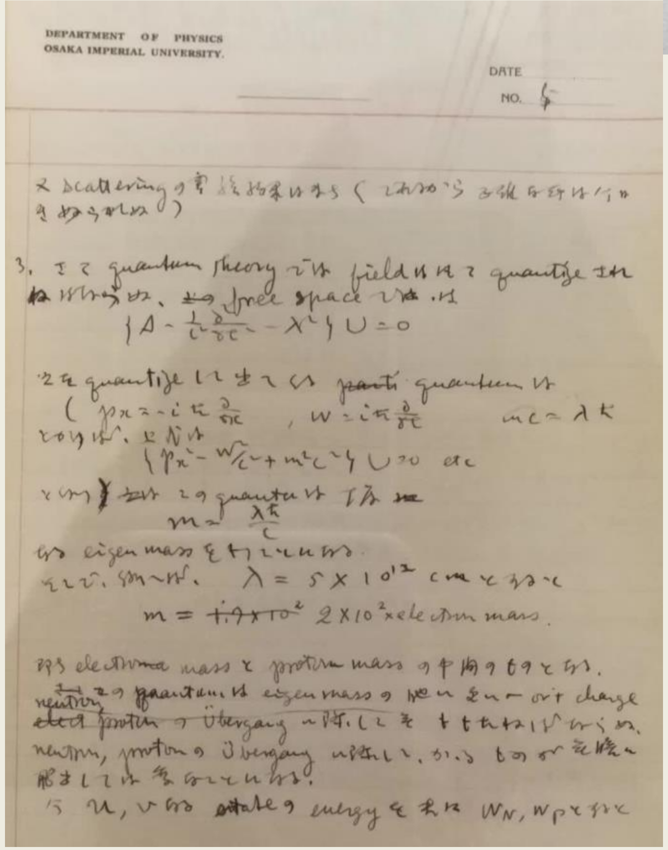
# Messager of QCD



- In October 1934**, Hideki Yukawa** predicated the existence of a “heavy quantum” meson, exchanging nuclear force between neutrons and protons.



# Messager of QCD

- It was discovered by **Cecil Powel** in 1949 in cosmic ray tracks in a photographic emulsion.

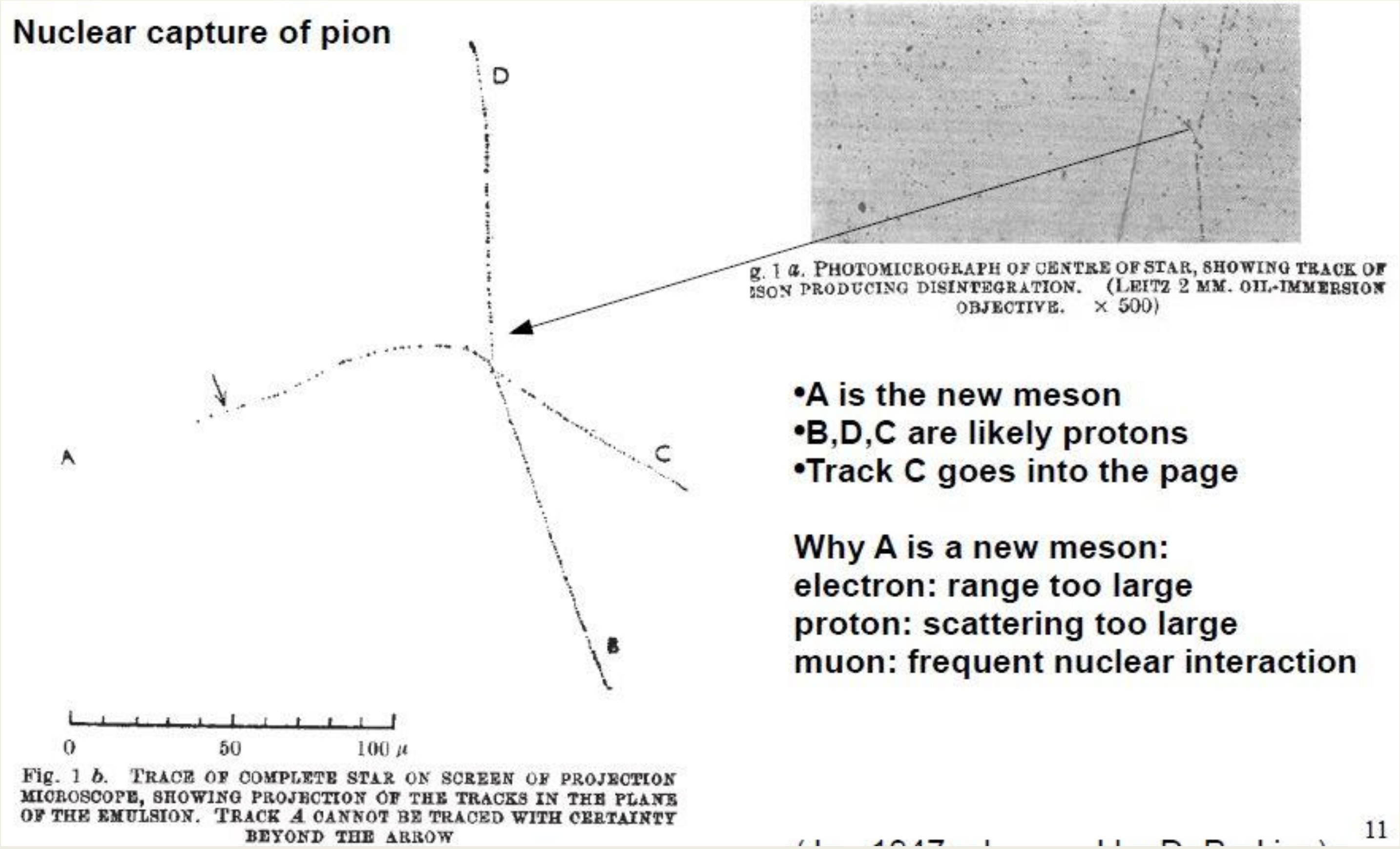



Fig. 1 b. TRACE OF COMPLETE STAR ON SCREEN OF PROJECTION MICROSCOPE, SHOWING PROJECTION OF THE TRACKS IN THE PLANE OF THE EMULSION. TRACK A CANNOT BE TRACED WITH CERTAINTY BEYOND THE ARROW


Mass measured in scattering
$\approx$ *250-350 $m_e$*

# Messager of QCD

- **Yoichiro Nambu** associated it with CSB in 1960.

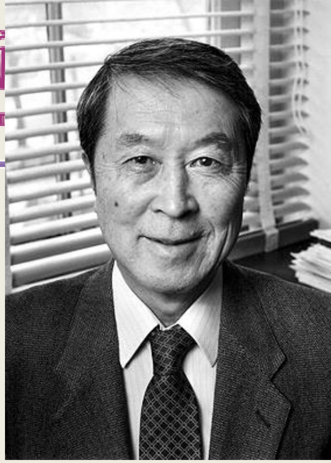
conveniently described by means of a coherent mixture of electrons and holes, which obeys the following

* Supported by the U. S. Atomic Energy Commission.
† Fulbright Fellow, on leave of absence from Instituto di Fisica dell' Universita, Roma, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy.
[1] A preliminary version of the work was presented at the Midwestern Conference on Theoretical Physics, April, 1960 (unpublished). See also Y. Nambu, Phys. Rev. Letters 4, 380 (1960);

$$E\psi_2 = -\sigma\cdot p\psi_2 + m\psi_1, \qquad (1.3)$$
$$E_p = \pm(p^2+m^2)^{\frac{1}{2}},$$

where $\psi_1$ and $\psi_2$ are the two eigenstates of the chirality operator $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$.

According to Dirac's original interpretation, the ground state (vacuum) of the world has all the electrons in the negative energy states, and to create excited states (with zero particle number) we have to supply an

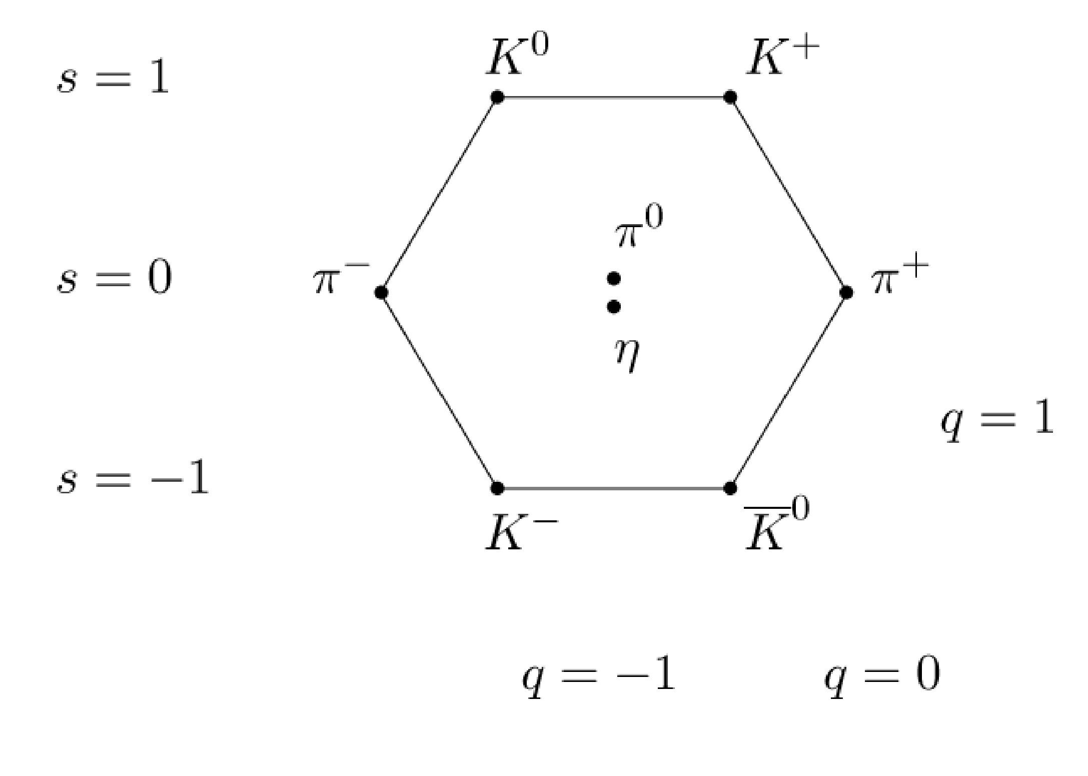


Nambu and Jona-Lasinio 1961 paper [9] was an amazing breakthrough. Before the word "quark" was invented, and one learned anything about quark masses, it postulated the notion of chiral symmetry and its spontaneously breaking. They postulated existence of 4-fermion interaction, with some coupling $G$, strong enough to make a superconductor-like gap even in fermionic vacuum. The second important parameter of the model was the cutoff $\Lambda \sim 1\,GeV$, below which their hypothetical attractive 4-fermion interaction operates.

E. Shuryak, arXiv:1908.10270

# Messager of QCD

- Pion was nicely accommodated in the Eight Fold way of **Murray Gell-Mann** in 1961.



# Measurements-I

pion electroproduction and elastic pion scattering

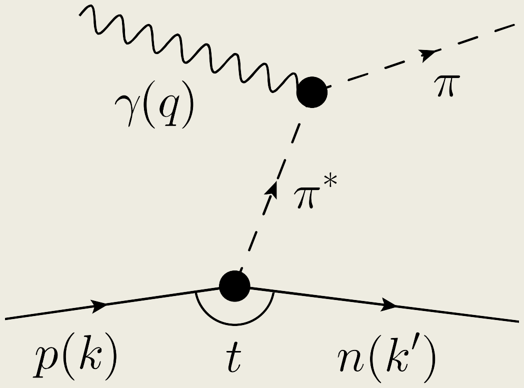
| Experiment | Process | $-t$ range (GeV$^2$) | $N_{\rm pts}$ |
|---|---|---|---|
| Brown 1973 [97] | $ep \to e'\pi^+n$ | 0.176 – 1.188 | 5 |
| Ackermann 1978 [98] | $ep \to e'\pi^+n$ | 0.35 | 1 |
| Bebek 1978 [99] | $ep \to e'\pi^+n$ | 0.18 – 9.77 | 21 |
| Brauel 1979 [100] | $ep \to e'\pi^+n$ | 0.7 | 1 |
| Volmer 2001 [101] | $ep \to e'\pi^+n$ | 0.6 – 1.6 | 4 |
| Huber 2008 [102] | $ep \to e'\pi^+n$ | 0.6 – 2.45 | 8 |
| Adylov 1977 [103] | $e$-$\pi$ scattering | 0.0138 – 0.0353 | 22 |
| Dally 1981 [104] | $e$-$\pi$ scattering | 0.0317 – 0.0705 | 20 |
| Dally 1982 [105] | $e$-$\pi$ scattering | 0.039 – 0.092 | 14 |
| Amendolia 1986 [106] | $e$-$\pi$ scattering | 0.015 – 0.253 | 45 |
| **Total** | – | 0.0138 – 9.77 | 141 |

arxiv:2508.15073

- At low $Q^2$, $F_\pi$ can be measured directly via high energy elastic $\pi^+$ scattering from the atomic electrons
  - ➢ CERN SPS used 300 GeV pions to measure form factor up to $Q^2$=0.25GeV$^2$
    (Amedolia et al, NPB277, 168 (1986))
  - ➢ These data used to constrain the pion charge radius: $r_\pi$=0.657±0.012 fm

- At larger $Q^2$, $F_\pi$ must be measured indirectly using the "pion cloud" of the proton in exclusive pion electroproduction: p(e, e' $\pi^+$)n
  - ➢ at small –t, the pion pole process dominates the longitudinal cross section, $\sigma_L$
    (L. Favart, et al, Eur. Phys. J. A 52 (2016) 158)
  - ➢ In the Born term model, $F_\pi$ appears as

$$\frac{d\sigma_L}{dt} \propto \frac{-t}{(t-m_\pi^2)} g_{\pi NN}^2(t)Q^2F_\pi^2(Q^2,t)$$




Sullivan process, in which a nucleon's pion cloud is used to provide access to the pion's elastic form factor

Experimental studies over the last decade have given confidence in the electroproduction method yielding the physical pion form factor----Tanja Horn

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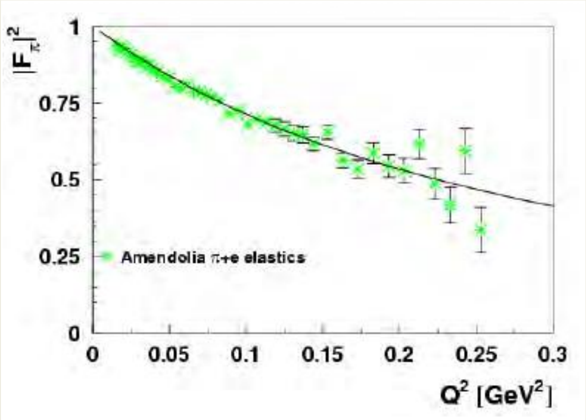


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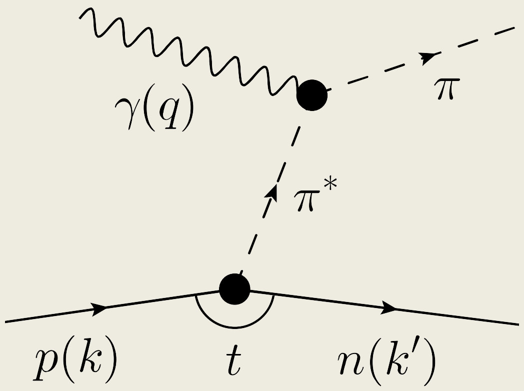


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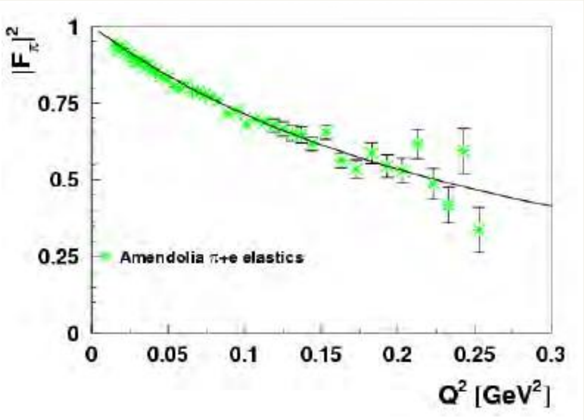


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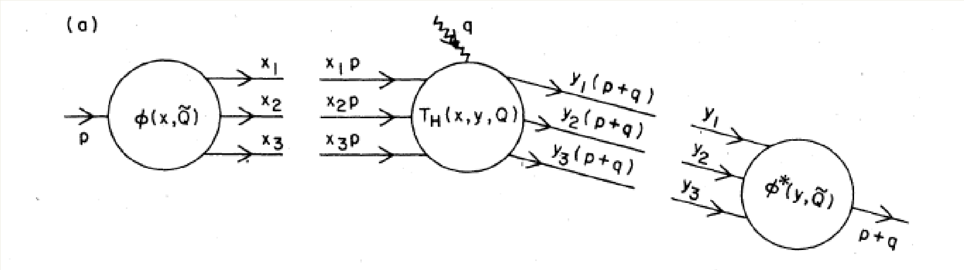


(G.R. Farrar and D.R.Jackson, PRL43 (1979) 246;
P. Lepage and S. Brodsky, PLB 87 (1979) 359)

$$Q^2 F_\pi(Q^2) \overset{Q^2\gg\Lambda^2_{QCD}}{\sim} 16\,\pi\, f_\pi^2\, \alpha_s(Q^2)\, w_\pi^2; \qquad w_\pi = \frac{1}{3}\int_0^1 dx\, \frac{1}{x}\,\varphi_\pi(x)$$

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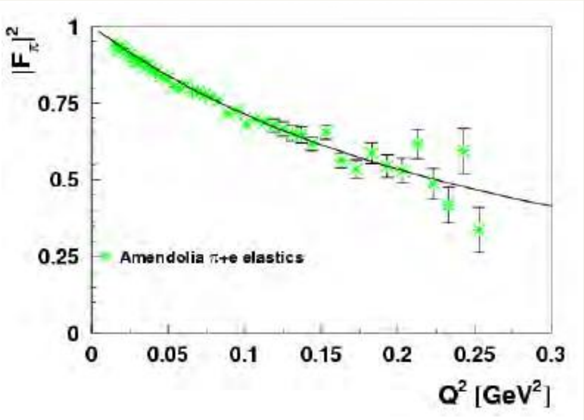
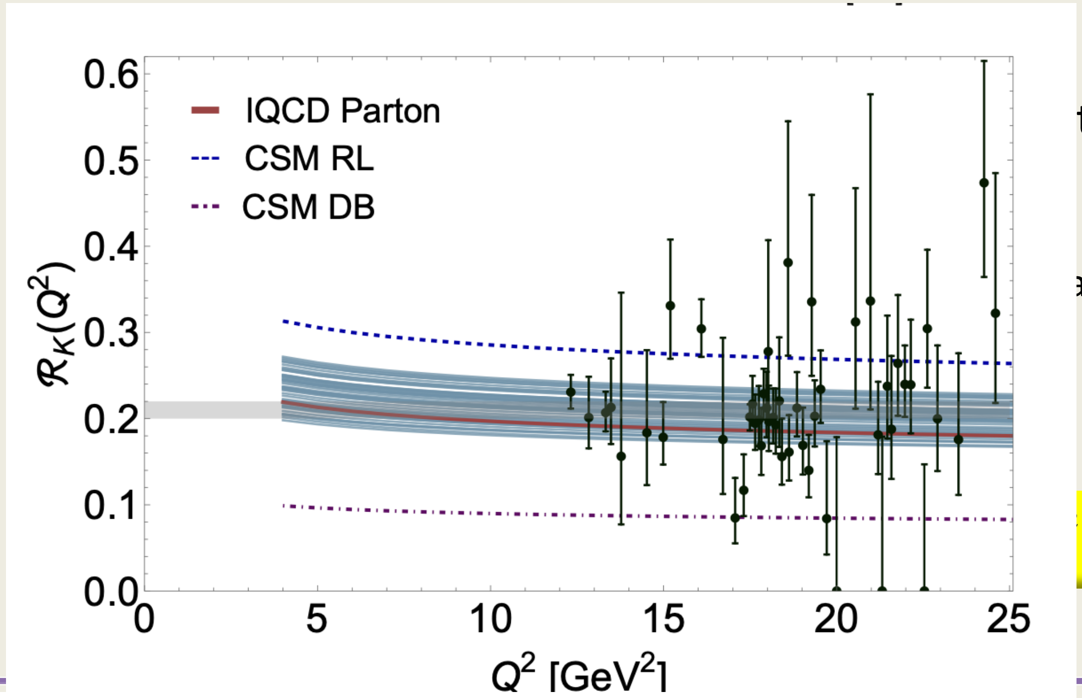
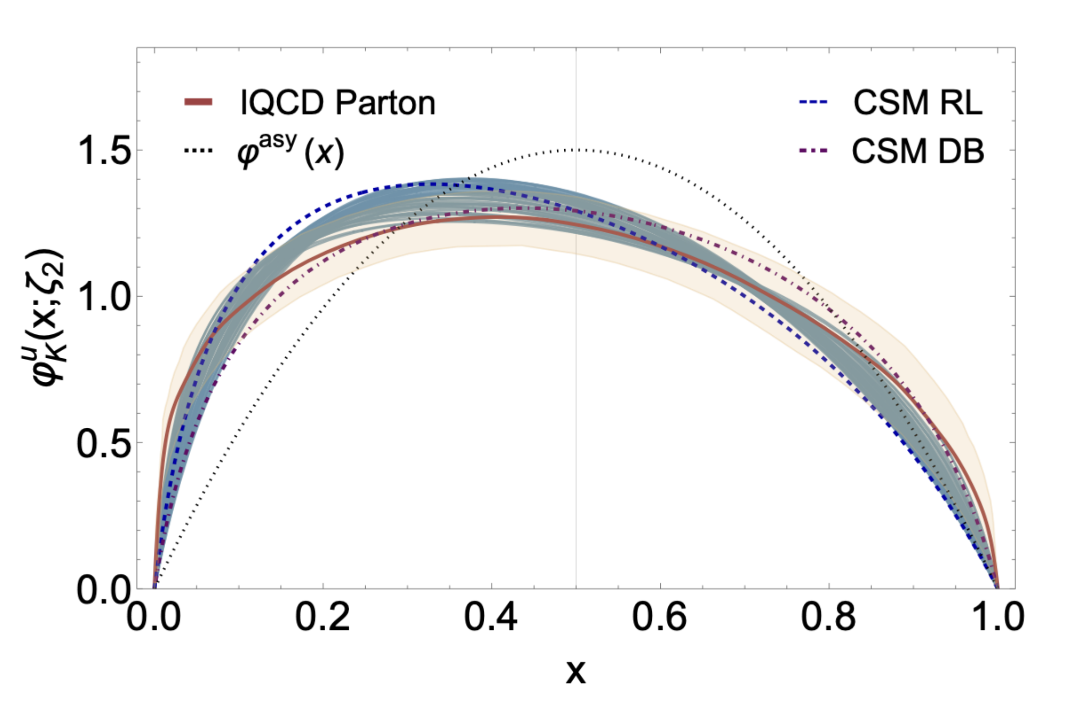



the “pion cloud”

ates the

$(Q^2,t)$




$$\mathcal{R}_K(Q^2) := \frac{|F_{K^0}(Q^2)|}{|F_{K^+}(Q^2)|} \overset{Q^2>Q_0^2}{\approx} \left|\frac{-\frac{1}{3}w_l^2(Q^2)+\frac{1}{3}w_s^2(Q^2)}{\frac{2}{3}w_l^2(Q^2)+\frac{1}{3}w_s^2(Q^2)}\right|$$

arxiv：2504.07372

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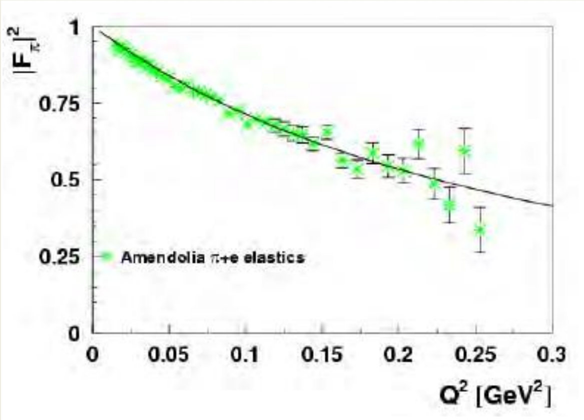


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Sullivan pro
access to th

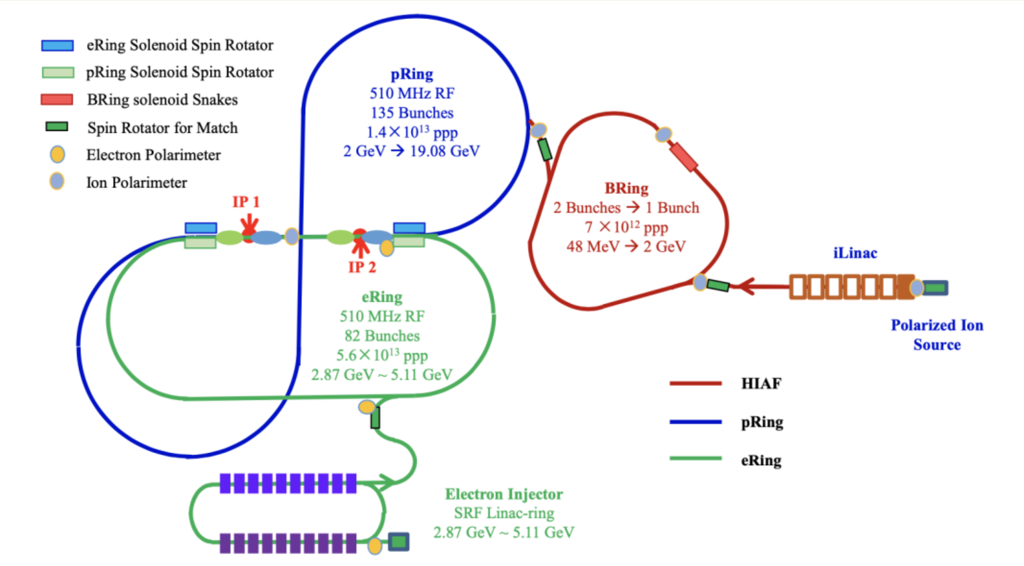
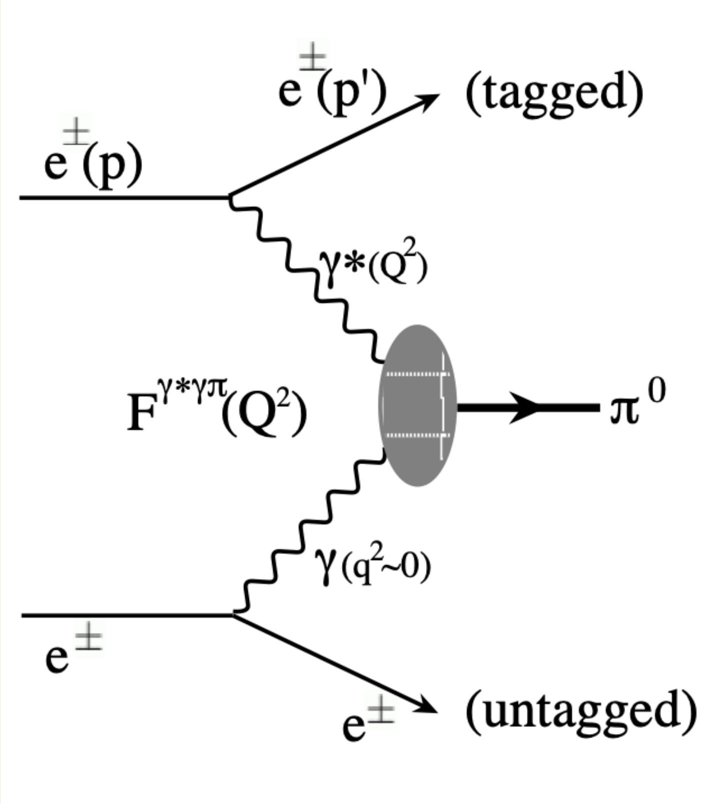


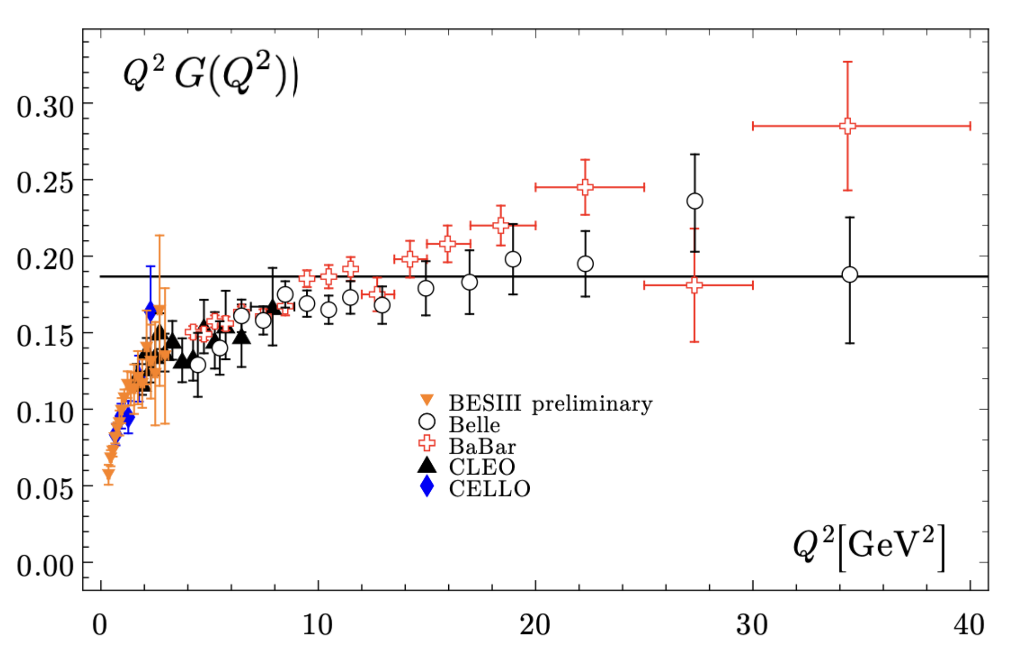

Figure 3.6: The polarization design of the EicC accelerator facility.

Experimental studies over the last decade have given confidence in the electroproduction method yielding the physical pion form factor----Tanja Horn

# Measurements-II

- Transition Form Factor



# Measurements-II

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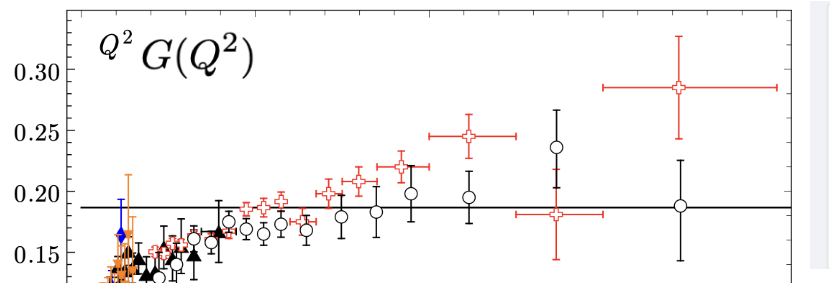


### Measurement of the gamma gamma* ---> pi0 transition form factor
#1

BaBar Collaboration • Bernard Aubert (Annecy, LAPP) et al. (May, 2009)

Published in: *Phys.Rev.D* 80 (2009) 052002 • e-Print: 0905.4778 [hep-ex]

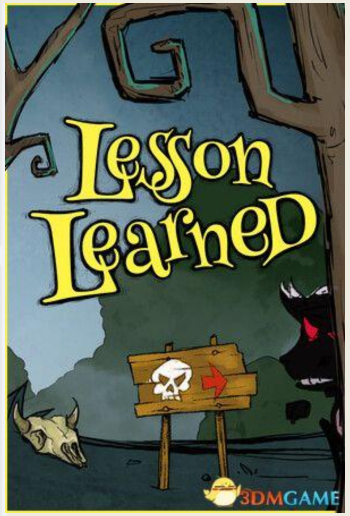
pdf links DOI cite claim reference search 462 citations

### Shape of Pion Distribution Amplitude
#26

A.V. Radyushkin (Old Dominion U. and Jefferson Lab and Dubna, JINR) (Jun, 2009)

Published in: *Phys.Rev.D* 80 (2009) 094009 • e-Print: 0906.0323 [hep-ph]

pdf links DOI cite claim reference search 167 citations




### On the Pion Distribution Amplitude Shape
#27

M.V. Polyakov (Ruhr U., Bochum and St. Petersburg, INP) (Jun, 2009)

Published in: *JETP Lett.* 90 (2009) 228-231 • e-Print: 0906.0538 [hep-ph]

pdf DOI cite claim reference search 124 citations

$$G(Q^2) \to \frac{4\pi^2 f_\pi^2}{Q^2} \int_0^1 d\alpha \frac{\varphi(\alpha)}{3(1-\alpha)},$$

Abstract: (arXiv)

We argue that the recent BaBar data on $\gamma \to \pi$ e.m. transition form factor at large photon virtuality supports the idea that pion distribution amplitude (DA) is close to unity with $\phi_\pi'(0)/6 \gg 1$ at a normalization point of $\mu = 0.6 \div 0.8$~GeV. Such pion DA can be obtained in the effective chiral quark model. The possible flat shape of the pion DA implies that the standard expansion of the DA in Gegenbauer polynomials can be divergent. On basis of chiral models we predict that the two-pion DA should be anomalously flat for pions in the S-wave and that such feature is absent for higher partial waves. The later implies that the $\rho$, $f_2$, etc. meson DAs have no anomalous endpoint behaviour. Possible implications of such pion DA for other hard exclusive processes are shortly discussed.

### Measurement of $\gamma\gamma^* \to \pi^0$ transition form factor at Belle
#1

Belle Collaboration • S. Uehara (KEK, Tsukuba) et al. (May, 2012)

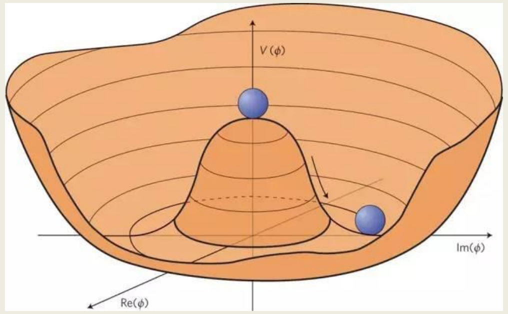
Published in: *Phys.Rev.D* 86 (2012) 092007 • e-Print: 1205.3249 [hep-ex]

pdf DOI cite claim reference search 310 citations

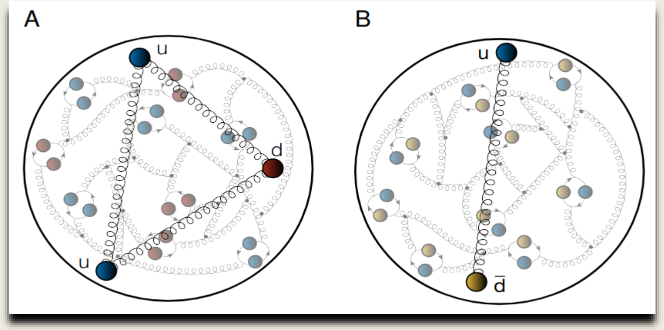
# Messager of QCD

PION's dichotomy

Goldstone Boson




Bound State




Bethe-Salpeter Equations

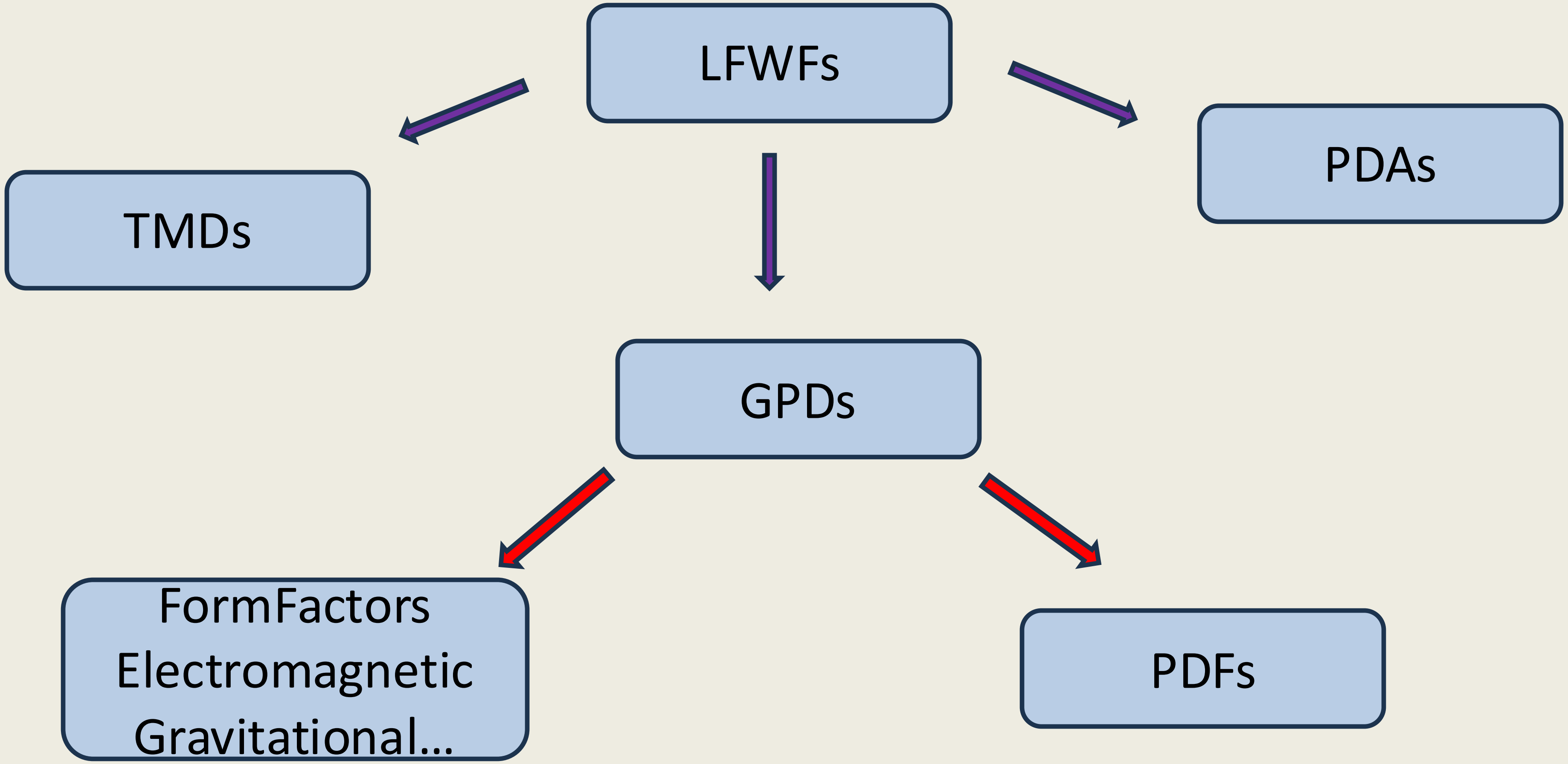

[1] Yuan-Ben Dai, Chao-Shang Huang, and Dong-Sheng Liu. Calculation of chiral symmetry breaking and pion properties as a Goldstone boson. Phys. Rev. D, 43:1717–1725,1991.

[2] H. J. Munczek. Dynamical chiral symmetry breaking, Goldstone's theorem and the consistency of the Schwinger-Dyson and Bethe-Salpeter Equations. Phys. Rev. D, 52:4736–4740, 1995.

[3] Pieter Maris, Craig D. Roberts, and Peter C. Tandy. Pion mass and decay constant. Phys. Lett. B, 420:267–273, 1998.

# Conventional way…if you like/can






Compute everything from LFWFs…

# However…Emergent Phenomena

- Confinement and DCSB are emergent phenomena

  Not revealed by any amount of staring at Lagrangian for quantum chromodynamics;

  They determine the character of the QCD's spectrum, the structure and interactions of bound states

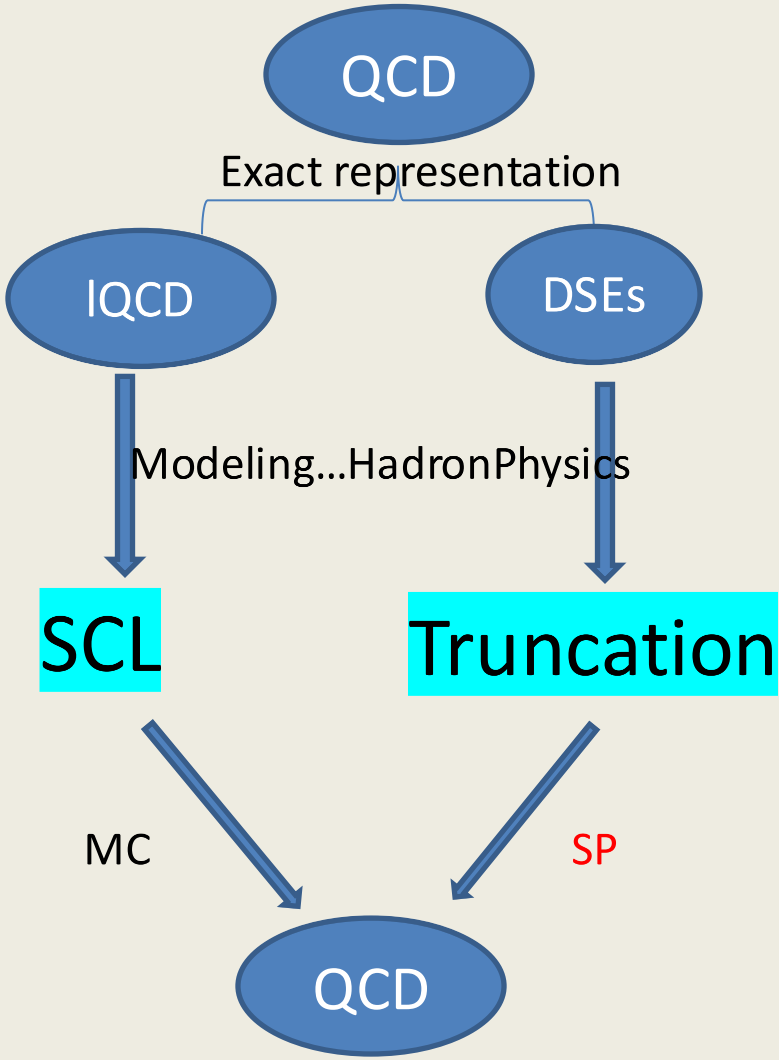
- Can one understand confinement and DCSB in terms of properties of the degrees-of-freedom used to formulate QCD?

  E.g., is it pointless to attempt to predict the pion's DF/FF on a domain that is not yet accessible?

Must develop nonperturbative calculational methods to define and tackle QCD

1) Lattice-regularized QCD
2) Continuum methods in quantum filed theory

# In the beginning…




**1. Introduction**

In order to help organize our thinking about QCD and our understanding of hadronic physics it may be useful to group some relevant issues into three broad categories. These categories are certainly not meant to be a complete set, but are meant to help guide us in our approach to the problems we must face when attempting to solve QCD.

In the first category we have what we might call exact representations of QCD, for example, the complete set of Schwinger-Dyson equations for QCD, or the continuum limit of lattice QCD. We shall also include in this category "exact" light-front theory, by which we mean light-front quantized QCD including all necessary effects of vacuum degrees of freedom (also known as "zero modes," though this phrase has several distinct meanings in light-front quantization). This theory has a nontrivial vacuum state due to the presence of zero longitudinal momentum particles. Correctly incorporating these into the theory from the beginning is a difficult problem, and is a subject of ongoing research efforts.

In the second category we have simple pictures of hadronic physics, each of which may roughly correspond to one or more of the exact representations. In this group we have, for example, truncation to the first Schwinger-Dyson equation, or the strong coupling limit of lattice QCD. Corresponding loosely to the exact light-front theory we have several simple pictures, among them the infinite momentum frame and the closely related parton

[1] Based on a talk presented by K.G. Wilson at "Theory of Hadrons and Light-Front QCD," Polana Zgorzelisko, Poland, August 1994.

K.G.Wilson, arXiv:hep-th/9411007

# Conventional way…if you like/can

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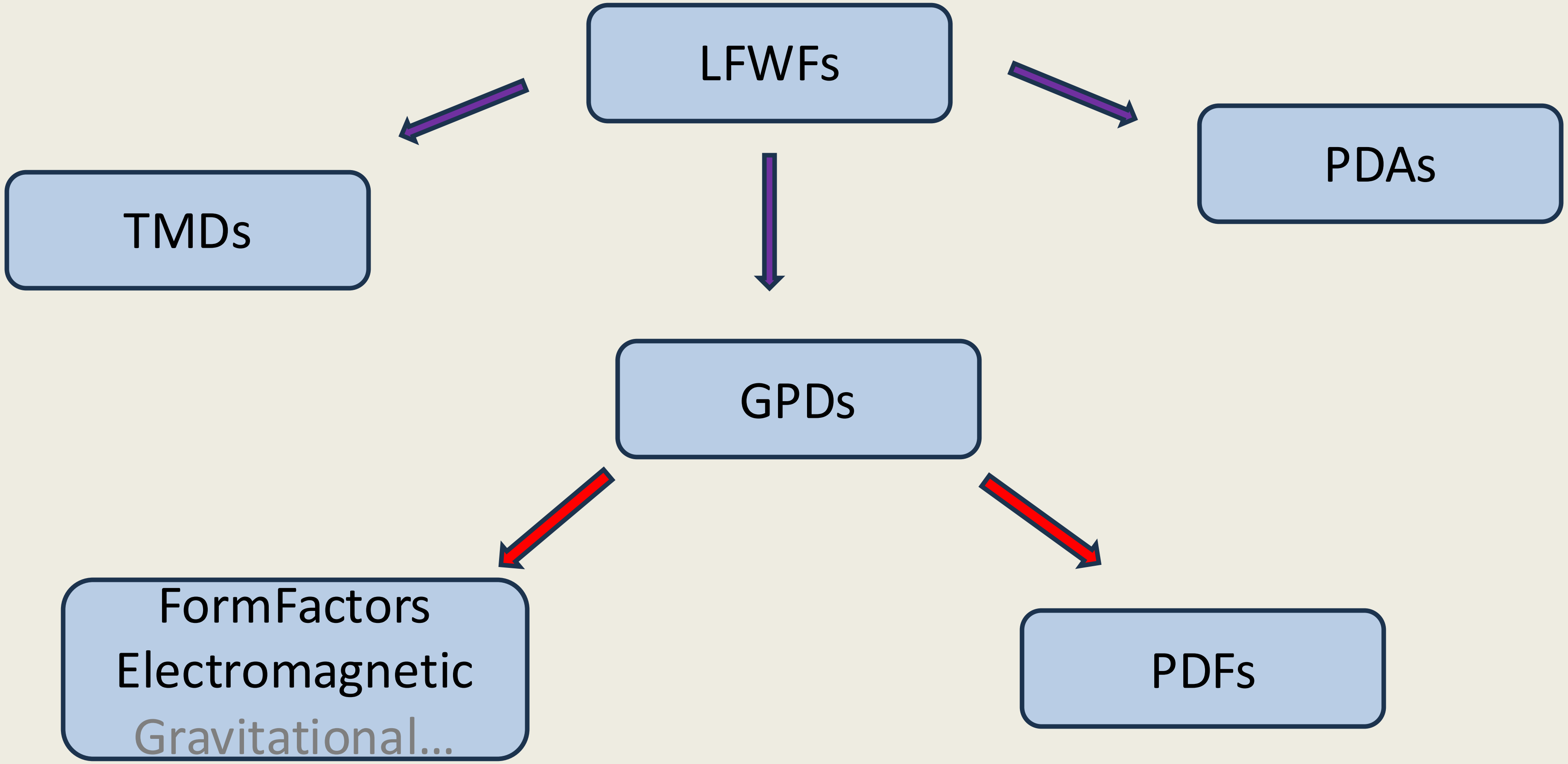



arXiv:2509.22016 [pdf, html, other]

**Dynamical gluon effects in the three-dimensional structure of pion**

Jiangshan Lan, Kaiyu Fu, Satvir Kaur, Zhimin Zhu, Chandan Mondal, Xingbo Zhao, James P. Vary

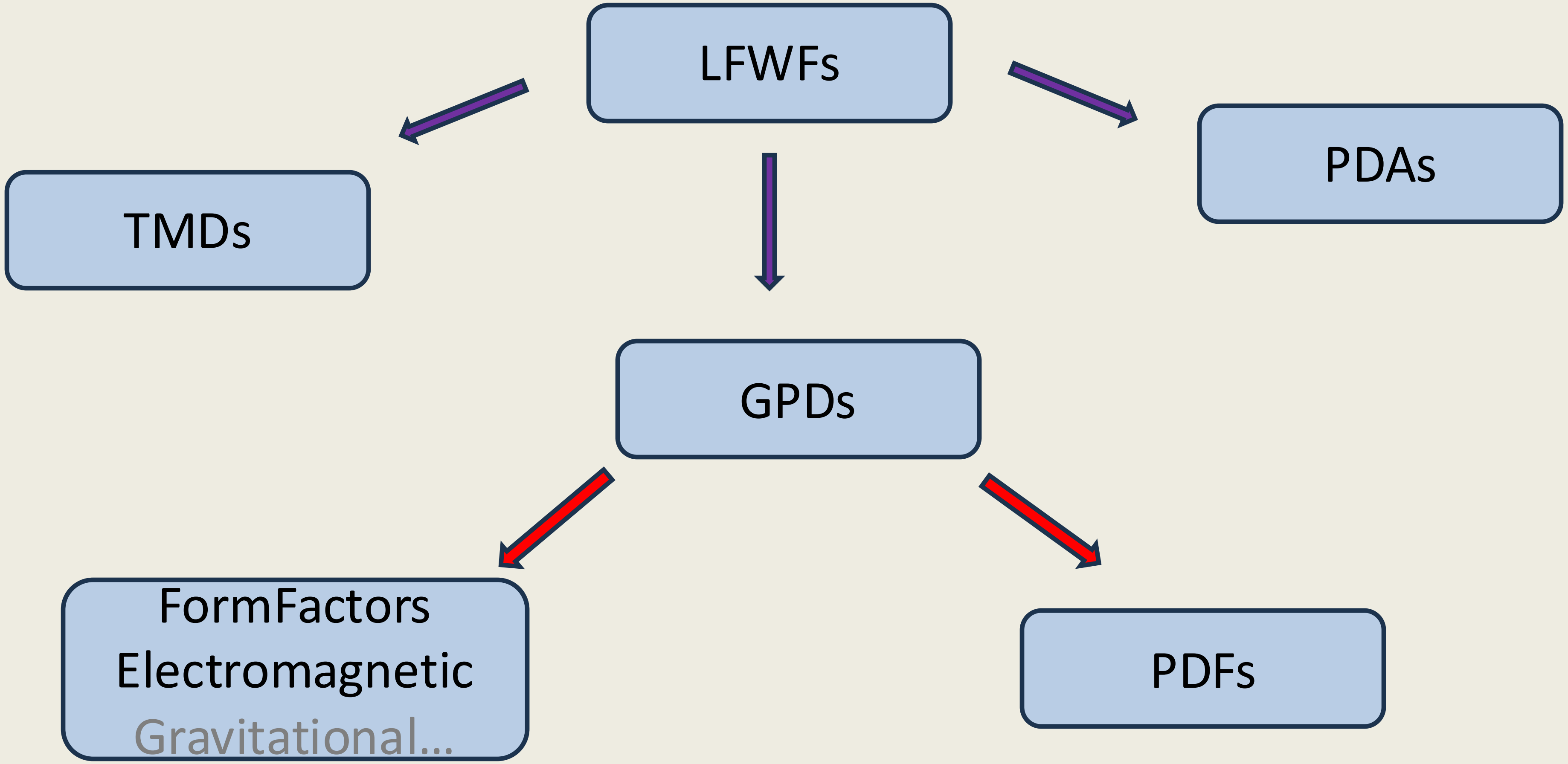
Comments: 23 pages, 9 figures

Subjects: **High Energy Physics - Phenomenology (hep-ph)**; High Energy Physics - Theory (hep-th); Nuclear Theory (nucl-th)

We investigate the internal structure of the pion, including the contributions from one dynamical gluon, using the basis light-front quantization (BLFQ) approach. By solving a light-front QCD Hamiltonian with a three-dimensional confining potential, we obtain the light-front wavefunctions (LFWFs) for both the quark-antiquark and quark-antiquark-gluon Fock sectors. These wavefunctions are then employed to compute the unpolarized generalized parton distributions (GPDs) and the transverse-momentum-dependent parton distributions (TMDs) of valence quarks and gluons. We also extract the transverse spatial distributions, providing the squared radii of quark and gluon densities in the impact-parameter space. This work contributes toward a three-dimensional understanding of the pion's internal structure in both momentum and coordinate space.

# Conventional way…if you like/can






**Convergence in charmonium structure: light-front wave functions from basis light-front quantization and Dyson-Schwinger equations** #2

Xianghui Cao (Hefei, CUST), Yang Li (Hefei, CUST), Chao Shi (NUAA, Nanjing), James P. Vary (Iowa State U.), Qun Wang (Hefei, CUST and Guangxi Normal U.) (Jul 23, 2025)

e-Print: 2507.17330 [hep-ph]

pdf cite claim reference search 0 citations

# Play games…Inverse Problem



LFWFs

TMDs

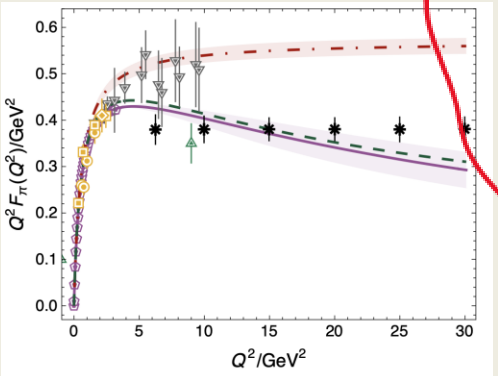
PDAs

2GeV !

GPDs

FormFactors
Electromagnetic
Gravitational…

PDFs



# Play games...Inverse Problem



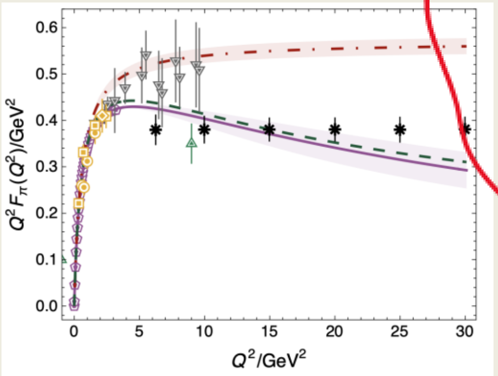
TMDs

PDAs

2GeV !

FormFactors
Electromagnetic
Gravitational...

PDFs



# Play games…Inverse Problem

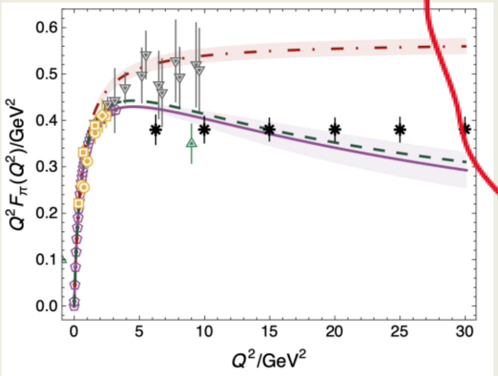


TMDs

PDAs

FormFactors
Electromagnetic
Gravitational…

PDFs




**Determination of the pion generalized parton distributions at zero skewness** #1

MMGPDs Collaboration • Muhammad Goharipour (IPM, Tehran) et al. (Aug 20, 2025)

e-Print: 2508.15073 [hep-ph]

pdf cite claim reference search 1 citation

# Play games…Inverse Problem

## FACT

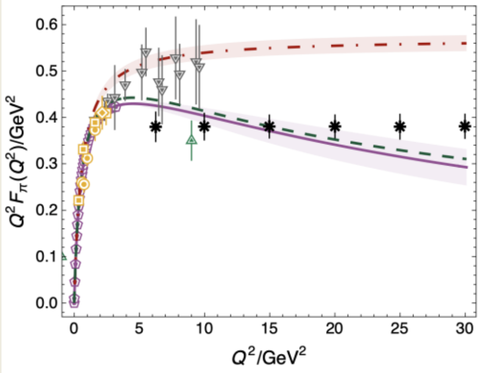
- VMD at low $Q^2$

$$\frac{m_v^2}{Q^2 + m_v^2}$$

- QCD prediction at high $Q^2$

$$Q^2 F_\pi(Q^2) \overset{Q^2 \gg \Lambda_{\rm QCD}^2}{\sim} 16\,\pi\, f_\pi^2\, \alpha_s(Q^2)\, w_\pi^2; \qquad w_\pi = \frac{1}{3}\int_0^1 dx\, \frac{1}{x}\,\varphi_\pi(x)$$

$$\sim \frac{m_v^2}{Q^2 + m_v^2}\ \alpha(Q^2)\omega^2(Q^2)$$




FormFactors
Electromagnetic
Gravitational…

# Play games...Inverse Problem

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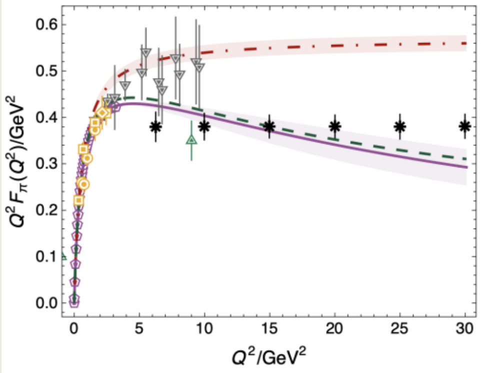
- Monople!

$$\frac{2\lambda}{Q^2 + 2\lambda}$$

- What we need?

$$F_\tau^q(t) = \int_0^1 dx\, H_v^q(x, \xi = 0, t)\,,$$

An integral representation!




FormFactors
Electromagnetic
Gravitational...

# A way

PHYSICAL REVIEW LETTERS **120,** 182001 (2018)

**Universality of Generalized Parton Distributions in Light-Front Holographic QCD**

Guy F. de Téramond,[1] Tianbo Liu,[2,3] Raza Sabbir Sufian,[2] Hans Günter Dosch,[4] Stanley J. Brodsky,[5] and Alexandre Deur[2]

(HLFHS Collaboration)

# Bridging (LFHmodel)

$$F_\tau(Q^2) = \frac{1}{N_\tau} B\left(\tau - 1, \tau_0 + \frac{Q^2}{4\lambda}\right),$$

where $N_\tau = \Gamma(\tau_0)\Gamma(\tau-1)/\Gamma(\tau+\tau_0-1)$ and

$$B(\alpha,\beta) = \int_0^1 (1-y)^{\alpha-1} y^{\beta-1} dy\,.$$

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$$B(\alpha,\beta) = \int_0^1 (1-y)^{\alpha-1} y^{\beta-1} dy\,.$$

- for integer tau

$$F_\tau(Q^2) \sim \frac{1}{(Q^2+M_0^2)\cdots(Q^2+M_{\tau-2}^2)},$$

where one can readily identify $M_n^2 = 4\lambda(n+1/2)$.

- tau=2

$$F(Q^2) \sim \frac{2\lambda}{Q^2+2\lambda}$$

# Bridging (LFHmodel)

$$F_\tau(Q^2) = \frac{1}{N_\tau} B\left(\tau - 1, \tau_0 + \frac{Q^2}{4\lambda}\right),$$

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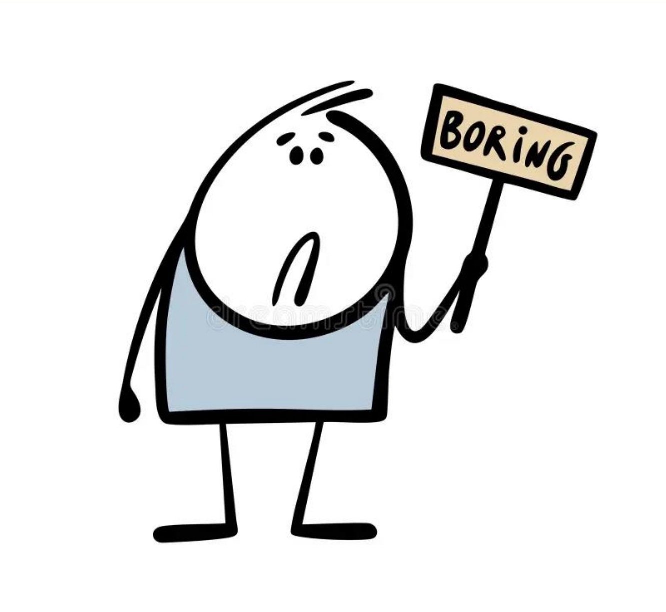
$$B(\alpha,\beta) = \int_0^1 (1-y)^{\alpha-1} y^{\beta-1} dy\,.$$

$$F_\tau^q(t) = \int_0^1 dx\, H_v^q(x, \xi = 0, t)\,,$$

$$y = x$$

$$\sim \frac{1}{\sqrt{x}} x^{\frac{Q^2}{4\lambda}}$$



# Complicating it



$$F_\tau(Q^2) = \frac{1}{N_\tau} B\left(\tau - 1, \tau_0 + \frac{Q^2}{4\lambda}\right),$$

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$$F_\tau^q(t) = \int_0^1 dx\, H_v^q(x, \xi = 0, t)\,,$$

a transformation $y = w_\tau(x, Q^2)$, provided the following conditions [33, 36–38]:

$$w_\tau(0, Q^2) = 0,\ \ w_\tau(1, Q^2) = 1,\ \ \frac{\partial w_\tau(x, Q^2)}{\partial x} \geq 0\,. \quad (5)$$

According to Ref. [33], we can assume that $w_\tau(x, Q^2)$ is independent of $Q^2$, working hereafter with $w_\tau(x)$. Thus, the EBF representation of the EFF can be cast as:

$$F_\tau(Q^2) = \frac{1}{N_\tau}\int_0^1 dx\,(1 - w_\tau(x))^{\tau-2} w_\tau(x)^{\frac{Q^2}{4\lambda} - \frac{1}{2}} \frac{\partial w_\tau(x)}{\partial x}\,.$$

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Let us now consider the valence-quark GPD, expressed in terms of its corresponding LFWF via the overlap representation [21, 22]:

$$H_\tau^q(x,Q^2) = \int d^2\mathbf{b}_\perp\, e^{i(1-x)\mathbf{b}_\perp\cdot\mathbf{Q}} |\psi_\tau^q(x,\mathbf{b}_\perp)|^2\,. \quad (10)$$

Here $\mathbf{b}_\perp$ denotes the distance between the struck parton, relative to the hadron's transversse center of momentum. Inverting the Fourier transform, one gets:

$$|\psi_\tau^q(x,\mathbf{b}_\perp)|^2 = \frac{(1-x)^2}{(2\pi)^2}\int d^2Q\, e^{-i(1-x)\mathbf{b}_\perp\cdot\mathbf{Q}} H_\tau^q(x,Q^2)\,.$$

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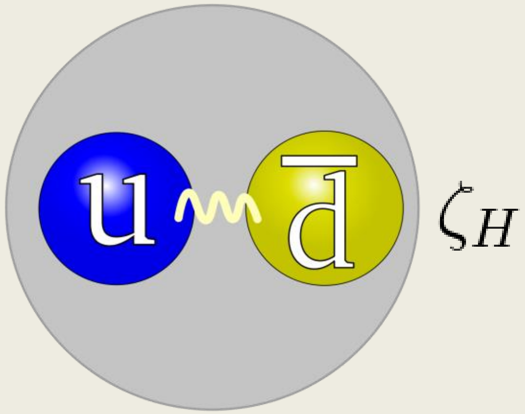
# Imagine the Hadron at the Hadronic Scale





Physics Letters B
Volume 93, Issue 3, 16 June 1980, Pages 313-317

Normalizing the renormalization group analysis of deep inelastic leptoproduction

R.L. Jaffe [1], G.G. Ross [2]

Regarding the Distribution of Glue in the Pion

Lei Chang (Nankai U.), Craig D. Roberts (Nanjing U.) (Jun 15, 2021)

Published in: *Chin.Phys.Lett.* 38 (2021) 8, 081101 • e-Print: 2106.08451 [hep-ph]

# Approximations

- Two-Quark Dominance: We assume that the LFHQCD model is defined at $\zeta_H$, so all the hadron properties are captured within its valence degrees of freedom. In line with this assumption, the LFWF is taken at leading twist and includes only valence-quark components.

monople

- Form Factor Representation: The twist-2 electromagnetic form factor admits a EBF representation, Eq. ($\boxed{1}$). This is a distinctive feature of the AdS/QCD and LFHQCD approaches. Together with the reparameterization invariance of the EBF, the sum rule in Eq. ($\boxed{4}$) allows these quantities to be expressed in terms of a profile function $w_\tau(x)$.

- Scale Independence: The profile function $w_\tau(x)$ is independent of $Q^2$. Since it is not determined from first principles but only requires to satisfy the conditions of Eq. ($\boxed{5}$), this common simplification is adopted.

$x \to 1 - x$ symmetry

In[1]:=
```
Clear[k02, y, s, Q2]
Clear[s]
Integrate[1/2^(1+s) 1/Gamma[1+s] Log[1/y]^s y^(Q2/(4 lambda) - 1/2), {y, 0, 1},
 Assumptions -> lambda > 0 && s > 0 && Q2 > 0]
```

Out[3]= $2^{1+s}\left(\frac{\text{lambda}}{2\,\text{lambda}+\text{Q2}}\right)^{1+s}$

s=0

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$x \to 1 - x$ symmetry

monople

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Out[3]= $2^{1+s}\left(\frac{\text{lambda}}{2\,\text{lambda} + \text{Q2}}\right)^{1+s}$

s=0

# Any other representation?

## DeepSeek's answer!

Question

函数(m2/(Q^2+m2))^{1+s}可以表示成如下积分形式Integrate[1/2^(1 + s) 1/ Gamma[1 + s] Log[1/y]^s y^(Q2/(4 m2) - 1/2),{y,0,1}]。请问，该函数是否还有独立于上表示的积分形式，且积分限从0到1。

好的，我们先整理一下已知条件，然后尝试寻找其他的积分表示。

Answer

寻找函数积分形式的另一种表示

**4. 检查正确性**

验证：令 $Q^2=0$，则

$$f(0)=1=\frac{1}{\Gamma(1+s)}\int_0^1(-\ln z)^s dz.$$

而 $\int_0^1(-\ln z)^s dz=\Gamma(1+s)$，成立。

所以另一种积分表示是：

$$\boxed{\left(\frac{m^2}{Q^2+m^2}\right)^{1+s}=\frac{1}{\Gamma(1+s)}\int_0^1\left(\ln\frac{1}{z}\right)^s z^{Q^2/m^2}dz}$$

其中 $z\in(0,1]$，$s>-1$。

这个表示比原已知表示更简洁，且独立于原表示（核函数不同）。

$$F_\tau(Q^2)=\frac{1}{N_\tau}B\left(\tau-1,\tau_0+\frac{Q^2}{4\lambda}\right), \tag{1}$$

where $N_\tau=\Gamma(\tau_0)\Gamma(\tau-1)/\Gamma(\tau+\tau_0-1)$ and

$$B(\alpha,\beta)=\int_0^1(1-y)^{\alpha-1}y^{\beta-1}dy. \tag{2}$$

Here $Q^2$ is the momentum transferred by the electromagnetic probe. The choice $\tau_0=1$ corresponds to the AdS/QCD framework, whereas $\tau_0=1/2$ defines the LH-FQCD case. While this distinction does not impact the subsequent content of the article, in the rest of the discussion it should be understood that we adopt the latter convention. In this case, the universal scale $\lambda$ is fixed

- Two types (corresponding to LFHmodel and Ads/QCD
- No difference in our discussion!

# Bridging (LFHmodel) at hadronic scale

$x \to 1 - x$ symmetry

Let us now consider the valence-quark GPD, expressed in terms of its corresponding LFWF via the overlap representation [21, 22]:

$$H_\tau^q(x,Q^2) = \int d^2\mathbf{b}_\perp\, e^{i(1-x)\mathbf{b}_\perp\cdot\mathbf{Q}}|\psi_\tau^q(x,\mathbf{b}_\perp)|^2 . \qquad (10)$$

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$$\psi(x,\mathbf{b}_\perp) = \psi(1-x,\mathbf{b}_\perp)$$
$$\Leftrightarrow H(x,Q^2) = H\left(1-x, \frac{(1-x)^2}{x^2}Q^2\right) .$$

$$H(x,Q^2) \sim q(x)\mathcal{M}(x)^{\frac{Q^2}{2\lambda}}$$

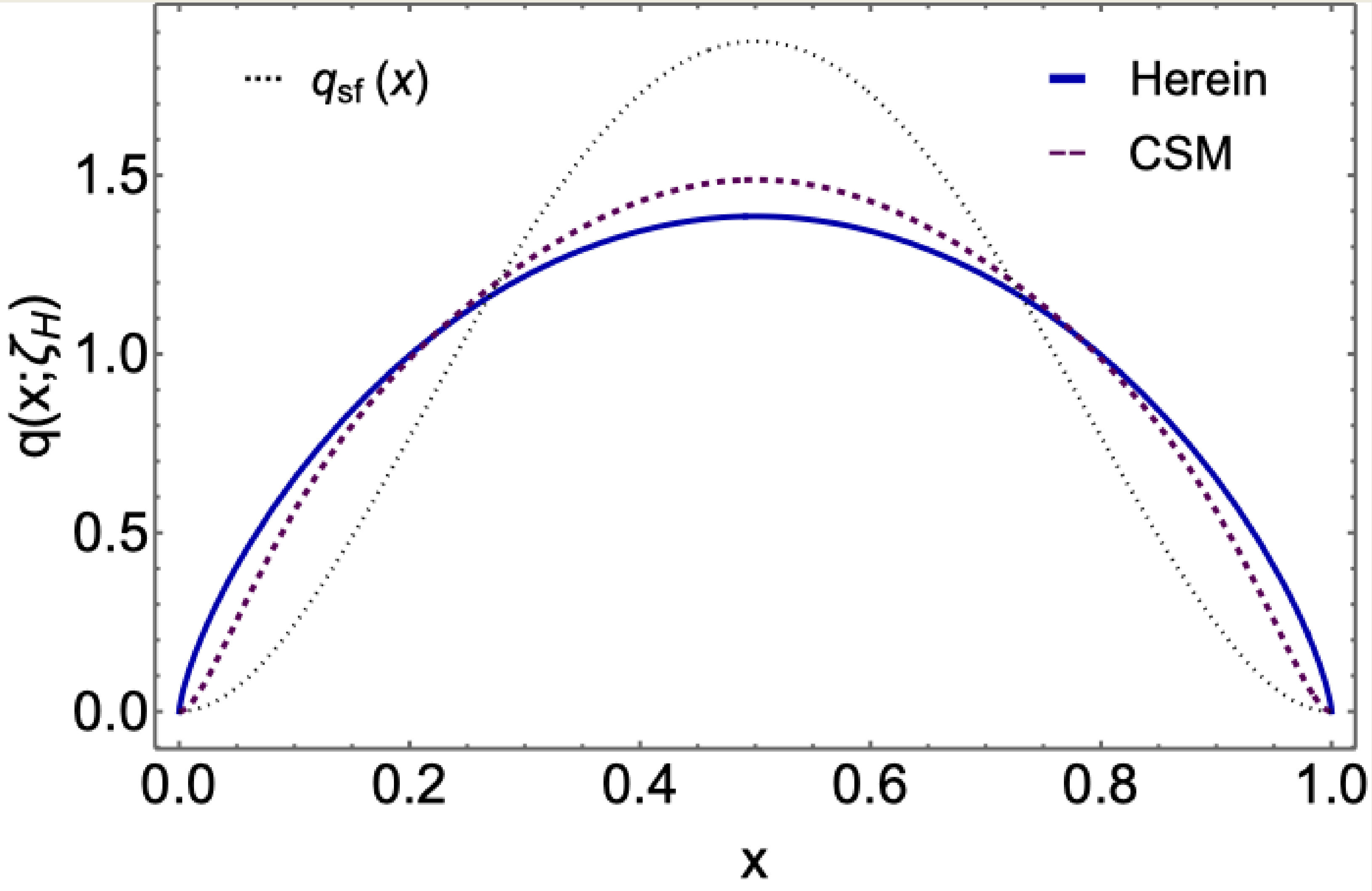
$$q(x) = \left[\frac{1}{2}w'(x)w(x)^{-\frac{1}{2}}\right] = \mathcal{M}'(x) , \ \mathcal{M}(x) := \sqrt{w(x)} .$$

$$\mathcal{M}(x) + \mathcal{M}(1-x) = 1 , \Leftarrow q(x) = q(1-x)$$

$$\mathcal{M}(x)^{x^2/(1-x)^2} = 1 - \mathcal{M}(x) .$$

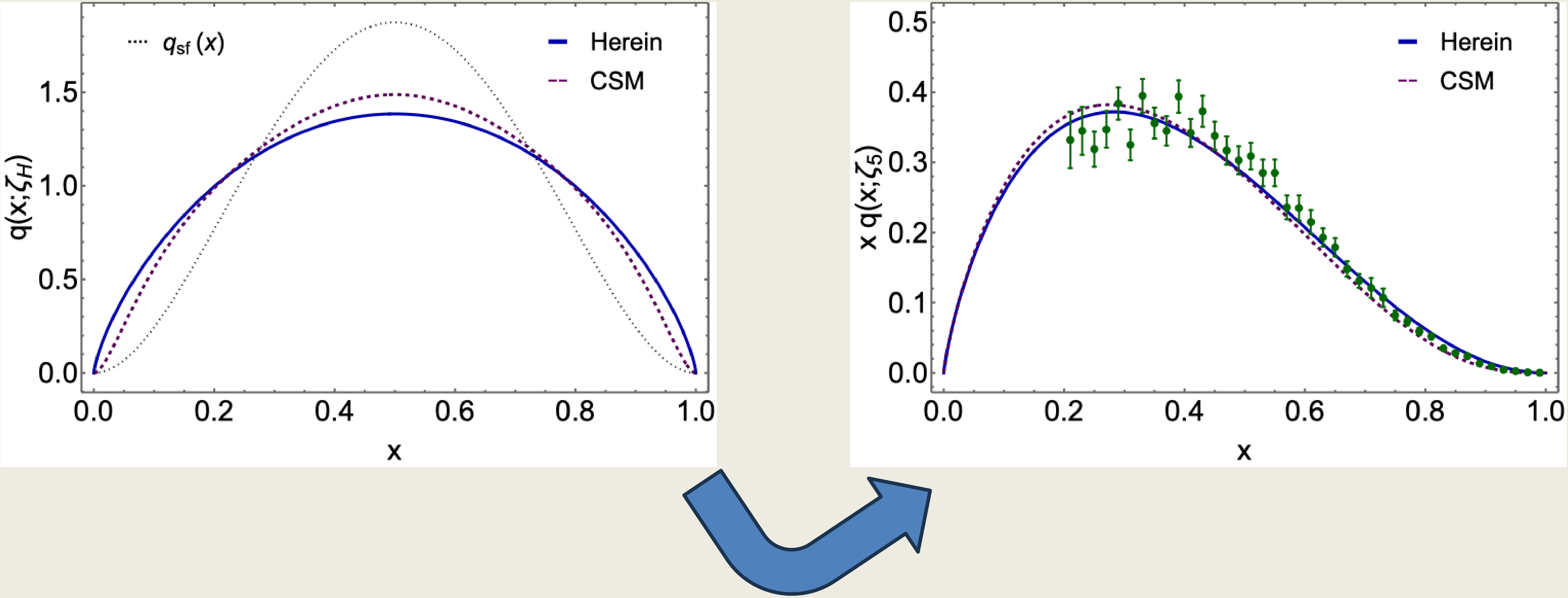
$$\mathcal{M}(x) \to q(x) \to H(x,Q^2) \to \psi(x,b) \to \varphi(x)$$

# Distribution function(DF at the hadronic scale)



# Distribution function(DF at 5.2GeV)

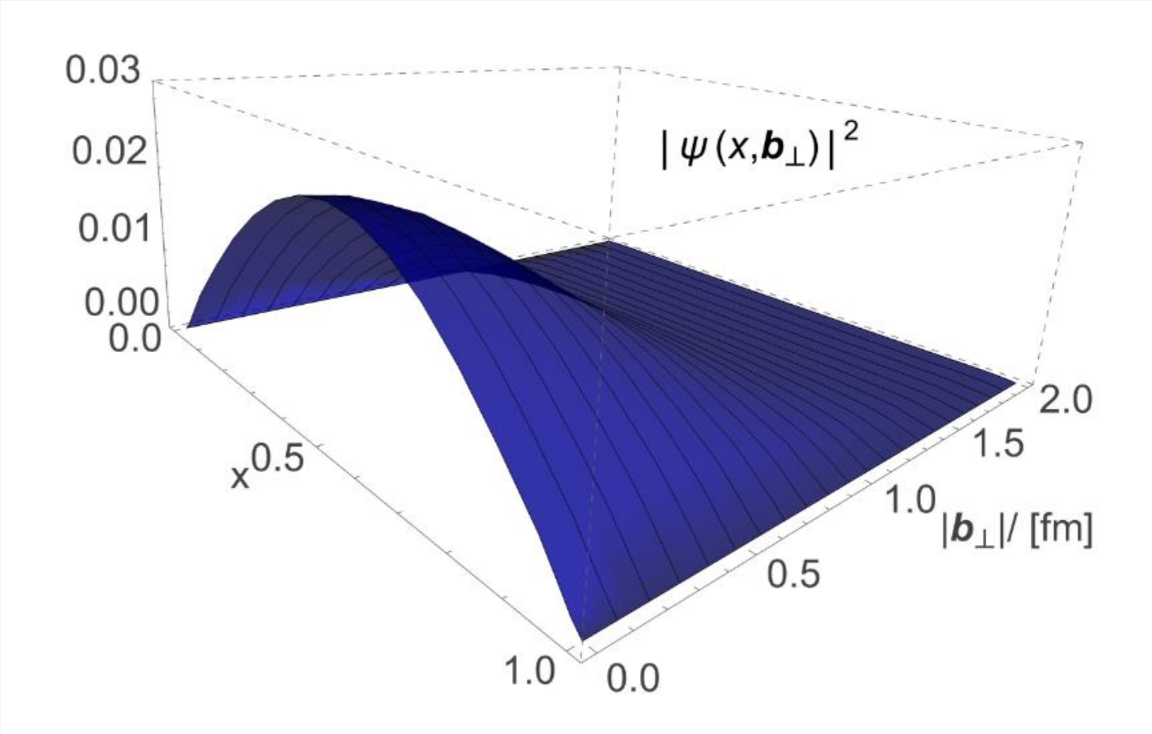


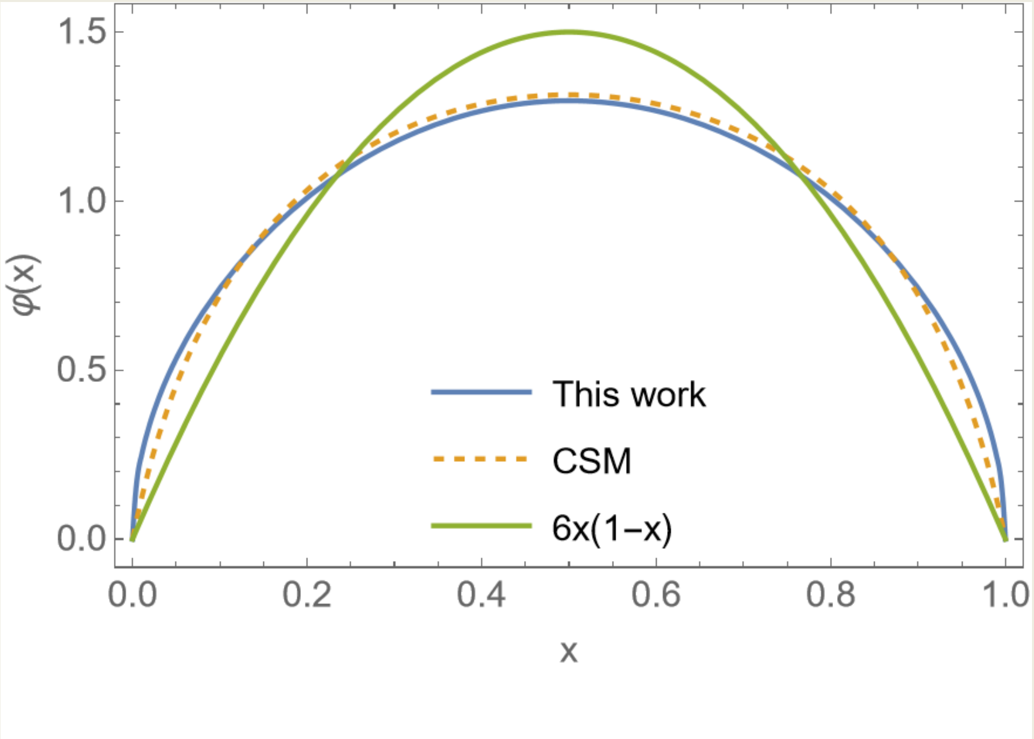



All orders DGLAP Evolution:Pepe’s talk

# Distribution amplitude(DA at hadronic scale)

GPDs → LFWFs → PDAs



# Distribution Amplitude(DA)...

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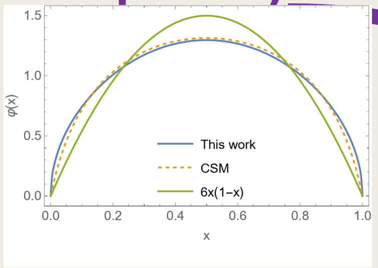
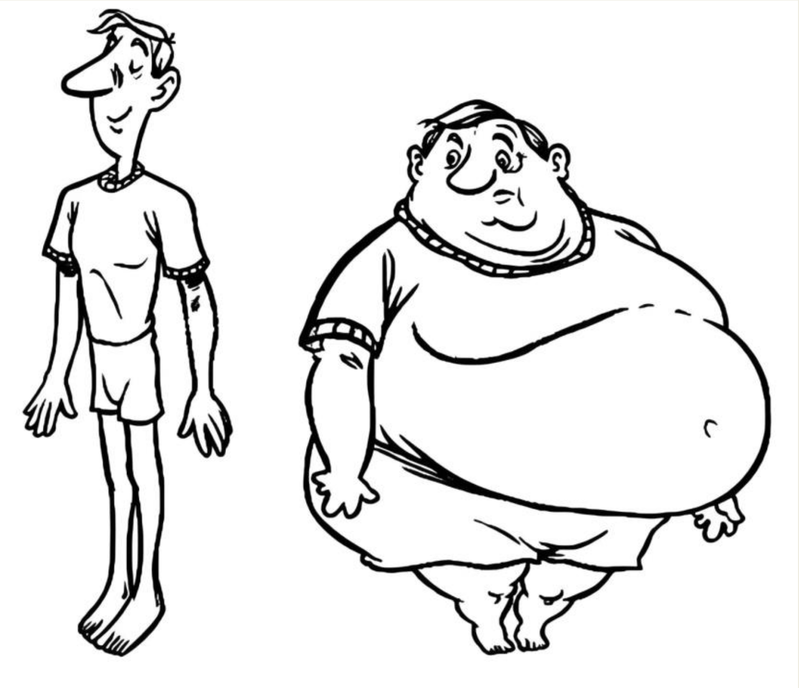
- $(x \to 1 - x$ symmetry,monopole)




$\langle (2x-1)^2 \rangle = 0.2 \;\; vs \;\; \langle (2x-1)^2 \rangle = 0.25$



**Broad(FAT)!**



CSM and lQCD： both at 2GeV? I will go back later

# Distribution Amplitude(DA)...

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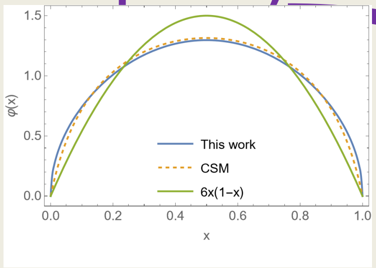
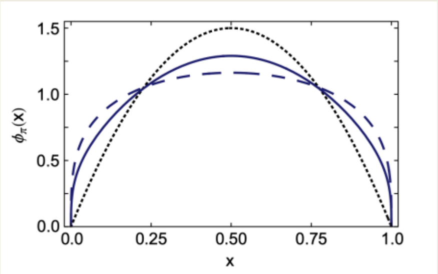



- CSM




- lQCD

lQCD DA moment at 2GeV

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LaMET
R. Zhang, et al., arXiv: 2005.13955 0.244(30)(20)
Jun Hua, et al., arXiv:2201.09173 0.300(41)
Xiang Gao, et al., arXiv: 2206.04084 0.287(6)(6)
Jack Holligan, et al., arXiv: 2301.10372 0.302(23)

Euclidean correlation functions
G. S. Bali, et al., arXiv: 1807.06671 0.3

Local twist-2 operator
G. S. Bali, et al., arXiv: 1903.08038 0.240(6)(2)(3)(2)
R. Arthur, et al., arXiv: 1011.5906 0.28(1)(2)

Lei Chang (NKU)

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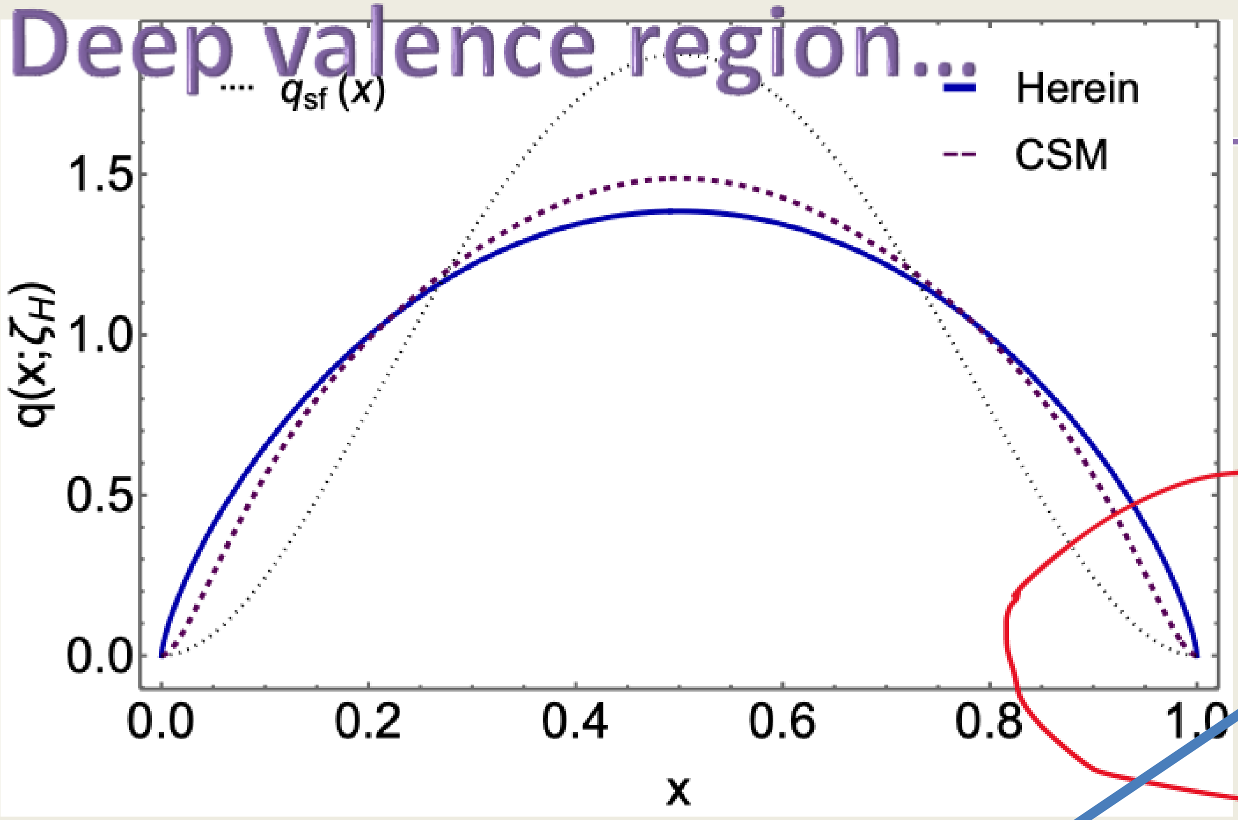
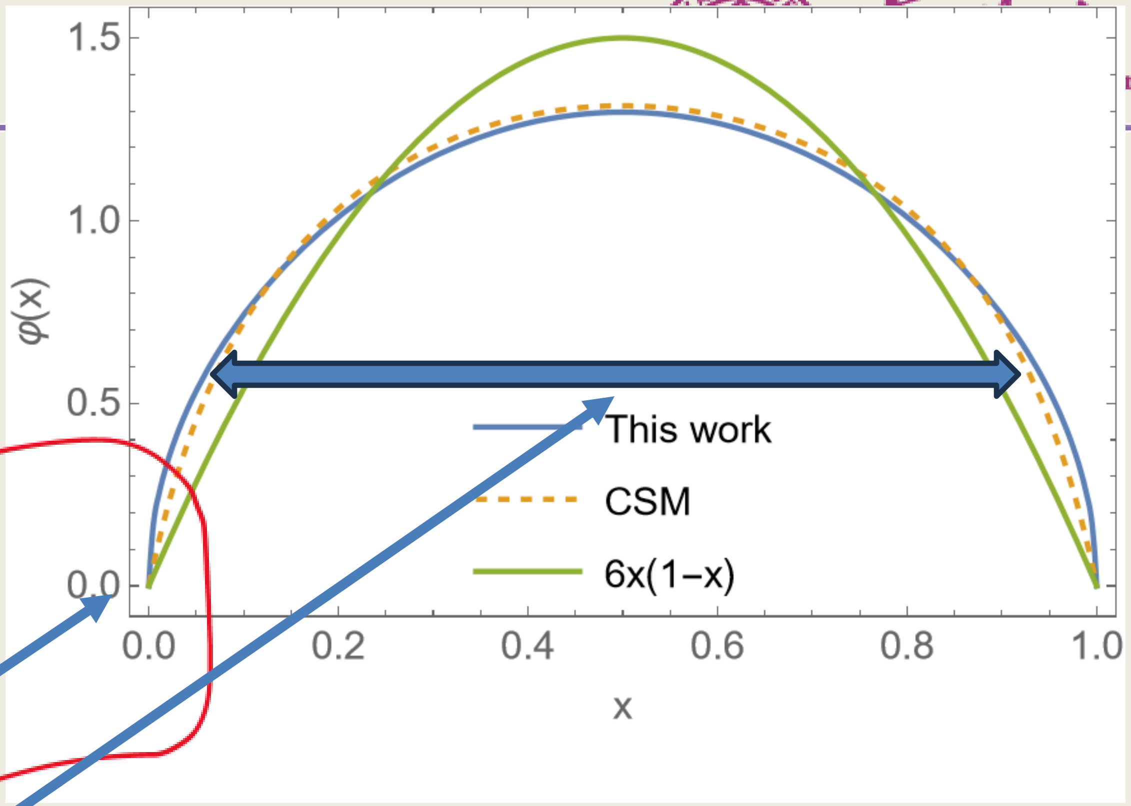
# Broad(FAT)!

CSM and lQCD： both at 2GeV? I will go back later

# Deep valence region...



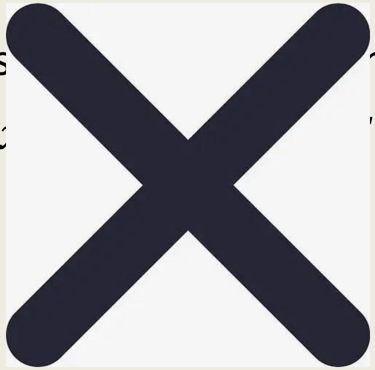

- What happen here?
- How fat is fat?

# Minding the interaction

# No interaction:

- In case of a system of two equal-mass non-interacting particles, $DA(x) \sim \delta\left(x - \frac{1}{2}\right) \sim DF(x)$;

## No interaction:

- In case of a sys  mass non-interacting particles, $DA($ … $(x)$;

## Can Never be true for pion!!!

2M+U=0

## turn on interaction:

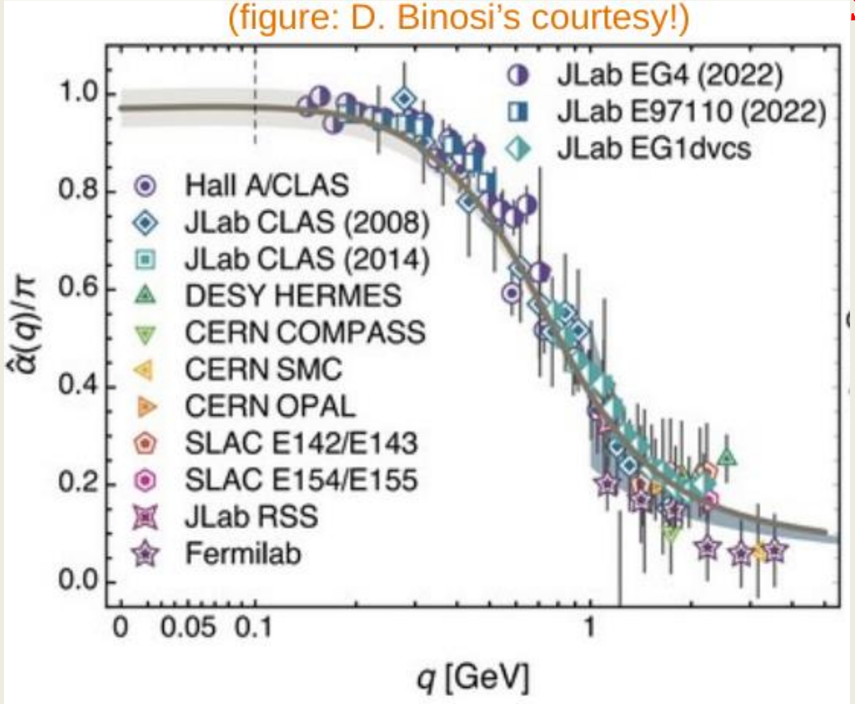
- When the interaction is switched on, the DA broadens. The width may be estimated as $\Gamma \sim E_{int}/m_q$;
- Special case: zero-range interaction…Picture: the probabilities for quark and antiquark are same whatever quarks carry how much momentum…
- Zero interaction also means that the wavelength of detecting is infinite…zero scale!

## DF=constant=DA

Recall the unconstrained maximum entropy condition…

# A rough picture(contact model)

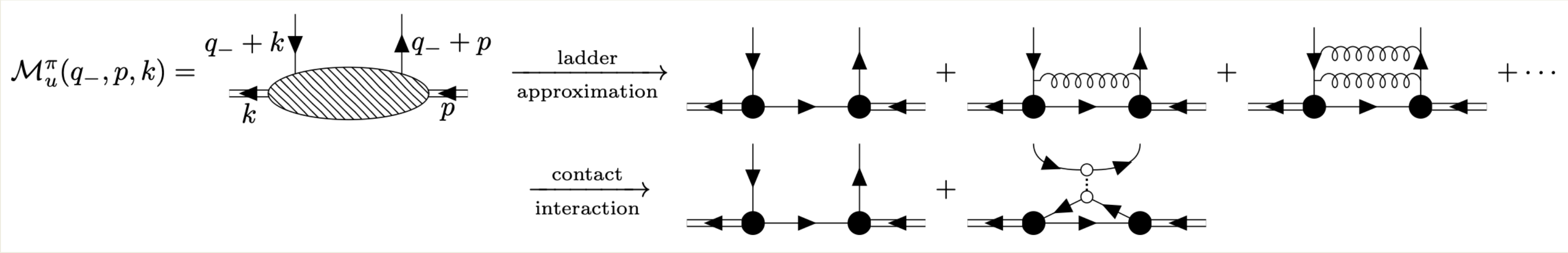
Zanbin Xing, et al., arXiv: 2301.02958




*Suppose* $\alpha(q) \to$ *constant*

- Effective interaction $\frac{4\pi\alpha(q=0)}{m_G^2}$ $(\alpha\sim\pi, m_G\sim0.5$
- Quark Mass/Bound state Amplitude momentum independent…
- $\alpha\sim0.36\pi, m_G\sim0.5$

- Calculate the pion-pion amplitude within symmetry-preserving way
- the polynomiality condition and sum rules are satisfied




In the chiral limit

$$H_u^{\mathrm{c.l.}}(x,\xi,0) = \frac{1}{2}\theta(\xi - x, x+\xi) + \theta(1-x, x-\xi)\,.$$

DF(xi=0)=1=DA(xi=1)

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Zanbin Xing, et al., arXiv: 2301.02958

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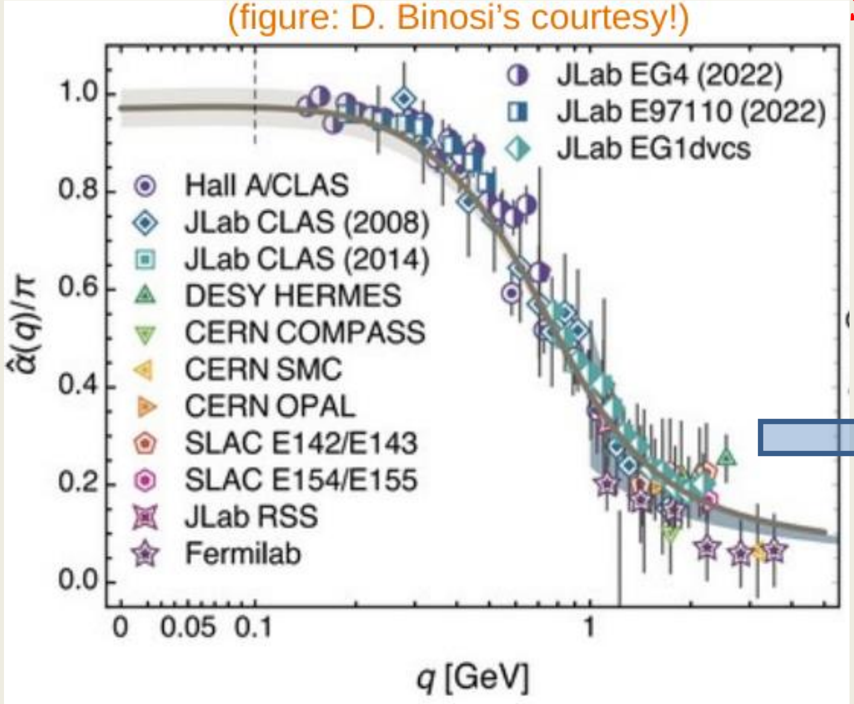


What happen here?

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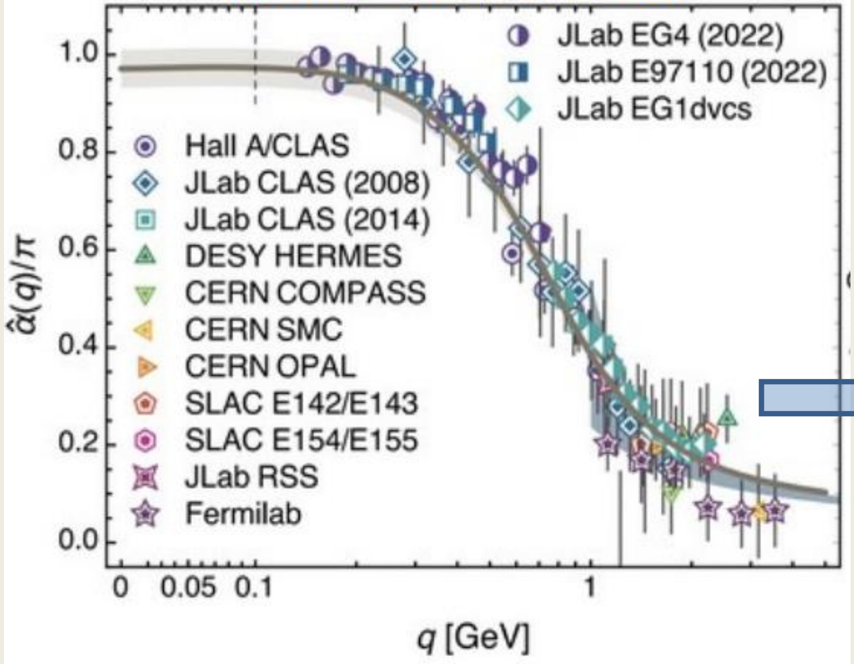
Zanbin Xing, et al., arXiv: 2301.02958

(figure: D. Binosi's courtesy!)

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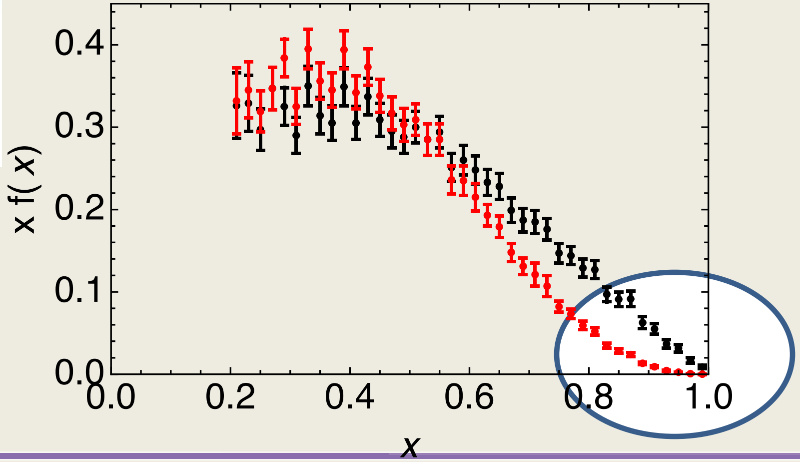


Model and Model

- Nambu – Jona-Lasinio model, translationally invariant regularisaion
  $q^\pi(x) \sim (1-x)^{0}$,
  which becomes "1" after evolving from a low resolution scale
- NJL models with a hard cutoff & also some duality arguments:
  $q^\pi(x) \sim (1-x)^{1}$
- Relativistic constituent quark models:
  $q^\pi(x) \sim (1-x)^{0\ldots2}$
  depending on the form of model wave function
- Instanton-based models
  $q^\pi(x) \sim (1-x)^{1\ldots2}$

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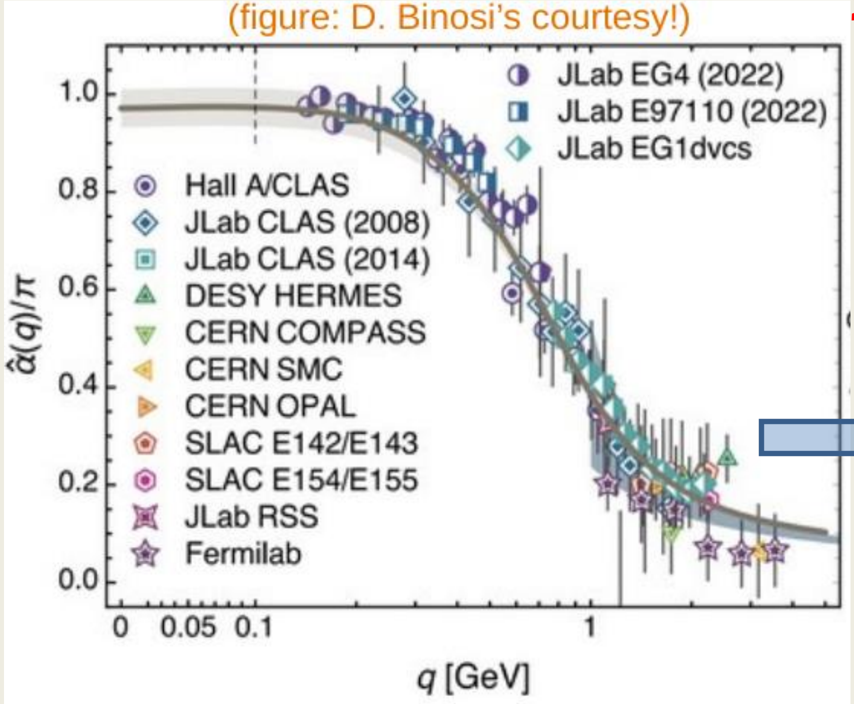
Zanbin Xing, et al., arXiv: 2301.02958

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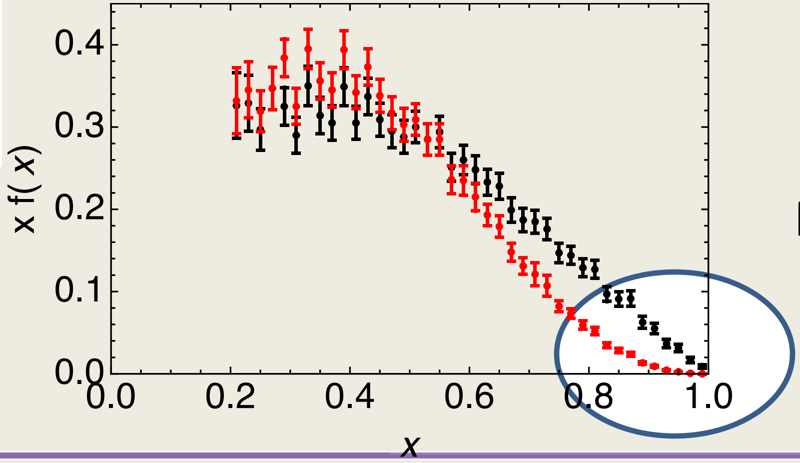
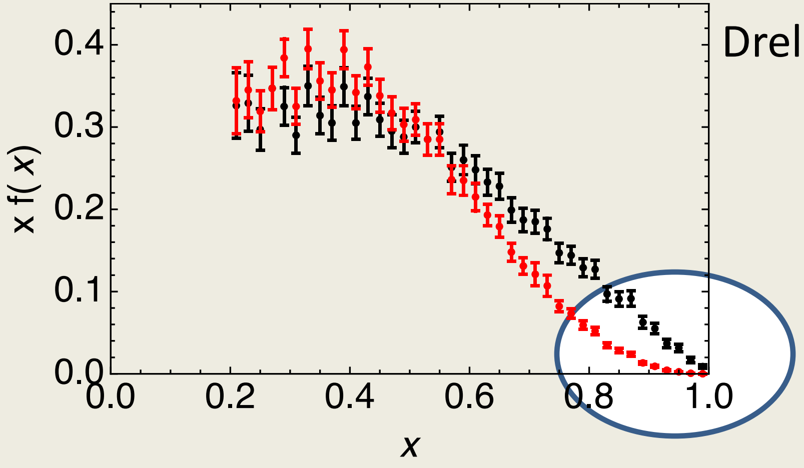



## Proved in QCD?

Modern language(xiaobin wang)

# A discussion about the relation




Drell-Yan West relation

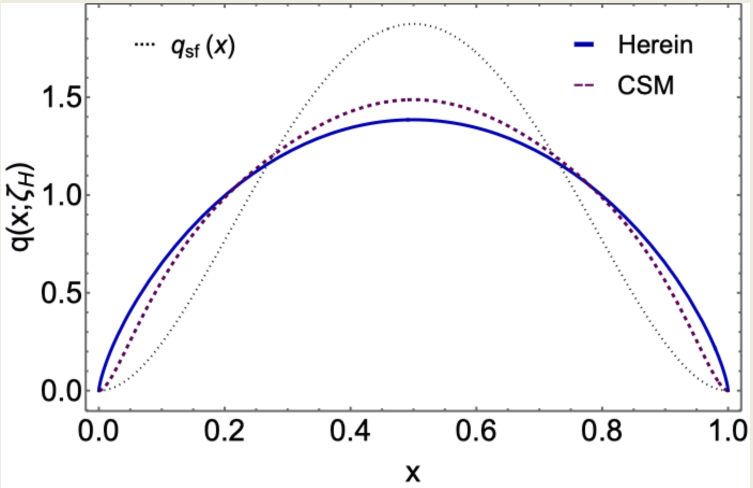
$$\lim_{x\to 1} q(x) \propto (1-x)^{n_H}$$

$$\lim_{Q^2\to\infty} F(Q^2) \propto \frac{1}{(Q^2)^{(n_H+1)/2}}$$

Perturbation QCD prediction

$$\lim_{x\to 1} q(x) \propto (1-x)^{2n_H-3+2|\Delta_s|}$$
$$\sim(1-x)^{2n_H-3+2} \text{（}for\ pion\text{）}$$

$$\lim_{Q^2\to\infty} F(Q^2) \propto \frac{1}{(Q^2)^{n_H-1}}$$




$$F(Q^2)\sim\frac{1}{Q^2}$$

$$\lim_{x\to 1} q(x) \propto (1-x)\log\frac{1}{1-x}$$

$$q(x)\sim x^{2/3}(1-x)^{2/3}$$

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# Infinite Scale

- Special case: one-loop evolution output at the infinity scale

$$DA = 6x(1-x)$$

# Zero Scale

- Special case: zero-range interaction…Picture: the probabilities for quark and antiquark are same whatever quarks carry how much momentum…

**DA=1**

# Chiral Symmetry and Bethe-Salpeter equation

Maris, Roberts and Tandy, **Phys. Lett. B420**(1998) 267-273

- Pion's Bethe-Salpeter amplitude Solution of the Bethe-Salpeter equation

$$\Gamma_{\pi^j}(k;P) = \tau^{\pi^j}\gamma_5\Big[iE_\pi(k;P) + \gamma\cdot P F_\pi(k;P) + \gamma\cdot k\, k\cdot P\, G_\pi(k;P) + \sigma_{\mu\nu}\, k_\mu P_\nu\, H_\pi(k;P)\Big]$$

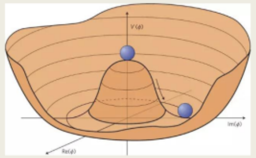
- Dressed-quark propagator $S(p) = \dfrac{1}{i\gamma\cdot p\, A(p^2) + B(p^2)}$

- Axial-vector Ward-Takahashi identity entails(chiral limit)

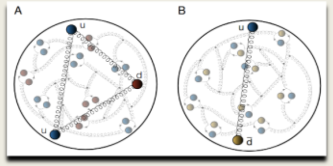
$$\boxed{f_\pi E(k;P|P^2=0) = B(k^2)} + (k\cdot P)^2 \frac{d^2 B(k^2)}{d^2 k^2} + \ldots$$

PION's dichotomy

Goldstone Boson



Bound State



Bethe-Salpeter Equations

# Boundaries of pion DF and moments at the hadronic scale

$$f_\pi E(k;P|P^2=0) = B(k^2) + (k\cdot P)^2 \frac{d^2 B(k^2)}{d^2 k^2} + \ldots$$

Nakanishi representation

$$E(k;P) \sim \int_{-1}^{1} dz \rho(z) \frac{M^2}{k^2 + z\, k\cdot P + M^2}$$

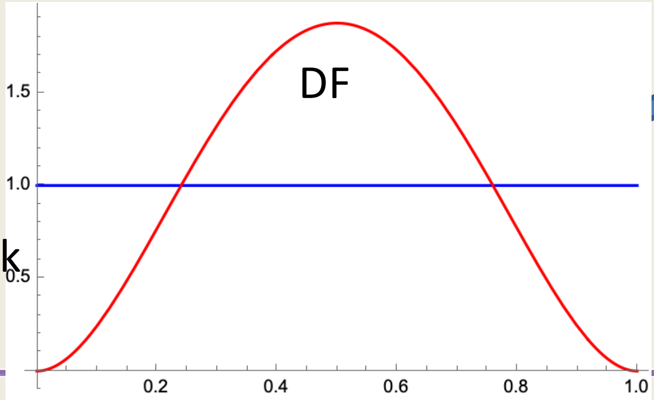
$\rho(z) \propto 1 - z^2$

$\rho(z) \propto \delta(1+z) + \delta(1-z)$

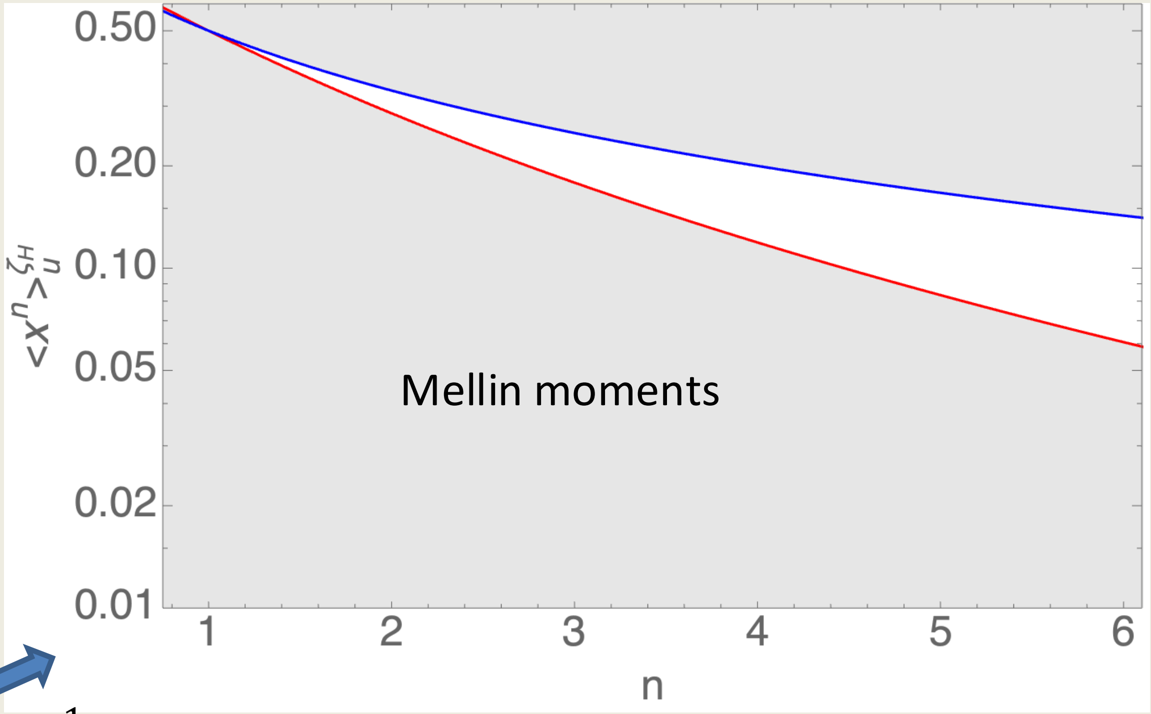
$DA \propto x(1-x)$

$DA \propto 1$

+Constituent Quark




$$\langle x^n \rangle \propto \int_0^1 dx\ x^n DF(x)$$




- Boundaries of Mellin moments at the hadronic scale
- Remember the massless properties of pion!

# My models for DA for two typical hadronic scale

## Infinite Scale

- Special case: one-loop evolution output at the infinity scale

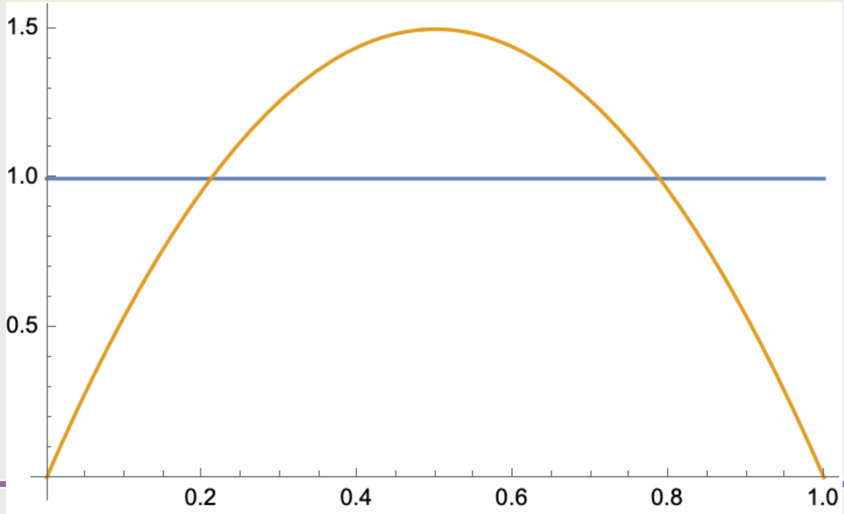
$$DA = 6x(1-x)$$

$$\langle (2x-1)^2 \rangle = 0.2$$

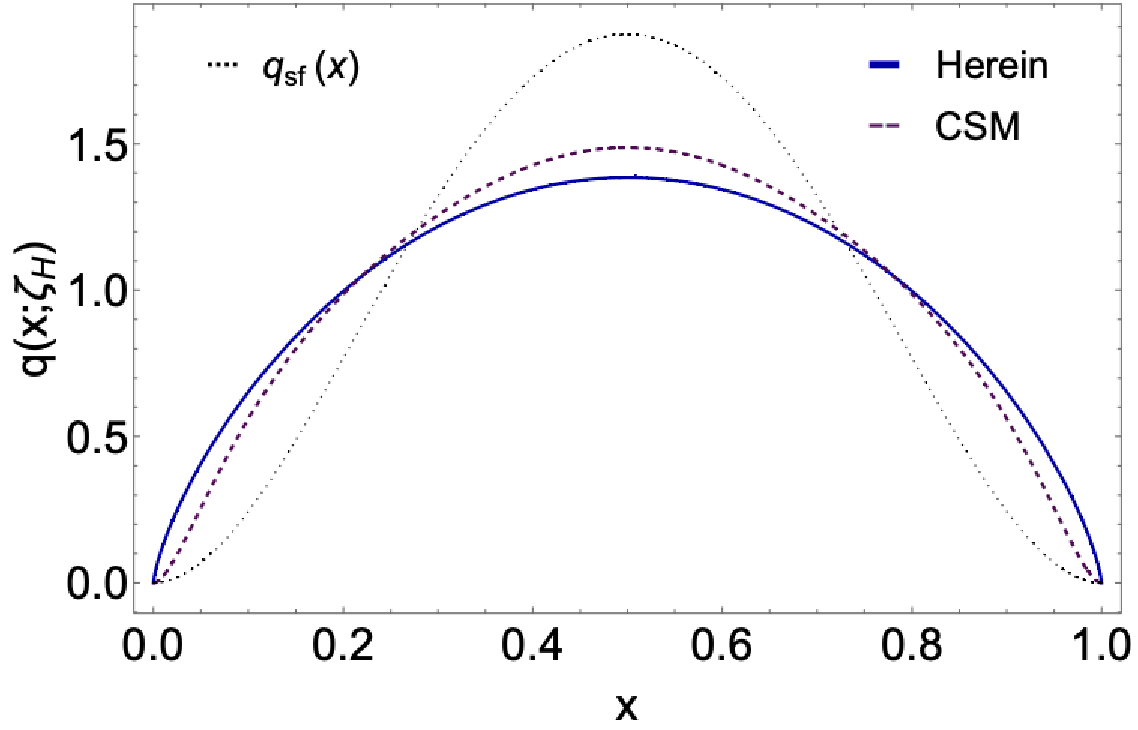
## Zero Scale

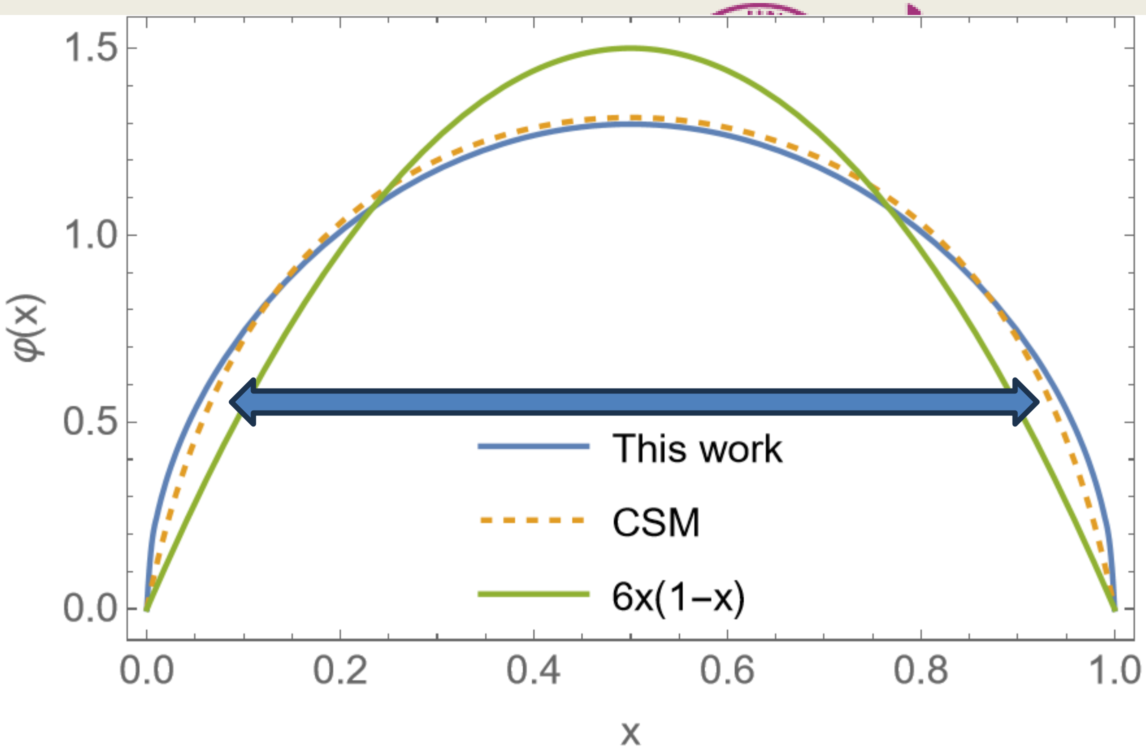
- Special case: zero-range interaction…Picture: the probabilities for quark and antiquark are same whatever quarks carry how much momentum…

$$DA=1$$




$$\langle (2x-1)^2 \rangle = 0.333\ldots$$

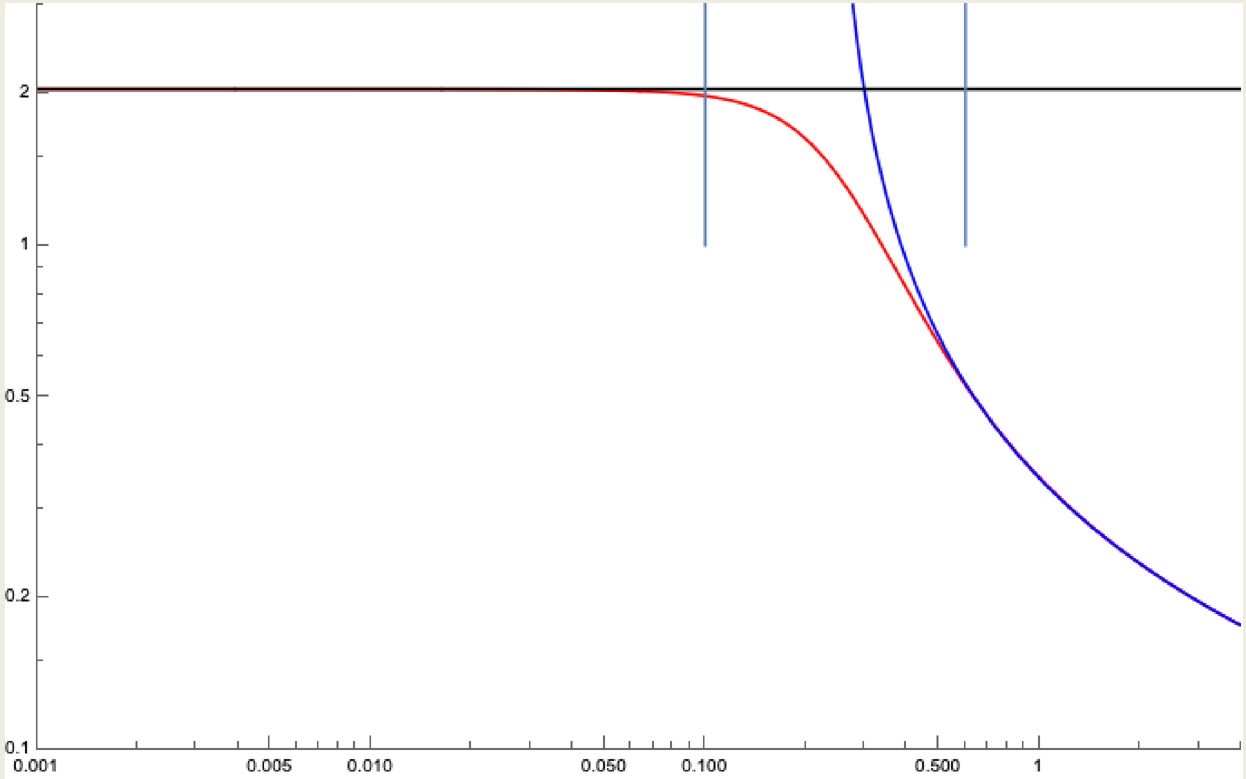



- Evolution from hadronic scale to 2GeV (Lattice data…)

# Zero Scale

# Infinite Scale




Suppose $\varphi(x) = 6x(1-x)(1+\sum_n' a_n C_n^{3/2}(2x-1))$ we can get

$$\frac{a_n(\zeta)}{a_n(\zeta_0)} = \mathrm{e}^{-\gamma_n \int_{\log \zeta_0^2}^{\log \zeta^2} dt \frac{\alpha(\mathrm{e}^t)}{4\pi}}$$

with $\gamma_n = -\frac{4}{3}\left(3 + \frac{2}{(n+1)(n+2)} - 4\sum_{k=1}^{n+1}\frac{1}{k}\right)$.

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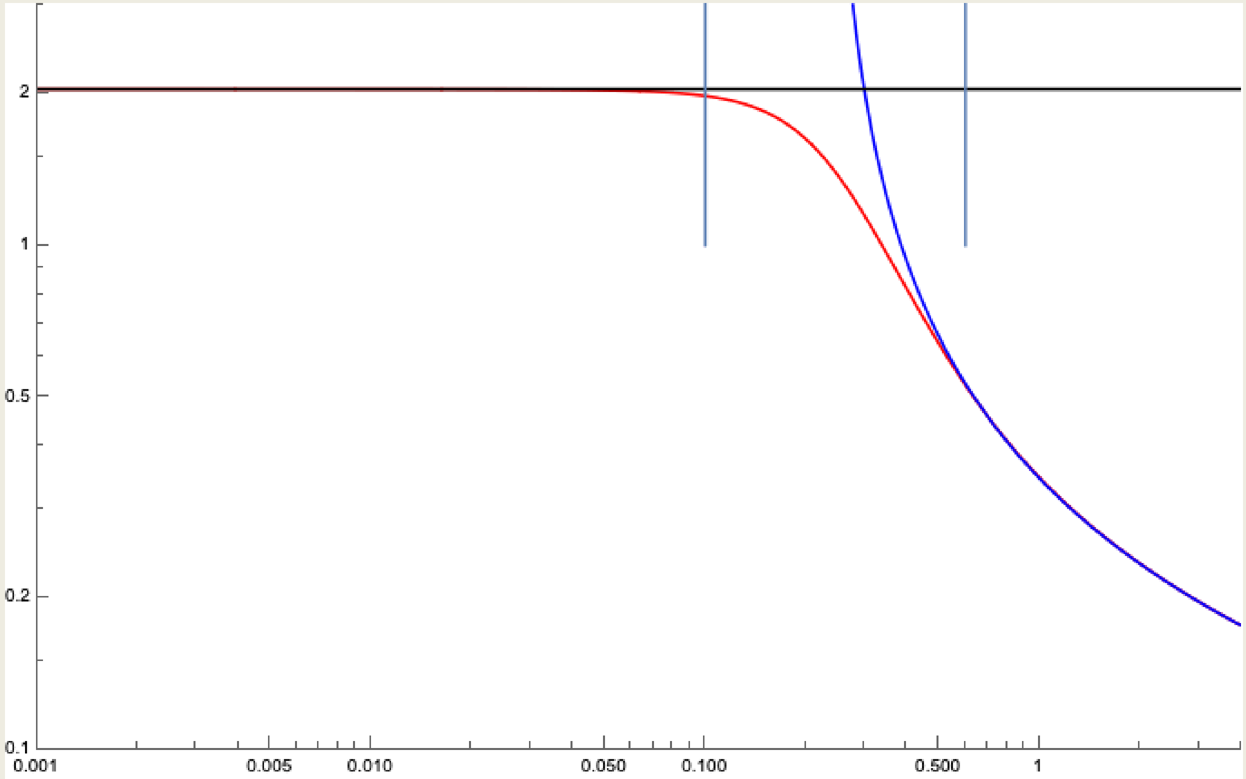
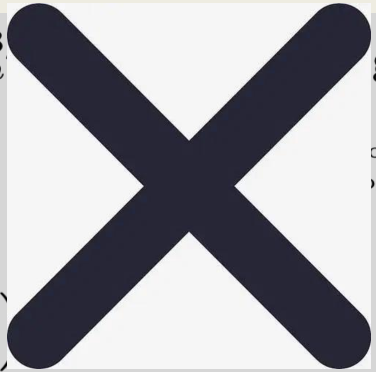
# Zero Scale




# Infinite Scale

Suppose $\varphi(x) = 6x(1-x)(1+\sum_n' a_n C_n^3$  get

$\log \zeta^2$ $\log \zeta_0^2$ $dt \, \frac{\alpha(e^t)}{4\pi}$

with $\gamma_n = -\frac{4}{3}\left(3 + \frac{2}{(n+1)(n+2)} - 4\sum_{k=1}^{n+1} \frac{1}{k}\right)$

# Guessing

$$\mu^2 \frac{\partial\, \varphi(x,\mu)}{\partial\, \mu^2} = \int_0^1 dy\ V(x,y)\ \varphi(y,\mu)$$

With the evolution kernel

$$V(x,y) = \sum_{n=0} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} [V_n(x,y)]_+$$

Modified LO BL kernel

$$V_0(x,y) = 2\mathrm{C}_F \left[\frac{1-x}{1-y}\left(1+\frac{1}{x-y}\right)\theta(x-y) + \frac{x}{y}\left(1+\frac{1}{y-x}\right)\theta(y-x)\right]$$

# Pion PDA: "one-loop" evolution practice

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$$\mu^2 \frac{\partial\, \varphi(x,\mu)}{\partial\, \mu^2} = \int_0^1 dy\; V(x,y)\; \varphi(y,\mu)$$

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PRL 110, 132001 (2013) PHYSICAL REVIEW LETTERS week ending 29 MARCH 2013

**Imaging Dynamical Chiral-Symmetry Breaking: Pion Wave Function on the Light Front**

Lei Chang,[1] I. C. Cloët,[2,3] J. J. Cobos-Martinez,[4,5] C. D. Roberts,[3,6] S. M. Schmidt,[7] and P. C. Tandy[4]

It is readily established that with Eqs. (7)–(9) in Eq. (3), one obtains the "asymptotic" distribution associated with a $(1/k^2)^\nu$ vector-exchange interaction, viz.,

$$\varphi_\pi(x) = \frac{\Gamma(2\nu+2)}{\Gamma(\nu+1)^2} x^\nu (1-x)^\nu. \qquad (12)$$

$$\varphi_\pi^{G_s}(x) = x^{\alpha_-}(1-x)^{\alpha_-}\left[1 + \sum_{2,4,\ldots}^{j_s} a_j^\alpha C_j^\alpha(2x-1)\right],$$

# Pion PDA: "one-loop" evolution practice

$$\mu^2 \frac{\partial\, \varphi(x,\mu)}{\partial\, \mu^2} = \int_0^1 dy\; V(x,y)\; \varphi(y,\mu)$$

With the evolution kernel

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We have nothing knowledge!

Introduce "anomaly-dimension" factor a!

$$V^a_{eff}(x,y) = 2\mathrm{C}_F \left[\left(\frac{1-x}{1-y}\right)^a\left(1+\frac{1}{x-y}\right)\theta(x-y) + \left(\frac{x}{y}\right)^a\left(1+\frac{1}{y-x}\right)\theta(y-x)\right]$$

# Conformal like solution



Suppose $\varphi(x) = \frac{\Gamma[2+2a]}{\Gamma[1+a]^2} x^a (1-x)^a (1 + \sum_n' b_n C_n^{a+\frac{1}{2}}(2x-1))$ we can get

$$\frac{b_n(\zeta)}{b_n(\zeta_0)} = \mathrm{e}^{-\int_{\log \zeta_0^2}^{\log \zeta^2} dt \frac{\alpha(e^t)}{4\pi} \gamma_n^a}$$

with $\boxed{\gamma_n^a = -\frac{4}{3}\left(-\frac{2}{a+1} + \frac{2a}{(n+a)(n+a+1)} - 4\sum_{k=0}^{n-1} \frac{1}{k+a+1}\right).}$

- At this order, the eigen values can be obtain properly and no interference between Gegenbauer polynomials coefficients;
- I just model this evoultion kernel to present the mass generation effect and staturation.

# Conformal like solution

Suppose $\varphi(x) = \frac{\Gamma[2+2a]}{\Gamma[1+a]^2} x^a (1-x)^a (1 + \sum_n' b_n C_n^{a+\frac{1}{2}}(2x-1))$ we can get

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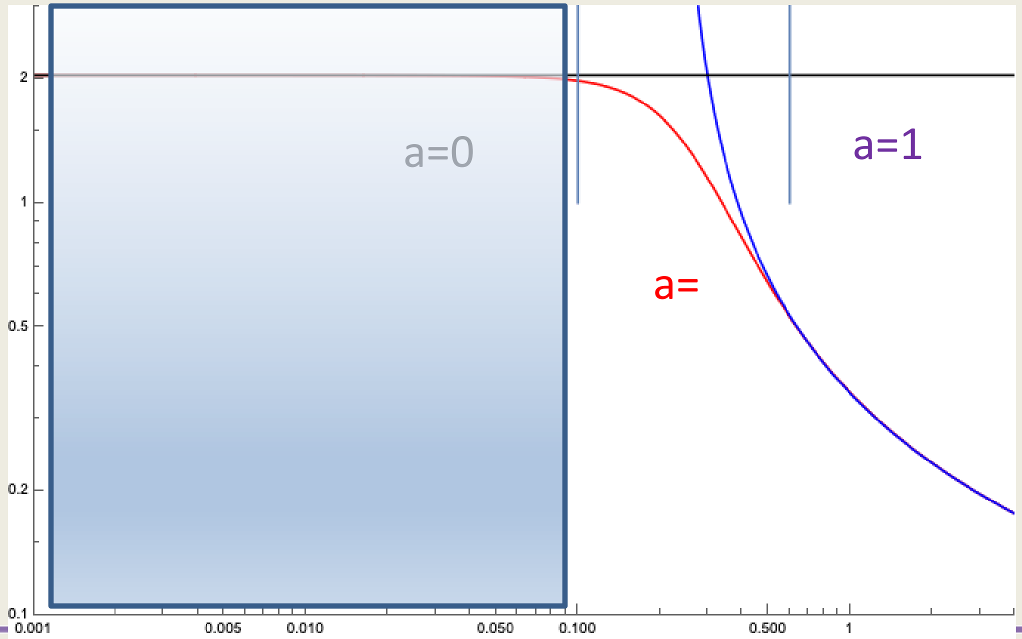


Image

- First stage, zeta<0.1. We have the constant DA at the zero scale at the present. If we want to hold this picture in this stage, the natrual choice is setting a=0.

$$Noting: b_{n>0} \equiv 0$$

# Conformal like solution

Suppose $\varphi(x) = \frac{\Gamma[2+2a]}{\Gamma[1+a]^2} x^a (1-x)^a (1 + \sum_n' b_n C_n^{a+\frac{1}{2}}(2x-1))$ we can get

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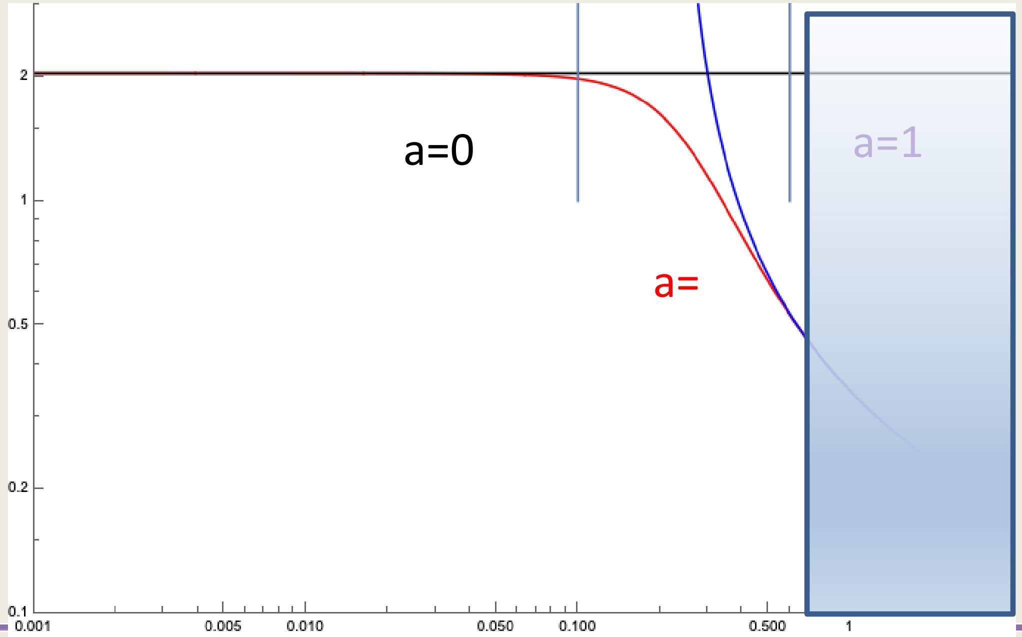


Image

- Third stage, zeta>0.7. I called this region is the perturbation region, with a=1 as usual.
- Why I choose 0.7$GeV \approx 3\Lambda_{QCD}$?
- 0.6-0.8GeV!

# Conformal like solution

Suppose $\varphi(x) = \frac{\Gamma[2+2a]}{\Gamma[1+a]^2} x^a (1-x)^a (1 + \sum_n' b_n C_n^{a+\frac{1}{2}}(2x-1))$ we can get

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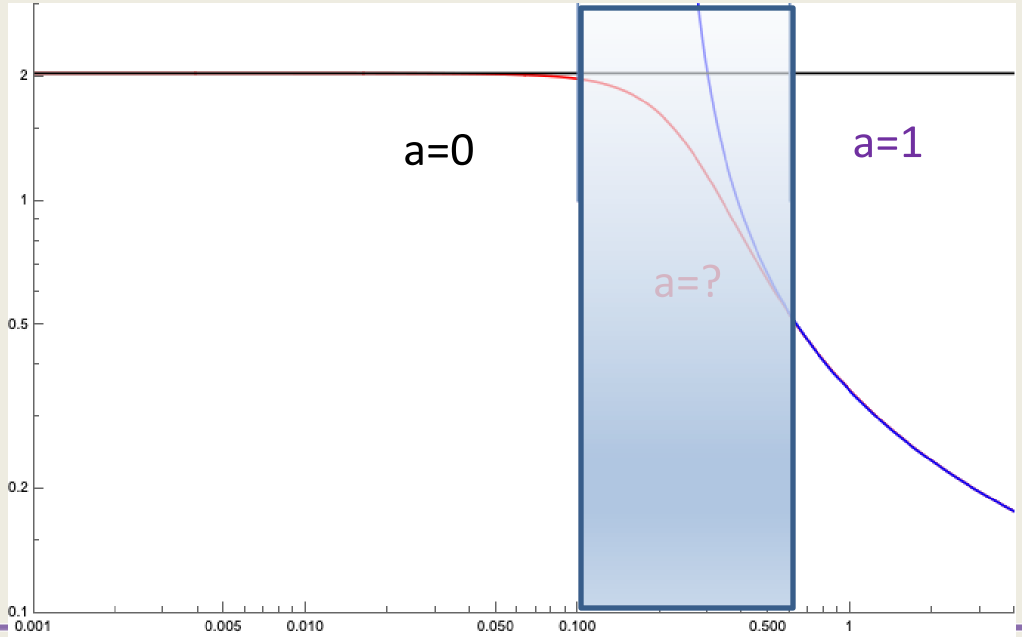


Image

- Second stage, the corssover region from the nonperturabtion to perturbation, dirty!
- a should depend on the scale and increases to 1 at the end of stage;
- I do not know the extact scale dependence of a, but I know that the DA should become narraower if a changing from 0 to 1 in this stage.

# Upper limit at 2GeV!

Suppose $\varphi(x) = \frac{\Gamma[2+2a]}{\Gamma[1+a]^2} x^a (1-x)^a (1 + \sum_n' b_n C_n^{a+\frac{1}{2}}(2x-1))$ we can get

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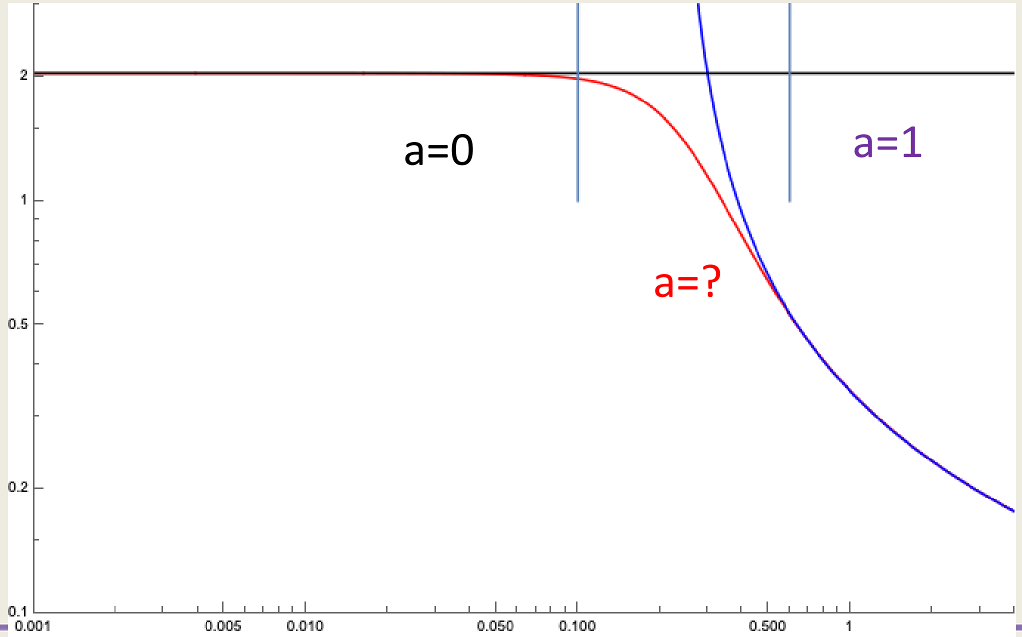
with $\gamma_n^a = -\frac{4}{3}\left(-\frac{2}{a+1} + \frac{2a}{(n+a)(n+a+1)} - 4\sum_{k=0}^{n-1} \frac{1}{k+a+1}\right)$.




Image

- From the arguments of the second stage, we can conclude that the fatest DA at 2GeV should be evolution of constant DA from 0.6-0.8GeV!
- The second moment(upper limit)

~0.28(from 0.6GeV)

~0.29(from 0.8GeV)

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## LaMET

| | |
|---|---|
| R. Zhang, et al., arXiv: 2005.13955 | 0.244(30)(20) |
| Jun Hua, et al., arXiv:2201.09173 | 0.300(41) |
| Xiang Gao, et al., arXiv: 2206.04084 | 0.287(6)(6) |
| Jack Holligan, et al., arXiv: 2301.10372 | 0.302(23) |

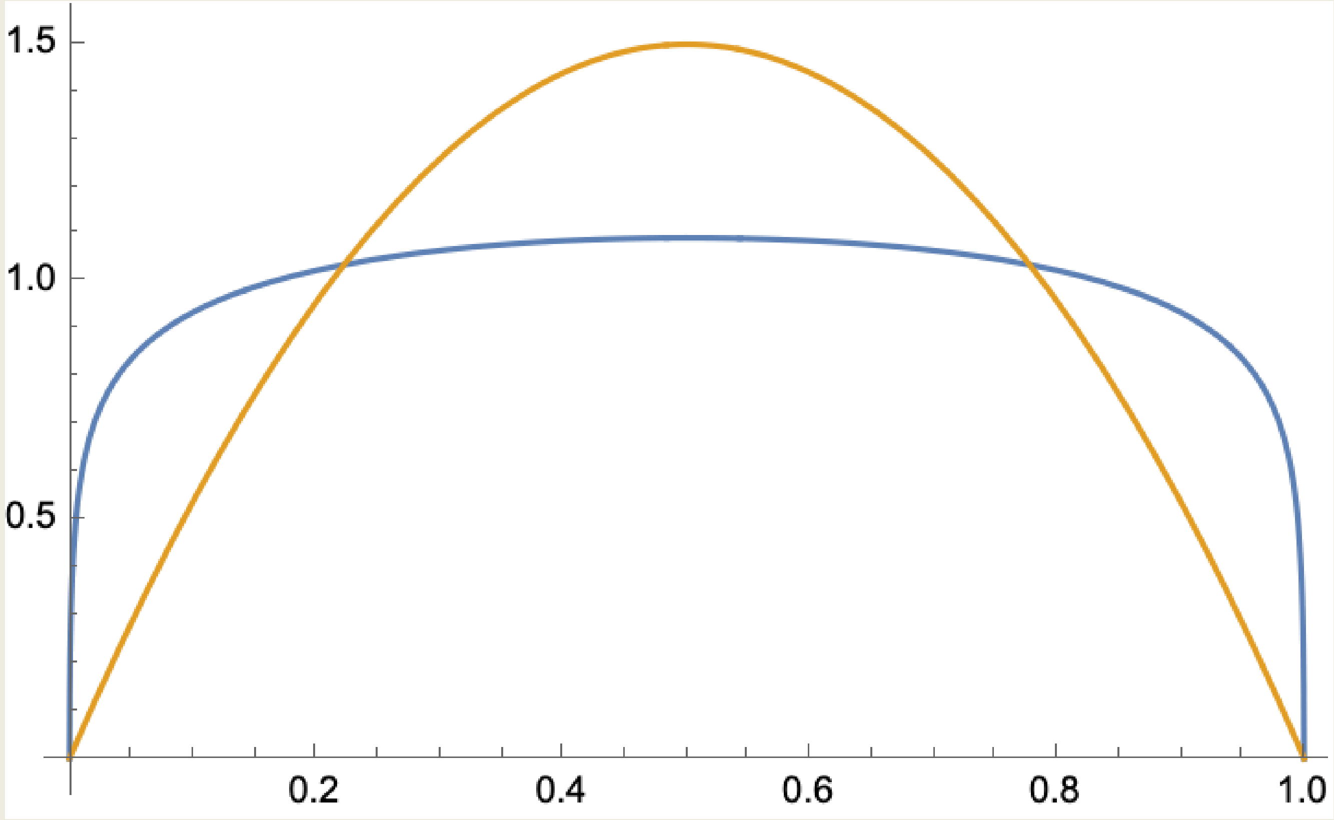
## Euclidean correlation functions

| | |
|---|---|
| G. S. Bali, et al., arXiv: 1807.06671 | 0.3 |

## Local twist-2 operator

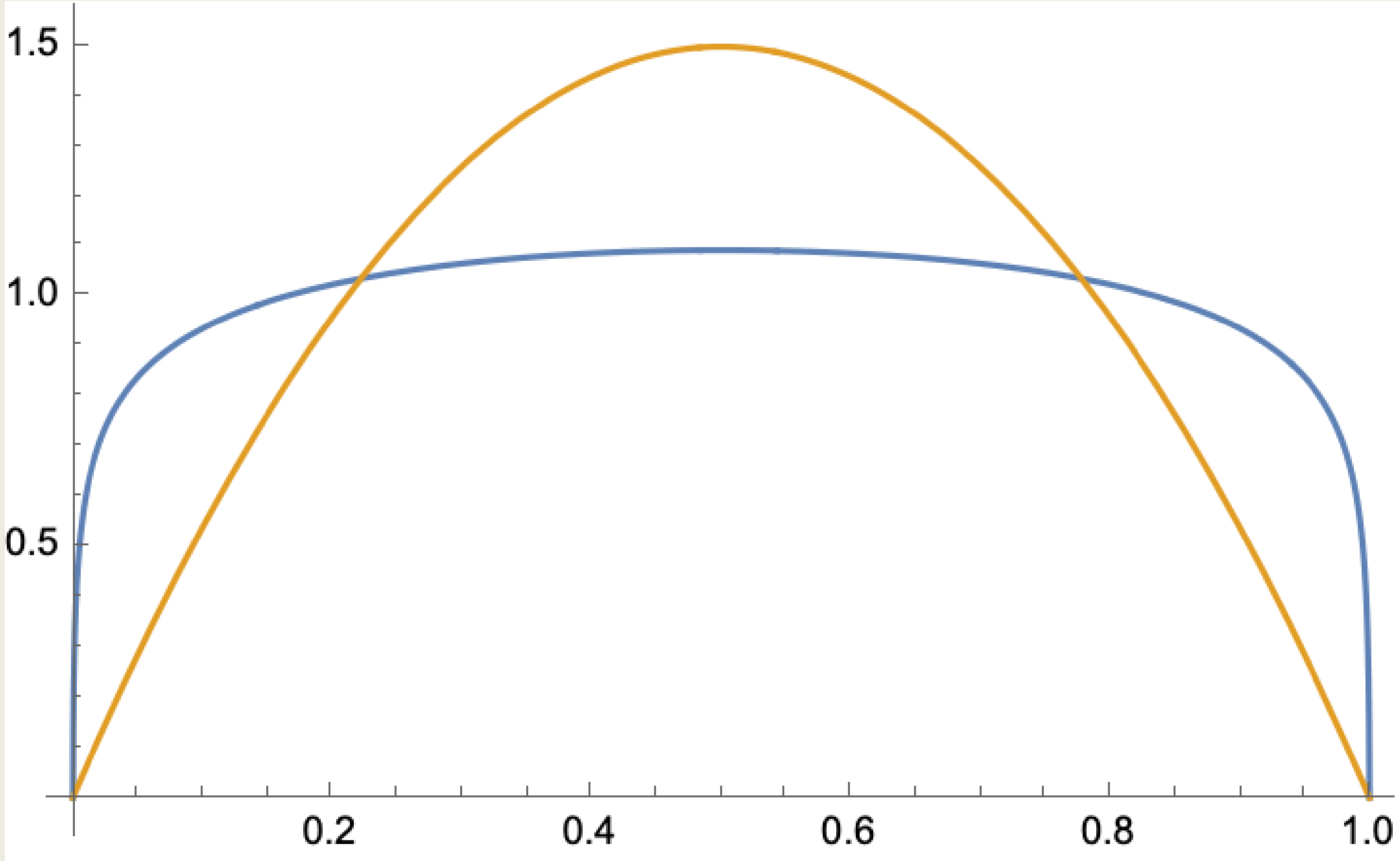
| | |
|---|---|
| G. S. Bali, et al., arXiv: 1903.08038 | 0.240(6)(2)(3)(2) |
| R. Arthur, et al., arXiv: 1011.5906 | 0.28(1)(2) |

# Upper Limit at 2GeV




$\langle \xi^2 \rangle < 0.29$

# Lower Limit at 2GeV???



# Lattice story updated

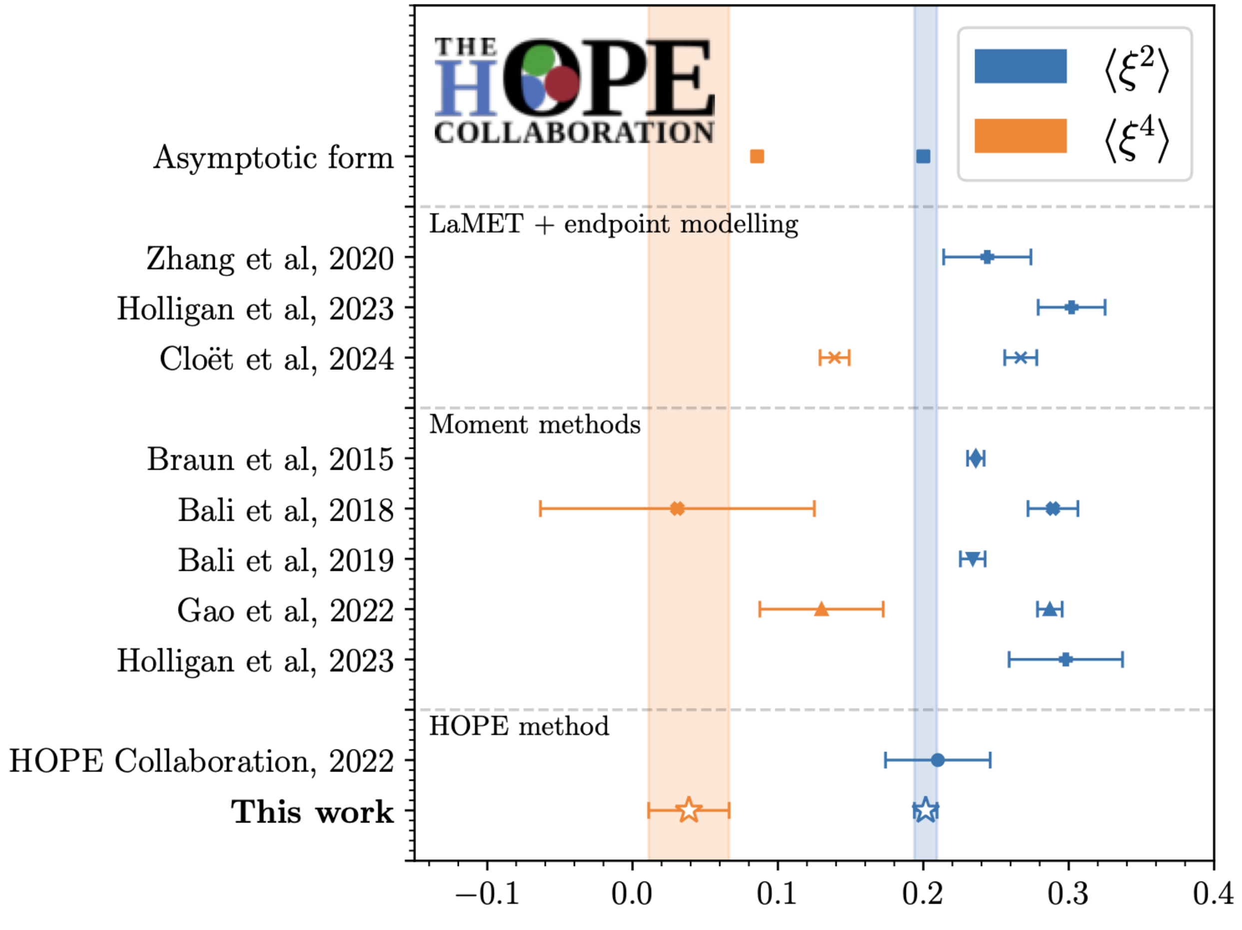



FIG. 10. Comparison of results obtained in this work to other determinations [19, 40, 61–64, 66] of the low Mellin moments of the pion LCDA.

- arxiv:2509.04799
- $m_\pi \approx 550MeV$
- at 2GeV

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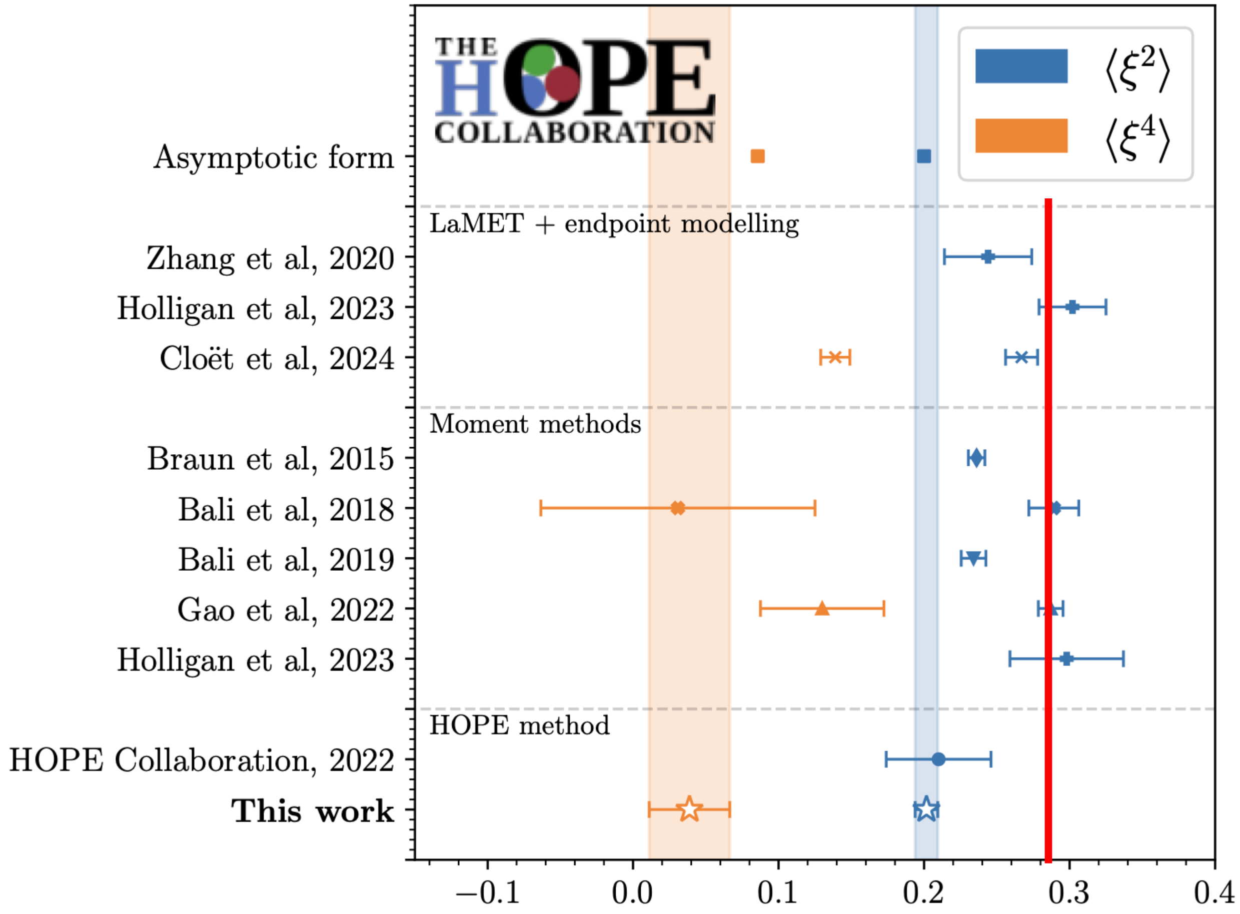


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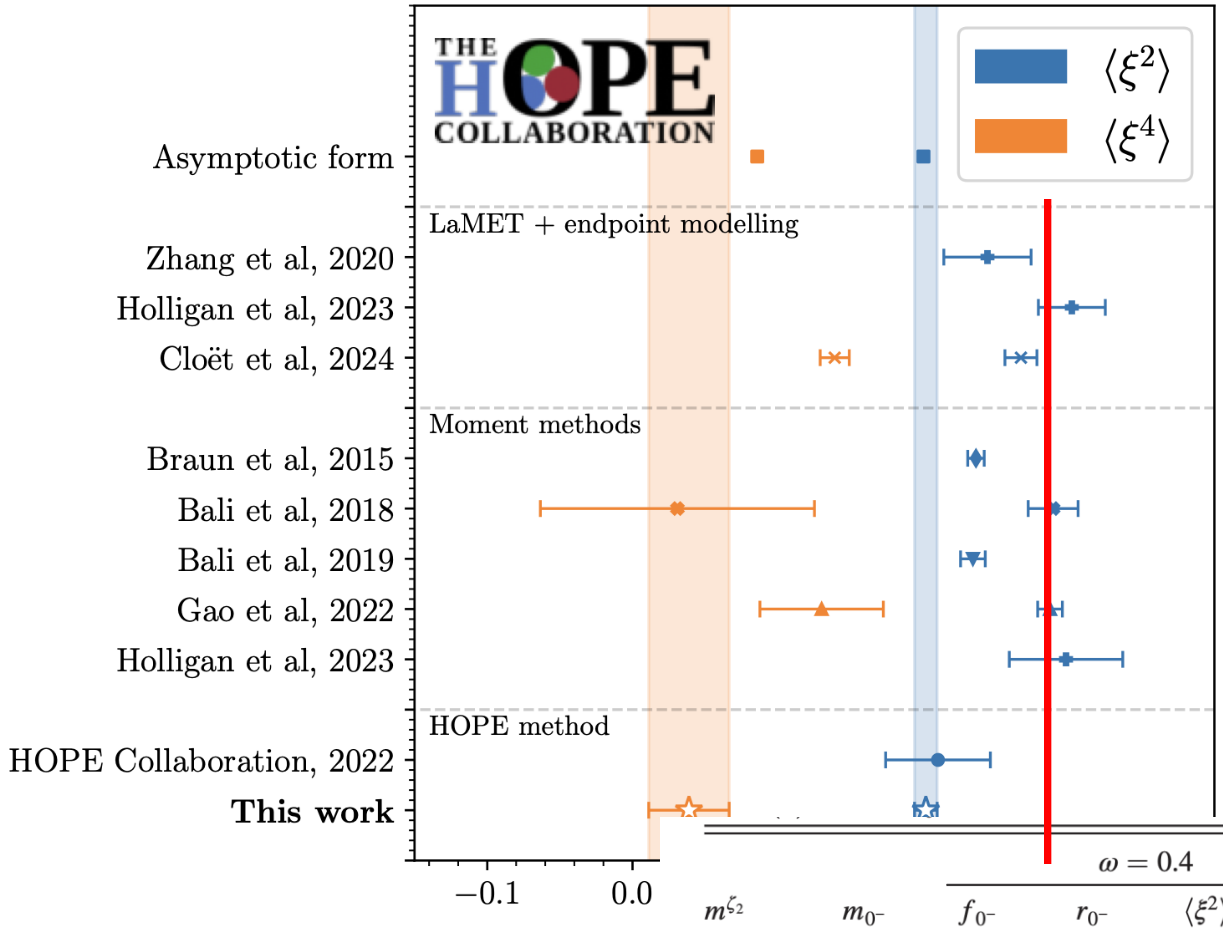


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PHYSICAL REVIEW D **98**, 091505(R) (2018)

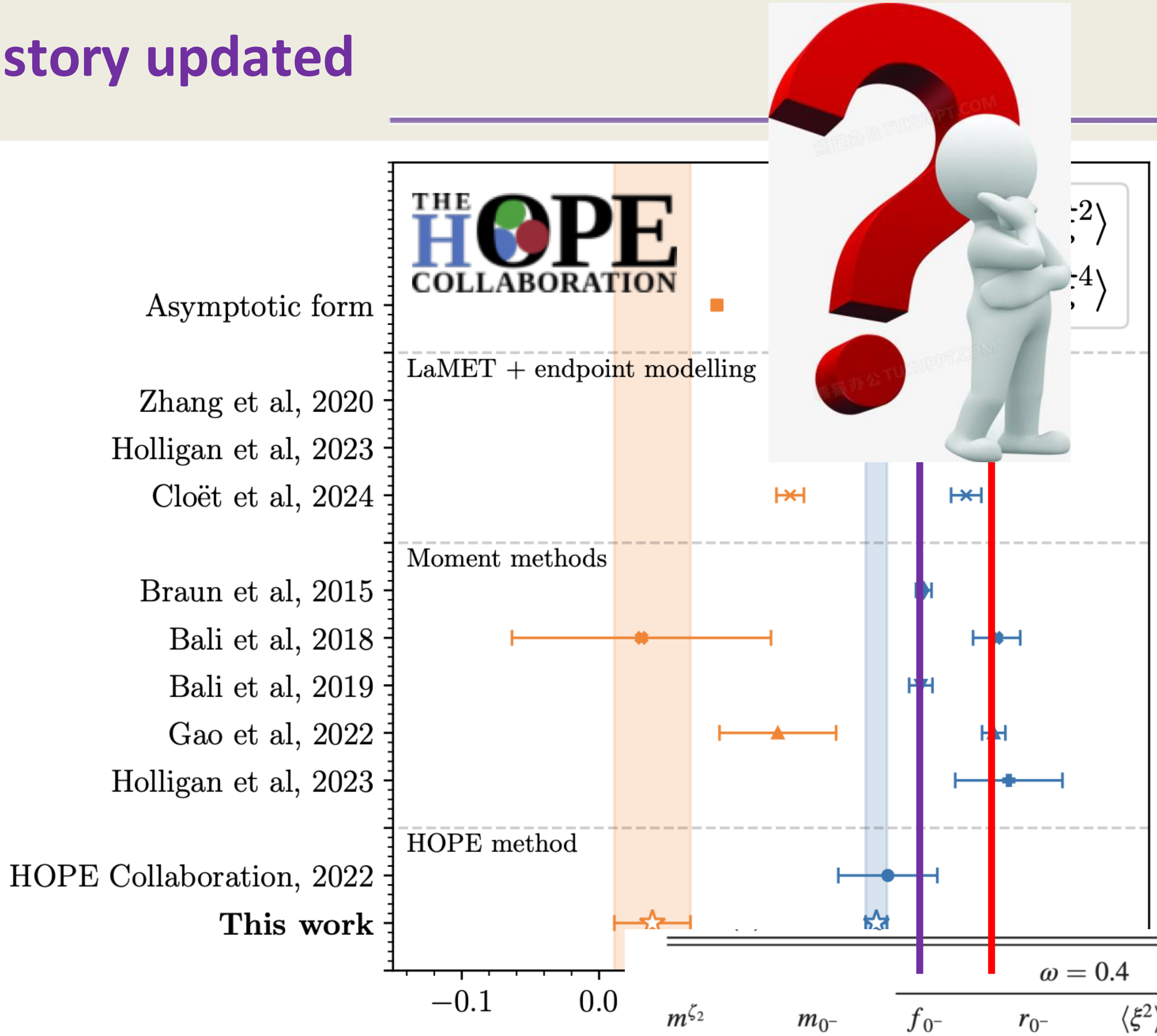
Rapid Communications

**Mass dependence of pseudoscalar meson elastic form factors**

Muyang Chen,[1] Minghui Ding,[1,2] Lei Chang,[1,*] and Craig D. Roberts[2,†]

| | | $\omega = 0.4$ | | | | $\omega = 0.5$ | | | | $\omega = 0.6$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m^{\zeta_2}$ | $m_{0^-}$ | $f_{0^-}$ | $r_{0^-}$ | $\langle\xi^2\rangle$ | $\alpha$ | $f_{0^-}$ | $r_{0^-}$ | $\langle\xi^2\rangle$ | $\alpha$ | $f_{0^-}$ | $r_{0^-}$ | $\langle\xi^2\rangle$ | $\alpha$ |
| 0.0046 | 0.14 | 0.097 | 0.63 | 0.255 | 0.46 | 0.094 | 0.66 | 0.265 | 0.39 | 0.092 | 0.68 | 0.273 | 0.33 |
| 0.053 | 0.47 | 0.117 | 0.53 | 0.217 | 0.80 | 0.115 | 0.55 | 0.226 | 0.71 | 0.115 | 0.56 | 0.229 | 0.68 |
| 0.107 | 0.69 | 0.135 | 0.47 | 0.196 | 1.05 | 0.133 | 0.49 | 0.207 | 0.92 | 0.133 | 0.49 | 0.211 | 0.87 |
| 0.152 | 0.83 | 0.147 | 0.43 | 0.180 | 1.28 | 0.145 | 0.45 | 0.193 | 1.09 | 0.145 | 0.45 | 0.200 | 1.00 |

# Lattice story updated

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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m^{\zeta_2}$ | $m_{0^-}$ | $f_{0^-}$ | $r_{0^-}$ | $\langle\xi^2\rangle$ | $\alpha$ | $f_{0^-}$ | $r_{0^-}$ | $\langle\xi^2\rangle$ | $\alpha$ | $f_{0^-}$ | $r_{0^-}$ | $\langle\xi^2\rangle$ | $\alpha$ |
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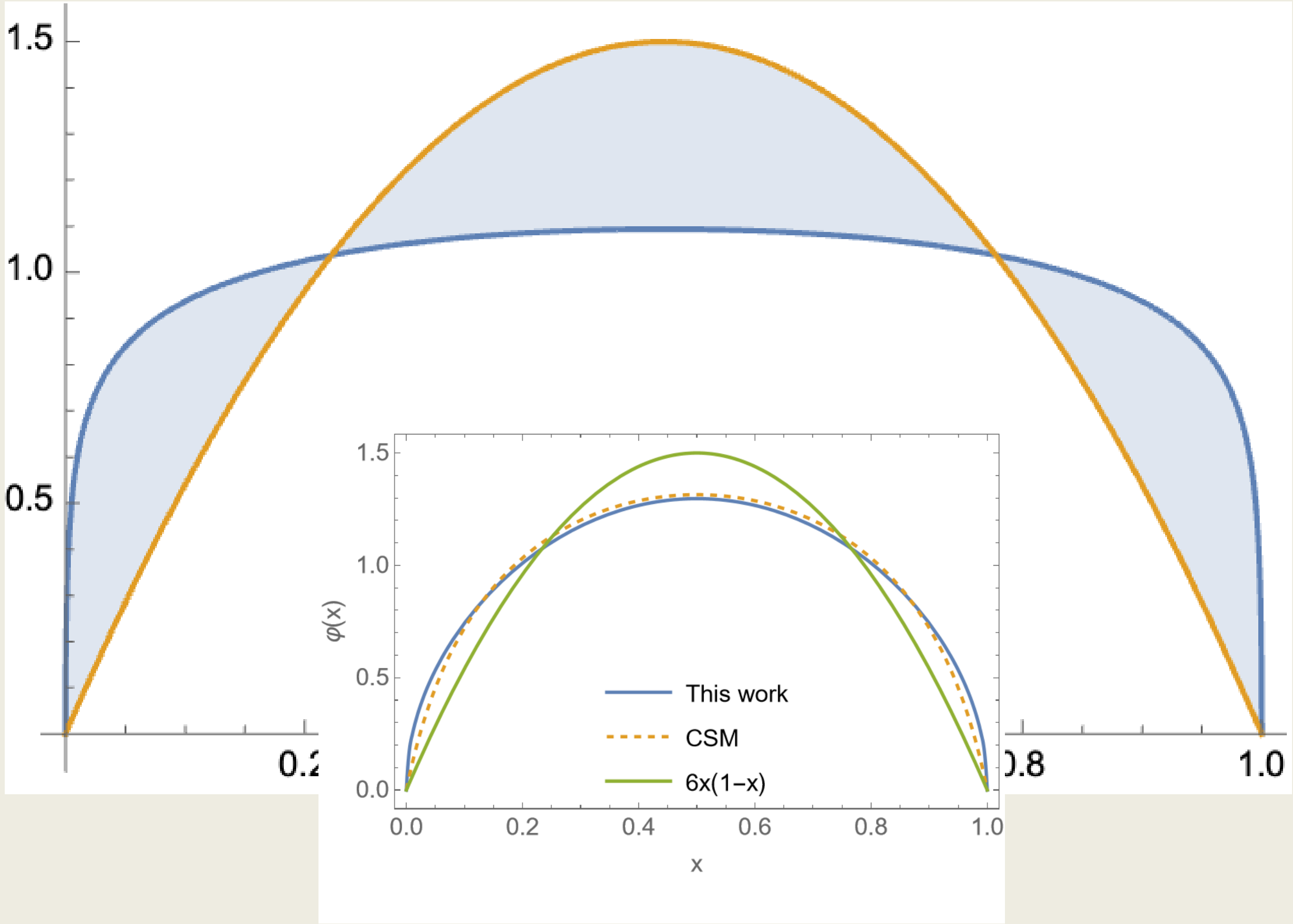
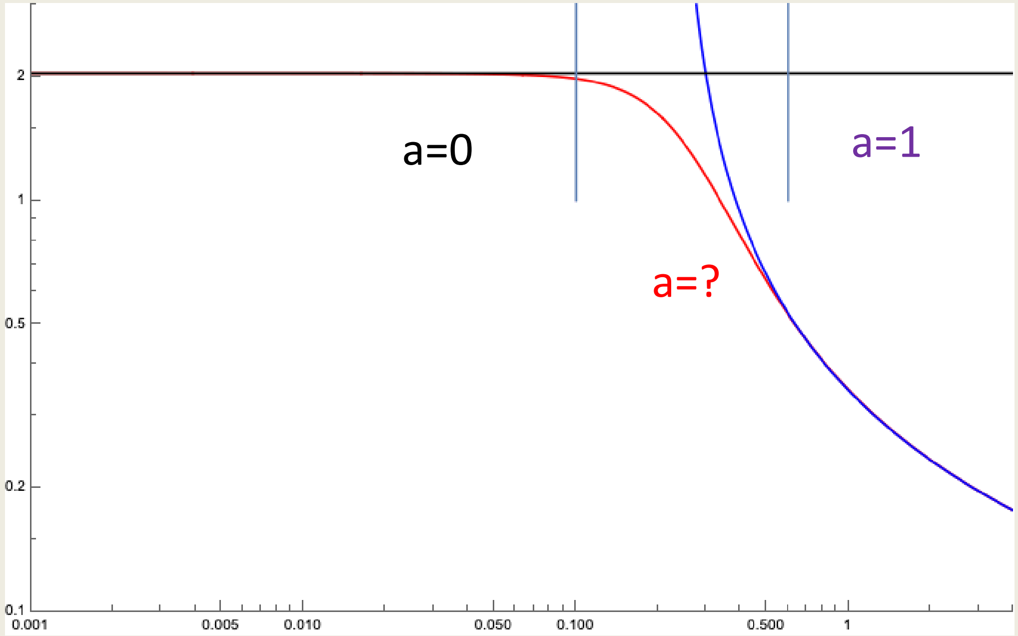
| 0.107 | 0.69 | 0.135 | 0.47 | 0.196 | 1.05 | 0.133 | 0.49 | 0.207 | 0.92 | 0.133 | 0.49 | 0.211 | 0.87 |
| 0.152 | 0.83 | 0.147 | 0.43 | 0.180 | 1.28 | 0.145 | 0.45 | 0.193 | 1.09 | 0.145 | 0.45 | 0.200 | 1.00 |

# Summary




- Based on the simplest approximation, we argue that the pion is broad(might be fattest) hadron in nature at the hadronic scale
- The evolution and compared to lQCD…unsolved…



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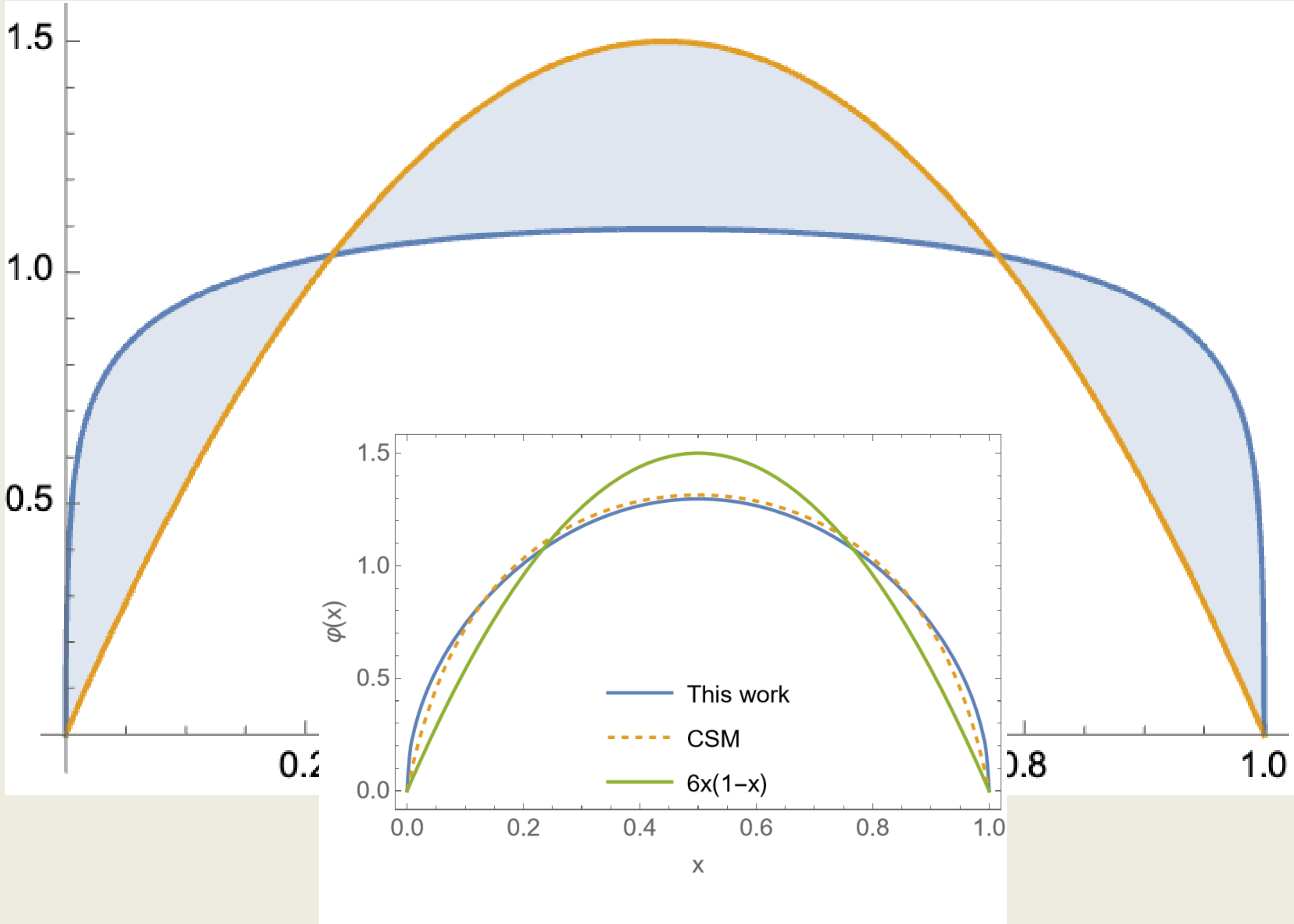


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