δn FORMALISM AND NON-PERTURBATIVE CURVATURE PERTURBATION DURING INFLATION..

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Work in progress with C. Chen.

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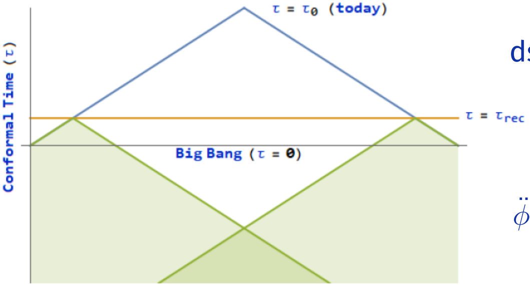
Outline

Motivation

Inflation (basics)

- In its simplest version, Inflation is exponential expansion of the universe driven by a scalar field

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] .$$



$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j ,$$

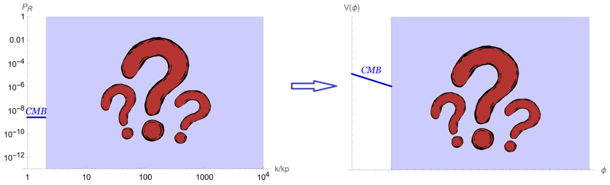
$$H^2 = \frac{\rho}{3M_P^2} = \frac{1}{3M_P^2} \left(V + \frac{\dot{\phi}^2}{2} \right) ,$$

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0 .$$

$$\epsilon_1 = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2H^2} \ll 1 , \qquad \epsilon_i = \frac{1}{H\epsilon_{i-1}} \frac{d\epsilon_{i-1}}{dt} .$$

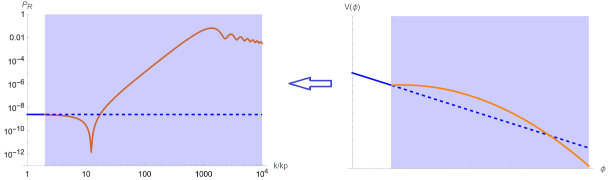
What about the potential?

- CMB observations on the spectral index and non-Gaussianities restrict the inflationary power spectrum and hence the inflationary potential, which must be compatible with SR inflation.
- However, the CMB is only accessible to a restricted range of scales.
- We know nothing about most of the inflationary potential.

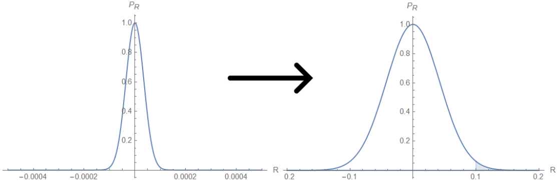


What about the potential?

- Any feature in the potential makes the power spectrum grow.



- A growth in the power spectrum makes large fluctuations more probable $\rightarrow$ PBH!



- Cosmological perturbation theory might not be enough to describe such large fluctuations. Alternatives?

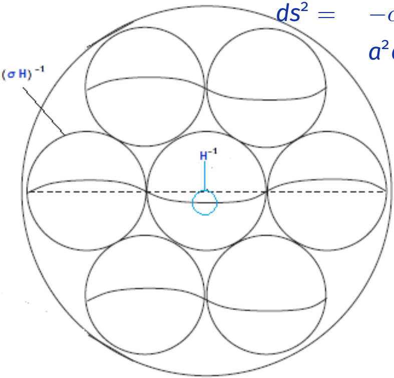
Review of δN formalism.

Gradient expansion

- The characteristic scale of inhomogeneities is larger that the Hubble horizon scale $L \gg H^{-1}$

$$L \sim \mathcal{O}\left(\frac{1}{\sigma}\right) \to L \sim \frac{1}{\sigma H}$$

with $\sigma \ll 1$



$$\begin{aligned} ds^2 = \quad & -\alpha^2 d\bar{t}^2 \\ & a^2 e^{2\zeta} \tilde{\gamma}_{ij} \left(d\bar{x}^i + \beta^i d\bar{t}\right) \left(d\bar{x}^j + \beta^j d\bar{t}\right) . \end{aligned}$$

$$ds^2 = -dt_p^2 + a(t_p)^2 \delta_{ij} dx_p^i dx_p^j .$$

$$H_p^2 = \rho_p \,,$$

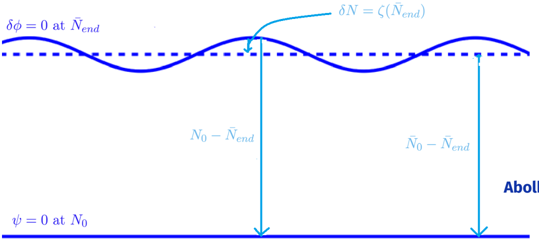
$$\frac{d^2}{dt_p^2}\phi_p + 3H_p \frac{d}{dt_p}\phi_p + V_{\phi_p} = 0$$

- The difference in the number of e-folds that each patch takes to reach some final comoving hypersurface is equal to the comoving curvature fluctuation ζ.

$$ds^2 = -\alpha^2 dt^2 + a^2 e^{2\zeta} \tilde{\gamma}_{ij} \left(d\bar{x}^i + \beta^i d\bar{t} \right) \left(d\bar{x}^j + \beta^j d\bar{t} \right)$$

$$N(t_0, t_{end}) \equiv -\frac{1}{3} \int_{t_0}^{t_{end}} K(t', \mathbf{x}) \alpha(t', \mathbf{x}) dt' \simeq \int_{t_0}^{t_{end}} \frac{\dot{a}}{a} dt + \zeta(t_{end}) - \zeta(t_0)$$

$$N(t_0, t_{end})^{comoving}_{flat} - \bar{N}(t_0, t_{end}) = \zeta(t_{end})$$



Salopek, Bond (1990)

Sasaki, Stewart (1996)

Sasaki, Tanaka (1998)

Sugiyama, Komatsu, Futamase (2013)

Abolhasani, Firouzjahi, Naruko, Sasaki (2019)

...

δN FORMALISM IN TERMS OF INITIAL CONDITIONS

- How can we compute δN?
- We will make use of the separate universe approach in which each patch (defined by $\mathbf{x}$) follows an homogeneous and isotropic evolution.

$$\frac{\partial \phi}{\partial N} = -\pi \,,$$
$$\frac{\partial \pi}{\partial N} - \left(3 - \frac{\pi^2}{2M_{PL}^2}\right)\left(\pi + \frac{M_{PL}^2}{V(\phi)}\frac{dV(\phi)}{d\phi}\right) = 0 \,,$$

- The solution is

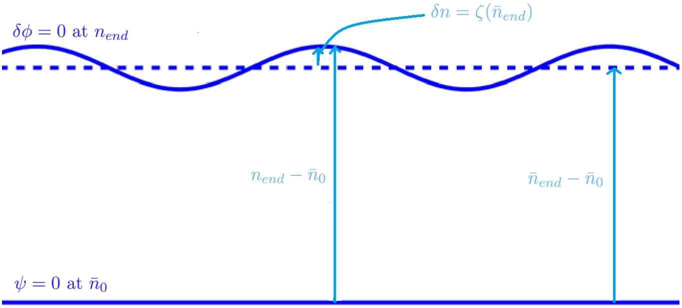
$$\phi\left(N - \bar{N}_{end}, \phi_{end}(\mathbf{x}), \pi_{end}(\mathbf{x})\right) \,, \qquad \pi\left(N - \bar{N}_{end}, \phi_{end}(\mathbf{x}), \pi_{end}(\mathbf{x})\right) \,,$$

- From there, taking into account that $\phi_{end}(\mathbf{x}) = \bar{\phi}_{end}$, we can invert to obtain

$$\zeta(\bar{N}_{end}) = \delta N = N\left(\bar{\phi}_0 + \delta\phi_0, \bar{\pi}_0 + \delta\pi_0\right) - \bar{N}\left(\bar{\phi}_0, \bar{\pi}_0\right) \,.$$

δn FORMALISM AND ITS SIMPLIFICATIONS

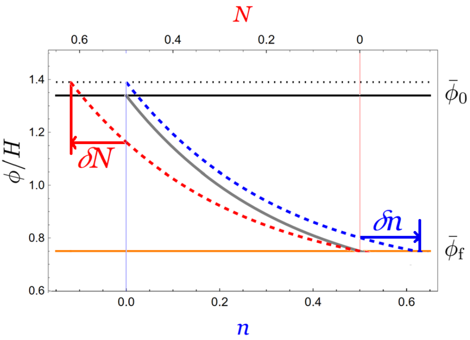
- We can do exactly the same by counting forward in time from some initial time $\bar{n}_0$.



- In this case we have

$$\zeta(\bar{n}_{end}) = \delta n = n_{end}\left(\bar{\phi}_0 + \delta\phi_0, \bar{\pi}_0 + \delta\pi_0\right) - \bar{n}_{end}\left(\bar{\phi}_0, \bar{\pi}_0\right) .$$

δN vs δn



- Exactly equivalent!
- Using conservation of probability, the PDF of δn (ζ) is

$$\mathbb{P}_{\delta n}(\delta n) = \int\int \mathbb{P}_{\delta\phi,\delta\pi}(\delta\phi_0, \delta\pi_0)\, \boldsymbol{\delta}\left(\delta n - \delta n(\delta\phi_0, \delta\pi_0)\right) d\delta\phi_0 d\delta\pi_0 \,,$$

- What are the advantages of counting forward in time?

Initial conditions for δn

$$\mathbb{P}_{\delta n}(\delta n) = \int \int \mathbb{P}_{\delta\phi,\delta\pi}(\delta\phi_0, \delta\pi_0)\, \boldsymbol{\delta}\left(\delta n - \delta n(\delta\phi_0, \delta\pi_0)\right) d\delta\phi_0 d\delta\pi_0 \,,$$

- Assuming that non-linearities in the field fluctuation are small, $\mathbb{P}_{\delta\phi,\delta\pi}(\delta\phi_0, \delta\pi_0)$ can be computed with linear perturbation theory.

$$\frac{\partial \delta\phi_k}{\partial n} = \delta\pi_k \,,$$

$$\frac{\partial \delta\pi_k}{\partial n} + (3 - \epsilon_1)\,\delta\pi_k + \left[\left(\frac{k}{a\bar{H}}\right)^2 + \left(-\frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_3\right)\right]\delta\phi_k = 0$$

$$\hat{\delta\phi}(\mathbf{x}, n) \equiv \int \frac{d^3k}{(2\pi)^3}\left[\delta\phi_k(n)\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \delta\phi_k^\star(n)\hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right]$$

- By construction, $\mathbb{P}_{\delta\phi,\delta\pi}(\delta\phi_0, \delta\pi_0)$ must be given at superhorizon scales.

Initial conditions for δn

$$\mathbb{P}_{\delta n}(\delta n) = \int\int \mathbb{P}_{\delta\phi,\delta\pi}(\delta\phi_0, \delta\pi_0)\, \boldsymbol{\delta}\left(\delta n - \delta n(\delta\phi_0, \delta\pi_0)\right) d\delta\phi_0 d\delta\pi_0 \,,$$

- Assuming that non-linearities in the field fluctuation are small, $\mathbb{P}_{\delta\phi,\delta\pi}(\delta\phi_0, \delta\pi_0)$ can be computed with linear perturbation theory.

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$$\frac{\partial \delta\pi_k}{\partial n} + (3 - \epsilon_1)\,\delta\pi_k + \left[\cancel{\left(\frac{k}{aH}\right)^2} + \left(-\frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_3\right)\right]\delta\phi_k = 0$$

$$\hat{\delta\phi}(\mathbf{x}, n) \equiv \int \frac{d^3k}{(2\pi)^3}\left[\delta\phi_k(n)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + \delta\phi_k^\star(n)\hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \simeq \delta\phi$$

- $\delta\phi$ and $\delta\pi$ are now random variables rather than quantum operators.

Initial conditions for δn

- $\mathbb{P}_{\delta\phi,\delta\pi}(\delta\phi_0, \delta\pi_0)$ can be written as

$$\mathbb{P}_{\delta\phi,\delta\pi}(\delta\phi_0, \delta\pi_0) = \mathcal{N}(0, \boldsymbol{\Sigma})$$

with

$$\boldsymbol{\Sigma} \equiv \begin{bmatrix} \sigma^2_{\delta\phi}(\bar{n}_0) & \sigma^2_{\delta\phi\delta\pi}(\bar{n}_0) \\ \sigma^2_{\delta\pi\delta\phi}(\bar{n}_0) & \sigma^2_{\delta\pi\delta\pi}(\bar{n}_0) \end{bmatrix}$$
$$= \begin{bmatrix} \mathcal{C}_{\delta\phi,\delta\phi}(k = \sigma a(\bar{n}_0)\bar{H}(\bar{n}_0), \bar{n}_0) & \mathrm{Re}\left[\mathcal{C}_{\delta\phi,\delta\pi}(k = \sigma a(\bar{n}_0)\bar{H}(\bar{n}_0), \bar{n}_0)\right] \\ \mathrm{Re}\left[\mathcal{C}_{\delta\pi,\delta\phi}(k = \sigma a(\bar{n}_0)\bar{H}(\bar{n}_0), \bar{n}_0)\right] & \mathcal{C}_{\delta\pi,\delta\pi}(k = \sigma a(\bar{n}_0)\bar{H}(\bar{n}_0), \bar{n}_0) \end{bmatrix},$$

where $\sigma \ll 1$ and

$$\mathcal{C}_{q,p}(k, \bar{n}) \equiv \frac{k^3}{2\pi^2} q_k(\bar{n}) p^\star_k(\bar{n})$$

Initial conditions for δn

- Whenever σ is small enough to use the separate universe approach (and hence the δn formalism), we have

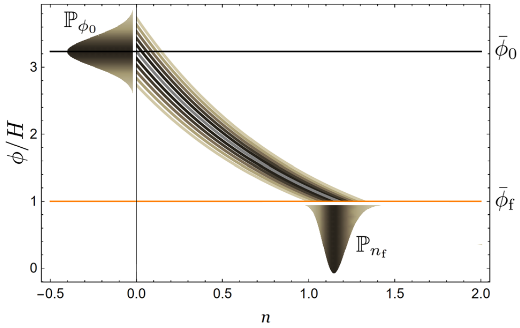
$$\frac{\det[\boldsymbol{\Sigma}]}{\sigma^2_{g,p}} \ll 1\,, \rightarrow \delta\phi_0 \text{ and } \delta\pi_0 \text{ are fully correlated!}$$

$$\mathbb{P}_{\delta\phi,\delta\pi}(\delta\phi_0,\delta\pi_0) = \boldsymbol{\delta}\left(\frac{\delta\pi_0}{\sigma_{\delta\pi}} \pm \frac{\delta\phi_0}{\sigma_{\delta\phi}}\right)\frac{1}{\sqrt{2\pi}\sigma_{\delta\phi}\sigma_{\delta\pi}}e^{-\frac{1}{8}\left(\frac{\delta\pi_0}{\sigma_{\delta\pi}}\mp\frac{\delta\phi_0}{\sigma_{\delta\phi}}\right)^2}$$

$$\mathbb{P}_{\delta n}(\delta n) = \mathbb{P}_{\delta\phi}(\delta\phi_0(\delta n))\left|\frac{d\delta\phi_0(\delta n)}{d\delta n}\right|\,.$$

- **Simplifications**
  - δn count forward in time and hence allows us to define the initial fluctuations $\delta\phi_0$ and $\delta\pi_0$ at some given time $\bar{n}_0$.
  - $\delta\phi_0$ and $\delta\pi_0$ are completely correlated $\rightarrow$ we only need to specify the trajectories of ϕ.
  - $\delta\phi_0(\delta n)$ is generically simpler to compute than $\delta n(\delta\phi_0)$

The simple formula



- Each trajectory follows

$$\phi\left(n-\bar{n}_0, \bar{\phi}_0+\delta\phi_0, \bar{\pi}_0+\delta\pi_0\right)=\phi\left(n-\bar{n}_0, \bar{\phi}_0+\delta\phi_0, \bar{\pi}_0 \pm \frac{\sigma_{\delta\pi,\delta\pi}(k_\sigma, \bar{n}_0)}{\sigma_{\delta\phi,\delta\phi}(k_\sigma, \bar{n}_0)}\delta\phi_0\right)$$

- At $\phi=\bar{\phi}_f$

$$\phi\left(\bar{n}_f+\delta n-\bar{n}_0, \bar{\phi}_0+\delta\phi_0, \bar{\pi}_0 \pm \frac{\sigma_{\delta\pi,\delta\pi}(k_\sigma, \bar{n}_0)}{\sigma_{\delta\phi,\delta\phi}(k_\sigma, \bar{n}_0)}\delta\phi_0\right)=\bar{\phi}_f=\phi\left(\bar{n}_f-\bar{n}_0, \bar{\phi}_0, \bar{\pi}_0\right)$$

- From

$$\phi\left(\bar{n}_{\mathrm{f}}+\delta n-\bar{n}_{\mathrm{o}},\bar{\phi}_{\mathrm{o}}+\delta\phi_{\mathrm{o}},\bar{\pi}_{\mathrm{o}}\pm\frac{\sigma_{\delta\pi,\delta\pi}(k_\sigma,\bar{n}_{\mathrm{o}})}{\sigma_{\delta\phi,\delta\phi}(k_\sigma,\bar{n}_{\mathrm{o}})}\delta\phi_{\mathrm{o}}\right)=\phi\left(\bar{n}_{\mathrm{f}}-\bar{n}_{\mathrm{o}},\bar{\phi}_{\mathrm{o}},\bar{\pi}_{\mathrm{o}}\right) \tag{1}$$

  we can solve $\delta\phi_{\mathrm{o}}(\delta n)$ and use

$$\mathbb{P}_{\delta n}(\delta n)=\mathbb{P}_{\delta\phi}(\delta\phi_{\mathrm{o}}(\delta n))\left|\frac{d\delta\phi_{\mathrm{o}}(\delta n)}{d\delta n}\right|$$

  to compute $\mathbb{P}_{\delta n}(\delta n)$.
- Note that equation (1) is nothing more than the full non-linear gauge transformation between $\delta\phi_{\mathrm{o}}$ and δn.
- Let us apply this formula!

Quadratic potential (Full)

- Equation of motion

$$\frac{\partial^2 \phi}{\partial n^2} + 3\frac{\partial \phi}{\partial n} + 3\eta\phi = 0\,,$$

- Solution

$$\bar{\phi}(n) = c^0_+ e^{-\lambda_+(n-\bar{n}_0)} + c^0_- e^{-\lambda_-(n-\bar{n}_0)}\,,$$

where $c^0_\pm = \mp\frac{\pi_0+\lambda_\mp\phi_0}{\lambda_+-\lambda_-}$ and $\lambda_\pm = \frac{1}{2}\left(3 \pm \sqrt{9-12\eta}\right)$

- Most general form of the variances

$$\sigma^2_{\delta\phi,\delta\phi}(k,n) = \left|C^0_-(k)e^{-\lambda_-(n-\bar{n}_0)} + C^0_+(k)e^{-\lambda_+(n-\bar{n}_0)}\right|^2$$

$$\sigma^2_{\delta\pi,\delta\pi}(k,n) = \left|\lambda_- C^0_-(k)e^{-\lambda_-(n-\bar{n}_0)} + \lambda_+ C^0_+(k)e^{-\lambda_+(n-\bar{n}_0)}\right|^2$$

- We have all the ingredients to solve

$$\phi\left(\bar{n}_{\mathrm{f}}+\delta n-\bar{n}_0, \bar{\phi}_0+\delta\phi_0, \bar{\pi}_0 \pm \frac{\sigma_{\delta\pi,\delta\pi}(k_\sigma, \bar{n}_0)}{\sigma_{\delta\phi,\delta\phi}(k_\sigma, \bar{n}_0)}\delta\phi_0\right)=\phi\left(\bar{n}_{\mathrm{f}}-\bar{n}_0, \bar{\phi}_0, \bar{\pi}_0\right)$$

- The solution is

$$\frac{\delta\phi_0}{\left|C_-^0(k)e^{-\lambda_-(n-\bar{n}_0)}+C_+^0(k)e^{-\lambda_+(n-\bar{n}_0)}\right|}=\frac{\bar{c}_-^{\,\mathrm{f}}(1-e^{\lambda_-\delta n})+\bar{c}_+^{\,\mathrm{f}}(1-e^{\lambda_+\delta n})}{\left|C_-^{\mathrm{f}}(k)e^{-\lambda_-\delta n}+C_+^{\mathrm{f}}(k)e^{-\lambda_+\delta n}\right|}$$

where $\bar{c}_\pm^{\,\mathrm{f}} \equiv \pm\frac{\bar{\pi}_0+\lambda_\pm\bar{\phi}_0}{\lambda_+-\lambda_-}e^{-\lambda_\pm(\bar{n_{\mathrm{f}}}-\bar{n}_0)}$ and $C_\pm^{\mathrm{f}}(k)=C_\pm^0(k)e^{-\lambda_\pm(\bar{n_{\mathrm{f}}}-\bar{n}_0)}$

- From where $\mathbb{P}_{\delta n}(\delta n)$ immediately follows

$$\mathbb{P}_{\delta n}(\delta n)=\frac{1}{\sqrt{2\pi}}\left|\frac{d}{d\delta n}\left(\frac{\bar{c}_-^{\,\mathrm{f}}(1-e^{\lambda_-\delta n})+\bar{c}_+^{\,\mathrm{f}}(1-e^{\lambda_+\delta n})}{\left|C_-^{\mathrm{f}}(k)e^{-\lambda_-\delta n}+C_+^{\mathrm{f}}(k)e^{-\lambda_+\delta n}\right|}\right)\right|e^{-\frac{1}{2}\left(\frac{\bar{c}_-^{\,\mathrm{f}}(1-e^{\lambda_-\delta n})+\bar{c}_+^{\,\mathrm{f}}(1-e^{\lambda_+\delta n})}{\left|C_-^{\mathrm{f}}(k)e^{-\lambda_-\delta n}+C_+^{\mathrm{f}}(k)e^{-\lambda_+\delta n}\right|}\right)^2}$$

- Impossible to get generically with the traditional method.
- Reduces to well-known PDFs in the case of SR, USR, CR...

The non-perturbative equation of motion for ζ (in progress).

- If we choose a gauge gauge in which $\phi = \bar{\phi}$ ($\delta\phi = 0$). The equations of motion in the separate universe approach can be written as

$$\left(\frac{\bar{H}+\dot{\zeta}}{\alpha}\right)^2 = \frac{1}{3M_{PL}^2}\left(\frac{\dot{\bar{\phi}}^2}{2\alpha^2} + V(\bar{\phi})\right) ,$$

$$\frac{1}{\alpha}\frac{d}{dt}\left(\frac{\bar{H}+\dot{\zeta}}{\alpha}\right) + \left(\frac{\bar{H}+\dot{\zeta}}{\alpha}\right)^2 = -\frac{1}{3M_{PL}^2}\left(\frac{\dot{\bar{\phi}}^2}{\alpha^2} - V(\bar{\phi})\right) ,$$

- Solving for the lapse function α we find a non linear equation of motion for ζ:

$$\ddot{\zeta} + \bar{H}\left(3+\epsilon_2\right)\dot{\zeta} + \left(6+\frac{3}{2}\epsilon_2\right)\dot{\zeta}^2 + \frac{1}{\bar{H}}\left(3+\frac{\epsilon_2}{2}\right)\dot{\zeta}^3 \simeq 0 \,.$$

The non-perturbative equation of motion for ζ

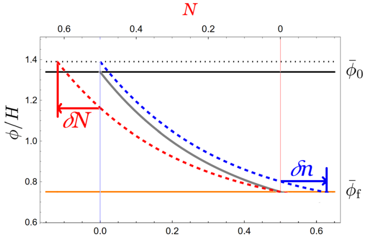
- Surprisingly (or maybe not), this equation of motion seems to be exactly equivalent to the δn formalism.
- In fact our novel solution for $\delta n = \zeta$ in case of a quadratic potential. i.e.

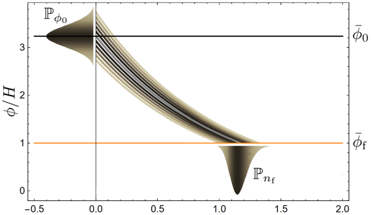
$$\frac{\delta\phi_0}{\left|C^0_-(k)e^{-\lambda_-(n-\bar{n}_0)} + C^0_+(k)e^{-\lambda_+(n-\bar{n}_0)}\right|} = \frac{\bar{c}^{\,f}_-(1-e^{\lambda_-\zeta(n)}) + \bar{c}^{\,f}_+(1-e^{\lambda_+\zeta(n)})}{\left|C^f_-(k)e^{-\lambda_-\zeta(n)} + C^f_+(k)e^{-\lambda_+\zeta(n)}\right|}$$

  satisfies the non-linear equation for ζ.
- This opens a new-possibility to study the non-perturbative curvature perturbation without need of the δn formalism.
- However, how should the initial conditions be given to our new equation of motion?

Conclusions

- The way of computing the PDF of ζ with the classical δN formalism can be greatly simplified due to
  - We can count forward in time
  - Initial $\delta\phi_0$ and $\delta\pi_0$ are completely correlated at some initial time $\bar{n}_0$.
  - No need to compute $\delta n(\delta\phi_0)$
- Due to this simplification, we have obtained for the first time the full $\mathbb{P}_{\delta n}(\delta n)$ for a quadratic potential.
- (In progress) A new non-perturbative equation of motion for ζ is found, which seems equivalent to the δn formalism.





Thank you!