Scalar wave emission from particles radially infalling in a Schwarzschild black hole

Giorgio Di Russo

Hangzhou Institute for Advanced Study, UCAS, Hangzhou

Talk based on a work in progress with Donato Bini and Andrea Geralico

October 31, 2025

Main ideas and motivations

- Study Black Holes (BH) perturbations due some field have been studied both numerically and analytically.
- Scalar emission by a massive scalar particle in radial infall into a Schwarzschild BH has been studied in:
  - Zerilli, PRD 2, 2141 (1970) $\Longrightarrow$ $\omega >> 1$;
  - Mitsou, 1012.2028, Silva, Tambalo et al, PDR 109, 024036 (2024), Barausse, Bert, Cardoso et al, 2106.09721 $\Longrightarrow$ numerical;
  - In the last paper ℓ summation leads to Log divergencies addressed to point-like probe approximation.
  - Analytic self force on circular orbits: 1305.4884,1312.2503,1503.02334, Bini, Damour, Geralico, Kavanagh.
  - Analytic self force on eccentric orbits Bini, Damour, Geralico 1511.04533.
  - Analytic self force on hyperbolic obits Geralico 2507.03442.
- Here I show analytical computations within the Post-Newtonian (PN) i.e. low velocities approximation.
- Both PN expanded and Mano-Suzuki-Takasugi (MST) solutions to the perturbation eqs will be employed.

Plan of the talk

- Recap on BHs perturbations theory in order to fix notation.
- Heun equations (HE) arising from the study of perturbations.
- The massive particle in radial fall in Schwarzschild geometry.
- Definition of energy loss at infinity and its computation with PN solutions: limitations and necessity of MST.
- MST method in a nutshell.
- Final results.
- Future directions: the gravitational case.

Black Holes Perturbation Theory: The Heun Equations

- Given the Einstein field equations
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,,$$
- the linear perturbation can be studied
$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \,.$$
- In some cases the eqs at first order in the perturbation $h_{\mu\nu}$ become ODEs of the second order.
- For simplicity, one could alternatively study the propagation of scalar waves in the curved background
$$\Box\Phi = \mu^2\Phi, \qquad \Box = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \,, \qquad g = |\det\left(g_{\mu\nu}\right)| \,,$$
- which in some cases allows the separation between angular and radial dynamics.
- In $d = 4$ the separation is achieved through the ansatz
$$\Phi(t, r, \theta, \phi) = e^{-i\omega t + i m_\phi \phi} R(r) S(\theta) \,.$$
- $S(\theta)$ are
  - Legendre polynomials in the spherically symmetric case (*e.g.* Schwarzschild);
  - Oblate spheroidal harmonics in the axially symmetric case (*e.g.* Kerr).
- In the vast majority of the cases, $R(r)$ are ODEs of Heun type (HE) with four regular singularities
$$\left[\frac{d^2}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{d}{dz} + \frac{\alpha\beta z - p}{z(z-1)(z-t)}\right] W(z) = 0 \,,$$
- where $\alpha + \beta + 1 = \gamma + \delta + \epsilon$. The normal (or canonical) form is achieved thanks to the redefinition
$$W(z) = z^{-\gamma/2}(1-z)^{-\delta/2}(t-z)^{-\epsilon/2}\psi(z) \qquad \Longrightarrow \qquad \left[\frac{d^2}{dz^2} + Q_{HE}\right]\psi(z) = 0.$$

- The $Q-$functions of the various confluences of the HE are (Bonelli, Iossa et al 2201.04491):

$$
\begin{aligned}
Q_{HE} &= \frac{2\gamma-\gamma^2}{4z^2}+\frac{2\delta-\delta^2}{4(z-1)^2}+\frac{2\epsilon-\epsilon^2}{4(z-t)^2}+\frac{-\gamma\epsilon+2p-\gamma\delta t-2\alpha\beta z+\gamma\delta z+\gamma z\epsilon+\delta z\epsilon}{2(z-1)z(t-z)}\\
Q_{CHE} &= -\frac{\epsilon^2}{4}+\frac{2\gamma-\gamma^2}{4z^2}+\frac{2\delta-\delta^2}{4(z-1)^2}+\frac{-\gamma\delta+\gamma\epsilon-2p+2\alpha z-\gamma z\epsilon-\delta z\epsilon}{2(z-1)z}\\
Q_{RCHE} &= \frac{2\gamma-\gamma^2}{4z^2}+\frac{2\delta-\delta^2}{4(z-1)^2}+\frac{\gamma\delta+2p}{2z}+\frac{2\beta-\gamma\delta-2p}{2(z-1)}\\
Q_{DCHE} &= -\frac{\delta^2}{4z^4}-\frac{(\gamma-2)\delta}{2z^3}+\frac{-\gamma^2+2\gamma-2\delta-4p}{4z^2}+\frac{2\alpha-\gamma}{2z}-\frac{1}{4}\\
Q_{RDCHE} &= \frac{\epsilon}{z^3}-\frac{p}{z^2}+\frac{\beta}{z}-\frac{1}{4}\\
Q_{DRDCHE} &= \frac{\epsilon}{z^3}-\frac{p}{z^2}+\frac{1}{z} \qquad\qquad C = Confluent,\ D = Doubly,\ R = Reduced
\end{aligned}
$$

- Physically the $Q-$function plays the role of effective potential thanks to the analogy with the non relativistic Schrödinger eq.
- The function $-Q$ always has a maximum, furthermore
  - Fuzzball geometries: an internal minimum (Bianchi, GDR 2212.07504);
  - Asymptotically AdS geometries: external minimum (Berti, Cardoso, Pani 0903.5311).

Black Holes Perturbation Theory: The Heun Equations

- Radial dynamics :
  - HE (4 regular): AdS-KN BH (Aminov, Arnaudo, Bonelli et al 2307.10141);
  - CHE (2 regular, 1 irregular): flat KN (Bonelli, Iossa et al 2105.04483, Bianchi, Consoli et al 2109.09804);
  - RCHE (2 regular, 1 irregular): CCLP, D1/D5 circular fuzzball, JMaRT (Bianchi, GDR 2212.07504, Bianchi, Di Benedetto, GDR et al 2305.00865);
  - DCHE (2 irregular) : eKN, D3-D3-D3-D3 BH;
  - RDCHE (2 irregular): eCCLP, BMPV;
  - DRDCHE a.k.a. Mathieu (2 irregular): D3, D1/D5 BHs (Bianchi, Consoli 2105.04245, Fioravanti, Gregori 2112.11434).
- Angular dynamics:
  - Spherically symmetric geometries: Spherical harmonics.
  - Axially symmetric geometries:
    - CHE: KN (Berti, Cardoso, Casalas 0511111);
    - RCHE: D1-D5 circular fuzzball, JMaRT.

(Bianchi, GDR 2203.14900)

- The classical observables of BH perturbation theory.
  - Quasi normal modes (QNMs) whose boundary conditions are:
    - BH: ingoing at horizon, outgoing at infinity;
    - Fuzzballs: regular at cap, outgoing at infinity;
    - AdS BH: ingoing at horizon, regular at infinity.
  - Tidal deformability (GDR, Fucito, Morales 2402.06621)
  - Amplification factor (Cipriani, Di Benedeto, GDR et al 2405.06566)
  - Wave form reconstruction (Bianchi, Bini, GDR 2407.10868, 2411.19612, 2502.21040, 2506.04876)

Geodesic motion

- The metric of Schwarzschild geometry

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad , \quad f(r) = \left(1 - \frac{2M}{r}\right) .$$

- The radial and equatorial motion implies $\theta = \frac{\pi}{2}$ and $\phi = 0$, so that for time-like geodesics

$$ds^2 = -1 = -f(r)dt^2 + \frac{dr^2}{f(r)} .$$

- Since the canonical momentum associated to t is conserved and interpreted as the energy

$$P_\mu = \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} \quad , \quad P_t = -f(r)\,\dot{t} = -E ,$$

- the equation of motion is

$$\frac{dr_p}{dt} = -\frac{1}{E}\left(1 - \frac{2M}{r_p(t)}\right)\sqrt{E^2 - 1 + \frac{2M}{r_p(t)}} .$$

- Considering particles initially at rest $E = 1$, the previous eq. admits an exact solution

$$\frac{t(r)}{2M} = -\frac{2}{3}\left(\frac{r}{2M}\right)^{3/2}\left(1 + \frac{6M}{r}\right) + \ln\left(\frac{\sqrt{\frac{r}{2M}} + 1}{\sqrt{\frac{r}{2M}} - 1}\right) .$$

- Setting $\eta = \frac{1}{c}$, the previous geodesic can be inverted in PN sense $r_p(t) \sim \frac{GM}{c^2} \sim \eta^2 \Longrightarrow M \to M\,\eta^2$

$$r_p(t) = t^{2/3}\alpha - \frac{8}{9}\alpha^3\eta^2 + \frac{16\alpha^5}{27t^{2/3}}\eta^4 + \frac{640\alpha^7}{2187t^{4/3}}\eta^6 - \frac{256\alpha^9}{10935t^2}\eta^8 + O(\eta^{10}) \quad , \quad \alpha = \left(\frac{9}{2}M\right)^{1/3} .$$

Scalar field sourced by a scalar charge

- The non-homogeneous wave equation to solve is

$$\Box\Phi = -4\pi\rho \,,$$

- where ρ is the scalar density and q is the scalar charge

$$\rho = q\int_{-\infty}^{\infty} \frac{1}{\sqrt{-g}}\delta^{(4)}(x^\alpha - x_p^\alpha(s))ds \quad , \quad x_p^\alpha(s) = \left(t_p(s), r_p(s), \frac{\pi}{2}, 0\right) \,.$$

- Integration of ρ is done in t and using the completeness of spherical harmonics

$$\rho = \frac{qf(r_p(t))}{r_p^2(t)}\delta(r - r_p(t))\sum_{\ell m} Y_{\ell m}(\theta,\phi)Y^*_{\ell m}\left(\frac{\pi}{2}, 0\right) \,.$$

- In terms of the energy-momentum tensor, the scalar density is defined

$$\rho = \sum_{\ell m}\int \frac{d\omega}{2\pi}e^{-i\omega t}\hat{T}_{\ell m}(\omega, r)Y_{\ell m}(\theta,\phi) = \sum_{\ell m} T_{\ell m}(t, r)Y_{\ell m}(\theta,\phi) \,.$$

- So the full non-homogeneous differential problem becomes

$$\Box\psi = \sum_{lm}\int \frac{d\omega}{2\pi}e^{-i\omega t}Y_{lm}(\theta,\phi)\left[f(r)R''_{lm\omega} + \frac{2(r-M)}{r^2}R'_{lm\omega} + \left(\frac{\omega^2}{f} - \frac{\ell(\ell+1)}{r^2}\right)\right]$$

$$= -4\pi\sum_{lm}\int \frac{d\omega}{2\pi}e^{-i\omega t}\widehat{T}_{lm}(\omega, r)Y_{lm}(\theta,\phi)Y^*_{lm}(\frac{\pi}{2}, 0) \,,$$

- where

$$\widehat{T}_{lm}(\omega, r) = q\int dt e^{i\omega t}\frac{f(r_p(t))}{r_p^2(t)}\delta(r - r_p(t))Y^*_{lm}(\frac{\pi}{2}, 0) = qY^*_{lm}(\frac{\pi}{2}, 0)\mathcal{F}(r,\omega) \,.$$

Scalar field sourced by a scalar charge

- The equation is solved with Green function method

$$\mathcal{L}_{(r)} G_{lm\omega}(r, r') \equiv \Big[\frac{d^2}{dr^2} + \frac{\Delta'(r)}{\Delta(r)} \frac{d}{dr} + \left(\frac{\omega^2 r^4}{\Delta^2(r)} - \frac{L}{\Delta(r)} \right) \Big] G_{lm\omega}(r, r') = \frac{\delta(r - r')}{r'(r' - 2M)} \, ,$$

- where

$$G_{lm\omega}(r, r') = \frac{1}{W_{lm\omega}} [R_{\rm in}(r) R_{\rm up}(r') H(r' - r) + R_{\rm in}(r') R_{\rm up}(r) H(r - r')]$$

$$W_{lm\omega} = \Delta(r)[R_{\rm in}(r) R'_{\rm up}(r) - R_{\rm up}(r) R'_{\rm in}(r)] \, .$$

- The particular solution for the radial equation reads

$$R_{lm\omega}(r) = -4\pi \int dr' \, G_{lm\omega}(r, r') r'^2 \, \widehat{T}_{lm}(\omega, r') \, .$$

- The radiated energy can be derived from the energy-momentum tensor which for massless complex scalars is

$$\frac{d^2 E}{dt d\Omega} = \lim_{r \to \infty} \left(r^2 T^{\rm scal\, r}{}_t \right) \quad , \quad 8\pi T^{\rm scal}_{\mu\nu} = \Phi^*_{,\mu} \Phi_{,\nu} + \Phi_{,\mu} \Phi^*_{,\nu} - g_{\mu\nu} \Phi^*_{,\alpha} \Phi^{,\alpha} \, .$$

- The final expression reads

$$\Delta E = \int dt \frac{dE}{dt} = \int \sin\theta d\theta d\phi dt r^2 \, T^{\rm scal\, r}{}_t = -\int_{2M}^{\infty} \frac{dt}{dr} dr \frac{r^2 f(r_p(t))}{4\pi} \sum_{\ell m} \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \omega e^{-i(\omega - \omega')t} \mathcal{J}_{\ell m \omega\omega'}|_{r = r_p(t)} \, ,$$

$$\mathcal{J}_{\ell m \omega\omega'}(r) = \mathrm{Im} \left(R_{\ell m\omega}(r) \frac{d}{dr} R^*_{\ell m\omega'} \right) \, .$$

PN solution of homogeneous equation

- The equation describing a massless scalar wave in Schwarzschild is

$$r(r-2M)R''(r)+2(r-2M)R'(r)+\left[\frac{\omega^2 r^3}{r-2M}-\ell(\ell+1)\right]R(r)=0\,.$$

- Recall that PN weights are

$$M\to M\eta^2\,,\qquad \omega\to\omega\eta\,,$$

- where for the frequency, the PN weight is justified by $\frac{\omega}{c}\sim\frac{v}{cr_p(t)}\sim\eta$.
- Solving order by order in η, we find IN and UP PN solutions

$$\begin{aligned}R^{PN}_{IN} &= r^\ell+\eta^2 r^{\ell-1}\left(\frac{r^3\omega^2}{2(2\ell+2)}-\ell M\right)+\eta^4\frac{r^{\ell-2}}{8}\left[\frac{r^6\omega^4}{(2\ell+5)(2\ell+3)}+\frac{8(\ell-1)^2\ell M^2}{2\ell-1}+\frac{4(\ell^2-5\ell-10)Mr^3\omega^2}{(\ell+1)(2\ell+3)}\right]\\ &+\eta^6\frac{r^\ell}{48}\left[\frac{r^6\omega^6}{(2\ell+3)(2\ell+5)(2\ell+7)}-\frac{2Mr^3\omega^4\left(3\ell^3-27\ell^2-142\ell-136\right)}{(\ell+1)(\ell+2)(2\ell+3)(2\ell+5)}\right.\\ &\left.+\omega^2\left(\frac{96M^2\left(15\ell^2+15\ell-11\right)}{(2\ell-1)(2\ell+1)^2(2\ell+3)}-\frac{96M^2\left(15\ell^2+15\ell-11\right)\log(r)}{(2\ell-1)(2\ell+1)(2\ell+3)}\right)-\frac{16M^3(\ell-2)^2(\ell-1)\ell}{r^3(2\ell-1)}\right]+O\left(\eta^8\right)\,,\\ R^{PN}_{UP}(r,\ell) &= R^{PN}_{IN}(r,-\ell-1)\,.\end{aligned}$$

- Notice that PN solutions are real, so that they do not contribute to energy loss since

$$\mathcal{J}_{\ell m\omega\omega'}(r)=\mathrm{Im}\left(R^{PN}_{\ell m\omega}(r)\frac{d}{dr}R^{PN\,*}_{\ell m\omega'}\right)=0\,.$$

MST solutions for homogeneous Teukolsky equation

- Original paper on MST solutions to homogeneous Teukolsky eq. (Kerr) (Mano, Suzuki, Takasugi 9603020).
- MST-type IN solution: ingoing boundary condition at $r = 2M$:

$$R(z) = e^{i\epsilon z}(-z)^{-i\epsilon} f(z) \quad , \quad \epsilon = 2M\omega \quad , \quad z = -\frac{r-2M}{2M} \, .$$

- The Teukolsky eq becomes

$$\begin{aligned} z(1-z)f''(z) + [1 - 2i\epsilon - (2-2i\epsilon)z]f'(z) + (\nu + i\epsilon)(\nu + 1 - i\epsilon)f(z) = \\ = 2i\epsilon[-z(1-z)f'(z) + zf(z)] + [\nu(\nu+1) - \ell(\ell+1) + \epsilon^2 - i\epsilon]f(z) \, , \end{aligned}$$

- where ν is the renormalized angular momentum (RAM) $\nu = \ell + \mathcal{O}(\epsilon)$.
- The solution can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} C_n f_{n+\nu}(z), \qquad f_{n+\nu}(z) = {}_2F_1\left(n+\nu+1-i\epsilon, -n-\nu-i\epsilon, 1-2i\epsilon, z\right) \, .$$

- As a consequence the ODE becomes a 3-terms recursion relation

$$\alpha_n C_{n+1} + \beta_n C_n + \gamma_n C_{n-1} = 0 \, ,$$

$$\alpha_n = \frac{i\epsilon\,(n+\nu+1+i\epsilon)^2\,(n+\nu+1-i\epsilon)}{(n+\nu+1)(2n+2\nu+3)} \quad , \quad \gamma_n = -\frac{i\epsilon\,(n+\nu+i\epsilon)\,(n+\nu-i\epsilon)^2}{(n+\nu)(2n+2\nu-1)} \, ,$$

$$\beta_n = -\ell(\ell+1) + (n+\nu)(n+\nu+1) + 2\epsilon^2 + \frac{\epsilon^4}{(n+\nu)(n+\nu+1)} \, .$$

- The consistency of the 3-terms recursion can be written in form of a continuous fraction

$$\beta_0 - \frac{\alpha_{-1}\gamma_0}{\beta_{-1} - \frac{\alpha_{-2}\gamma_{-1}}{\beta_{-2} - \frac{\alpha_{-3}\gamma_{-2}}{\beta_{-3} - \ldots}}} - \frac{\alpha_0\gamma_1}{\beta_1 - \frac{\alpha_1\gamma_2}{\beta_2 - \frac{\alpha_2\gamma_3}{\beta_3 - \ldots}}} = 0 \, .$$

MST solutions for homogeneous Teukolsky equation

- Continuous fraction can be solved perturbatively introducing the RAM

$$\nu = \ell + \sum_{n=1}^{\infty} \nu_n (M\omega)^n \, .$$

- UP solution: outgoing b.c.s at $R(r) \underset{r\to\infty}{\sim} e^{i\omega r}$. Introduce $z = \omega(r - 2M)$ and $R(z) = z^{-1} g(z)$.

$$z^2 g''(z) + [z^2 + 2\epsilon z - \nu(\nu+1)] g(z) = -\epsilon z (g''(z) + g(z)) + \epsilon g'(z) - \frac{\epsilon(1+\epsilon^2)}{z} g(z) + [\lambda - \nu(\nu+1) - 3\epsilon^2] g(z) \, .$$

- As for the IN solution we can manage the RHS of the previous eq as a (small) source.
- Solution of the LHS can be written in terms of Coulomb functions

$$g(z) = \sum_{n=-\infty}^{\infty} \tilde{C}_n g_{n+\nu}(z) \quad , \quad g_{n+\nu}(z) = e^{-iz} (2z)^{n+\nu} z \frac{\Gamma(n+\nu+1+i\epsilon)}{\Gamma(2n+2\nu+2)} \Phi(n+\nu+1+i\epsilon, 2n+2\nu+2, 2iz) \, .$$

- Analogously to the IN, plugging the previous ansatz in the ODE, we have again a 3-terms recursion

$$\tilde{\alpha}_n \tilde{C}_{n+1} + \tilde{\beta}_n \tilde{C}_n + \tilde{\gamma}_n \tilde{C}_{n-1} = 0 \, .$$

- IN and UP are matched if the coefficients of the recursions coincide

$$\alpha_n = \tilde{\alpha}_n \quad , \quad \beta_n = \tilde{\beta}_n \quad , \quad \gamma_n = \tilde{\gamma}_n \, .$$

- This happens automatically with IN and UP constructed in such way.
- In Schwarzschild ($a = s = 0$), RAM is $\nu_{2n+1} = 0$ for $n \in Z$ and

$$\nu_2 = -\frac{2\ell^2 \left(15\ell^2 + 15\ell - 11\right)}{(2\ell-1)(2\ell+1)(2\ell+3)}$$

$$\nu_4 = -\frac{2(\ell(\ell+1)(5\ell(\ell+1)(21\ell(\ell+1)(8\ell(\ell+1)(22\ell(\ell+1)-125)+1479)-16525)+8733)+3240)}{\ell(\ell+1)(2\ell-3)(2\ell-1)^3(2\ell+1)^3(2\ell+3)^3(2\ell+5)}$$

$$\cdots$$

MST solutions

$$R_{IN}^{\ell=0}(r)=\frac{7}{9}-\frac{7}{54}r^2\omega^2\eta^2-\frac{7}{27}i(12\gamma_E-1)M\omega\eta^3+\frac{7r\omega^2(r^3\omega^2-200M)\eta^4}{1080}+\frac{7iM\omega(72M+(12\gamma_E-1)r^3\omega^2)\eta^5}{162r}$$
$$-\frac{\eta^6\omega^2\left(7r^6\omega^4-258720M^2\log\left(\frac{2M}{r}\right)+96M^2\left(2940\gamma_E^2-1226-490\gamma_E-245\pi^2\right)-6664Mr^3\omega^2\right)}{45360}+O\left(\eta^7\right),$$

$$R_{UP}^{\ell=0}=-\frac{7i}{18r\omega}+\frac{7\eta}{18}+\frac{7i}{18}\left(\frac{r\omega}{2}-\frac{M}{r^2\omega}\right)\eta^2-\frac{7(r^3\omega^2+12M(1+\gamma_E-i\pi))}{108r}\eta^3$$
$$-\frac{7i\eta^4\left(32M^2+12(13-4\gamma_E)Mr^3\omega^2-96Mr^3\omega^2\log(2r\omega)+r^6\omega^4\right)}{432r^3\omega}+O\left(\eta^5\right),$$

$$R_{IN}^{\ell=1}=\frac{r}{M}-\left(\frac{r^3\omega^2}{10M}+1\right)\eta^2+i(5-4\gamma_E)r\omega\eta^3+\frac{r^2\omega^2}{280}\left(\frac{r^3\omega^2}{M}-196\right)\eta^4+\frac{i}{10}(4\gamma_E-5)\omega(10M+r^3\omega^2)\eta^5$$
$$-\frac{\eta^6r\omega^2}{27291600M}\Big[1008\left(463303-541500\gamma_E+216600\gamma_E^2-18050\pi^2\right)M^2-1635330Mr^3\omega^2+1805r^6\omega^4$$
$$-69138720M^2\log\left(\frac{2M}{r}\right)\Big]+O\left(\eta^7\right),$$

$$R_{UP}^{\ell=1}=-\frac{i}{2r^2\omega^2}-\frac{i}{4}\left(\frac{4M}{r^3\omega^2}+1\right)\eta^2+\left(\frac{r\omega}{6}+\frac{M(1-\gamma_E+i\pi)}{r^2\omega}\right)\eta^3-\frac{i\eta^4}{80r^4\omega^2}\left(80Mr^3\omega^2+144M^2-5r^6\omega^4\right)$$
$$+\eta^6\Big[-\frac{16iM^3}{5r^5\omega^2}+\frac{i\left(185333+54150\gamma_E^2+108300i\pi-63175\pi^2-3610i\gamma_E(30\pi-49i)\right)M^2}{54150r^2}$$
$$-\frac{iM\left(19M-10r^3\omega^2\right)\log(2r\omega)}{15r^2}+\frac{1}{72}i(24\gamma_E-61)Mr\omega^2-\frac{1}{288}ir^4\omega^4\Big]+O\left(\eta^7\right),$$

...

Some technicalities and final result

- The energy loss at infinity

$$\Delta E = -\int_{2M}^{\infty} \frac{dt}{dr} dr \frac{r^2 f(r_p(t))}{4\pi} \sum_{\ell m} \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \omega e^{-i(\omega-\omega')t} \mathrm{Im}\left(R_{\ell m\omega}(r) \frac{d}{dr} R^*_{\ell m\omega'} \right) .$$

- Starting from MST solution, the procedure is the following:
  - Compute the (constant) Wronskian and the Green function;
  - Compute the radial function $R_{\ell m\omega}(r)$;
  - Compute $Im(R\partial_r R^*)$ on the orbit $r = r_p(t)$;
  - Fourier transform in ω and ω' and integration in t.

$$\int_{-\infty}^{\infty} e^{i\omega t} t^s dt = \frac{i^s}{|\omega|^{s+1}} \Gamma(1+s)(\mathrm{sgn}(\omega) - 1)\sin(\pi s) ,$$

$$\int_{-\infty}^{\infty} e^{i\omega t} \log(t) t^s dt = \frac{i^s}{2|\omega|^{s+1}} \Gamma(1+s)(\mathrm{sgn}(\omega)-1)\left[(i\pi - log(\omega^2) + 2\psi^{(0)}(s+1))\sin(\pi s) + 2\pi\cos(\pi s)\right] .$$

- The final result involving $\ell = 0, 1, 2$ MST contributions

$$\Delta E = \frac{q^2\eta^3}{\alpha^3}\left[\frac{143}{20} - \frac{33}{5}\gamma_E + \left(\frac{43081}{280} - 84\gamma_E\right)\eta^2 + \left(-\frac{83271}{5600} + \frac{63}{4}\gamma_E\right)\sqrt{3}\pi\eta^3 \right.$$
$$\left. + \left(-\frac{2727625}{1008} + \frac{644453}{1680}\gamma_E\right)\eta^4 + \left(\frac{403824}{6125} - \frac{933}{5}\gamma_E\right)\sqrt{3}\pi\eta^5\right] + O\left(\eta^8\right) .$$

- Reconstructed field up to $O\left(\eta^8\right)$ (physically non relevant)

$$\Phi(t) = \frac{qM^{2/3}}{t^{\frac{5}{3}}}\eta^3\left[-\frac{2\sqrt[3]{2}}{3^{\frac{5}{3}}} - \frac{4M^{\frac{1}{3}}}{3t^{\frac{1}{3}}}\eta + \frac{320\, 2^{\frac{2}{3}} M^{\frac{2}{3}}}{81\sqrt[3]{3}t^{\frac{2}{3}}}\eta^2 - \frac{M\left(704\log\left(\frac{4M}{3t}\right) + 3\pi\left(704i + 416\sqrt{3} + 183\pi\right) + 6400\right)}{324\, 6^{\frac{2}{3}}\, t}\eta^3\right] .$$

Outlooks: the gravitational perturbations setup

- The presence of the particle in the background reflects in the presence of a energy momentum tensor

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}\,, \quad G_{\mu\nu}(\tilde{g}_{\mu\nu}) = G_{\mu\nu}(g_{\mu\nu}) + \delta G_{\mu\nu}(h_{\mu\nu}) + O(h^2), \quad \tilde{T}_{\mu\nu} = T_{\mu\nu} + \delta T_{\mu\nu}\,.$$

- Since the background is Ricci flat, the perturbations of Einstein equations are

$$f_\mu = h_{\mu\alpha}{}^{;\alpha} \quad , \quad -\frac{1}{2} h_{\mu\nu;\alpha}{}^{;\alpha} + f_{(\mu;\nu)} - R_{\alpha\mu\beta\nu} h^{\alpha\beta} - \frac{1}{2} h_{;\mu;\nu} - \frac{1}{2} g_{\mu\nu}(f_\lambda{}^{;\lambda} - h_{;\lambda}{}^{;\lambda}) = 8\pi\delta T_{\mu\nu}\,.$$

- The separation of radial and angular equations is done through the decomposition in tensorial spherical harmonics

$$\begin{aligned}\mathbf{h} = e^{-i\omega t}\sum_{\ell m}\Bigg[& f(r) H_{0\ell m}(r)\mathbf{a}^{(0)}_{\ell m} - i\sqrt{2} H_{1\ell m}(r)\mathbf{a}^{(1)}_{\ell m} + \frac{1}{f(r)} H_{2\ell m}(r)\mathbf{a}_{\ell m} - \frac{i}{r}\sqrt{2\ell(\ell+1)}\, h^{(e)}_{0\ell m}(r)\mathbf{b}^{(0)}_{\ell m} \\ &+ \frac{\sqrt{2\ell(\ell+1)}}{r} h^{(e)}_{1\ell m}(r)\mathbf{b}_{\ell m} + \sqrt{\frac{\ell(\ell+1)(\ell-1)(\ell+2)}{2}}\, G_{\ell m}(r)\mathbf{f}_{\ell m} + \left(\sqrt{2} K_{\ell m}(r) - \frac{\ell(\ell+1)}{\sqrt{2}} G_{\ell m}(r)\right)\mathbf{g}_{\ell m} \\ &- \frac{\sqrt{2\ell(\ell+1)}}{r} h_{0\ell m}(r)\mathbf{c}^{(0)}_{\ell m} + \frac{i\sqrt{2\ell(\ell+1)}}{r} h_{1\ell m}(r)\mathbf{c}_{\ell m} + \frac{\sqrt{2\ell(\ell+1)(\ell-1)(\ell+2)}}{2r^2} h_{2\ell m}(r)\mathbf{d}_{\ell m}\Bigg]\,,\end{aligned}$$

$$\delta\mathbf{T} = \sum_{\ell m}\left[A^{(0)}_{\ell m}\mathbf{a}^{(0)}_{\ell m} + A^{(1)}_{\ell m}\mathbf{a}^{(1)}_{\ell m} + A_{\ell m}\mathbf{a}_{\ell m} + B^{(0)}_{\ell m}\mathbf{b}^{(0)}_{\ell m} + B_{\ell m}\mathbf{b}_{\ell m} + Q^{(0)}_{\ell m}\mathbf{c}^{(0)}_{\ell m} + Q_{\ell m}\mathbf{c}_{\ell m} + D_{\ell m}\mathbf{d}_{\ell m} + G^{(s)}_{\ell m}\mathbf{g}_{\ell m} + F_{\ell m}\mathbf{f}_{\ell m}\right],$$

- where $\{\mathbf{a}^{(0)}_{\ell m}, \mathbf{a}^{(1)}_{\ell m}, \mathbf{a}_{\ell m}, \mathbf{b}^{(0)}_{\ell m}, \mathbf{b}_{\ell m}, \mathbf{c}^{(0)}_{\ell m}, \mathbf{c}_{\ell m}, \mathbf{d}_{\ell m}, \mathbf{g}_{\ell m}, \mathbf{f}_{\ell m}\}$ are the tensorial spherical harmonics.
- Explicit expressions for tensorial harmonics are available in Nakano, Sago, Sasaki, gr-qc/0208060.
- Odd sector $(-1)^{\ell+1}$: $\{\mathbf{c}^{(0)}_{\ell m}, \mathbf{c}_{\ell m}, \mathbf{d}_{\ell m}\}$; Even sector $(-1)^{\ell}$: $\{\mathbf{a}^{(0)}_{\ell m}, \mathbf{a}^{(1)}_{\ell m}, \mathbf{a}_{\ell m}, \mathbf{b}^{(0)}_{\ell m}, \mathbf{b}_{\ell m}, \mathbf{g}_{\ell m}, \mathbf{f}_{\ell m}\}$.
- In the even sector Regge-Wheeler gauge can be imposed $H_{2\ell m}(r) = 0$.

Outlooks: the gravitational perturbations setup

- The sourced depend on the specific geodesic we are considering.
- To be concrete for radial falling geodesics, the odd sector (Regge-Wheeler) is not modified by the sources

$$r(r-2M)R''_{odd}(r)+2MR'_{odd}(r)+\left(\frac{\omega^2 r^3}{r-2M}+\frac{6M}{r}-\ell(\ell+1)\right)R_{odd}(r)=0\,.$$

- The even sector is described by Zerilli eq.

$$\left(1-\frac{2M}{r}\right)^2 Z''(r)+\frac{2M}{r^2}\left(1-\frac{2M}{r}\right)Z'(r)+\left(\omega^2-V_{Zerilli}(r)\right)Z(r)=S_Z(r)\,,$$

$$V_{Zerilli}(r)=\left(1-\frac{2M}{r}\right)\frac{72M^3+36M^2 r(\ell-1)(\ell+2)+r^2(\ell-1)^2(\ell+2)^2\left(6M+r\ell^2+r\ell\right)}{r^3(r(\ell+2)(\ell-1)+6M)^2}\,,$$

$$S_Z(r)=\frac{16\pi(r-2M)^2}{r(\ell+2)(\ell-1)+6M}A_{\ell m}(r)+\frac{16\sqrt{2}M\pi(2M-r)(6M-(\ell^2+\ell+4)r)}{r\omega(r(\ell+2)(\ell-1)+6M)^2}A^{(1)}_{\ell m}(r)$$
$$+\frac{8\sqrt{2}\pi(r-2M)^2}{\omega(r(\ell+2)(\ell-1)+6M)}\frac{dA^{(1)}_{\ell m}(r)}{dr}\,.$$

- The sources specialized to radial fall are

$$A_{\ell m}(r,t)=\mu\,u_t(t)\left(\frac{dr_p(t)}{dt}\right)^2\frac{\delta(r-r_p(t))}{(r-2M)^2}Y^*_{\ell m}\left(\frac{\pi}{2},0\right)\,,$$

$$A^{(0)}_{\ell m}(r,t)=\mu\,u_t(t)\left(1-\frac{2M}{r}\right)^2\frac{\delta(r-r_p(t))}{r^2}Y^*_{\ell m}\left(\frac{\pi}{2},0\right)\,,$$

$$A^{(1)}_{\ell m}(r,t)=\sqrt{2}i\mu\,u_t(t)\frac{dr_p(t)}{dt}\frac{\delta(r-r_p(t))}{r^2}Y^*_{\ell m}\left(\frac{\pi}{2},0\right)\,.$$

Outlooks: the gravitational perturbations setup

- Via the Chandrasekhar transformation Zerilli equation can be transformed to Regge-Wheeler

$$R_{even}(r) = \mathcal{A}_{\mathbf{1}}(r)Z(r) + \mathcal{A}_{\mathbf{2}}(r)\frac{dZ(r)}{dr} \quad , \quad S_{RW}(r) = \mathcal{A}_{\mathbf{1}}(r)S_Z(r) + \mathcal{A}_{\mathbf{2}}(r)\frac{dS_Z(r)}{dr} \, ,$$

$$\mathcal{A}_{\mathbf{1}}(r) = \Lambda + \frac{9M^2\left(1-\frac{2M}{r}\right)}{r(\lambda r + 3M)} \quad , \quad \mathcal{A}_{\mathbf{2}} = -3M\left(1-\frac{2M}{r}\right) \quad , \quad \Lambda = \lambda(\lambda+1) \quad , \quad \lambda = \frac{(\ell-1)(\ell+2)}{2} \, .$$

- After this transformation, homogeneous equations of both even and odd sectors are identical.
- So that PN and MST solutions are the same.
- In the even sector there are non-homogeneous terms due to the sources.
- These results are obtained from Nakano, Sago, Sasaki, gr-qc/0208060.
- In the previous paper most (not ALL!) of the results appearing in Zerilli, PRD 2, 2141 (1970) are corrected.

Conclusions

- Computations done in this framework are of valuable interest for people in amplitudes community.
- Contrarily to these computations, amplitudes are non perturbative in η and exact in G.
- Hyperbolic motion requires both PN (small η) and PM (small G) expansions.
- However high PM computations (many loops) are accessible.

Thank you for attention!