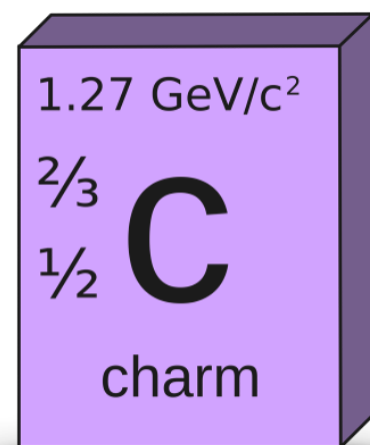




粲味强子单举衰变的理论进展

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高能理论论坛 (HETH-Forum)
中科院高能所理论室, 2025年10月24日

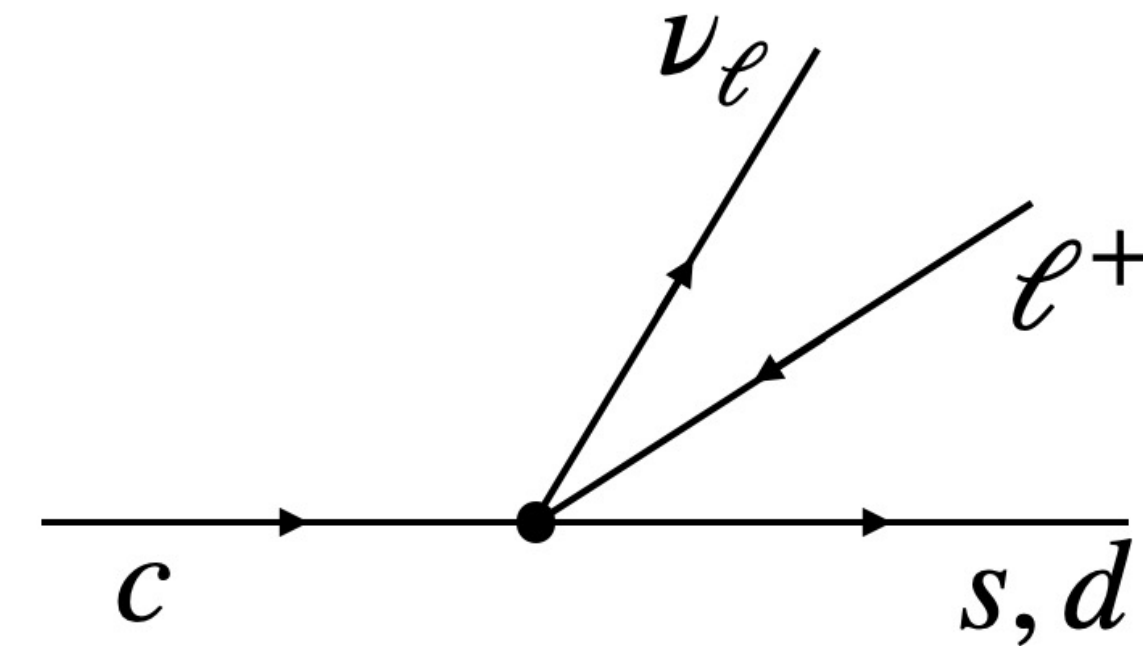


Why inclusive charm decays?

❖ Semileptonic inclusive D meson decays

➡ $D \rightarrow e^+ X$ ($D \rightarrow e^+ \nu_e X$, only e^+ is detected)

➡ $X = X_s + X_d$



❖ Compared to exclusive decays

➡ Better theoretical control

More important with more powerful experiments (BESIII, STCF)

❖ Compared to beauty decays

➡ Special to new dynamics attached with up-type quarks

➡ More sensitive to power corrections

Determination by charm, application in beauty

Why inclusive charm decays?

❖ Resolve (or at least give hints to) current flavor puzzles/anomalies

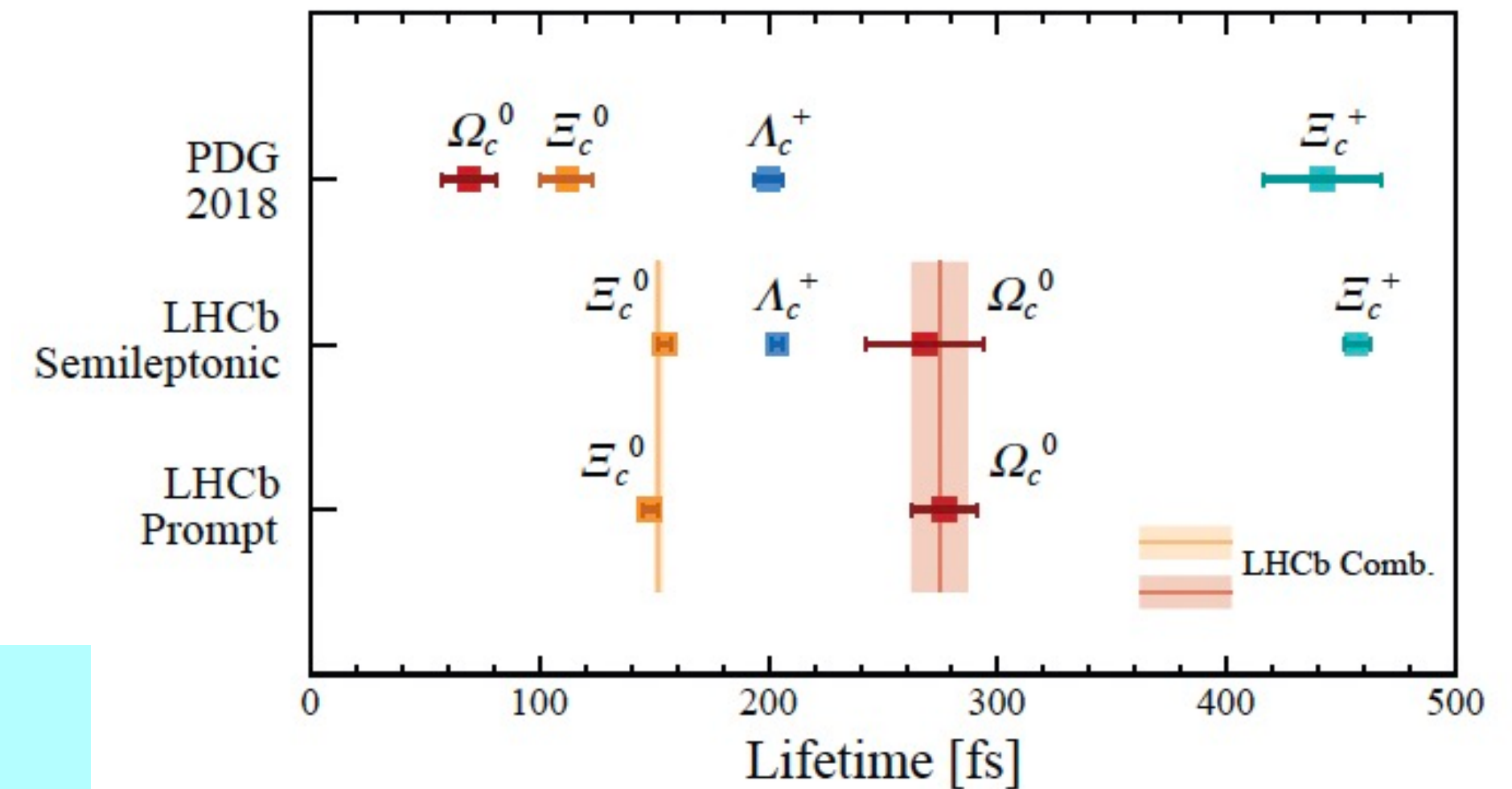
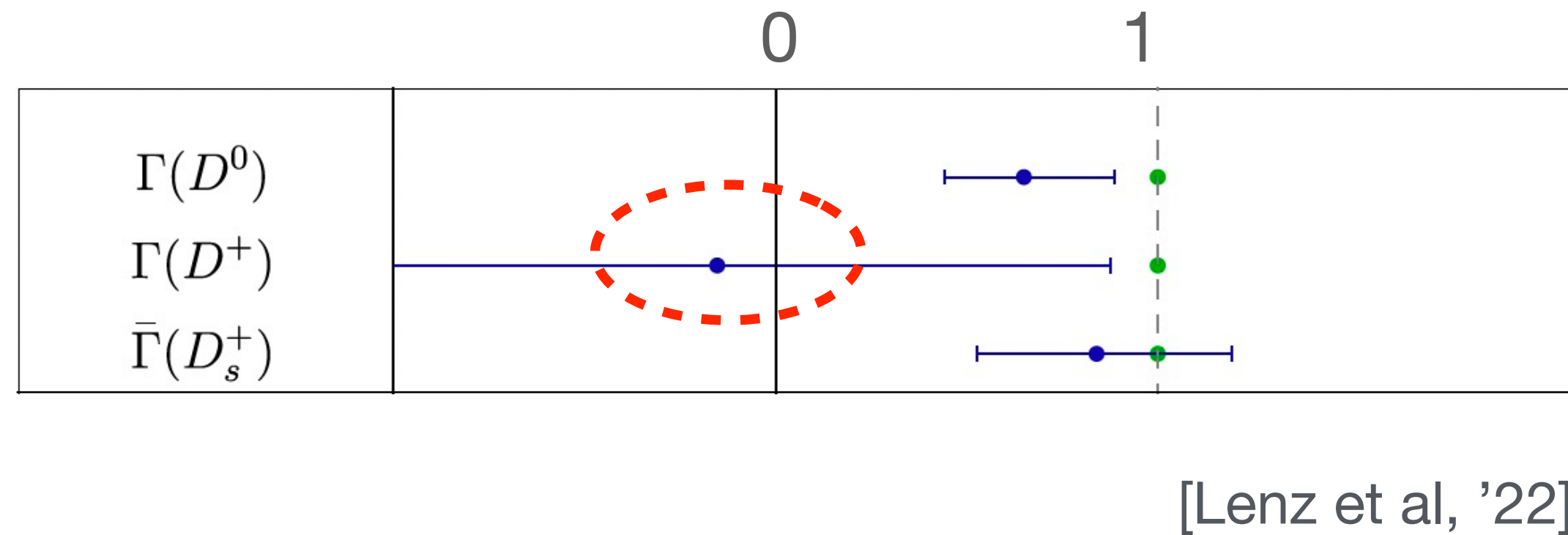
➡ Puzzles in **charmed hadron lifetimes**: theory vs experiment

➡ V_{cb} , V_{ub} **puzzles**: inclusive vs exclusive

➡ $b \rightarrow s$ **anomalies**: P'_5 in $B \rightarrow K^* \ell \ell$

Why inclusive charm decays?

❖ Flavor puzzle 1. Charmed hadron lifetimes: theory vs experiment



- ❖ **Key issue:** Nonperturbative power corrections with large/unknown uncertainties
- ❖ **Solution:** Extraction in the inclusive decay spectrum
 - ➡ Spectrum and lifetime share **identical** HQE parameters

$$\begin{aligned}
 \mathcal{O}(1/m_c^3) &\Rightarrow \tau(\Xi_c^+) > \tau(\Lambda_c^+) > \tau(\Xi_c^0) > \tau(\Omega_c^0), \\
 \mathcal{O}(1/m_c^4) &\Rightarrow \tau(\Omega_c^0) > \tau(\Xi_c^+) > \tau(\Lambda_c^+) > \tau(\Xi_c^0), \\
 \mathcal{O}(1/m_c^4) \text{ with } \alpha &\Rightarrow \tau(\Xi_c^+) > \tau(\Omega_c^0) > \tau(\Lambda_c^+) > \tau(\Xi_c^0).
 \end{aligned}$$

[Cheng, '21]

Again a more precise experimental determination of μ_π^2 from fits to semileptonic D^+ , D^0 and D_s^+ meson decays – as it was done for the B^+ and B^0 decays – would be very desirable.

[Lenz et al, '22]

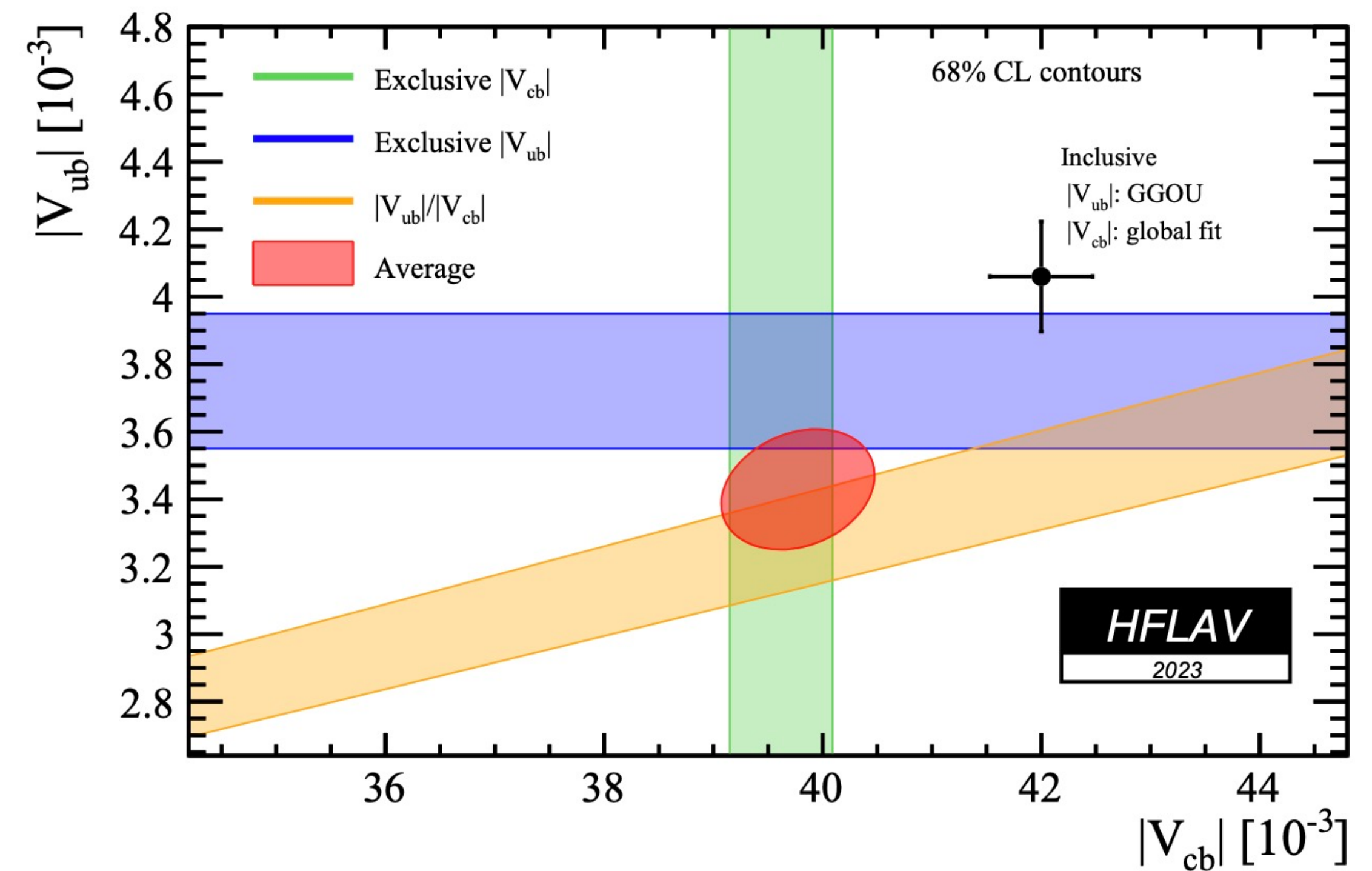
Why inclusive charm decays?

❖ **Flavor puzzle 2.** V_{cb} , V_{ub} : inclusive vs exclusive

◆ **Key issue:** Systematic uncertainties from theoretical inclusive and exclusive frameworks?

◆ **Solution:** Test V_{cd} , V_{cs} : inclusive vs exclusive

➡ **Await the first inclusive values**



Why inclusive charm decays?

❖ Flavor puzzle 3. $b \rightarrow s$ anomalies: P'_5 in $B \rightarrow K^* \ell \ell$

◆ **Key issue:** First-principle calculation of long-distance penguin in this channel is still missing

➡ Only achieved in $B \rightarrow \gamma \gamma$

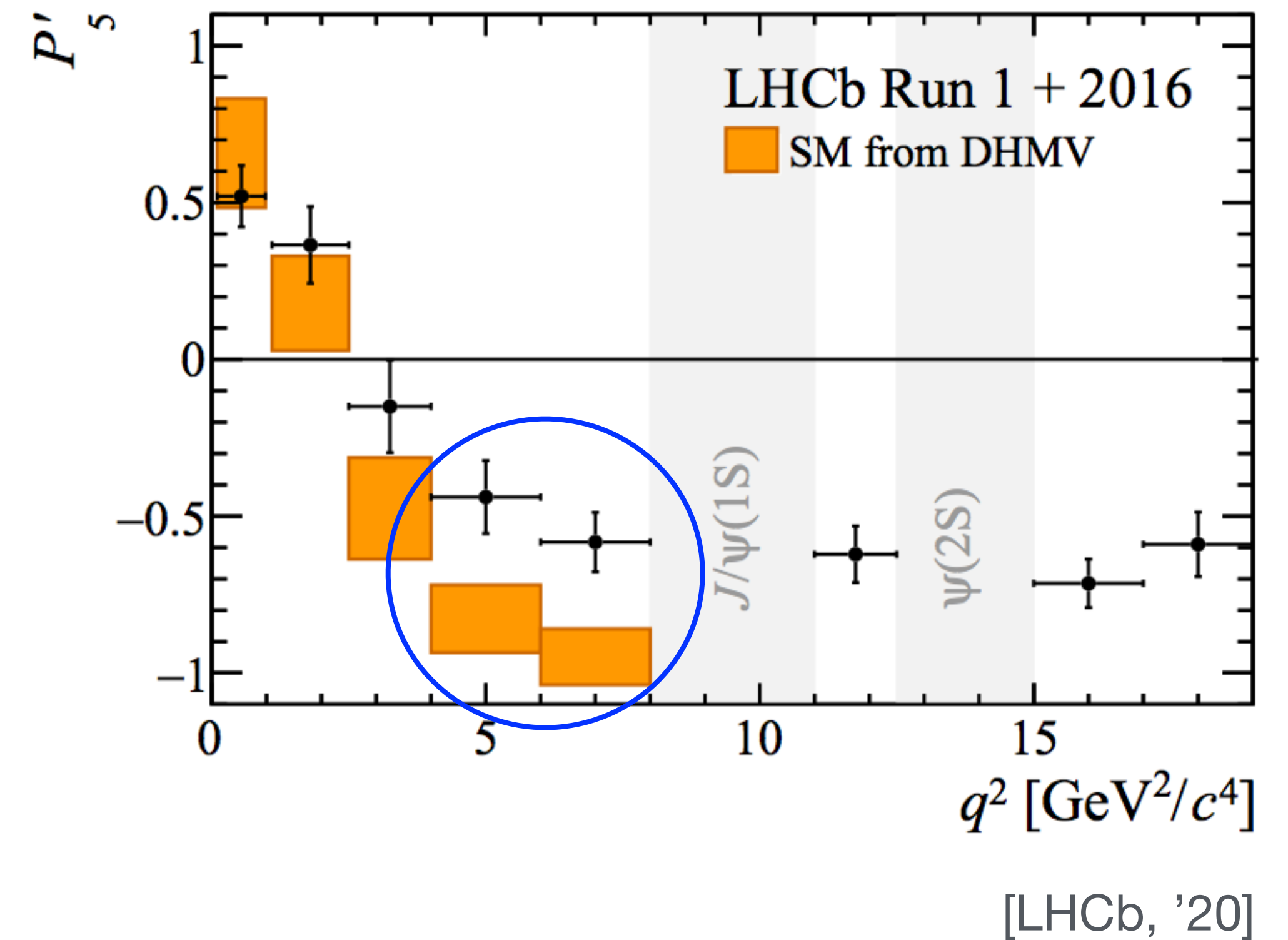
[QQ, Shen, Wang, Wang, PRL, '23]

◆ **Solution:** Test FCNC inclusive decays

➡ Inclusive $B \rightarrow X_{d,s} \ell \ell$

[Huber, Hurth, Jenkins, Lunghi, QQ, Vos, JHEP, '19, '20, '24]

➡ Inclusive $D \rightarrow X_u \ell \ell$



Theoretical Framework and Developments

Theoretical framework

❖ Optical theorem

$$\sum \langle D | H | X \rangle \langle X | H | D \rangle \propto \text{Im} \int d^4x e^{-ip_D \cdot x} \langle D | T\{H(x)H(0)\} | D \rangle$$

❖ Operator product expansion (OPE)

➡ Short distance $x \sim 1/m_c$

➡ Dynamical fluctuation in D meson $\sim \Lambda_{\text{QCD}}$

$$T\{H(x)H(0)\} = \sum_n C_n(x) O_n(0) \rightarrow 1 + \frac{\Lambda_{\text{QCD}}}{m_c} + \frac{\Lambda_{\text{QCD}}^2}{m_c^2} + \dots$$

Systematic OPE in HQET.

Theoretical framework

- Heavy quark effective theory

$$h_v(x) \equiv e^{-im_c v \cdot x} \frac{1 + \gamma \cdot v}{2} c(x) \quad v = (1, 0, 0, 0)$$

Subtract the big intrinsic momentum,
Leave only $\sim \Lambda_{\text{QCD}}$ degrees of freedom.

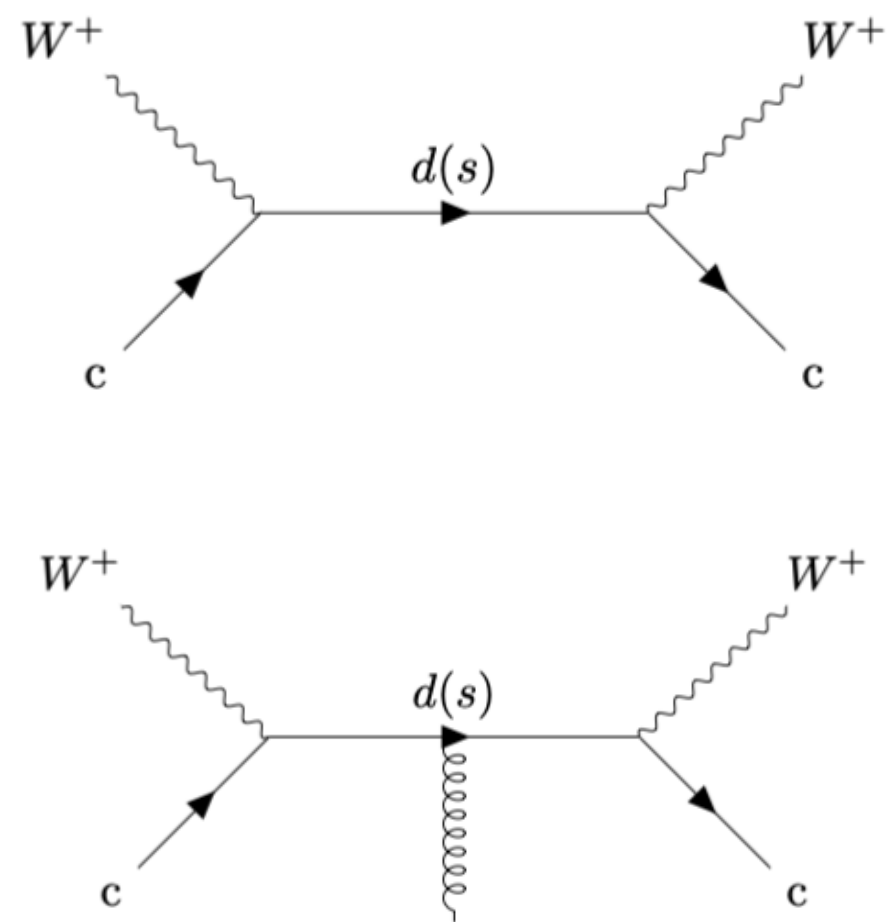
$$L \ni \bar{h}_v i v \cdot D h_v - \bar{h}_v \frac{D_\perp^2}{2m_c} h_v - a(\mu) g \bar{h}_v \frac{\sigma \cdot G}{4m_c} h_v + \dots$$

$$\text{Similar to } \frac{m}{\sqrt{1 - v^2}} = m + \frac{1}{2} m v^2 + \dots$$

Theoretical framework

❖ **OPE and Perturbative QCD:** Systematical expansion of $\alpha_s(m_c)$ and Λ_{QCD}/m_c

$$\langle T\{H(x)H(0)\} \rangle = \sum_n C_n(x) \langle O_n(0) \rangle$$



★ LO: $\alpha_s^0(m_c)$

★ NLO: $\alpha_s(m_c)$

★ NNLO: $\alpha_s^2(m_c)$

★ ...

★ Dim-3: $\bar{h}_\nu h_\nu (\bar{c} \gamma^\mu c)$ **partonic decay rate**

★ Dim-5: $\bar{h}_\nu D_\perp^2 h_\nu, g \bar{h}_\nu \sigma \cdot G h_\nu$.

★ Dim-6: $\bar{h}_\nu D_\mu (\nu \cdot D) D^\mu h_\nu, (\bar{h}_\nu \Gamma_1 q)(\bar{q} \Gamma_2 h_\nu), \dots$

★ ...

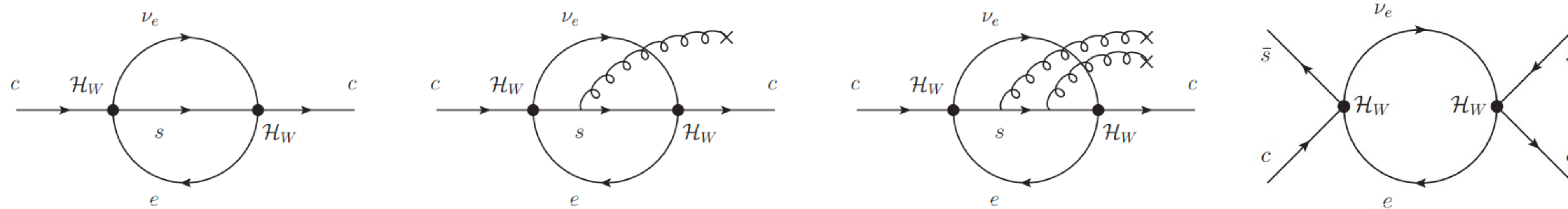
Question: convergent expansion of $\alpha_s(m_c)$ and Λ_{QCD}/m_c ?

Heavy quark expansion

❖ Heavy quark expansion up to dimension-7 operators (LO)

[Fael, Mannel, Vos, '19]

➡ QCD and HQET operator matching by calculating quark-gluon diagrams

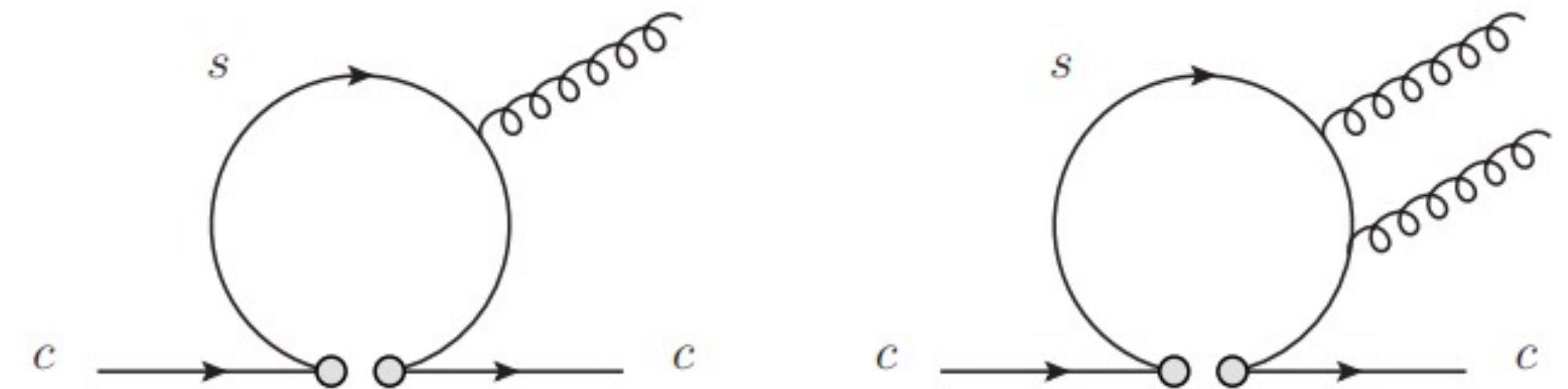


➡ Free to choose HQET operator basis, but RPI ones are favored

➡ $c \rightarrow d$ is like $b \rightarrow u$, $c \rightarrow s$ is **not** like $b \rightarrow c$: m_s appears in operators, not WCs

➡ **Operator mixing at LO:** 4q operators to 2q operators

$$C_i^{2q}(\mu) = C_i^{2q}(m_c) + \log\left(\frac{\mu}{m_c}\right) \sum_j \hat{\gamma}_{ij}^T C_j^{4q}(m_c)$$



Heavy quark expansion

❖ Heavy quark expansion up to dimension-7 operators (LO)

[Fael,Mannel,Vos, '19]

➡ LO results for decay widths, $\langle E_e^n \rangle$, $\langle (q^2)^n \rangle$ are given ($q^2 \equiv (p_e + p_\nu)^2$)

$$\frac{\Gamma(D \rightarrow X_s \ell \nu)}{\Gamma_0} = (1 - 8\rho - 10\rho^2) \mu_3 + (-2 - 8\rho) \frac{\mu_G^2}{m_c^2} + 6 \frac{\tilde{\rho}_D^3}{m_c^3} \\ + \frac{16}{9} \frac{r_G^4}{m_c^4} + \frac{32}{9} \frac{r_E^4}{m_c^4} - \frac{34}{3} \frac{s_B^4}{m_c^4} + \frac{74}{9} \frac{s_E^4}{m_c^4} + \frac{47}{36} \frac{s_{qB}^4}{m_c^4} + \frac{\tau_0}{m_c^3}$$

$$\mathcal{Y}_1 \equiv E_e/m_c$$

$$\mathcal{Y}_1 = \frac{3}{5} - 6\rho - 23\rho^2 - (1 + 16\rho) \frac{\mu_G^2}{m_c^2} + \frac{139}{15} \frac{\rho_D^3}{m_c^3} + \frac{3}{5} \frac{\rho_{LS}^3}{m_c^3} + \frac{503}{90} \frac{\delta\rho_D^3}{m_c^3} + \frac{3}{10} \frac{\delta\rho_{LS}^3}{m_c^3} \\ + \frac{1271}{180} \frac{r_G^4}{m_c^4} - \frac{208}{45} \frac{r_E^4}{m_c^4} - \frac{682}{45} \frac{s_B^4}{m_c^4} + \frac{203}{15} \frac{s_E^4}{m_c^4} + \frac{283}{180} \frac{s_{qB}^4}{m_c^4} + \frac{1}{4} \frac{\delta G_2^4}{m_c^4} + \frac{\tau_0}{m_c^3} + \frac{\tau_m}{m_c^4} + \frac{\tau_\epsilon}{m_c^4}$$

❖ Leading-order Dim-7 operator WCs ($\Lambda_{\text{QCD}}^3/m_c^3$ correction)

[Finauri, '25]

➡ only for $b \rightarrow c$ spectrum

Perturbative expansion

❖ α_s corrections to Dim-3 operator WCs (c -quark decay)

➡ NLO for $b \rightarrow c$ spectrum; zero-mass limit to $b \rightarrow u$

[Fazio,Neubert, '99;
Capdevila,Gambino,Nandi, '21]

➡ NNLO for $b \rightarrow u$ (total width and spectrum)

[Ritbergen '99; Brucherseifer,Caola,Melnikov, '13]

➡ **NNLO** for $c \rightarrow s/d$ total width

⦿ Analytical by expansion of $\delta = (1 - m_q/m_c)$ (10% uncertainty) [Fael,Schonwald,Steinhauser, '20]

⦿ Analytical at leading color (95% contribution)

[Chen,Li,Li,Wang,Wu, '23]

⦿ Numerical full contribution (AMFlow)

[Chen,Chen,Guan,Ma, '23]

Perturbative expansion

❖ α_s corrections to Dim-5 and Dim-6 operator WCs ($\Lambda_{\text{QCD}}^2/m_c^2$ correction)

➡ NLO for $b \rightarrow c$ spectrum; zero-mass limit to $b \rightarrow u$

[Alberti,(Ewerth),Gambino,Nandi, '12,'13;
Capdevila,Gambino,Nandi, '21]

➡ NLO for $c \rightarrow s/d$ Darwin operator (total width)

[Moreno '21]

Phenomenology

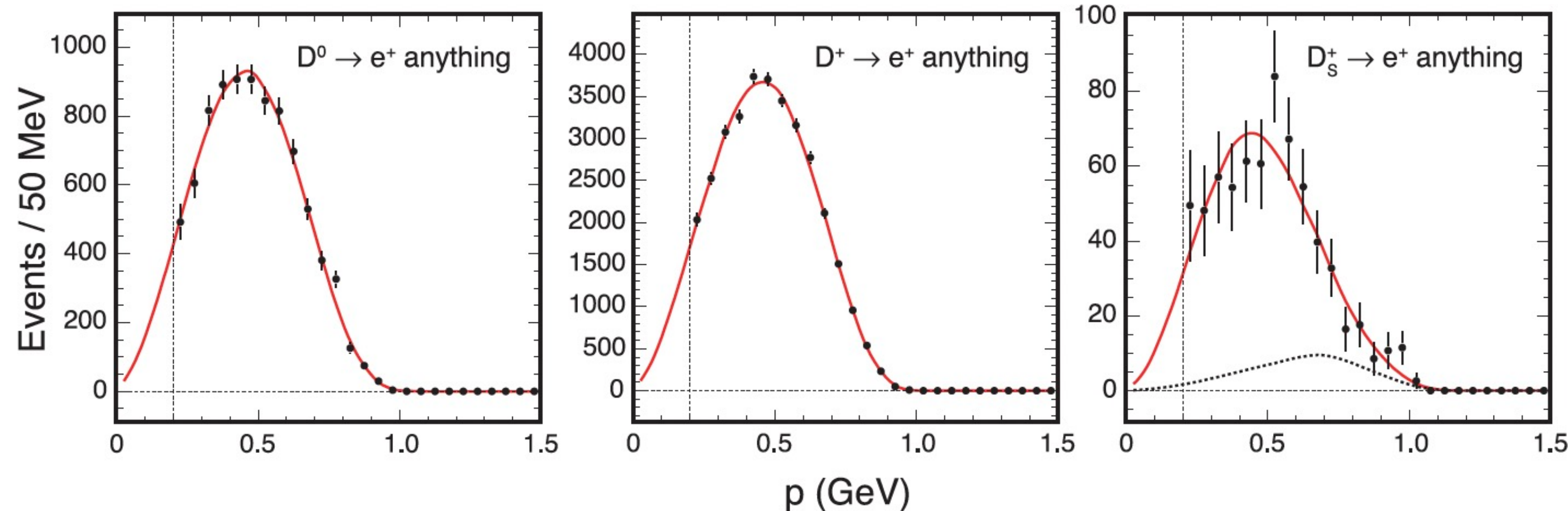
Experimental status

CLEO measurements

$$D^0 \rightarrow e^+ X$$

$$D^+ \rightarrow e^+ X$$

$$D_s^+ \rightarrow e^+ X$$

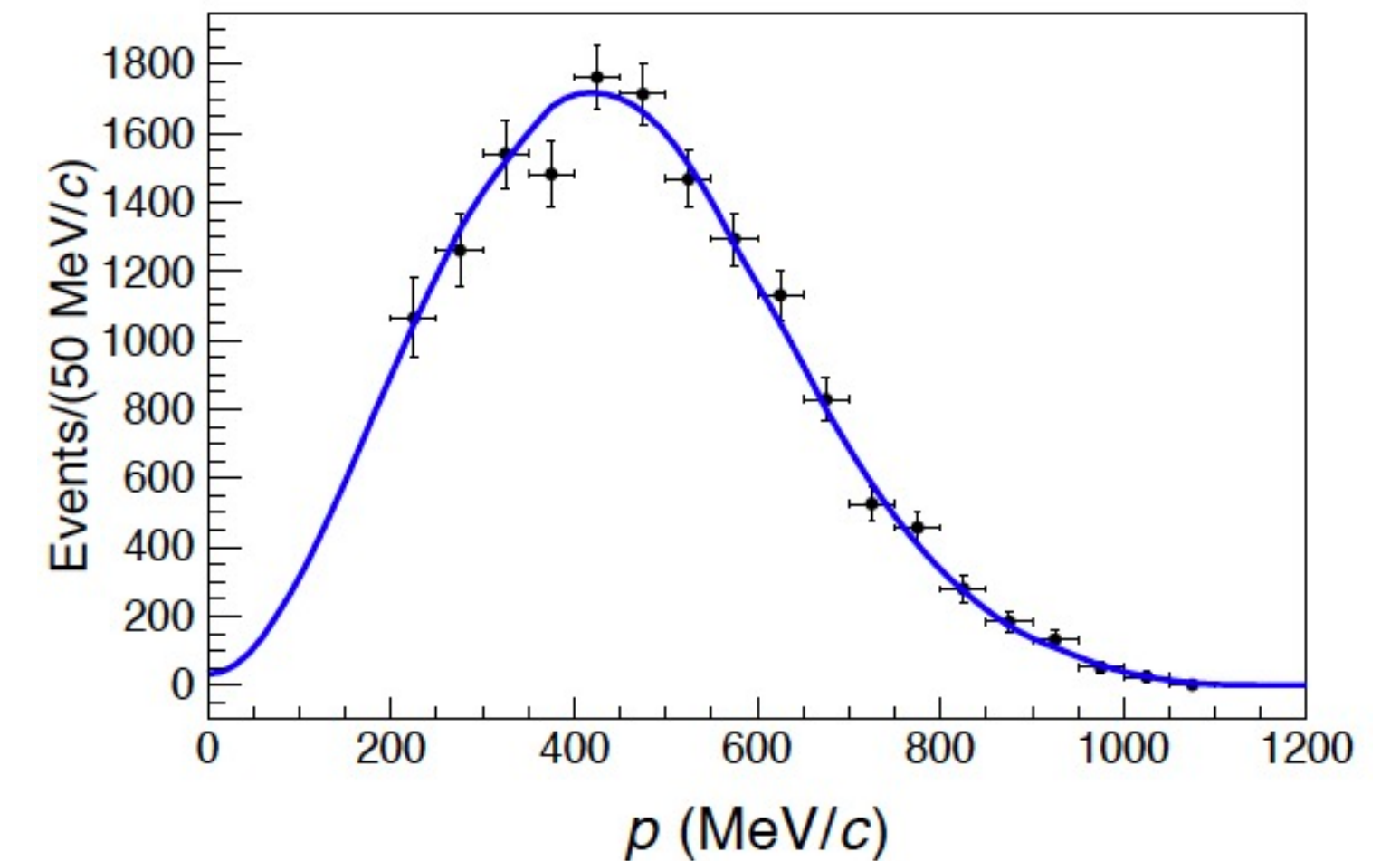


$$\begin{aligned}\mathcal{B}(D^0 \rightarrow X e^+ \nu_e) &= (6.46 \pm 0.09 \pm 0.11)\%, \\ \mathcal{B}(D^+ \rightarrow X e^+ \nu_e) &= (16.13 \pm 0.10 \pm 0.29)\%, \\ \mathcal{B}(D_s^+ \rightarrow X e^+ \nu_e) &= (6.52 \pm 0.39 \pm 0.15)\%,\end{aligned}$$

[CLEO, '09]

BESIII measurements

$$D_s^+ \rightarrow e^+ X$$



$$B(D_s^+ \rightarrow X e^+ \nu_e) = (6.30 \pm 0.13 \pm 0.10) \%$$

[BESIII, '21]

2% precision!

Phenomenological status

PHYSICAL REVIEW D **104**, 012003 (2021)

**Measurement of the absolute branching fraction of inclusive semielectronic
 D_s^+ decays**

(BESIII Collaboration)

$$\frac{\Gamma(D_s^+ \rightarrow X e^+ \nu_e)}{\Gamma(D^0 \rightarrow X e^+ \nu_e)} = 0.790 \pm 0.016(\text{stat.}) \pm 0.011(\text{syst.}) \pm 0.016(\text{ext.}).$$

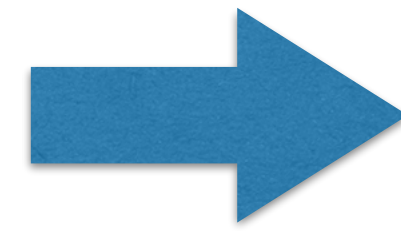
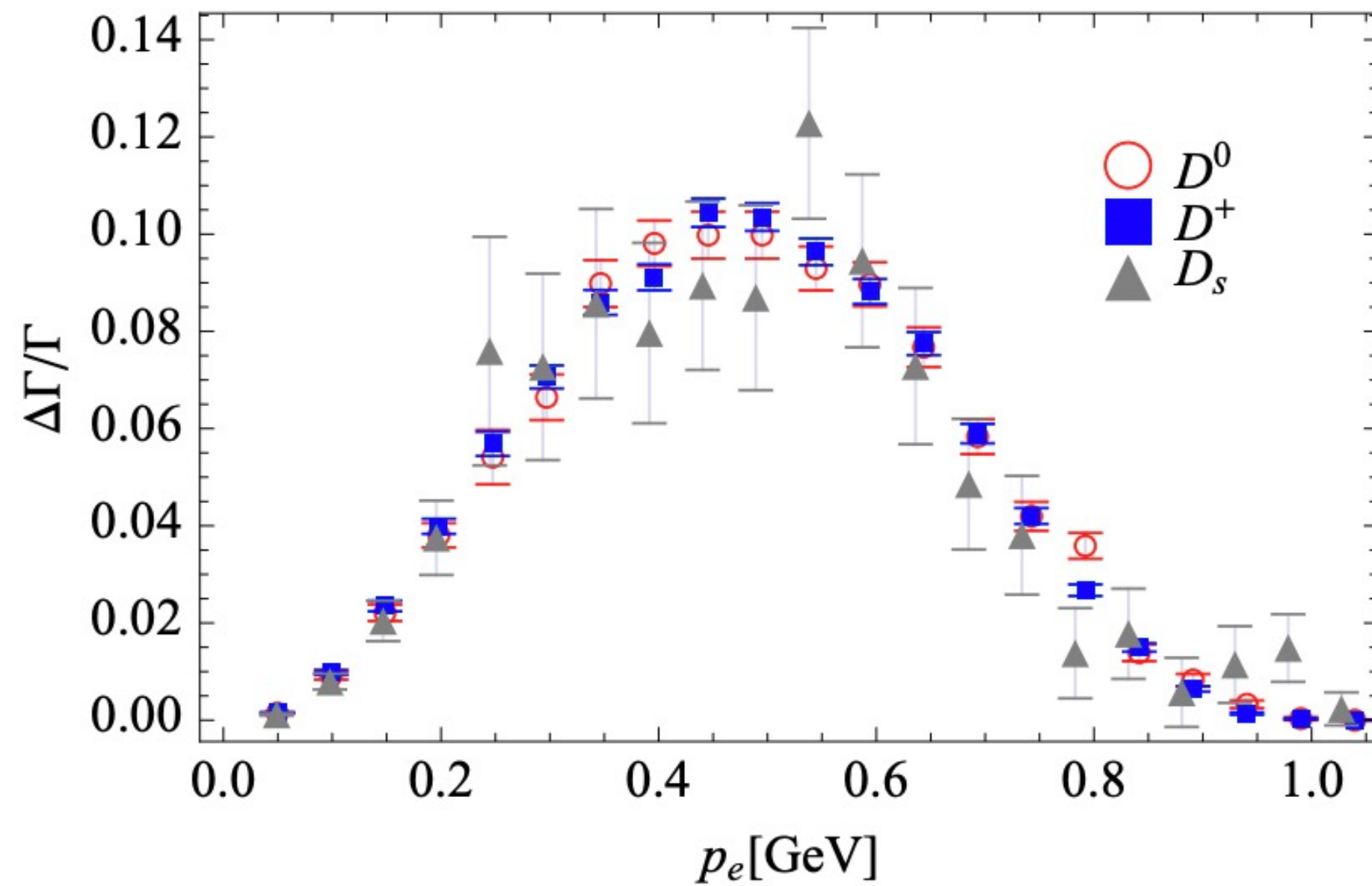
where the external systematic uncertainty includes the total uncertainty from $\mathcal{B}(D^0 \rightarrow X e^+ \nu_e)$, the D^0 and D_s^+ lifetimes, and $\mathcal{B}(D_s^+ \rightarrow \tau^+ \nu_\tau)$. This result is in agreement with the prediction of $\frac{\Gamma(D_s^+ \rightarrow X e^+ \nu)}{\Gamma(D^0 \rightarrow X e^+ \nu)} = 0.813$ from Ref. [18],

[18] M. Gronau and J. L. Rosner, *Phys. Rev. D* **83**, 034025 (2011).

Phenomenological Progress

❖ 4-quark operator matrix elements were extracted from electron energy spectrum

[Gambino,Kamenik, '10; Ligeti,Luke,Manohar, '10]



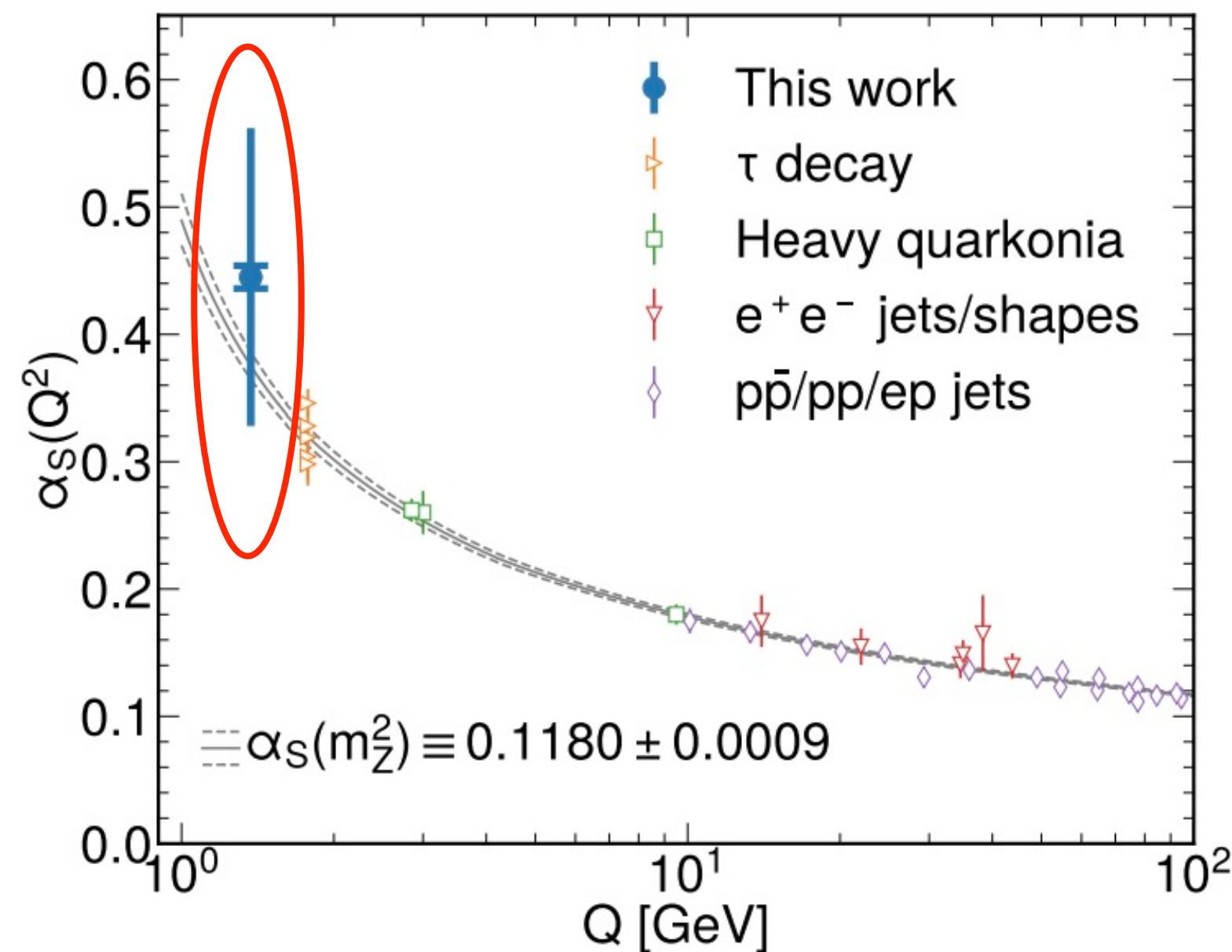
$$\begin{aligned}
 B_{\text{WA}}^{(0),s}(D^0) &= -0.001(3)\text{GeV}^3, & B_{\text{WA}}^{(0),s}(D^+) &= -0.001(3)\text{GeV}^3, \\
 \bar{B}_{\text{WA}}^{(1),s}(D^0) &= -0.0001(6)\text{GeV}^3, & \bar{B}_{\text{WA}}^{(1),s}(D^+) &= -0.0001(6)\text{GeV}^3, \\
 \bar{B}_{\text{WA}}^{(2),s}(D^0) &= -0.0001(10)\text{GeV}^3, & \bar{B}_{\text{WA}}^{(2),s}(D^+) &= -0.0002(10)\text{GeV}^3, \\
 \bar{B}_{\text{WA}}^{(\sigma),s}(D^0) &= -0.0000(7)\text{GeV}^3, & \bar{B}_{\text{WA}}^{(\sigma),s}(D^+) &= -0.0000(7)\text{GeV}^3,
 \end{aligned}$$

2-quark operator matrix elements were assumed to be identical to the B meson ones, which is questionable.

Phenomenological Progress

- ❖ Even strong coupling constant α_s was extracted from electron energy spectrum

[Wu,Lou,Che,Li,Huang,Ruan,Ye, '24]



Again, non-perturbative HQE parameters were assumed to be identical to the B meson ones, which is questionable.

We should come **down-to-earth**: start with the Dim-5 operators first!

Their matrix elements in D decays have never been determined.

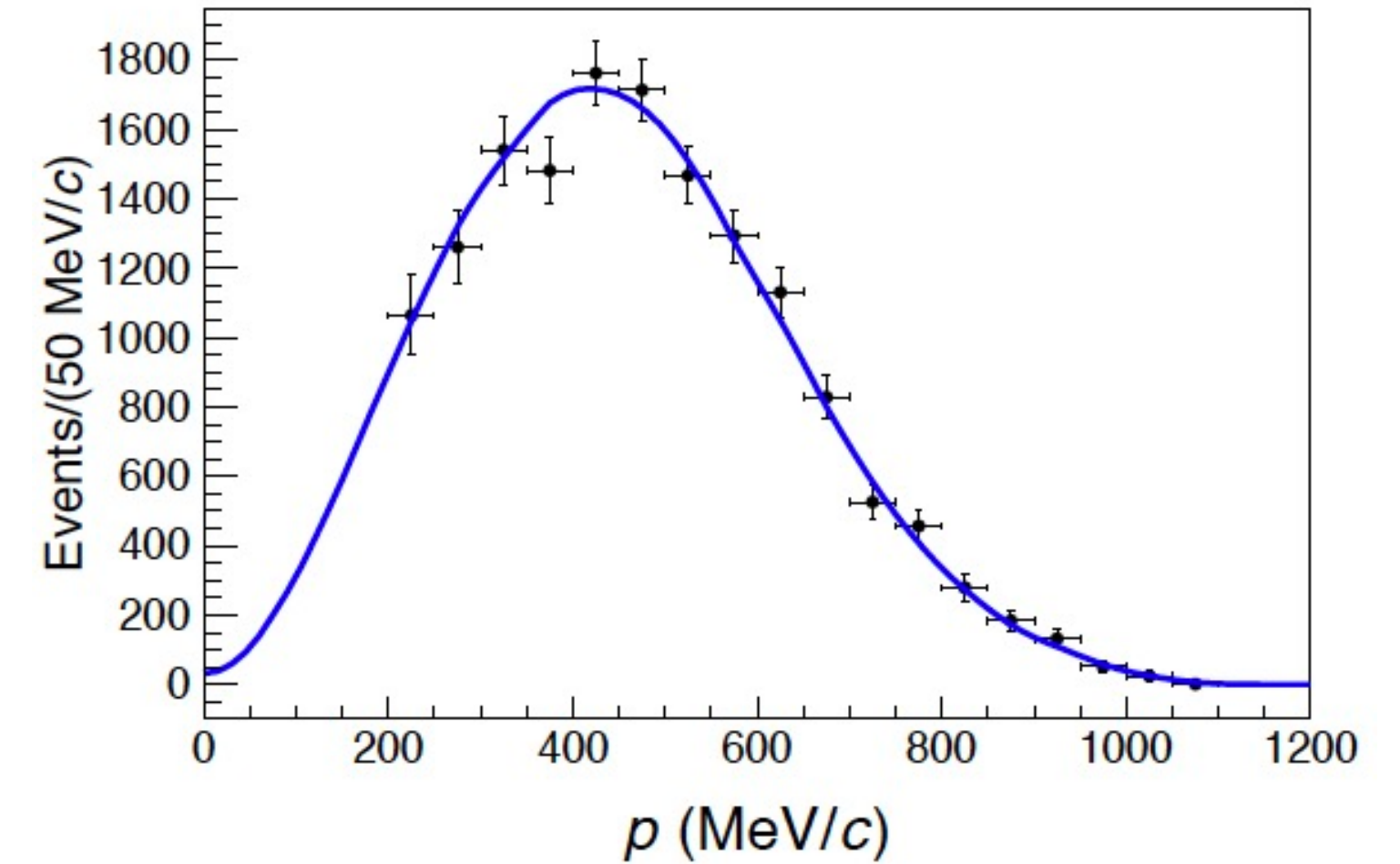
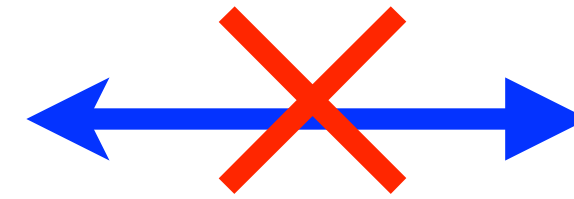
Pheno-1. Determine the HQE parameters

[Shao,Huang,QQ,2502.05901]

Fitting strategy

❖ Electron energy spectrum ($y \equiv 2E_e/m_c$)

$$\begin{aligned} \frac{1}{\Gamma_0} \frac{d\Gamma}{dy} = & 12(1-y)y^2\theta(1-y) \\ & + \frac{2\mu_\pi^2}{m_c^2} \left[-10y^3\theta(1-y) + 2\delta(1-y) \right] \\ & - \frac{2\mu_G^2}{3m_c^2} \left[6y^2(6-5y)\theta(1-y) \right] + \mathcal{O}(\alpha_s, \frac{\Lambda^3}{m_c^3}) \end{aligned}$$



❖ Up to finite power, the obtained spectrum is **NOT** the experimental spectrum

➡ Observables require integration over final states

➡ $\Gamma = \int \frac{d\Gamma}{dy} dy$, $\langle E_\ell^n \rangle = \frac{1}{\Gamma} \int \frac{d\Gamma}{dy} E_\ell^n dy$ (n=1,2,3,4) are the **observables**

➡ Shape function — — infinite power summation?

[Neubert, '93]

Theoretical results

❖ Theoretical results for total decay rate and energy moments (NNLO & $\Lambda_{\text{QCD}}^3/m_c^3$)

NLO analytical integration

NNLO numerical results provided by
authors of [Chen,Chen,Guan,Ma,'23]

$$\Gamma_{D_i \rightarrow X_q} = \hat{\Gamma}_0 |V_{cq}|^2 m_c^5 \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \frac{2}{3} \left(\frac{25}{4} - \pi^2 \right) + \frac{\alpha_s^2(\mu)}{\pi^2} \left[\frac{\beta_0}{4} \frac{2}{3} \left(\frac{25}{4} - \pi^2 \right) \log \left(\frac{\mu^2}{m_c^2} \right) + 2.14690 n_l - 29.88311 \right] \right. \\ \left. - 8\rho\delta_{sq} - \frac{1}{2} \frac{\mu_\pi^2(D_i)}{m_c^2} - \frac{3}{2} \frac{\mu_G^2(D_i)}{m_c^2} + \left(6 + 8 \log \left(\frac{\mu^2}{m_c^2} \right) \right) \frac{\rho_D^3(D_i)}{m_c^3} + \frac{\tau_0(D_i \rightarrow X_q)}{m_c^3} + \dots \right\}$$

Dim-5, $\Lambda_{\text{QCD}}^2/m_c^2$ Dim-6, $\Lambda_{\text{QCD}}^3/m_c^3$

❖ For Dim-6 4-quark operator contributions, practically difficult to extract

➡ Vacuum-Insertion-Approximation (VIA), $\tau_0 = 0$

➡ HQET Sum Rules [King, Lenz, Piscopo, Rauh,'19]

Theoretical results

❖ Theoretical results for total decay rate and energy moments (NNLO & $\Lambda_{\text{QCD}}^3/m_c^3$)

$$\langle E_e \rangle_{D_i} = \frac{\hat{\Gamma}_0}{\Gamma_{D_i}} \sum_{q=d,s} |V_{cq}|^2 m_c^6 \left[\frac{3}{10} + \frac{\alpha_s}{\pi} a_1^{(1)} + \frac{\alpha_s^2}{\pi^2} a_1^{(2)} - 3\rho\delta_{sq} - \frac{1}{2} \frac{\mu_G^2(D_i)}{m_c^2} + \frac{139}{30} \frac{\rho_D^3(D_i)}{m_c^3} + \frac{3}{10} \frac{\rho_{LS}^3(D_i)}{m_c^3} + \dots \right],$$

$$\langle E_e^2 \rangle_{D_i} = \frac{\hat{\Gamma}_0}{\Gamma_{D_i}} \sum_{q=d,s} |V_{cq}|^2 m_c^7 \left[\frac{1}{10} + \frac{\alpha_s}{\pi} a_2^{(1)} + \frac{\alpha_s^2}{\pi^2} a_2^{(2)} - \frac{6}{5} \rho\delta_{sq} + \frac{1}{12} \frac{\mu_\pi^2(D_i)}{m_c^2} - \frac{11}{60} \frac{\mu_G^2(D_i)}{m_c^2} + \frac{17}{6} \frac{\rho_D^3(D_i)}{m_c^3} + \frac{7}{30} \frac{\rho_{LS}^3(D_i)}{m_c^3} + \dots \right],$$

$$\langle E_e^3 \rangle_{D_i} = \frac{\hat{\Gamma}_0}{\Gamma_{D_i}} \sum_{q=d,s} |V_{cq}|^2 m_c^8 \left[\frac{1}{28} + \frac{\alpha_s}{\pi} a_3^{(1)} + \frac{\alpha_s^2}{\pi^2} a_3^{(2)} - \frac{1}{2} \rho\delta_{sq} + \frac{1}{14} \frac{\mu_\pi^2(D_i)}{m_c^2} - \frac{1}{14} \frac{\mu_G^2(D_i)}{m_c^2} + \frac{223}{140} \frac{\rho_D^3(D_i)}{m_c^3} + \frac{1}{7} \frac{\rho_{LS}^3(D_i)}{m_c^3} + \dots \right],$$

$$\langle E_e^4 \rangle_{D_i} = \frac{\hat{\Gamma}_0}{\Gamma_{D_i}} \sum_{q=d,s} |V_{cq}|^2 m_c^9 \left[\frac{3}{224} + \frac{\alpha_s}{\pi} a_4^{(1)} + \frac{\alpha_s^2}{\pi^2} a_4^{(2)} - \frac{3}{14} \rho\delta_{sq} + \frac{3}{64} \frac{\mu_\pi^2(D_i)}{m_c^2} - \frac{13}{448} \frac{\mu_G^2(D_i)}{m_c^2} + \frac{481}{560} \frac{\rho_D^3(D_i)}{m_c^3} + \frac{9}{112} \frac{\rho_{LS}^3(D_i)}{m_c^3} + \dots \right],$$

Mass scheme

$$\Gamma = m_c^5(\Gamma^{(0)} + \alpha_s \Gamma^{(1)} + \alpha_s^2 \Gamma^{(2)}) = \left(\bar{m}_c(1 + \alpha_s \bar{m}^{(1)} + \alpha_s^2 \bar{m}^{(2)}) \right)^5 (\Gamma^{(0)} + \alpha_s \Gamma^{(1)} + \alpha_s^2 \Gamma^{(2)})$$

❖ **Pole mass scheme** (suffering from renormalon)

$$\Gamma/\Gamma_{\text{LO}} = 1 - 0.77\alpha_s - 2.38\alpha_s^2 - 10.73\alpha_s^3 \approx 1 - 30\% - 36\% - 62\%$$

Not convergent,
negative at NNNLO!

❖ **$\overline{\text{MS}}$ mass scheme**

$$\Gamma/\Gamma_{\text{LO}} = 1 + 1.35\alpha_s + 3.02\alpha_s^2 + 7.69\alpha_s^3 \approx 1 + 52\% + 46\% + 44\%$$

[Melnikov, van Ritbergen, '99]

Convergent,
but very slowly!

❖ **1S mass scheme** (half of J/ψ mass)

[Hoang, Ligeti, Manohar, '98; Hoang, Teubner, '99]

$$\Gamma/\Gamma_{\text{LO}} \approx 1 - 13.1\% - 4.8\% + 1.8\%$$

Answer: convergent expansion of $\alpha_s(m_c)$!

❖ **Kinetic mass scheme**

[Fael, Schönwald, Steinhauser, '20]

Not work for charm, because it requires expansion of μ^2/m_c^2 and $\alpha_s(\mu)$

Experimental data

$$\frac{d\Gamma}{dy} = ay^2(1 + by)(1 - y)$$

Original
data

Extrapolation

[Gambino, Kamenik, '10]

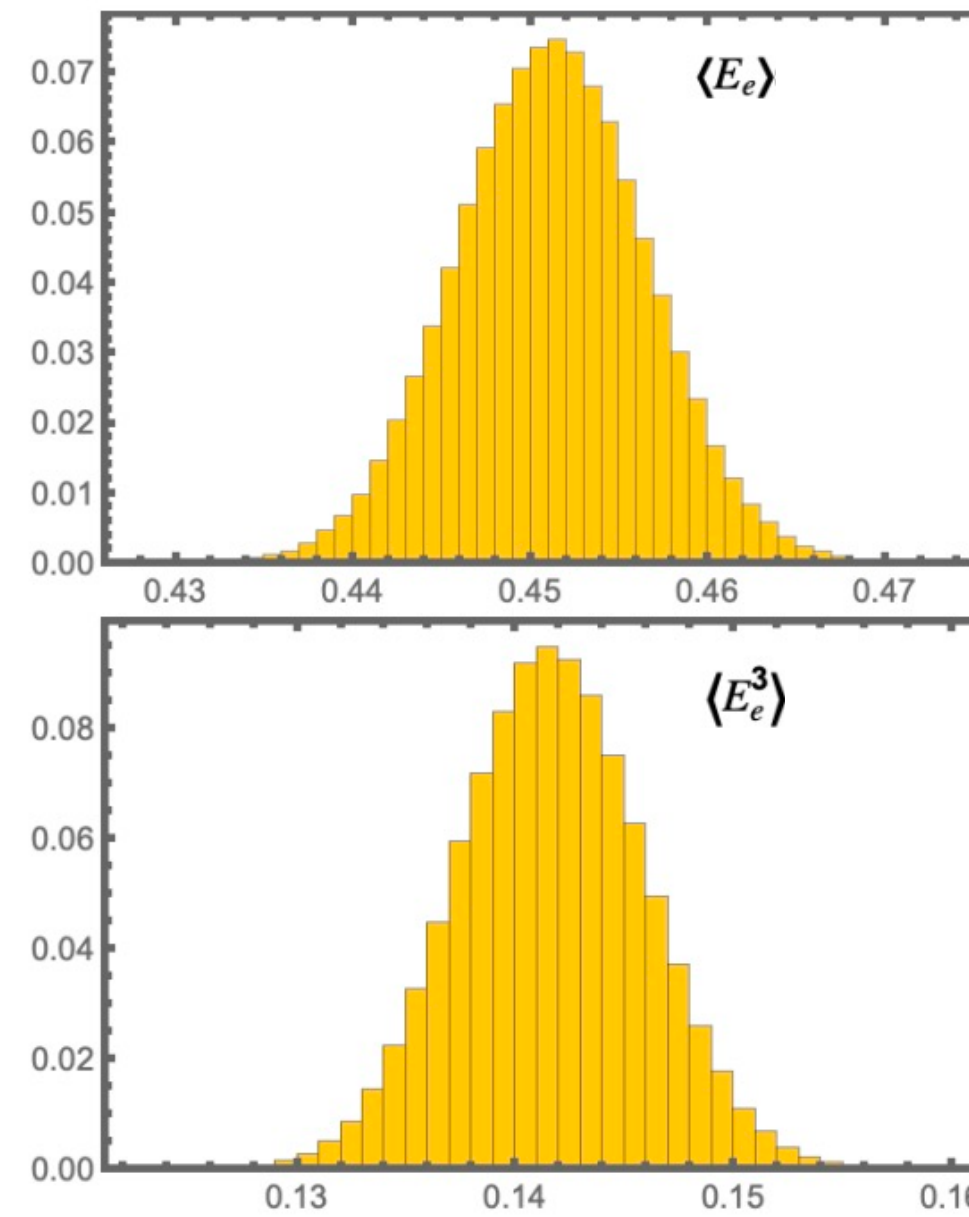
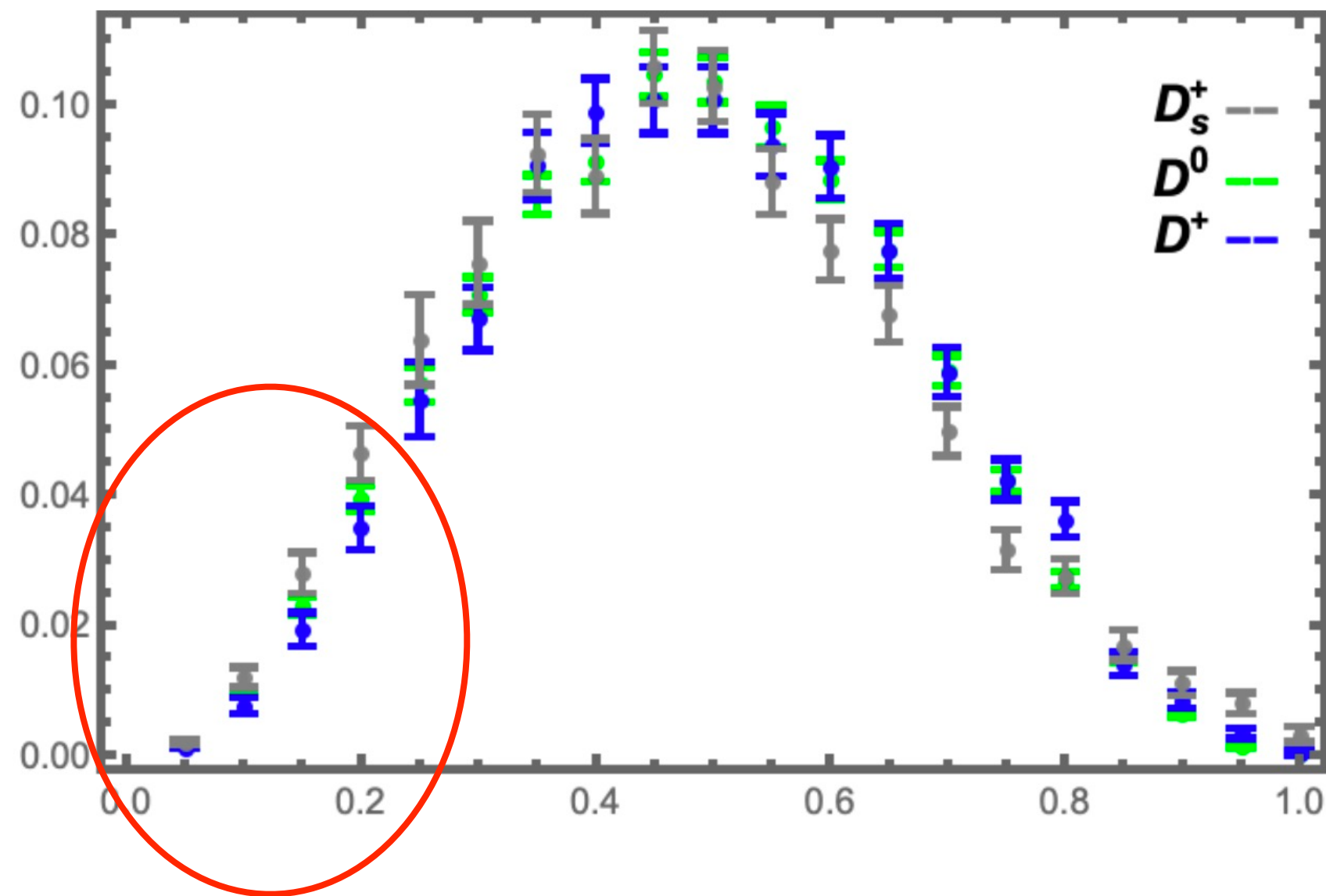
All-bin
distribution

Lorentz boost

Rest
frame

MC Simulation

Electron energy
moments



$$\begin{aligned} \langle E_e \rangle_{exp}^{D_s} &= 0.437(6) \text{ GeV}, & \langle E_e^2 \rangle_{exp}^{D_s} &= 0.220(5) \text{ GeV}^2 \\ \langle E_e \rangle_{exp}^{D^0} &= 0.462(5) \text{ GeV}, & \langle E_e^2 \rangle_{exp}^{D^0} &= 0.242(5) \text{ GeV}^2 \\ \langle E_e \rangle_{exp}^{D^+} &= 0.455(4) \text{ GeV}, & \langle E_e^2 \rangle_{exp}^{D^+} &= 0.236(4) \text{ GeV}^2 \\ \langle E_e^3 \rangle_{exp}^{D_s} &= 0.121(4) \text{ GeV}^3, & \langle E_e^4 \rangle_{exp}^{D_s} &= 0.072(3) \text{ GeV}^4 \\ \langle E_e^3 \rangle_{exp}^{D^0} &= 0.138(4) \text{ GeV}^3, & \langle E_e^4 \rangle_{exp}^{D^0} &= 0.084(3) \text{ GeV}^4 \\ \langle E_e^3 \rangle_{exp}^{D^+} &= 0.134(3) \text{ GeV}^3, & \langle E_e^4 \rangle_{exp}^{D^+} &= 0.081(3) \text{ GeV}^4 \end{aligned}$$

Statistical uncertainties uncorrelated,
Systematic uncertainties fully correlated.

Global fit

❖ Global fit in two mass schemes, each with **Scenario 1** ($\Lambda_{\text{QCD}}^2/m_c^2$) and **Scenario 2** ($\Lambda_{\text{QCD}}^3/m_c^3$)

$\overline{\text{MS}}$ scheme	$\chi^2/\text{d.o.f.}$	D_i	μ_π^2/GeV^2	μ_G^2/GeV^2	ρ_D^3/GeV^3	ρ_{LS}^3/GeV^3
Scenario 1	4.5	$D^{0,+}$	0.09 ± 0.01	0.27 ± 0.14	-	-
		D_s	0.09 ± 0.02	0.39 ± 0.12	-	-
Scenario 2	2.1	$D^{0,+}$	0.11 ± 0.02	0.26 ± 0.14	-0.002 ± 0.002	0.003 ± 0.002
		D_s	0.12 ± 0.02	0.38 ± 0.13	-0.003 ± 0.002	0.005 ± 0.002

4-q operator contributions
vanish under VIA, $\tau_0 = 0$.

1S scheme	$\chi^2/\text{d.o.f.}$	D_i	μ_π^2/GeV^2	μ_G^2/GeV^2	ρ_D^3/GeV^3	ρ_{LS}^3/GeV^3
Scenario 1	4.9	$D^{0,+}$	0.04 ± 0.01	0.33 ± 0.02	-	-
		D_s	0.06 ± 0.02	0.44 ± 0.02	-	-
Scenario 2	0.33	$D^{0,+}$	0.09 ± 0.02	0.32 ± 0.02	-0.003 ± 0.002	0.004 ± 0.002
		D_s	0.11 ± 0.02	0.43 ± 0.02	-0.004 ± 0.002	0.005 ± 0.002

**Reliable perturbative
calculation ensures a
good fit!**

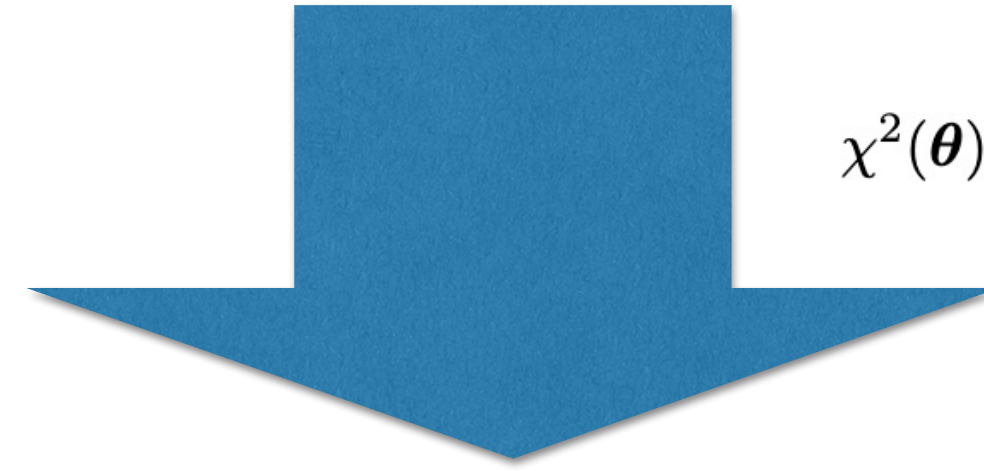
Differences between Scenarios 1 and 2 as systematic uncertainties.

Weak-annihilation Uncertainty

❖ Redo the fits, adopting the HQET SR calculation for the weak annihilation contributions

$$\tau_0(D_d \rightarrow X_d) = \tau_0(D_s \rightarrow X_s) = \tau_{\text{val}} = (-0.18 \pm 0.65) \text{ GeV}^3 \quad [\text{King, Lenz, Piscopo, Rauh, '19}]$$

$$\tau_0(D_u \rightarrow X_{d,s}) = \tau_0(D_d \rightarrow X_s) = \tau_0(D_s \rightarrow X_d) = \tau_{\text{nonval}} = (0.45 \pm 2.10) \text{ GeV}^3$$



$$\chi^2(\theta) = \sum_{l=u,d,s} \sum_{i=1}^5 \sum_{j=1}^5 (y_{i,l} - \eta_{i,l}) V_{ij,l}^{-1} (y_{j,l} - \eta_{j,l}) + \left(\frac{\tau_{\text{nonval}} - (-0.18)}{0.65} \right)^2 + \left(\frac{\tau_{\text{val}} - 0.45}{2.10} \right)^2$$

$$\begin{aligned} \mu_\pi^2(D^{0,+}) &= 0.08 \text{ GeV}^2, \quad \mu_G^2(D^{0,+}) = 0.33 \text{ GeV}^2, \quad \rho_D^3(D^{0,+}) = -0.003 \text{ GeV}^3, \quad \rho_{LS}^3(D^{0,+}) = 0.004 \text{ GeV}^3 \\ \mu_\pi^2(D_s) &= 0.15 \text{ GeV}^2, \quad \mu_G^2(D_s) = 0.38 \text{ GeV}^2, \quad \rho_D^3(D_s) = -0.005 \text{ GeV}^3, \quad \rho_{LS}^3(D_s) = 0.006 \text{ GeV}^3, \\ \tau_{\text{val}} &= -0.11 \text{ GeV}^3, \quad \tau_{\text{nonval}} = 0.002 \text{ GeV}^3. \end{aligned}$$

The best-fit values only slightly change.

Global fit

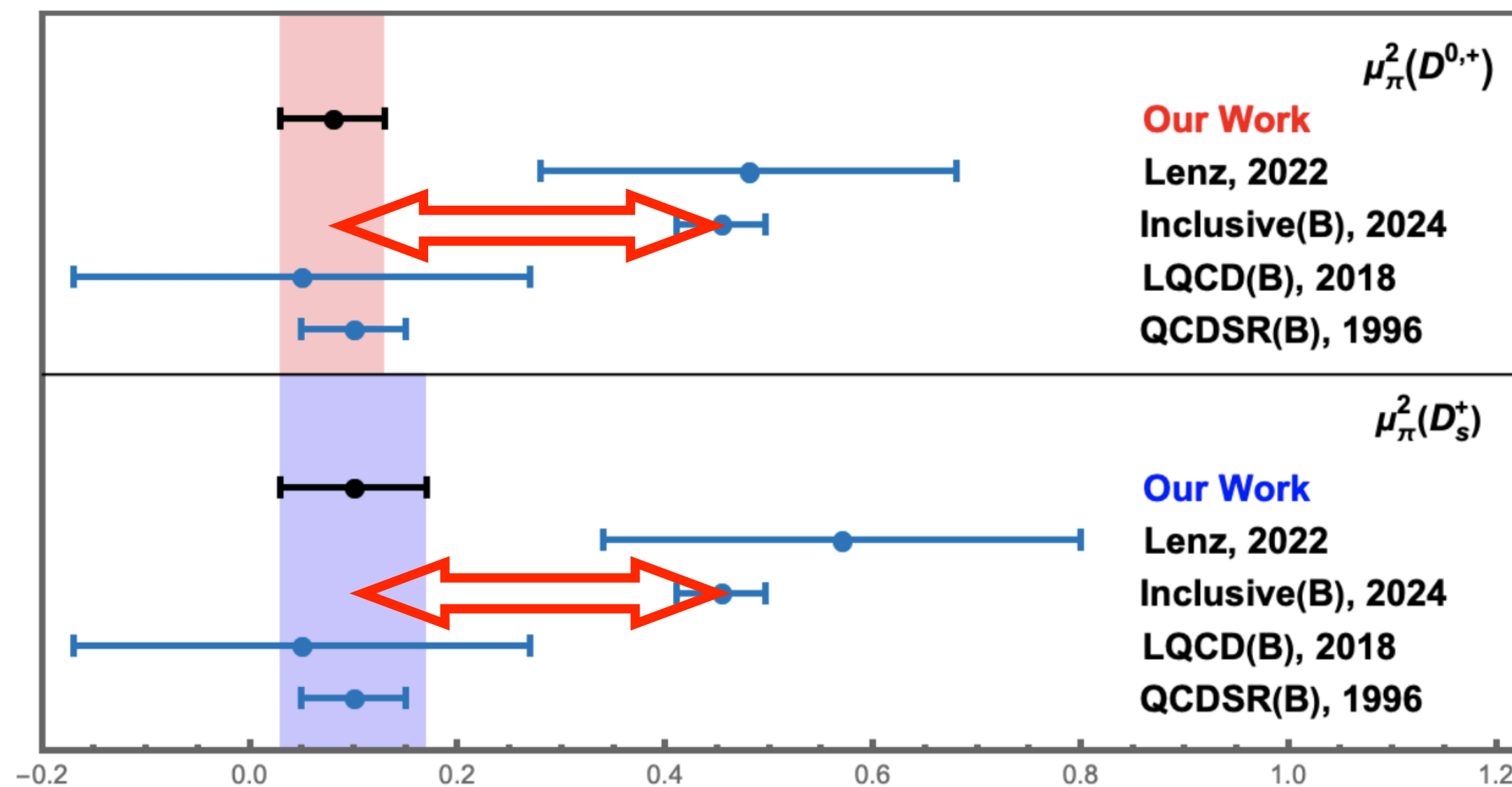
❖ Extracted nonperturbative HQE parameters

$$\begin{aligned}\mu_\pi^2(D^{0,+}) &= (0.08 \pm 0.05)\text{GeV}^2, \\ \mu_G^2(D^{0,+}) &= (0.33 \pm 0.03)\text{GeV}^2, \\ \rho_D^3(D^{0,+}) &= (-0.003 \pm 0.002)\text{GeV}^3, \\ \rho_{LS}^3(D^{0,+}) &= (0.004 \pm 0.002)\text{GeV}^3,\end{aligned}$$

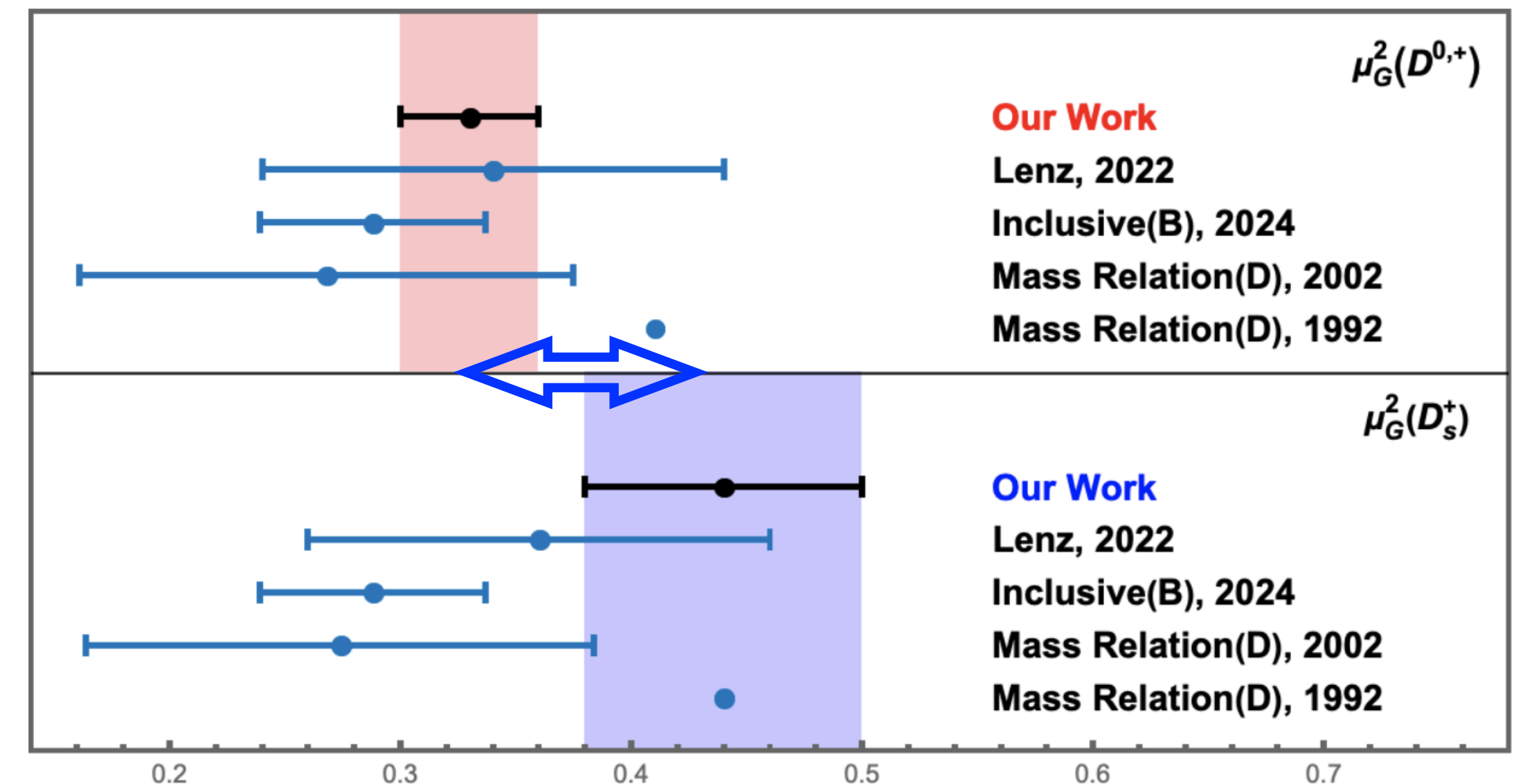
$$\begin{aligned}\mu_\pi^2(D_s^+) &= (0.10 \pm 0.07)\text{GeV}^2, \\ \mu_G^2(D_s^+) &= (0.44 \pm 0.06)\text{GeV}^2, \\ \rho_D^3(D_s^+) &= (-0.004 \pm 0.002)\text{GeV}^3, \\ \rho_{LS}^3(D_s^+) &= (0.005 \pm 0.002)\text{GeV}^3,\end{aligned}$$

Sizable breaking effects of
flavor SU(3) symmetry and
heavy quark symmetry.

μ_π^2



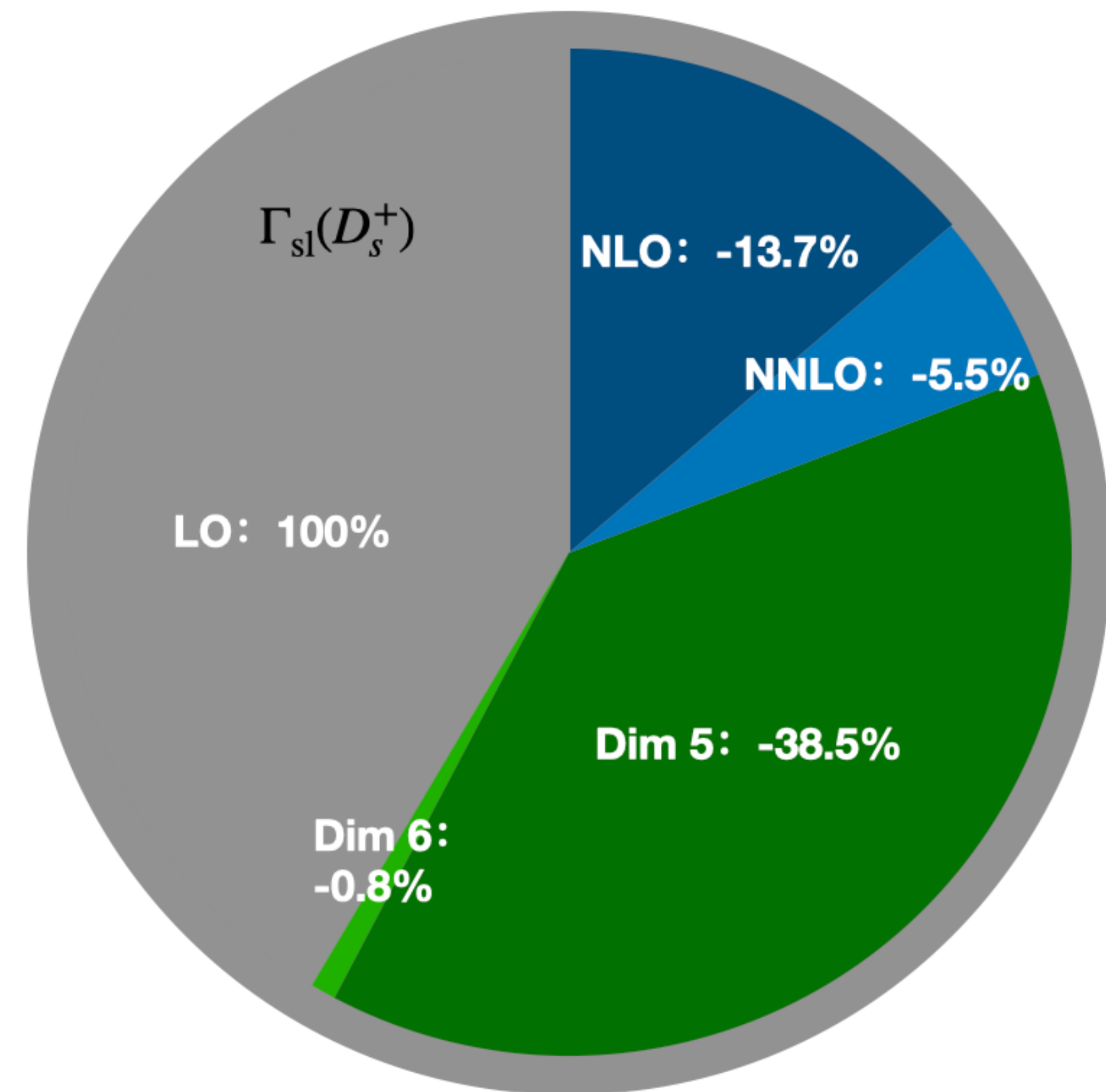
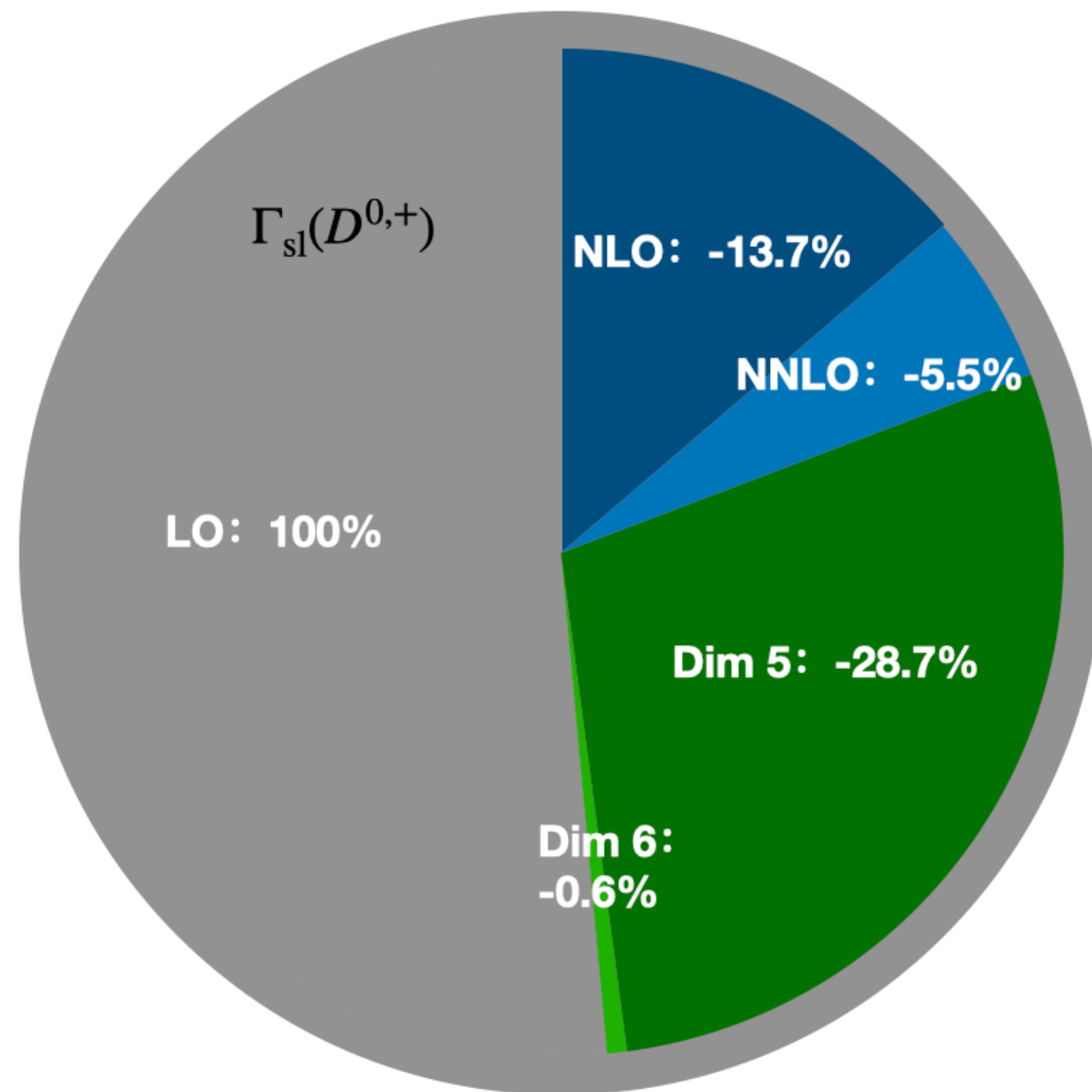
μ_G^2



Inappropriate to use beauty parameters for charm!

Convergence

❖ Various contributions to inclusive $D^{0,+}$ and D_s^+ decay widths



Convergent expansions of $\alpha_s(m_c)$ and Λ_{QCD}/m_c !

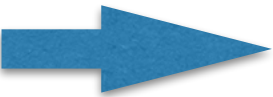
Pheno-2. Determine the CKM matrix elements

[Shao,Feng,Liu,**QQ**,Sun,Yu,2509.11404]

Determine the CKM matrix elements

- ❖ Treat the $V_{cs,cd}$ as unknown parameters, instead of input
- ❖ Difficult for inclusive measurements to separate X_s and $X_d \Rightarrow$ **Sum of exclusive channels**

D^+ decays		D^0 decays		D_s decays	
Mode	BR(%)	Mode	BR(%)	Mode	BR(%)
$D^+ \rightarrow \bar{K}^0 e^+ \nu_e$	8.72 ± 0.09	$D^0 \rightarrow K^- e^+ \nu_e$	3.549 ± 0.026	$D_s \rightarrow \phi e^+ \nu_e$	2.34 ± 0.12
$(K^- \pi^+)_{\bar{K}^*(892)^0} e^+ \nu_e$	3.54 ± 0.09 [23]	$(\bar{K}^0 \pi^-)_{S\text{-wave}} e^+ \nu_e$	0.079 ± 0.017	$D_s \rightarrow \eta e^+ \nu_e$	2.27 ± 0.06
$(K^- \pi^+)_{S\text{-wave}} e^+ \nu_e$	0.228 ± 0.011	$D^0 \rightarrow K^*(892)^- e^+ \nu_e$	2.04 ± 0.047 [24]	$D_s \rightarrow \eta' e^+ \nu_e$	0.81 ± 0.04
$D^+ \rightarrow \bar{K}_1(1270)^0 e^+ \nu_e$	0.230 ± 0.026	$D^0 \rightarrow \bar{K}_1(1270)^- e^+ \nu_e$	0.101 ± 0.018	$D_s \rightarrow f_0(980) e' \nu_e$	0.164 ± 0.013
$D^+ \rightarrow \eta e^+ \nu_e$	0.111 ± 0.007	$D^0 \rightarrow \pi^- e^+ \nu_e$	0.291 ± 0.004	$D_s \rightarrow K^0 e^+ \nu_e$	0.288 ± 0.026
$D^+ \rightarrow \pi^0 e^+ \nu_e$	0.372 ± 0.017	$D^0 \rightarrow \rho(770)^- e^+ \nu_e$	0.145 ± 0.007	$D_s \rightarrow K^*(892)^0 e^+ \nu_e$	0.205 ± 0.020
$D^+ \rightarrow \pi^+ \pi^- e^+ \nu_e$	0.245 ± 0.008	$D^0 \rightarrow a(980)^- e^+ \nu_e$	$0.0133^{+0.0034}_{-0.0030}$		
$D^+ \rightarrow \pi^0 \pi^0 e^+ \nu_e$	0.0315 ± 0.0027 [25]				
$D^+ \rightarrow \omega e^+ \nu_e$	0.169 ± 0.011				
$D^+ \rightarrow \eta' e^+ \nu_e$	0.020 ± 0.004				
$D^+ \rightarrow a(980)^0 e^+ \nu_e$	0.017 ± 0.008				



	D^+	D^0	D_s
$X_s e^+ \nu_e$	$14.60 \pm 0.16\%$	$5.81 \pm 0.06\%$	$5.58 \pm 0.14\%$
$X_d e^+ \nu_e$	$0.96 \pm 0.03\%$	$0.45 \pm 0.01\%$	$0.49 \pm 0.03\%$

$$\begin{aligned}
 B(D^0 \rightarrow X_{d+s} e^+ \nu_e) &= 0.0636(15), \\
 B(D^+ \rightarrow X_{d+s} e^+ \nu_e) &= 0.1602(32), \\
 B(D_s \rightarrow X_{d+s} e^+ \nu_e) &= 0.0631(14),
 \end{aligned}$$

Compatible within 1-2 σ

Determine the CKM matrix elements

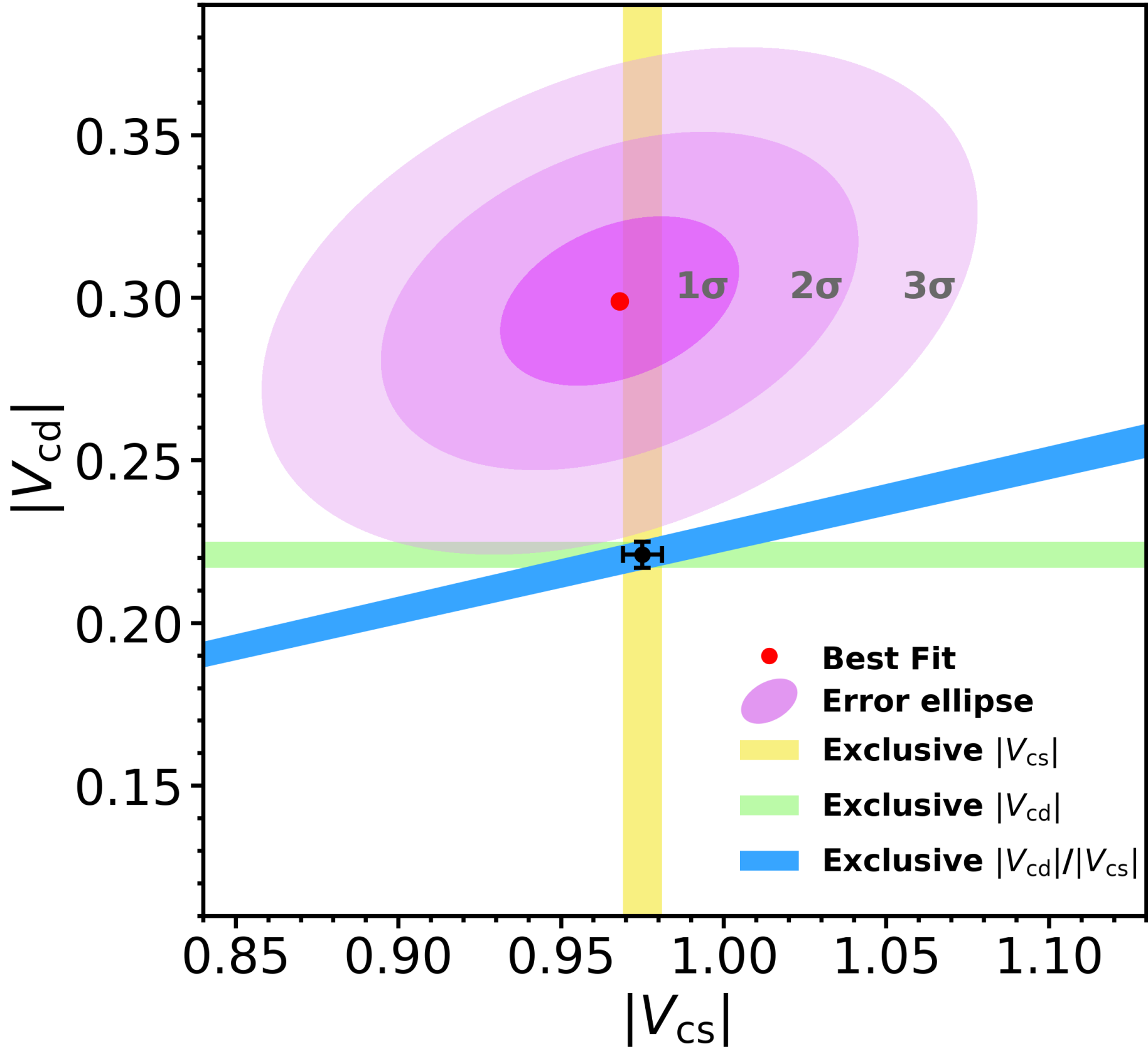
- ❖ Data choice strategy: all inclusive X_{s+d} data are used
 - ➡ **S1**: Sum-of-exclusive X_s data, plus sum-of-exclusive X_d data,
 - ➡ **S2**: Sum-of-exclusive X_s data
 - ➡ **S3**: Sum-of-exclusive X_d data
- ❖ For each strategy, two scenarios for weak-annihilation contributions
 - ➡ **Scenario 1**: VIA, $\tau_0 = 0$
 - ➡ **Scenario 2**: HQET SR, input the previous best-fit values $\tau_{\text{val}} = -0.11\text{GeV}^3$, $\tau_{\text{nonval}} = 0.002\text{GeV}^3$.

Determine the CKM matrix elements

S1 ($X_{s,d}$)	$\chi^2/\text{d.o.f.}$	$ V_{cs} $	$ V_{cd} $
Scenario 1	3.24	$0.977 \pm 0.022 \pm 0.026$	$0.252 \pm 0.006 \pm 0.006$
Scenario 2	1.12	$0.969 \pm 0.022 \pm 0.026$	$0.253 \pm 0.006 \pm 0.006$
S2 (X_s)	$\chi^2/\text{d.o.f.}$	$ V_{cs} $	$ V_{cd} $
Scenario 1	0.43	$0.974 \pm 0.022 \pm 0.026$	$0.279 \pm 0.031 \pm 0.007$
Scenario 2	0.29	$0.960 \pm 0.021 \pm 0.026$	$0.278 \pm 0.031 \pm 0.006$
S3 (X_d)	$\chi^2/\text{d.o.f.}$	$ V_{cs} $	$ V_{cd} $
Scenario 1	4.08	$0.982 \pm 0.023 \pm 0.027$	$0.253 \pm 0.006 \pm 0.005$
Scenario 2	1.24	$0.974 \pm 0.022 \pm 0.026$	$0.254 \pm 0.006 \pm 0.006$

$$|V_{cs}| = 0.968 \pm 0.022 \pm 0.026 \pm 0.014$$

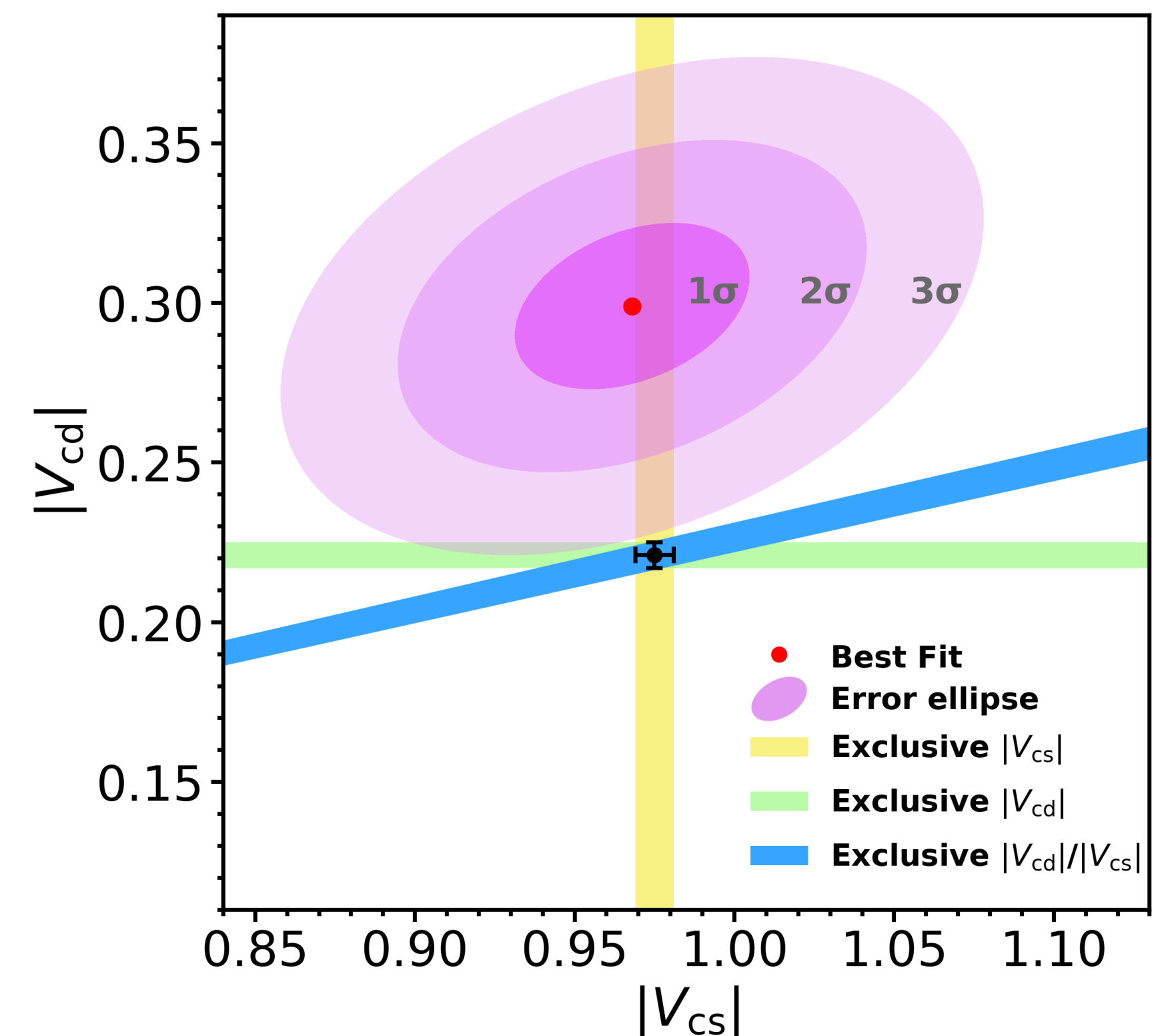
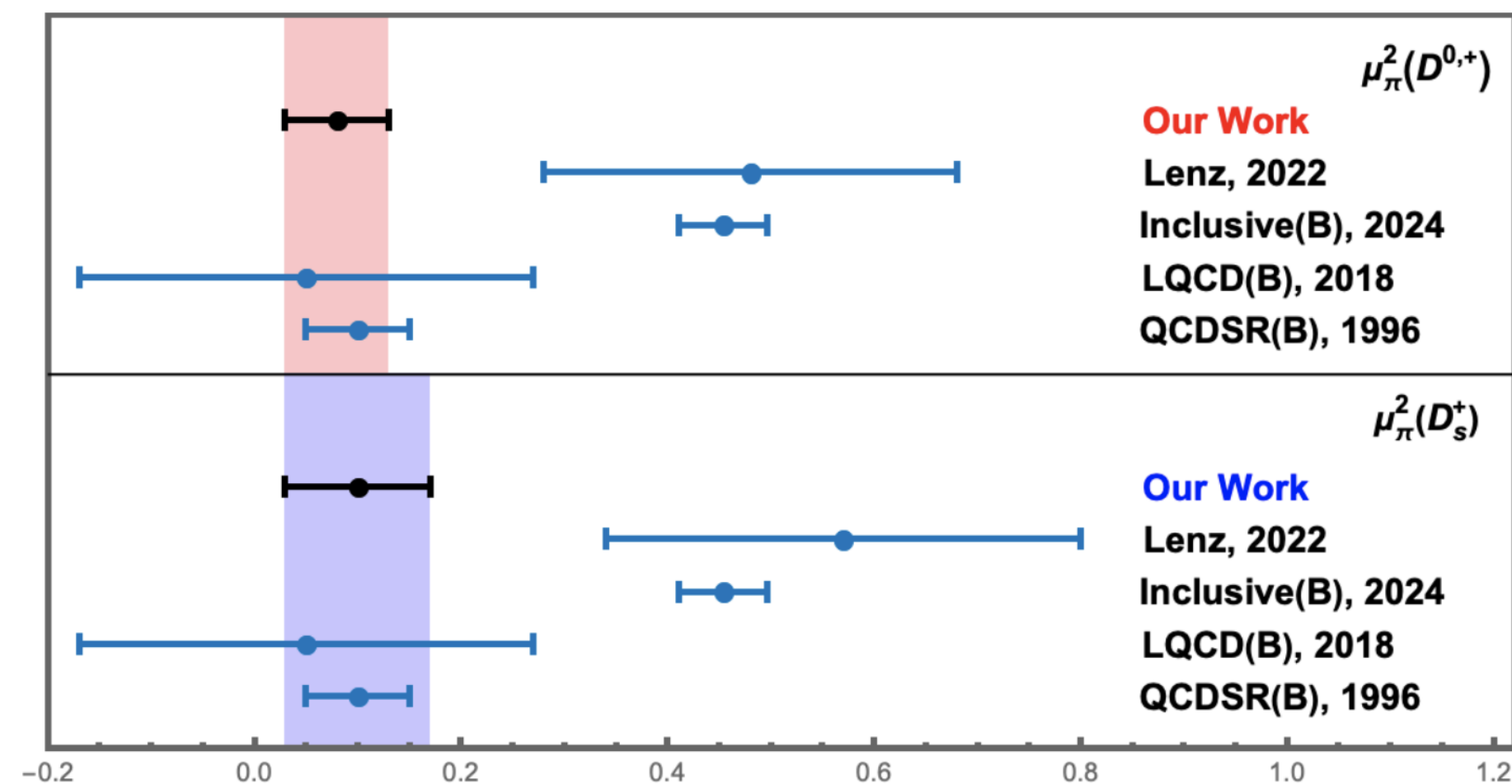
$$|V_{cd}| = 0.299 \pm 0.025 \pm 0.007 \pm 0.002$$



Inclusive vs exclusive: 3σ tension!

Summary and Prospect

- ❖ Comprehensive theoretical achievements for inclusive charm decays, **theoretical framework**, **power corrections**, and α_s **corrections**, lay a solid foundation for phenomenological study
- ❖ α_s -**expansion** and **heavy quark expansion** are **valid** in inclusive charm decays (with an proper mass scheme chosen)
- ❖ For the **first time**, we determine
 - ➡ The **HQE parameters** for D mesons
 - ➡ The **CKM matrix elements** $V_{cs,cd}$ (inclusive)



Summary and Prospect

❖ Possible theoretical improvements

- ➡ Include higher order radiative corrections, $\mathcal{O}(\alpha_s^3)$
- ➡ Include higher power corrections, complete dimension-6 and -7 operator
- ➡ Include charmed baryons in the study
- ➡

❖ Possible experimental improvements

- ➡ Laboratory frame $\rightarrow$ rest frame
- ➡ Direct measurements of the moments
- ➡ Separate X_d , X_s

Thank you!

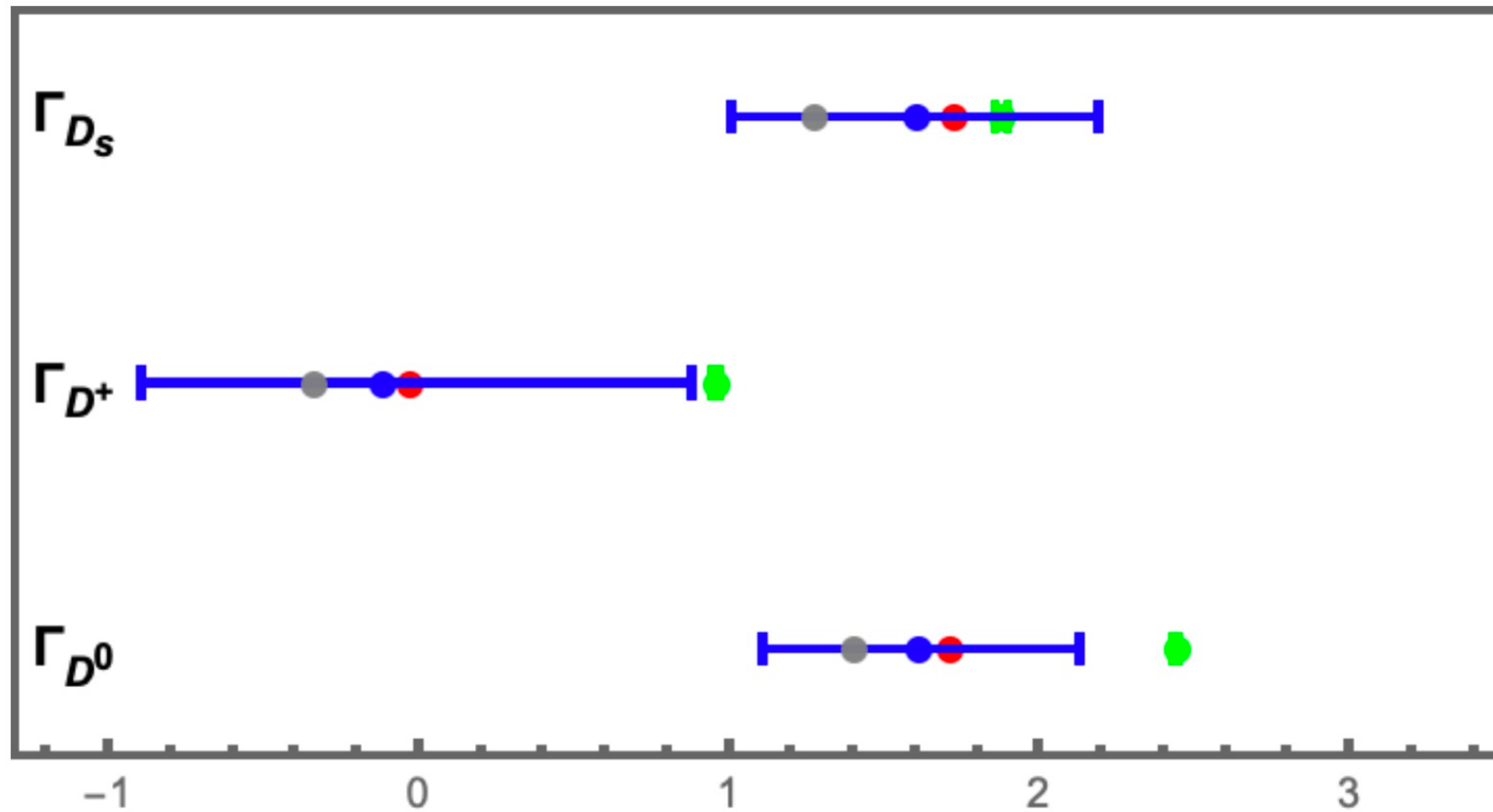
Backup

Mass scheme transformation

$$m_c = \bar{m}_c(\mu) \left[1 + \frac{\alpha_s(\mu)}{\pi} \left(\frac{4}{3} + \log \left(\frac{\mu^2}{\bar{m}_c^2} \right) \right) + \frac{\alpha_s^2(\mu)}{\pi^2} \frac{1}{288} \left(112\pi^2 + 2905 + 16\pi^2 \log(4) - 48\zeta(3) \right. \right. \\ \left. \left. - 12(2n_f - 45) \log^2 \left(\frac{\mu^2}{\bar{m}_c^2} \right) - 4(26n_f - 519) \log \left(\frac{\mu^2}{\bar{m}_c^2} \right) - 2(71 + 8\pi^2) n_f \right) + \mathcal{O}(\alpha_s^3) \right]$$

$$m_c = m_{c,1S} + m_{c,1S} \frac{\alpha_s(\mu)^2 C_F^2}{8} \left\{ 1 + \frac{\alpha_s}{\pi} \left[\left(-\log(\alpha_s(\mu) m_{c,1S} C_F / \mu) + \frac{11}{6} \right) \beta_0 - 4 + \frac{\pi}{8} C_F \alpha_s \right] + \dots \right\}$$

Modification of D Lifetime Prediction



Robust test of the fit

