Renormalization of the Cyclic Wilson Loop

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T30f
Theoretische Teilchen-
und Kernphysik

Loop functions and Wilson loops

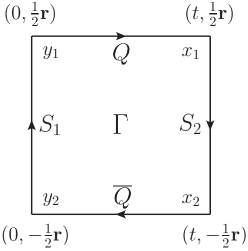
Wilson line

$$U(C) = \mathcal{P}\exp\left[ig\int_{C(x,y)} \mathrm{d}z^\mu A_\mu(z)\right]$$

Wilson loop

$$W(t,\mathbf{r}) = \left\langle \widetilde{\mathrm{Tr}}\,\mathcal{P}\exp\left[ig\oint_\Gamma dx^\mu A_\mu(x)\right]\right\rangle$$
$$= \left\langle \widetilde{\mathrm{Tr}}\,U(S_1)U(\bar{Q})U(S_2)U(Q)\right\rangle$$

with path



(notation: $\widetilde{\mathrm{Tr}}A = \frac{1}{N_c}\mathrm{Tr}A$)

Loop functions

$$L(\Gamma_1,\Gamma_2,\dots) = \left\langle \widetilde{\mathrm{Tr}}\left[U(\Gamma_1)\right]\widetilde{\mathrm{Tr}}\left[U(\Gamma_2)\right]\cdots\right\rangle$$

Motivation

- vacuum Wilson loop related to static quark-antiquark potential
- at finite T Polyakov loop related to static quark free energy
- correlator of two Polyakov loops related to static $Q\,\overline{Q}$ free energy
- loop functions are important tool for definition of finite T potential
- operators of interest:

Polyakov loop correlator

$$P_c = \left\langle \widetilde{\mathrm{Tr}}\left[U(\bar{Q})\right]\widetilde{\mathrm{Tr}}\left[U(Q)\right]\right\rangle$$

Cyclic Wilson loop

$$W_c = \widetilde{\mathrm{Tr}}\left\langle U(S)U(\bar{Q})U^\dagger(S)U(Q)\right\rangle$$

quark lines extend from imaginary time 0 to $\beta = 1/T$ (periodic)
strings are inverse: same position, opposite direction

- cyclic Wilson loop found to have unusual divergence structure[1]

[1] [Burnier, Y. and Laine, M. and Vepsäläinen, M. 2010]

Divergences [2]

Where do divergences come from?
Compare:

Covariant gauge gluon propagator in configuration space

$$D_{\mu\nu}^{ab}(x,y) = \frac{\delta^{ab}}{4\pi^2(x-y)^2}\left[\frac{1+\xi}{2}\delta_{\mu\nu} + (1-\xi)\frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2}\right]$$

$\rightarrow$ UV divergences arise, when vertices get close

- internal vertices (not on the contour) lead to usual UV divergences (self energy, vertex corrections, etc.)
- vertices on contour (line vertices) lead to additional UV divergences (e.g. cusp divergences)

[2] [Dotsenko, V. S. and Vergeles, S. N. 1980], [Brandt, R. A. and Neri, F. and Sato, M. 1981]

Divergences

Superficial degree of divergence for line vertices

at smooth point: $\omega = 1 - N_{ex}$

at singular point: $\omega = -N_{ex}$

- N_{ex}: number of external lines (leading to vertices at a finite distance)
- smooth point: contour Γ is differentiable
- singular point: Γ is not differentiable

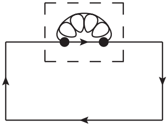
3 types of divergences

- loop mass: all vertices contracted at a smooth point
- line vertex: contraction at a smooth point with one external line
- cusp/intersection: all vertices contracted at a singular point

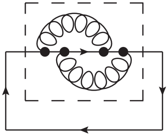
(1st is linear, 2nd and 3rd are logarithmic)

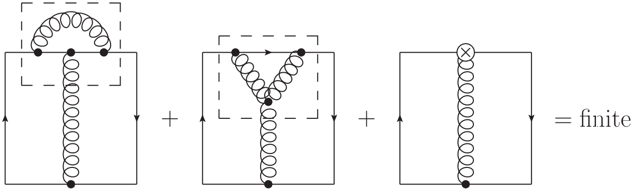
Divergences

- linear divergences:
  - exponentiate and factor out
  - exponent is proportional to contour length Λ
  - automatically removed in DR



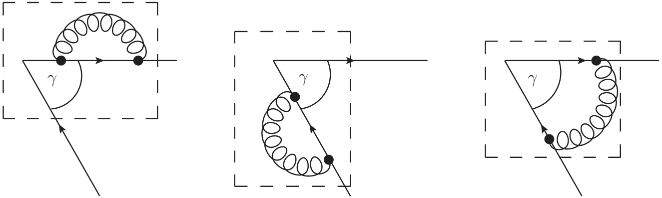
- line vertex divergences:
  - correspond to line vertex corrections
  - are removed through charge renormalization
  - line vertex renormalization constant $Z_{gA} = \frac{Z_1}{Z_3}$ and counterterm $Z_{gA} - 1$ (in Feynman gauge)





Divergences

- cusp divergences:
  - exponentiate and factor out
  - removed through multiplicative constant
  - renormalization constant depends only on cusp angle



- Cusp divergence at $\mathcal{O}(\alpha_s)^3$: $\dfrac{C_F\alpha_s}{2\pi\varepsilon}\left(1+(\pi-\gamma)\cot\gamma\right)$
- rectangular Wilson loop has 4 cusps with $\gamma=\frac{\pi}{2}$ so

$$Z_c=\exp\left[-\frac{2C_F\alpha_s}{\pi\bar{\varepsilon}}+\dots\right]$$

[3] [Korchemsky, G. P. and Radyushkin, A. V. 1987]

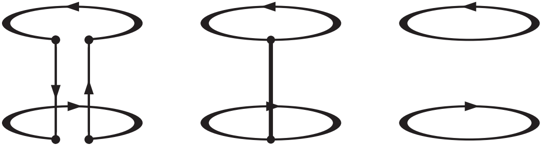
Divergences

- intersection divergences:
  - cannot be removed through a single multiplicative constant
  - set of associated loops mix under renormalization

  - same contour, but different path ordering at intersection
  - disconnected loops are traced separately
  - renormalization matrix depends only on intersection angles
- general case:
  1 renormalization constant / matrix for every cusp / intersection

Divergence of the cyclic Wilson loop

- periodic boundary conditions: $\tau = 0$ and $\tau = \beta$ are identified
- cusps turn into intersections:



- only intersections at string endpoints relevant (angles 0 and π not divergent)
- alternate path orderings lead to Polyakov loop correlator (finite)
- renormalization matrices at the 2 intersections must be identical

Renormalization formula (compact)

$$\begin{pmatrix} W_c^{(R)} \\ P_c \end{pmatrix} = \begin{pmatrix} Z & (1-Z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} W_c \\ P_c \end{pmatrix}$$

Renormalized Result

Renormalized Cyclic Wilson loop at $\mathcal{O}(\alpha_s^2)$

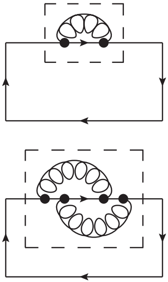
$$\ln W_c^{(R)} = \frac{C_F\alpha_s}{rT}\left\{1+\frac{\alpha_s}{4\pi}\left[\left(\tfrac{31}{9}C_A-\tfrac{20}{9}T_Fn_f\right)+\left(\tfrac{11}{3}C_A-\tfrac{4}{3}T_Fn_f\right)\left(\ln\mu^2r^2+2\gamma_E\right)\right]+\right.$$
$$\left.+\frac{C_A\alpha_s}{\pi}\left[\cancel{\frac{1}{\bar{\varepsilon}}}+1+2\gamma_E-\ln 4+\ln\mu^2r^2+\sum_{n=1}^{\infty}\frac{2(-1)^n\zeta(2n)}{n(4n^2-1)}(rT)^{2n}\right]\right\}+$$
$$+\frac{4\pi C_F\alpha_s}{T}\int_k\frac{e^{i\mathbf{r}\cdot\mathbf{k}}-1}{(\mathbf{k}^2)^2}\left(-\Pi_{00}^{(T)}(0,\mathbf{k})\right)+C_FC_A\alpha_s^2\quad+\mathcal{O}\left(\alpha_s^3\right)$$

- for more details see JHEP 03 (2013) 069 [arXiv:1212.4413]
- renormalization group improved result
- treatment of divergences for each diagram up to $\mathcal{O}(\alpha_s^3)$
- discussion for large distance r

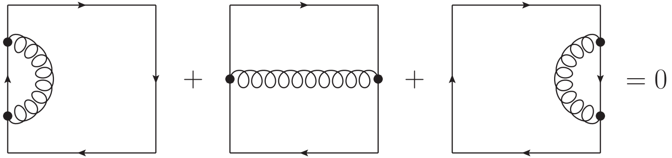
Linear Divergences

- can be neglected in DR
- proportional to the length of the Wilson line
- show up as $1/a$ terms in lattice calculations
- with general UV cutoff Λ:

$$P_c^{(R)} = \exp\left[-K\,\frac{2\Lambda}{T}\right] P_c$$



- what happens to the linear divergences from the strings?



Alternate form of $W_c - P_c$

- use the identity

$$U^\dagger(C)T^aU(C) = U_A^{ab}(C)T^b = T^bU_A^{\dagger\,ba}(C)$$

with U_A a Wilson line in the adjoint representation; $(T_A^c)_{ab} = -if^{abc}$

- split up a Polyakov line into components:

$$P(\mathbf{r}) = \mathcal{P}\exp\left[ig\int_0^\beta \mathrm{d}\tau\, A_0(\tau,\mathbf{r})\right] = P_1(\mathbf{r})1_{N_c} + P_8^a(\mathbf{r})T^a$$

with $P_1 = \mathrm{Tr}[P]/N_c$ and $P_8^a = \mathrm{Tr}[PT^a]/T_F$

- with that we can rewrite

$$P_c = \left\langle P_1(\mathbf{r})P_1^\dagger(\mathbf{0})\right\rangle \qquad W_c - P_c = \frac{T_F}{N_C}\left\langle P_8^a(\mathbf{r})U_A^{ab}(S)P_8^{\dagger\,b}(\mathbf{0})\right\rangle$$

Full renormalized expressions

- with these expressions we can see the behaviour of linear divergences
- arise from gluonic diagrams in a colour singlet configuration, i.e. proportional to δ_{ij} (fundamental) or δ^{ab} (adjoint)
- linear divergence from adjoint string factors out and exponentiates, analogously to fundamental Wilson lines
- coefficient of linear divergence may depend on the representation
- the full expressions for the renormalized loop functions are

$$P_c^{(R)} = \exp\left[-K_F\,\frac{2\Lambda}{T}\right]\left\langle P_1(\mathbf{r})P_1^\dagger(\mathbf{0})\right\rangle$$

$$W_c^{(R)} - P_c^{(R)} = \exp\left[-K_F\,\frac{2\Lambda}{T} - K_A\Lambda r\right] Z_{int}\frac{T_F}{N_C}\left\langle P_8^a(\mathbf{r})U_A^{ab}(S)P_8^{\dagger\,b}(\mathbf{0})\right\rangle$$

Conclusions

- in the vacuum the Wilson loop gives the static potential
- cusp divergences can be removed by a multiplicative constant
- at finite T Polyakov loop correlator gives static quark free energy
- cyclic Wilson loop has intersection instead of cusp divergences
- it mixes with the Polyakov loop correlator under renormalization
- $W_c - P_c$ gives a multiplicatively renormalizable quantity
- comparison to lattice is under way

References

Burnier, Y. and Laine, M. and Vepsäläinen, M.
Dimensionally regularized Polyakov loop correlators in hot QCD
JHEP 01, 2010

Dotsenko, V. S. and Vergeles, S. N.
Renormalizability of Phase Factors in the Nonabelian Gauge Theory
Nucl.Phys. B169, 1980

Brandt, R. A. and Neri, F. and Sato, M.
Renormalization of Loop Functions for All Loops
Phys.Rev. D24, 1981

Korchemsky, G. P. and Radyushkin, A. V.
Renormalization of the Wilson Loops Beyond the Leading Order
Nucl.Phys. B283, 1987

Thank you for your attention!

Perturbative expansion and divergences

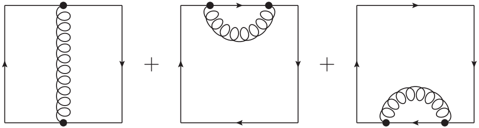
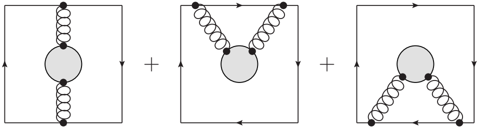
- in the following perturbation theory and DR will be used throughout
- calculations can be simplified using **exponentiation theorem**[4]

$$\langle U(\Gamma)\rangle = \sum_{n=0}^{\infty}(ig)^n \sum_{\gamma_n} C(\gamma_n)\gamma_n = \exp\left[\sum_{n=2}^{\infty}(ig)^n \sum_{\gamma_n\in 2\mathrm{PI}} \widetilde{C}(\gamma_n)\gamma_n\right]$$

- string operators are inverse to each other
  $\rightarrow$ many cancellations of diagrams ("**cyclicity cancellation**")
- exponentiation and cancellations reduce number of relevant diagrams
- in addition, in **Coulomb gauge** many diagrams vanish

[4] [Gatheral, J. G. M. 1983, Frenkel, J. and Taylor, J. C. 1984]

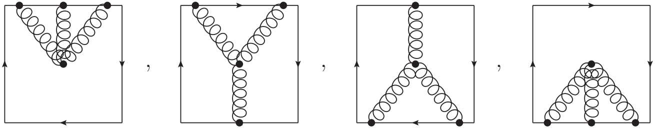
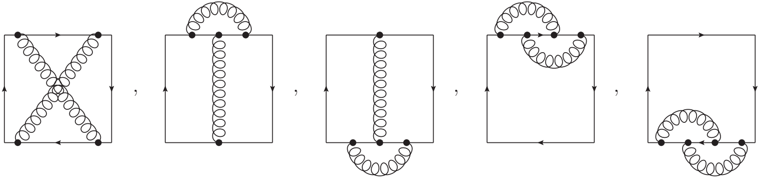
Contributions by diagram

-  $= \frac{\alpha_s C_F}{rT}$
  - 2 diagrams on the right vanish in any gauge
- 

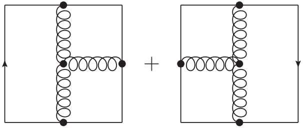
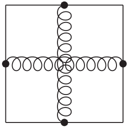
$$= \frac{\alpha_s^2 C_F}{4\pi r T}\left[\left(\frac{31}{9}C_A - \frac{20}{9}T_F n_f\right) + \left(\frac{11}{3}C_A - \frac{4}{3}T_F n_f\right)\left(\ln \mu^2 r^2 + 2\gamma_E\right)\right] +$$
$$+ \frac{4\pi\alpha_s C_F}{T}\int_k \frac{e^{-i\mathbf{r}\cdot\mathbf{k}} - 1}{(\mathbf{k}^2)^2}\left(-\Pi_{00}^{(T)}(0,\mathbf{k})\right)$$

  - 2 diagrams on the right only contribute to thermal part
  - IR divergences in thermal parts cancel
  - no analytic expression for thermal part, but UV finite

Contributions by diagram

-  $= 0$
  - diagrams vanish in Coulomb and Feynman gauge because of 3-gluon vertex
-  $= 0$
  - in Coulomb gauge all diagrams vanish

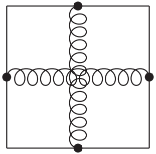
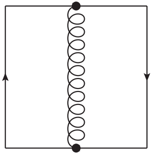
Contributions by diagram

-  $= \alpha_s^2 C_F C_A$
-  (only source of divergence in Coulomb gauge)

$$= \frac{\alpha_s^2 C_F C_A}{\pi r T} \left[\frac{1}{\varepsilon} + 1 + 2\gamma_E + \ln 4\pi^2 + 2 \ln \mu^2 r^2 + \sum_{n=1}^{\infty} \frac{2(-1)^n \zeta(2n)}{n(4n^2-1)} (rT)^{2n} \right]$$

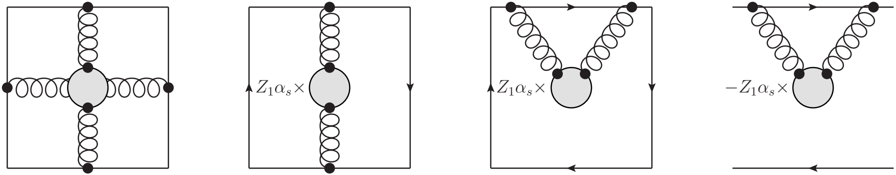
- series expansion for thermal part only valid for $rT \leq 1$
- divergence depends on physical parameters r, T
  $\rightarrow$ **cannot be removed by a multiplicative constant**

Renormalization at $\mathcal{O}(\alpha_s^2)$

- renormalization matrix for cyclic Wilson loop depends on only a single constant Z
- expand Z in orders of α_s: $Z = 1 + Z_1\alpha_s + Z_2\alpha_s^2 + \ldots$
- Polyakov loop correlator $P_c = 1 + \mathcal{O}(\alpha_s^2)$
  $\Rightarrow$ only new contribution comes from Z_1 times the tree level cyclic Wilson loop diagram
-  $+\, Z_1\alpha_s \times$  $= \text{finite}$

  defines value of $Z_1 = -\dfrac{C_A}{\pi\bar{\varepsilon}}$ in $\overline{\text{MS}}$-scheme

Trivial cancellations at $\mathcal{O}(\alpha_s^3)$

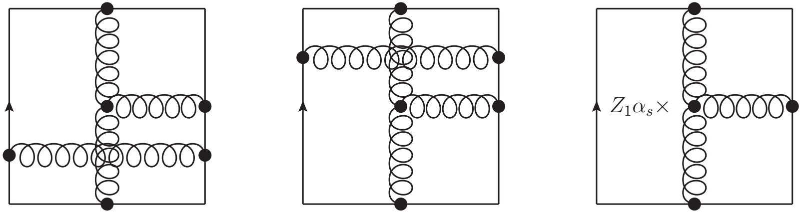


- the divergences of the two diagrams on the left cancel in completely the same way as for the $\mathcal{O}(\alpha_s^2)$ diagrams without the selfenergy
- the two diagrams on the right are exactly equal, so they cancel.

Compare: $ZW_c + (1-Z)P_c = P_c + Z(W_c - P_c)$

- in the difference $W_c - P_c$ all diagrams equal for W_c and P_c drop out, so when multiplying with Z one only has to consider diagrams where the colour factors differ between W_c and P_c

Cancellations at $\mathcal{O}(\alpha_s^3)$

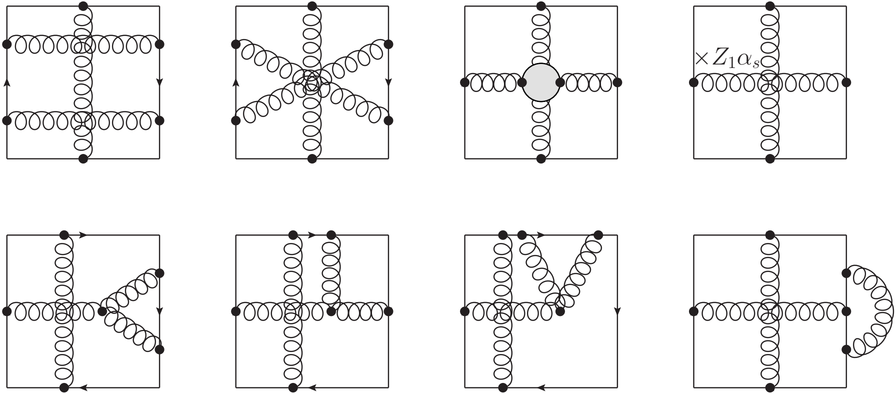


- left and middle diagram have the same colour factors
- so their integration regions can be combined
- divergent subdiagram factors out just like at $\mathcal{O}(\alpha_s^2)$
- colour-connected coefficients also fit

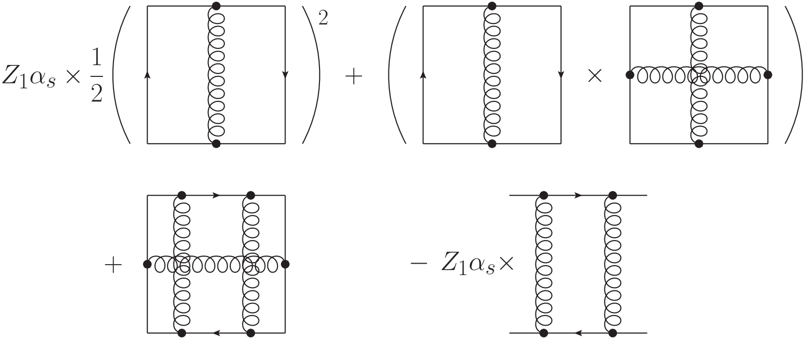
$$-\frac{C_F C_A^2 \alpha_s^3}{2\pi\bar{\varepsilon}} = -Z_1\alpha_s\left(-\frac{1}{2}C_F C_A \alpha_s^2\right)$$

Cancellations at $\mathcal{O}(\alpha_s^3)$

All of these divergences together must be canceled by $Z_2\alpha_s^2$ times the tree level diagram and thus determine the value of Z_2.



Cancellations at $\mathcal{O}(\alpha_s^3)$



The sum of the W_c diagrams gives $\left(C_F^2 - \frac{1}{2}C_F C_A\right)\frac{\alpha_s^2}{2r^2T^2}\frac{C_A\alpha_s}{\pi\varepsilon}$

This is exactly canceled by the contribution from P_c!