String calculation of the long range $Q\bar{Q}$ potential

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Outline

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Motivation

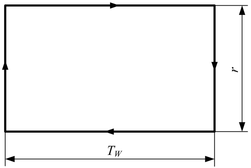
The static quark-antiquark potential is given by

$$V^{(0)}(r) = \lim_{T,T_W \to \infty} \frac{i}{T} \ln\langle W_\Box \rangle$$

where

$$W_\Box \equiv \mathrm{P}\exp\left\{-ig \oint_{r\times T_W} dz^\mu A_\mu(z)\right\},$$

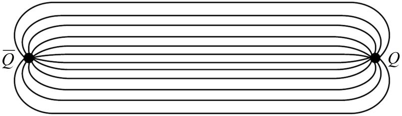
is the rectangular Wilson loop



In the long distance limit this leads to a static potential with a linear dependence in r

$$V^{(0)} = \sigma r$$

with force lines



The string hypothesis or **Effective String Theory (EST)** states that at long distances ($r\Lambda_{QCD} >> 1$)

$$\lim_{T_W\to\infty}\langle 0|W_\square(T_W,r)|0\rangle = Z\int \mathcal{D}\xi^1\mathcal{D}\xi^2\, e^{iS_{string}(\xi^1,\xi^2)}$$

where

$$S_{string} = \int dtdz\, \mathcal{L}(\partial^\mu \xi^l) = -\sigma\int dtdz\,\left(1-\frac{1}{2}\partial_\mu\xi^l\partial^\mu\xi^l\right),$$

and σ is the **string tension**.

Using these relations **Luscher et al.** *(Nucl. Phys. B173, 365, 1980)* obtained a non trivial correction to the static potential

$$V^{(0)}(r) = \sigma r + \mu - \frac{\pi}{12r}$$

This result agrees with lattice data for $\sigma \simeq 0.21\,\mathrm{GeV}^2$ (**Luscher et al.** *JHEP, 07, 2002)*

From the matching between **NRQCD** and **pNRQCD** **Brambilla Pineda Soto and Vairo** *(PRD63, 014023, 2000)* obtained the relativistic corrections to the static potential at the order of $1/m$:

$$V = V^{(0)} + V^{(1,0)}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \dots$$

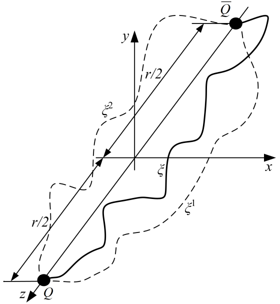
given by

$$V^{(1,0)}(r) = -\frac{1}{2}\lim_{T\to\infty}\int_0^T dt\, t\, \langle\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_1(0)\rangle\rangle_c,$$

where $\langle\langle\dots\rangle\rangle \equiv \langle\dots W_\Box\rangle/\langle W_\Box\rangle$ and

$$\langle\langle O_1(t_1)O_2(t_2)\rangle\rangle_c = \langle\langle O_1(t_1)O_2(t_2)\rangle\rangle - \langle\langle O_1(t_1)\rangle\rangle\langle\langle O_2(t_2)\rangle\rangle$$

Using symmetry considerations the following mapping between the EST and the operator insertions can be applied (**Perez-Nadal and Soto,** *PRD79, 114002, 2009* & **Kogut and Parisi** *PRL47, 1089, 1981)*



$$\psi^\dagger(t)\mathbf{E}^l(t,\frac{\mathbf{r}}{2})\psi(t) \mapsto \Lambda^2\partial_z\xi^l(t,\frac{r}{2})$$
$$\chi^\dagger(t)\mathbf{E}^l(t,-\frac{\mathbf{r}}{2})\chi(t) \mapsto -\Lambda^2\partial_z\xi^l(t,-\frac{r}{2})$$
$$\psi^\dagger(t)\mathbf{B}^l(t,\frac{\mathbf{r}}{2})\psi(t) \mapsto \Lambda'\epsilon^{lm}\partial_t\partial_z\xi^m(t,\frac{r}{2})$$
$$\chi^\dagger(t)\mathbf{B}^l(t,-\frac{\mathbf{r}}{2})\chi(t) \mapsto \Lambda'\epsilon^{lm}\partial_t\partial_z\xi^l(t,-\frac{r}{2})$$
$$\psi^\dagger(t)\mathbf{E}^3(t,\frac{\mathbf{r}}{2})\psi(t) \mapsto \Lambda''^2$$
$$\chi^\dagger(t)\mathbf{E}^3(t,-\frac{\mathbf{r}}{2})\chi(t) \mapsto -\Lambda''^2$$
$$\psi^\dagger(t)\mathbf{B}^3(t,\frac{\mathbf{r}}{2})\psi(t) \mapsto \Lambda'''\epsilon^{lm}\partial_t\partial_z\xi^l(t,\frac{r}{2})\partial_z\xi^m(t,\frac{r}{2})$$
$$\psi^\dagger(t)\mathbf{B}^3(t,-\frac{\mathbf{r}}{2})\psi(t) \mapsto \Lambda'''\epsilon^{lm}\partial_t\partial_z\xi^l(t,-\frac{r}{2})\partial_z\xi^m(t,-\frac{r}{2}).$$

Using this mapping for the $1/m$ correction to the potential we get

$$\langle\langle \mathbf{E}^l(t,\frac{\mathbf{r}}{2})\mathbf{E}^m(0,\frac{\mathbf{r}}{2})\rangle\rangle_c \mapsto \Lambda^4\partial_z\partial_{z'}\langle\xi^{\mathbf{l}}(\mathbf{t},\mathbf{r}/2)\xi^{\mathbf{m}}(\mathbf{0},\mathbf{r}/2)\rangle = \Lambda^4\partial_z\partial_{z'}\mathbf{G}_{\mathbf{F}}^{\mathbf{lm}}(\mathbf{t},\mathbf{r}/2;\mathbf{0},\mathbf{r}/2)$$

where the mapping into the EST is expressed in terms of the two point correlator of the string degrees of freedom:

$$G_F^{lm}(it,z;it',z') = \frac{\delta_{lm}}{4\pi\sigma}\ \ln\left(\frac{\cosh(\frac{\pi}{r}(t-t')) + \cos(\frac{\pi}{r}(z+z'))}{\cosh(\frac{\pi}{r}(t-t')) - \cos(\frac{\pi}{r}(z-z'))}\right)$$

then the potential is given by

$$\begin{aligned} V^{(1,0)}(r) \quad &\mapsto \quad -\frac{g^2\Lambda^4}{2}\lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty} dt\, t\,\partial_z\partial_{z'}G_F^{ll}(t,r/2;0,r/2) \\ &= \quad \frac{g^2\Lambda^4}{\sigma\pi}\lim_{\epsilon\to 0^+}\int_{\frac{\pi\epsilon}{2r}}^{\infty} dt\,\frac{t}{\sinh^2(t)} \end{aligned}$$

$$V^{(1,0)} = \frac{g^2\Lambda^4}{\sigma\pi}\ln(\sqrt{\sigma}r) + \text{Infinite constant}$$

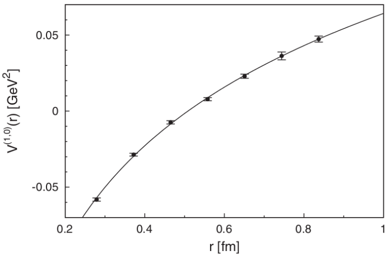


FIG. 1. The lattice data for $V^{(1,0)}(r)$, fitted to the EST prediction $V^{(1,0)}(r) = a \log r + b$.

(**Perez-Nadal and Soto,** *PRD79, 114002, 2009)*

$\mathcal{O}(1/m^2)$ corrections to the potential

The $\mathcal{O}(1/m^2)$ corrections are organized as follows

$$V = V^{(0)} + V^{(1,0)}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \frac{V^{(2,0)}}{m_1} + \frac{V^{(0,2)}}{m_2} + \frac{V^{(1,1)}}{m_1 m_2}$$

where

$$\begin{aligned} V^{(2,0)} &= V^{(2,0)}_{SD} + V^{(2,0)}_{SI}, \\ V^{(0,2)} &= V^{(0,2)}_{SD} + V^{(0,2)}_{SI}, \end{aligned}$$

$$V^{(2,0)}_{SI} = \frac{1}{2}\left\{\mathbf{p}_1, V^{(2,0)}_{p^2}(r)\right\} + \frac{V^{(2,0)}_{L^2}(r)}{r^2}\mathbf{L}_1^2 + V^{(2,0)}_r(r),$$

and

$\mathbf{p}_1 = -i\nabla_{x_1}$ and $\mathbf{L}_1 = r \times \mathbf{p}_1$

For the spin-dependent part we have

$$V_{SD}^{(2,0)} = V_{LS}^{(2,0)}(r)\,\mathbf{L}_1\cdot\mathbf{S},$$

where $\mathbf{S}$ is the total spin operator. The same kind of decomposition is made for the $V^{(1,1)}$ potential

$$V^{(1,1)} = V_{SD}^{(1,1)} + V_{SI}^{(1,1)},$$

where

$$V_{SI}^{(1,1)} = -\frac{1}{2}\left\{\mathbf{p_1}\cdot\mathbf{p_2}, V_{p^2}^{(1,1)}(r)\right\} - \frac{V_{L^2}^{(1,1)}(r)}{2r^2}(\mathbf{L_1}\cdot\mathbf{L_2}+\mathbf{L_2}\cdot\mathbf{L_1}) + V_r^{(1,1)}(r)$$

$$V_{SD}^{(1,1)} = V_{L_1S_2}^{(1,1)}(r)\mathbf{L_1}\cdot\mathbf{S_2} - V_{L_2S_1}^{(1,1)}(r)\mathbf{L_2}\cdot\mathbf{S_1} + V_{S_1S_2}^{(1,1)}(r)\mathbf{S_1}\cdot\mathbf{S_2} + V_{S_{12}}^{(1,1)}(r)\mathbf{S_{12}}\cdot\hat{\mathbf{r}},$$

$$\mathbf{S_{12}}(\hat{\mathbf{r}}) = 3\,\hat{\mathbf{r}}\cdot\sigma_1\hat{\mathbf{r}}\cdot\sigma_2 - \sigma_1\cdot\sigma_2$$

and

$$V_{L_1S_2}^{(1,1)}(r) = V_{L_2S_1}^{(1,1)}(r, m_1 \leftrightarrow m_2).$$

Let us consider for instance

$$V_{SD}^{(2,0)} = V_{LS}^{(2,0)}(r)\,\mathbf{L}_1 \cdot \mathbf{S},$$

where

$$V_{LS}^{(2,0)}(r) = -\frac{c_F^{(1)}}{r^2} i\mathbf{r} \cdot \lim_{T\to\infty} \int_0^T dt \langle\langle g\mathbf{B}_1(t) \times g\mathbf{E}_1(0)\rangle\rangle + \frac{c_S^{(1)}}{2r^2}\mathbf{r} \cdot (\nabla_{\mathbf{r}} \mathbf{V}^{(\mathbf{0})}),$$

The mapping leads to

$$\langle\langle \mathbf{B}(t, \frac{\mathbf{r}}{2}) \times \mathbf{E}(0, \frac{\mathbf{r}}{2})\rangle\rangle_3 \mapsto 2\Lambda^2\Lambda' \partial_t \partial_z \partial_{z'} G_F(t, r/2; 0, r/2)$$

and therefore

$$\begin{aligned} V_{LS}^{(2,0)}(r) &= -i\frac{2g^2 c_F^{(1)}}{\sigma r^2}\Lambda'\Lambda^2 \lim_{\epsilon\to 0+} \int_{\frac{\epsilon\pi}{2r}}^{\infty} dt\, t \frac{\cos(t)}{\sin^3(t)} + \frac{c_S^{(1)}\sigma}{2r} \\ &= -\frac{g^2 c_F^{(1)}\Lambda^2\Lambda'}{kr^2} + \frac{4c_F^{(1)} g^2 \Lambda^2 \Lambda'}{\epsilon_{UV}\pi r} + \frac{c_S^{(1)}\sigma}{2r} \end{aligned}$$

Renormalized the potential reads

$$V_{LS}^{(2,0)}(r) = -\frac{g^2 c_F^{(1)} \Lambda' \Lambda^2}{\sigma r^2} - \frac{\mu_c}{r}$$

Poincaré invariance implies

$$\frac{1}{2r}\frac{dV^{(0)}}{dr} + V_{LS}^{(2,0)} - V_{L_2 S_1}^{(1,1)} = 0,$$

Gromes, *Z. Phys. C 26, 401 (1984)*

$$\frac{r}{2}\frac{dV^{(0)}}{dr} + 2V_{\mathbf{L}^2}^{(2,0)} - V_{\mathbf{L}^2}^{(1,1)} = 0.$$

Brambilla et al. *Nuovo Cimiento A 103, 59 (1990)*

Which translates in to the parameters as

$$\begin{aligned} \mu_c &= \sigma/2 \\ g\Lambda^2 &= \sigma \end{aligned}$$

So the unknown constant we have introduce in order to renormalize $V_{LS}^{(2,0)}(r)$ is now fixed.

$$
\begin{aligned}
V_r^{(2,0)}(r) &= \frac{\pi C_f \alpha_s c_D^{(1)\prime}}{2}\delta^{(3)}(\mathbf{x}_1 - \mathbf{x}_2) \\
&- \frac{i c_F^{(1)\,2}}{4}\lim_{T\to\infty}\int_0^T dt\,\langle\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_1(0)\rangle\rangle_c + \frac{1}{2}(\nabla_{\mathbf{r}}^{\mathbf{2}}\mathbf{V}_{\mathbf{p^2}}^{(\mathbf{2},\mathbf{0})}) \\
&- \frac{i}{2}\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3\,(t_2 - t_3)^2\langle\langle g\mathbf{E}_1(t_1)\cdot g\mathbf{E}_1(t_2) g\mathbf{E}_1(t_3)\cdot g\mathbf{E}_1(0)\rangle\rangle_c \\
&+ \frac{1}{2}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2\,(t_1 - t_2)^2\langle\langle g\mathbf{E}_1^i(t_1) g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\right) \\
&- \frac{i}{2}\left(\boldsymbol{\nabla}_r^i V^{(0)}\right)\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2\,(t_1 - t_2)^3\langle\langle g\mathbf{E}_1^i(t_1) g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c \\
&+ \frac{1}{4}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt\, t^3\langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_1^j(0)\rangle\rangle_c(\boldsymbol{\nabla}_r^j V^{(0)})\right) \\
&- \frac{i}{12}\lim_{T\to\infty}\int_0^T dt\, t^4\langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_1^j(0)\rangle\rangle_c(\boldsymbol{\nabla}_r^i V^{(0)})(\boldsymbol{\nabla}_r^j V^{(0)}) \\
&- d_3^{(1)\prime} f_{abc}\int d^3\mathbf{x}\lim_{T_W\to\infty} g\langle\langle G^a_{\mu\nu}(x) G^b_{\mu\alpha}(x) G^c_{\nu\alpha}(x)\rangle\rangle\,,
\end{aligned}
$$

The potentials in the EST read

$$V^{(0)}(r) = \sigma r + \mu - \frac{\pi}{12r}\,,$$

$$V^{(1,0)}(r) = \frac{g^2\Lambda^4}{\sigma\pi}\ln(\sqrt{\sigma}r)$$

$$V^{(1,1)}_{p^2}(r) = 0\,,$$

$$V^{(2,0)}_{p^2}(r) = 0\,,$$

$$V^{(2,0)}_{L^2}(r) = -\frac{g^2\Lambda^4 r}{6\sigma}\,,$$

$$V^{(1,1)}_{L^2}(r) = \frac{g^2\Lambda^4 r}{6\sigma}$$

$$V^{(2,0)}_{LS}(r) = -\frac{g^2 c_F^{(1)}\Lambda^2\Lambda'}{\sigma\, r^2} - \frac{\mu_c}{r}$$

$$V^{(1,1)}_{L_2 S_1}(r) = -\frac{g^2 c_F^{(1)}\Lambda^2\Lambda'}{\sigma\, r^2}\,,$$

$$V^{(1,1)}_{L_1 S_2}(r) = -\frac{g^2 c_F^{(1)} g^2\Lambda^2\Lambda'}{\sigma\, r^2}\,,$$

$$V^{(1,1)}_{S^2}(r) = \frac{2g^2 c_F^{(1)} c_F^{(2)}\Lambda''' \pi^3}{45\sigma^4\, r^5}$$

$$V^{(1,1)}_{S_{12}}(r) = \frac{1}{4}V^{(1,1)}_{S^2}$$

$$V^{(2,0)}_r(r) = -\frac{2.404\, g^4\Lambda^8\, r}{\sigma^2\pi^3} - \frac{g^2 c_F^{(1)\,2}\Lambda'^2}{\sigma\pi\epsilon^3_{UV}} + \frac{3g^2\Lambda^4}{2\pi^3\sigma}\zeta(3)(\nabla^i_r\, r^2\,\nabla^i_r\, V^{(0)}) - \frac{g^2\Lambda^4\, r^3}{45\sigma}(\nabla^i_r V^{(0)})(\nabla^i_r V^{(0)})$$

$$V^{(1,1)}_r(r) = -\frac{0.601\, g^4\Lambda^8 r}{\sigma^2\,\pi^3} - \frac{9g^2\Lambda^4\zeta(3)}{4\pi^3\,\sigma}\nabla^i_r(r^2\,\nabla^i_r V^{(0)}) + \frac{7g^2\Lambda^4 r^3}{180\sigma}(\nabla^i_r V^{(0)})(\nabla^i_r V^{(0)})$$

The Spectrum

We will only consider the leading terms up to $1/r$ in the equal mass case

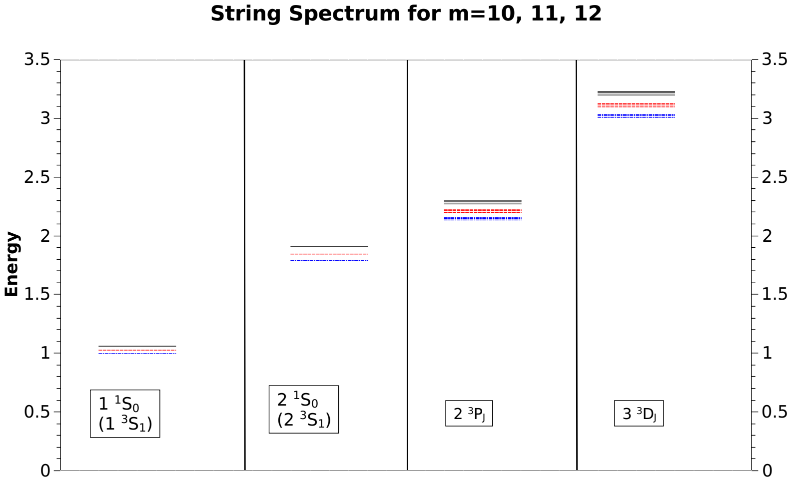
$$\begin{aligned} V^{(0)}(r) &= \sigma r \\ V^{(1,0)}(r) &= \frac{\sigma}{\pi}\ln(\sqrt{\sigma}r) \\ V^{(1,1)}_{p^2}(r) &= 0\,, \\ V^{(2,0)}_{p^2}(r) &= 0\,, \\ V^{(2,0)}_{L^2}(r) &= -\frac{\sigma r}{6}\,, \\ V^{(1,1)}_{L^2}(r) &= \frac{\sigma r}{6} \end{aligned}$$

$$\begin{aligned} V^{(2,0)}_{LS}(r) &= -\frac{\sigma}{2r} \\ V^{(1,1)}_{L_2 S_1}(r) &= 0 \\ V^{(1,1)}_{L_1 S_2}(r) &= 0 \\ V^{(1,1)}_{S^2}(r) &= 0 \\ V^{(1,1)}_{S_{12}}(r) &= 0 \end{aligned}$$

$$\begin{aligned} V^{(2,0)}_{r}(r) &= \sigma^2 r(0.155 - 0.044\,\sigma r^2) \\ V^{(1,1)}_{r}(r) &= \sigma^2 r(-0.368 + 0.078\,\sigma r^2) \end{aligned}$$

Then the long range potential simplifies to

$$\begin{aligned} V(r) &= \sigma r + \frac{2}{m} V^{(1,0)} + \frac{1}{m^2}\left\{ \frac{V_{L^2}^{(2,0)}}{r^2}\mathbf{L}^2 + V_{LS}^{(2,0)}\mathbf{L}\cdot\mathbf{S} + 2V_r^{(2,0)} + V_r^{(1,1)} \right\} \\ &= \sigma r + \frac{1}{m}\left\{ \frac{2\sigma}{\pi}\ln(\sqrt{\sigma} r) \right\} + \frac{1}{m^2}\left\{ -\frac{\sigma}{6r}\mathbf{L}^2 - \frac{\sigma}{2r}\mathbf{L}\cdot\mathbf{S} - 0.058\,\sigma^2 r - 0.011\,\sigma^3 r^3 \right\} \end{aligned}$$


String Spectrum for m=10, 11, 12
Energy
3.5
3
2.5
2
1.5
1
0.5
0
1 ¹S₀ (1 ³S₁)
2 ¹S₀ (2 ³S₁)
2 ³P_J
3 ³D_J

Outlook:

- Lattice QCD

  Yoshiaki Koma and Miho Koma: **Heavy quarkonium spectroscopy in pNRQCD with lattice QCD input**

  arxiv:1211.6795

- Phenomenology of quarkonium radiative transitions

  Brambilla, Pietrulewicz, Vairo: **Model-independent Study of Electric Dipole Transitions in Quarkonium**

  Phys. Rev. D 85, 094005 (2012)

Thanks for your attention!

Backup

$$
\begin{aligned}
V^{(1,0)}(r) &= -\frac{1}{2}\lim_{T\to\infty}\int_0^T dt\, t\, \langle\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\,,\\
V^{(2,0)}_{\mathbf{p}^2}(r) &= \frac{i}{2}\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\lim_{T\to\infty}\int_0^T dt\, t^2\, \langle\langle g\mathbf{E}^i_1(t) g\mathbf{E}^j_1(0)\rangle\rangle_c\,,\\
V^{(2,0)}_{\mathbf{L}^2}(r) &= \frac{i}{4}\left(\delta^{ij}-3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\right)\lim_{T\to\infty}\int_0^T dt\, t^2\, \langle\langle g\mathbf{E}^i_1(t) g\mathbf{E}^j_1(0)\rangle\rangle_c\,,\\
V^{(2,0)}_{LS}(r) &= -\frac{c_F^{(1)}}{r^2}i\mathbf{r}\cdot\lim_{T\to\infty}\int_0^T dt\, t\, \langle\langle g\mathbf{B}_1(t)\times g\mathbf{E}_1(0)\rangle\rangle + \frac{c_S^{(1)}}{2r^2}\mathbf{r}\cdot(\nabla_{\mathbf{r}}\mathbf{V}^{(\mathbf{0})})\,,\\
V^{(2,0)}_{\mathbf{p}^2}(r) &= \frac{i}{2}\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\lim_{T\to\infty}\int_0^T dt\, t^2\, \langle\langle g\mathbf{E}^i_1(t) g\mathbf{E}^j_1(0)\rangle\rangle_c\,,\\
V^{(2,0)}_{\mathbf{L}^2}(r) &= \frac{i}{4}\left(\delta^{ij}-3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\right)\lim_{T\to\infty}\int_0^T dt\, t^2\, \langle\langle g\mathbf{E}^i_1(t) g\mathbf{E}^j_1(0)\rangle\rangle_c\,,\\
V^{(1,1)}_{L_2S_1}(r) &= -\frac{c_F^{(1)}}{r^2}i\mathbf{r}\cdot\lim_{T\to\infty}\int_0^T dt\, t\, \langle\langle g\mathbf{B}_1(t)\times g\mathbf{E}_2(0)\rangle\rangle\\
V^{(1,1)}_{S^2}(r) &= \frac{2c_F^{(1)}c_F^{(2)}}{3}i\lim_{T\to\infty}\int_0^T dt\, \langle\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_2(0)\rangle\rangle - 4(d_{sv}+d_{vv}C_f)\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2)\,,\\
V^{(1,1)}_{\mathbf{S}_{12}}(r) &= \frac{c_F^{(1)}c_F^{(2)}}{4}i\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\lim_{T\to\infty}\int_0^T dt\left[\langle\langle g\mathbf{B}^i_1(t) g\mathbf{B}^j_2(0)\rangle\rangle - \frac{\delta^{ij}}{3}\langle\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_2(0)\rangle\rangle\right],
\end{aligned}
$$

and. . .

$$
\begin{aligned}
V_r^{(2,0)}(r) &= \frac{\pi C_f \alpha_s c_D^{(1)\prime}}{2}\delta^{(3)}(\mathbf{x}_1 - \mathbf{x}_2) \\
&- \frac{i c_F^{(1)\,2}}{4} \lim_{T\to\infty}\int_0^T dt \langle\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_1(0)\rangle\rangle_c + \frac{1}{2}(\nabla_{\mathbf{r}}^{\mathbf{2}} \mathbf{V}_{\mathbf{p^2}}^{(\mathbf{2},\mathbf{0})}) \\
&- \frac{i}{2}\lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3\, (t_2 - t_3)^2 \langle\langle g\mathbf{E}_1(t_1)\cdot g\mathbf{E}_1(t_2) g\mathbf{E}_1(t_3)\cdot g\mathbf{E}_1(0)\rangle\rangle_c \\
&+ \frac{1}{2}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2\, (t_1 - t_2)^2 \langle\langle g\mathbf{E}_1^i(t_1) g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\right) \\
&- \frac{i}{2}\left(\boldsymbol{\nabla}_r^i V^{(0)}\right)\lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2\, (t_1 - t_2)^3 \langle\langle g\mathbf{E}_1^i(t_1) g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c \\
&+ \frac{1}{4}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt\, t^3 \langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_1^j(0)\rangle\rangle_c (\boldsymbol{\nabla}_r^j V^{(0)})\right) \\
&- \frac{i}{12}\lim_{T\to\infty}\int_0^T dt\, t^4 \langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_1^j(0)\rangle\rangle_c (\boldsymbol{\nabla}_r^i V^{(0)})(\boldsymbol{\nabla}_r^j V^{(0)}) \\
&- d_3^{(1)\prime} f_{abc}\int d^3\mathbf{x} \lim_{T_W\to\infty} g\langle\langle G_{\mu\nu}^a(x) G_{\mu\alpha}^b(x) G_{\nu\alpha}^c(x)\rangle\rangle ,
\end{aligned}
$$

$$
\begin{aligned}
V_r^{(1,1)}(r) = & -\frac{1}{2}(\boldsymbol{\nabla}_r^2 V_{\mathbf{p}^2}^{(1,1)}) \\
& -i \lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3\, (t_2-t_3)^2 \langle\langle g\mathbf{E}_1(t_1)\cdot g\mathbf{E}_1(t_2) g\mathbf{E}_2(t_3)\cdot g\mathbf{E}_2(0)\rangle\rangle_c \\
& +\frac{1}{2}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\langle g\mathbf{E}_1^i(t_1) g\mathbf{E}_2(t_2)\cdot g\mathbf{E}_2(0)\rangle\rangle_c\right) \\
& +\frac{1}{2}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\langle g\mathbf{E}_2^i(t_1) g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c\right) \\
& -\frac{i}{2}\left(\boldsymbol{\nabla}_r^i V^{(0)}\right) \lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 (t_1-t_2)^3 \langle\langle g\mathbf{E}_1^i(t_1) g\mathbf{E}_2(t_2)\cdot g\mathbf{E}_2(0)\rangle\rangle_c \\
& -\frac{i}{2}\left(\boldsymbol{\nabla}_r^i V^{(0)}\right) \lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 (t_1-t_2)^3 \langle\langle g\mathbf{E}_2^i(t_1) g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\rangle_c \\
& +\frac{1}{4}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt\, t^3 \left\{\langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_2^j(0)\rangle\rangle_c + \langle\langle g\mathbf{E}_2^i(t) g\mathbf{E}_1^j(0)\rangle\rangle_c\right\} (\boldsymbol{\nabla}_r^j V^{(0)})\right) \\
& -\frac{i}{6}\lim_{T\to\infty}\int_0^T dt\, t^4 \langle\langle g\mathbf{E}_1^i(t) g\mathbf{E}_2^j(0)\rangle\rangle_c (\boldsymbol{\nabla}_r^i V^{(0)})(\boldsymbol{\nabla}_r^j V^{(0)}) + (d_{ss}+d_{vs}C_f)\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2),
\end{aligned}
$$

$$
\begin{aligned}
V_r^{(2,0)}(r) &= -\frac{2.404\,\sigma^2 r}{\pi^3} + \frac{6\sigma^2 r}{\pi^3}\zeta(3) - \frac{2\sigma^3 r^3}{45} \\
V_r^{(1,1)}(r) &= -\frac{0.601\sigma^2 r}{\pi^3} - \frac{9\sigma^2 r}{\pi^3}\zeta(3) + \frac{7\sigma^3 r^3}{90}
\end{aligned}
$$