The pole mass and static self-energy at large orders in perturbation theory

Based on

Bauer, Bali, Pineda: Phys.Rev.Lett. 108 (2012) 242002

Bali, Bauer, Pineda, Torrero: arXiv:1303.3279

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Motivation

m_Q. Fundamental parameter in heavy quarkonium physics.

Usual definitions:

- $m_{\overline{\rm MS}} \rightarrow$ short distance mass.
- $m_{\rm OS} \rightarrow$ natural definition for heavy quark physics.

$$m_{\rm OS} = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1} \,,$$

Renormalon (OPE) analysis predict $r_n \sim n!$.

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Renormalon (OPE) analysis predict $r_n \sim n!$.

$$M_B = m_{\rm OS} + \bar{\Lambda}_B + \mathcal{O}(1/m_{\rm OS})\,, \quad m_{\tilde{G}} = m_{\tilde{g},{\rm OS}} + \Lambda_H + \mathcal{O}\left(1/m_{\tilde{g},{\rm OS}}\right)$$

M_B is renormalon free. Therefore $m_{\rm OS}$ suffers from renormalon ambiguities:

$$m_{\rm OS} = m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots)$$

with $B_n \sim n!$. In other words

$$\delta_{np}^{\rm (pert.)} m_{\rm OS} = \delta_{np}^{\rm (pert.)} m_{\overline{\rm MS}}(1 + B_1\alpha_s + B_2\alpha_s^2 + \cdots) \sim \Lambda_{\rm QCD}!$$

$$m_{\rm OS} = m_{\overline{\rm MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1} \,,$$

$$m_{\rm OS} = m_{\overline{\rm MS}} + \int\limits_0^\infty {\rm d}t \, e^{-t/\alpha_s} \, B[m_{\rm OS}](t) \,, \qquad B[m_{\rm OS}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!} .$$

The behavior of the perturbative expansion at large orders is dictated by the closest singularity to the origin of its Borel transform ($u = \frac{\beta_0 t}{4\pi}$).

$$B[m_{\rm OS}](t) = N_m \nu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots \right) + (\text{analytic term}),$$

Next renormalon at $u = 1$.

$$r_n \stackrel{n\to\infty}{=} N_m \, \nu \, \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{b}{(n+b)} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots \right) .$$

$$b = \frac{\beta_1}{2\beta_0^2} \,, \qquad c_1 = \frac{1}{4\, b \beta_0^3} \left(\frac{\beta_1^2}{\beta_0} - \beta_2 \right) \,, \qquad \cdots$$

Over the years a lot of evidence in favour of the existence of the renormalon. Particularly important for heavy quark physics.

Two that I specially like:

- Static potential: $2m + V_s$ is renormalon free
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$$r_n \overset{n\to\infty}{\sim} m_{\overline{\rm MS}} \left(\frac{\beta_0}{2\pi}\right)^n n! N_m \sum_{s=0}^{n} \frac{\ln^s[\nu/m_{\overline{\rm MS}}]}{s!} \sim \nu$$

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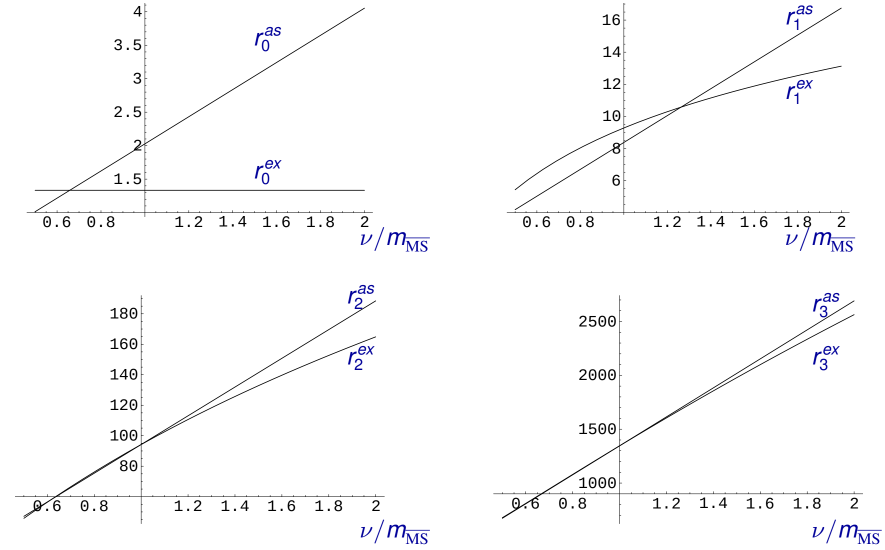

Figure : Plots of the exact (r_n^{ex}) and asymptotic (r_n^{as}) value of $r_n(\nu)$ at different orders in perturbation theory as a function of $\nu/m_{\overline{\rm MS}}$. From hep-lat/0509022.

Yet...

- Not possible to compute using known semiclassical analysis.
- Based on few orders in perturbation theory ($\sim 3, 4$)
- Against renormalon existence (Suslov), or against renormalon dominance (Zakharov and followers).

We would like to have a proof (at the same level of existing proofs of a linear potential at long distances), beyond any reasonable doubt, of the existence of the renormalon in QCD (and in heavy quarkonium physics).

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POLYAKOV LOOP versus δm (and m)

Possible to compute the energy of an static source in the lattice: δm of HQET.
We use Numerical Stochastic Perturbation Theory

$$\delta m = \frac{1}{a}\sum_{n=0}^{\infty} c_n^{(3,\rho)}\alpha^{n+1}(1/a)\ (\text{fundamental}), \quad \delta m_{\tilde{g}} = \frac{1}{a}\sum_{n=0}^{\infty} c_n^{(8,\rho)}\alpha^{n+1}(1/a)\ (\text{adjoint})$$

$$\lim_{n\to\infty} c_n^{(R,\rho)} = r_n(\nu)/\nu$$

$$L^{(R)}(N_S,N_T) = \frac{1}{N_S^3}\sum_{\mathbf{n}}\frac{1}{d_R}\mathrm{tr}\left[\prod_{n_4=0}^{N_T-1}U_4^R(n)\right] \qquad U_\mu^R(n) \approx e^{iA_\mu^R[(n+1/2)a]}$$

We implement triplet and octet representations R ($d_R = 3, 8$).

$$P^{(R,\rho)}(N_S,N_T) = -\frac{\ln\langle L^{(R,\rho)}(N_S,N_T)\rangle}{aN_T} = \sum_{n=0}^{\infty} c_n^{(R,\rho)}(N_S,N_T)\alpha^{n+1},$$

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	$\mathcal{O}(\alpha^4)$	$\mathcal{O}(\alpha^{20})$	$\mathcal{O}(\alpha^{32})$
$N_S(N_T)$	$4(4)$	$8(8,10,12,14)$	$4(8)$

Table : *The first arrow states to which order in α the coefficients of $c_n^{(R)}(N_T, N_S)$ have been computed for each specific lattice volume for PBC.*

$\mathcal{O}(\alpha^3)$	$N_S(N_T)$	$5(5,6,7,8,10)$			
$\mathcal{O}(\alpha^4)$	$N_S(N_T)$	$4(5,6,7,8,10,12,16,20,24)$	$12(16,20)$		
$\mathcal{O}(\alpha^{12})$	$N_S(N_T)$	$6(6,8,10,12,16)$	$8(12,16)$		
$\mathcal{O}(\alpha^{12})$	$N_S(N_T)$	$10(8,12,16,20)$	$16(12,16,20)$		
$\mathcal{O}(\alpha^{20})$	$N_S(N_T)$	$7(7,8)$	$8(8,10)$	$9(12)$	$10(10)$
$\mathcal{O}(\alpha^{20})$	$N_S(N_T)$	$11(16)$	$12(12)$	$14(14)$	

Table : *The first column states to which order in α the coefficients of $c_n^{(R)}(N_T, N_S)$ and the associated ratios have been computed for each specific lattice volume for TBC.*

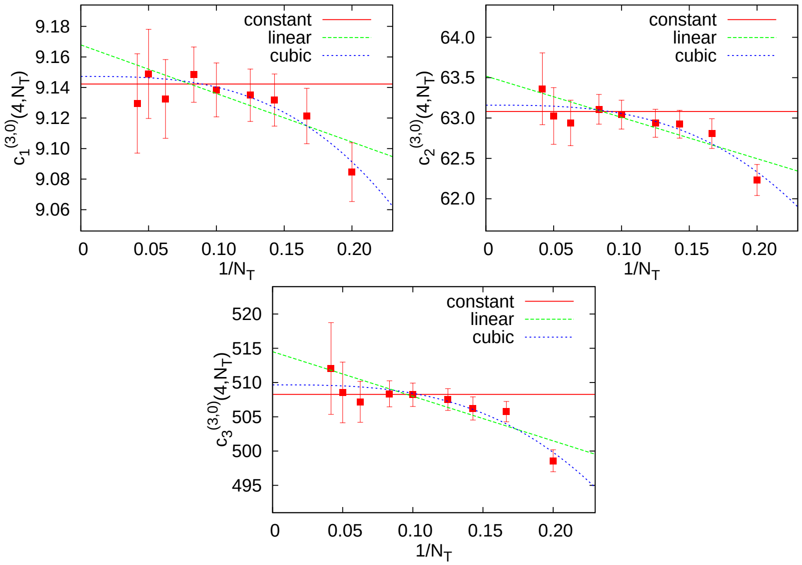

Figure : $c_{1,2,3}^{(3,0)}(4, N_T)$ *as a function of* $1/N_T$*, in comparison to a constant plus linear fit, a constant plus cubic fit, and a constant fitted only to the* $N_T > 10$ *points.*

$$\delta m(N_S) = \lim_{N_T \to \infty} P(N_S, N_T) \qquad \text{and} \qquad c_n(N_S) = \lim_{N_T \to \infty} c_n(N_S, N_T)\,.$$

For large N_S, we write

$$c_n(N_S) = c_n - \frac{f_n(N_S)}{N_S} + \mathcal{O}\left(\frac{1}{N_S^2}\right)\,.$$

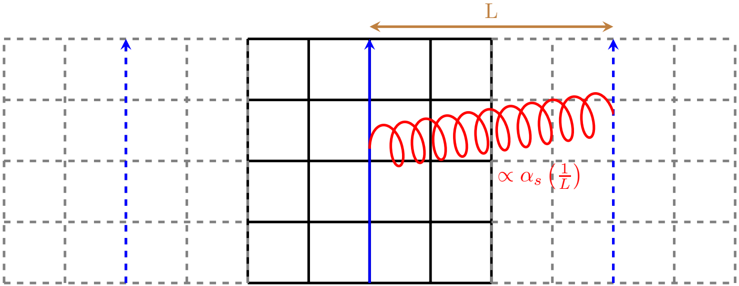

Figure : *Self-interactions with replicas producing* $1/L = 1/(aN_S)$ *Coulomb terms.*

$$\delta m(N_S) = \delta m - \frac{1}{aN_S}\sum_{n=0}^{\infty} f_n \alpha^{n+1}\left((aN_S)^{-1}\right) + \mathcal{O}\left(\frac{1}{N_S^2}\right).$$

Therefore, the coefficient $f_n(N_S)$ is a polynomial of $\ln(N_S)$:

$$f_n(N_S) = \sum_{i=0}^{n} f_n^{(i)} \ln^i(N_S)\,,$$

$f_n^{(0)} = f_n$ and the coefficients $f_n^{(i)}$ for $i > 0$ are determined by f_m with $m < n$ and β_j with $j \leq n-1$.

$$f_1(N_S) = f_1 + f_0 \frac{\beta_0}{2\pi} \ln(N_S)\,,$$

$$f_2(N_S) = f_2 + \left[2f_1 \frac{\beta_0}{2\pi} + f_0 \frac{\beta_1}{8\pi^2}\right] \ln(N_S) + f_0 \left(\frac{\beta_0}{2\pi}\right)^2 \ln^2(N_S)\,,$$

and so on.

$$P \propto \int_{1/(aN_S)}^{1/a} dk\, \alpha(k) \sim \frac{1}{a}\sum_n c_n \alpha^{n+1}\left(a^{-1}\right) - \frac{1}{aN_S}\sum_n c_n \alpha^{n+1}\left((aN_S)^{-1}\right)\,,$$

$$c_n \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n n!\,, \qquad f_n^{(i)}(N_S) \simeq N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{n!}{i!}\,.$$

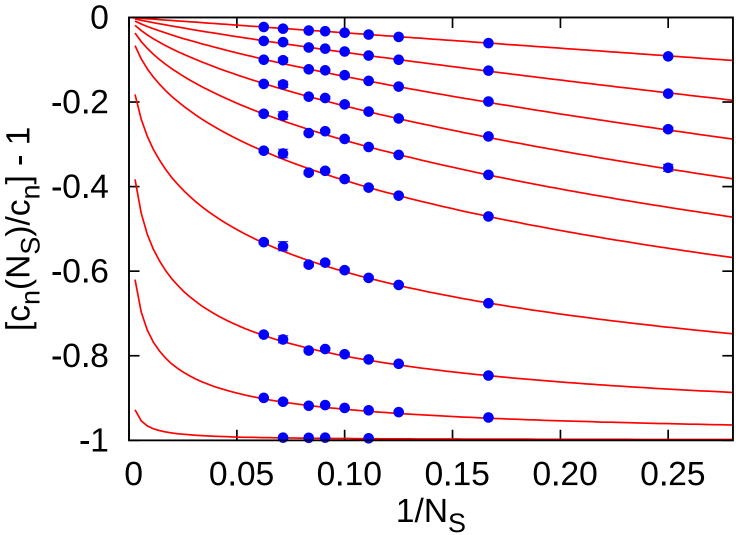

Figure : $c_n^{(3,0)}(N_S)/c_n^{(3,0)} - 1$ *for* $n \in \{0, 1, 2, 3, 4, 5, 7, 9, 11, 15\}$ *(top to bottom). For each value of* N_S *we have plotted the data point with the maximum value of* N_T*. The curves represent the global fit.* $-(1/N_S) f_{0,\rm DLPT}^{(3,0)}/c_{0,\rm DLPT}^{(3,0)}$ *is shown for* $n = 0$.

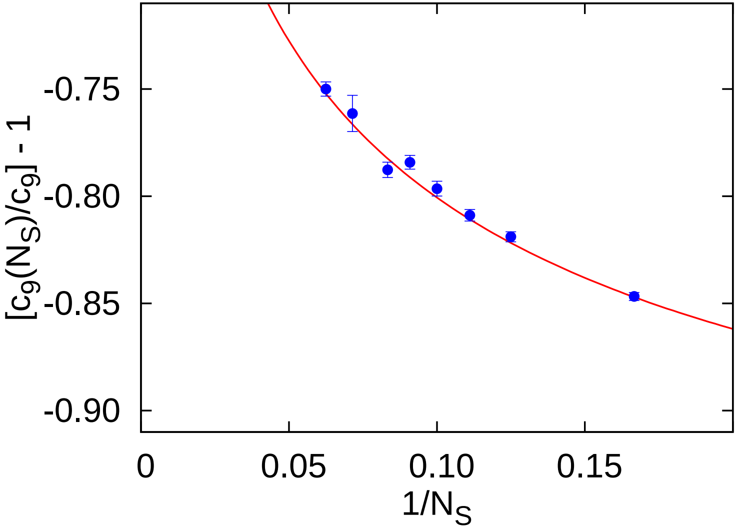

Figure : *Zoom of previous Figure for* $n = 9$.

$$
\frac{c_n^{(3,\rho)}}{c_{n-1}^{(3,\rho)}}\frac{1}{n}=\frac{c_n^{(8,\rho)}}{c_{n-1}^{(8,\rho)}}\frac{1}{n}
$$
$$
=\frac{\beta_0}{2\pi}\left\{1+\frac{b}{n}-\frac{bs_1}{n^2}+\frac{1}{n^3}\left[b^2s_1^2+b(b-1)(s_1-2s_2)\right]+\mathcal{O}\left(\frac{1}{n^4}\right)\right\}\ .
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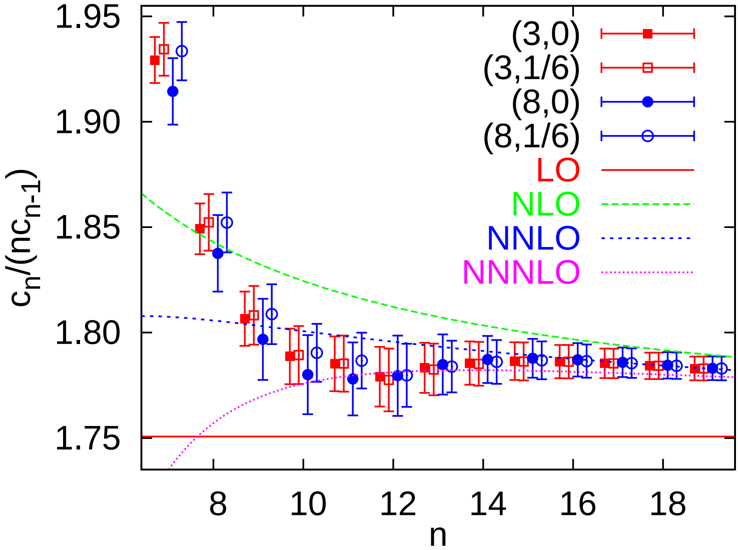

Figure : *The ratios $c_n/(nc_{n-1})$ for the smeared and unsmeared, triplet and octet fundamental static self-energies, compared to the prediction for the LO, next-to-leading order (NLO), NNLO and NNNLO of the $1/n$ expansion.*

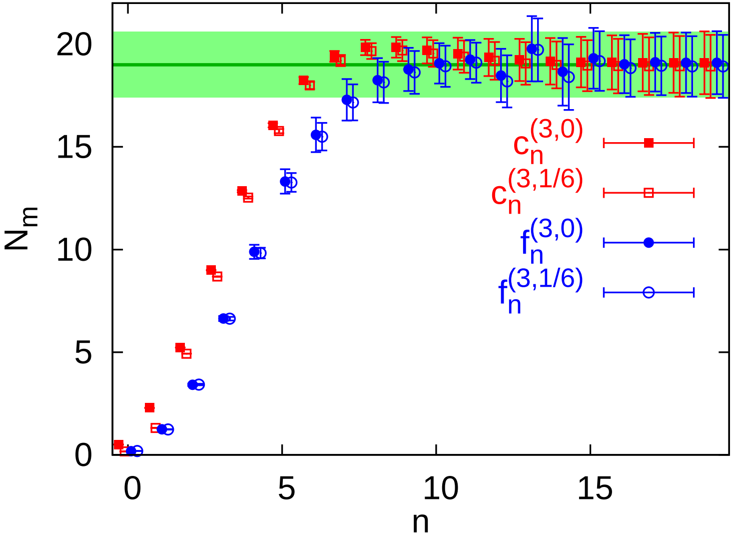

Figure : *N_m, determined via r_n truncated at NNLO, from the coefficients $c_n^{(3,0)}$, $c_n^{(3,1/6)}$, $f_n^{(3,0)}$ and $f_n^{(3,1/6)}$. The horizontal band is our final result.*

$$\alpha_{\overline{\rm MS}}(\mu) = \alpha_{\rm latt}(\mu)\left(1 + d_1\alpha_{\rm latt}(\mu) + d_2\alpha^2_{\rm latt}(\mu) + d_3\alpha^3_{\rm latt}(\mu) + \mathcal{O}(\alpha^4_{\rm latt})\right) \,,$$

$$N^{\overline{\rm MS}}_{m,m_{\tilde g}} = N^{\rm latt}_{m,m_{\tilde g}}\Lambda_{\rm latt}/\Lambda_{\overline{\rm MS}}\,, \quad \text{where} \quad \Lambda_{\overline{\rm MS}} = e^{\frac{2\pi d_1}{\beta_0}}\Lambda_{\rm latt} \approx 28.809338139488\,\Lambda_{\rm latt}\,.$$

This yields the numerical values

$$N^{\overline{\rm MS}}_m = 0.660(56)\,, \quad C_F/C_A\,N^{\overline{\rm MS}}_{m_{\tilde g}} = -C_F/C_A\,N^{\overline{\rm MS}}_{\Lambda} = 0.649(62)\,.$$

Other combinations of interest are

$$N^{\overline{\rm MS}}_{V_s} = -1.32(11)\,, \quad N^{\overline{\rm MS}}_{V_o} = 0.14(18)\,.$$

Assuming that

$$c_{3,\overline{\rm MS}} \simeq N^{\overline{\rm MS}}_m\left(\frac{\beta_0}{2\pi}\right)^3\frac{\Gamma(4+b)}{\Gamma(1+b)}\left(1 + \frac{b}{(3+b)}s_1 + \frac{b(b-1)}{(3+b)(2+b)}s_2 + \cdots\right),$$

and using our central value $c^{(3,0)}_{3,\rm latt} = 794.5$, we obtain

$$d_3 \simeq 365\,, \qquad \beta^{\rm latt}_3 \simeq -1.7\times 10^6\,.$$

$$\alpha_{\overline{\rm MS}}(\mu) = \alpha_{\rm latt}(\mu)\left(1 + d_1\alpha_{\rm latt}(\mu) + d_2\alpha^2_{\rm latt}(\mu) + d_3\alpha^3_{\rm latt}(\mu) + \mathcal{O}(\alpha^4_{\rm latt})\right) ,$$

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CONCLUSIONS

For the first time, it was possible to follow the factorial growth of the coefficients over many orders, from around α^9 up to α^{20}, vastly increasing the credibility of the prediction.

$$N_m^{\rm latt} = 19.0 \pm 1.6\,, \quad C_F/C_A\, N_\Lambda^{\rm latt} = -18.7 \pm 1.8\,,$$
$$N_m^{\overline{\rm MS}} = 0.660 \pm 0.056\,, \quad C_F/C_A\, N_\Lambda^{\overline{\rm MS}} = -0.649 \pm 0.062\,.$$

Completely consistent with continuum-like determinations.

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