QQQ singlet static potential and singlet static energy

Nora Brambilla

based on the paper

N. Brambilla, F. Karbstein, A. Vairo

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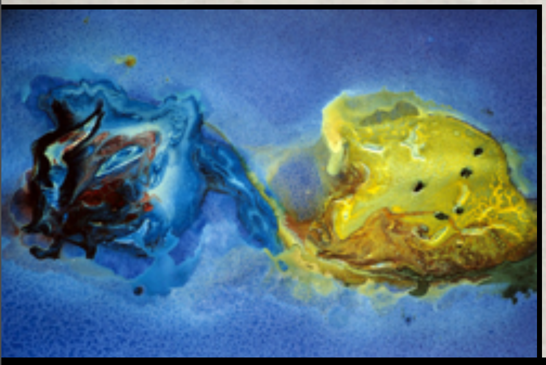

Physik Department TUM
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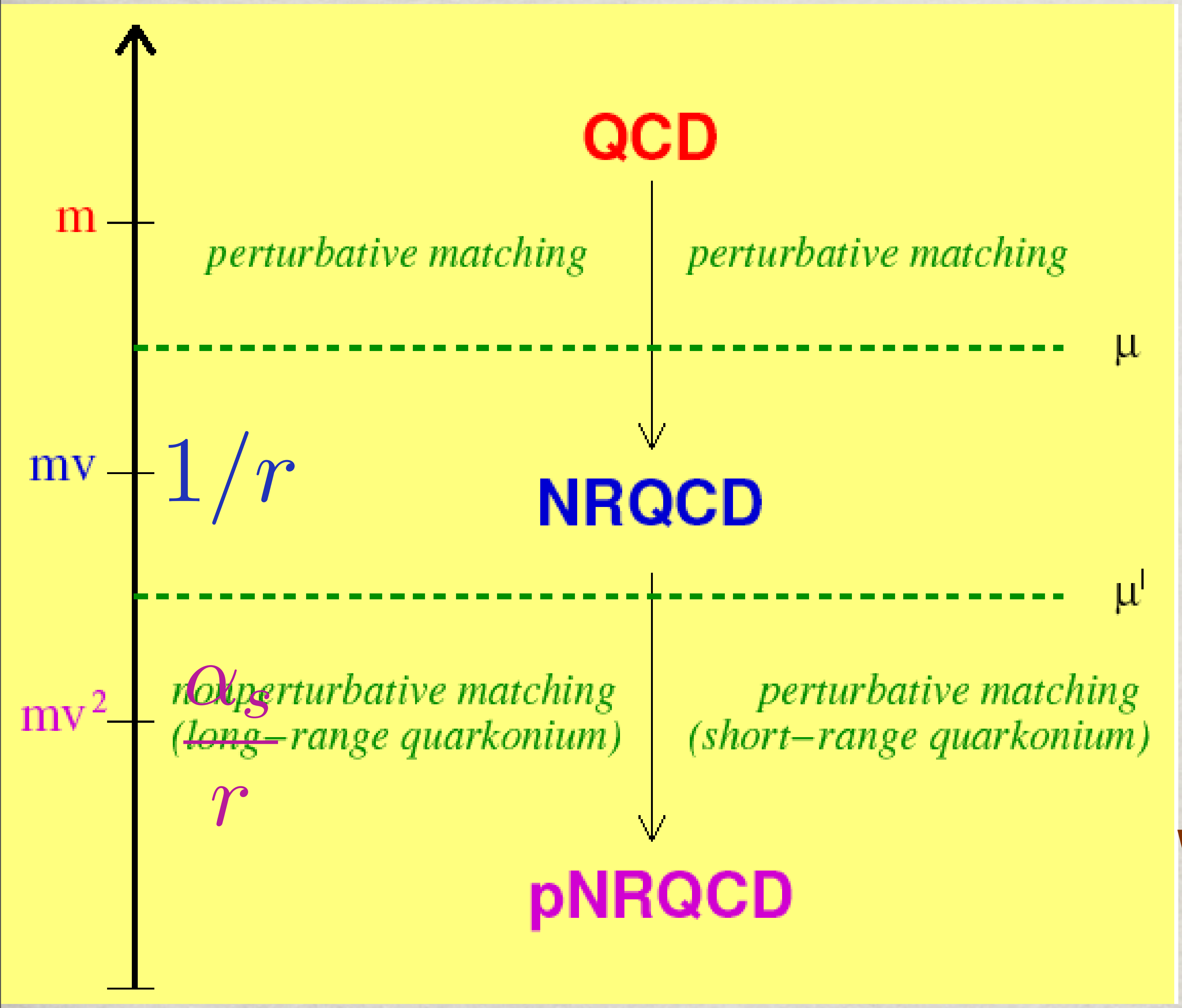
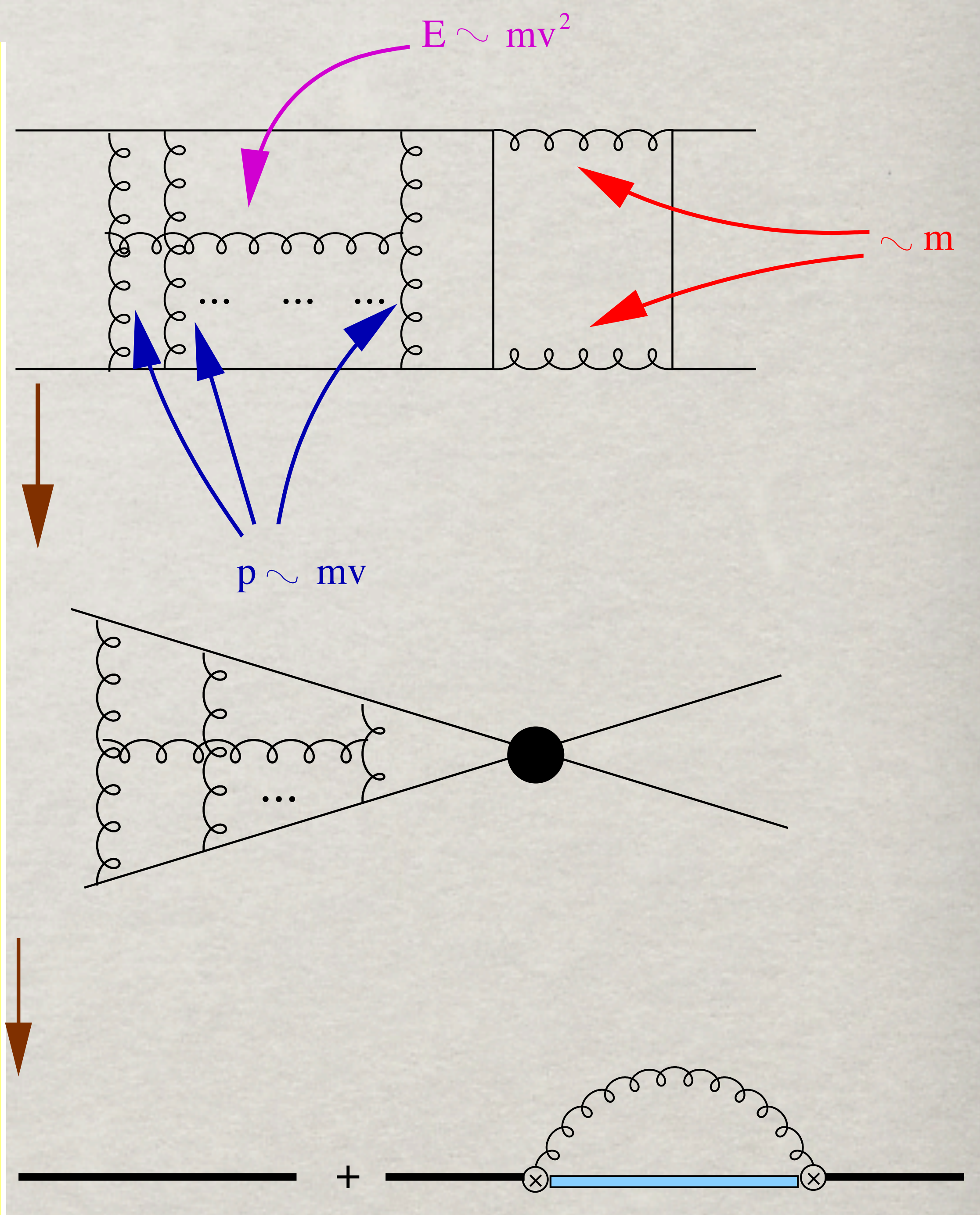
The QQbar singlet static potential and the QQbar singlet static energy are fundamental quantities calculated in perturbation theory and the lattice since the beginning of QCD

A proper definition of these quantities is given in nonrelativistic effective field theories

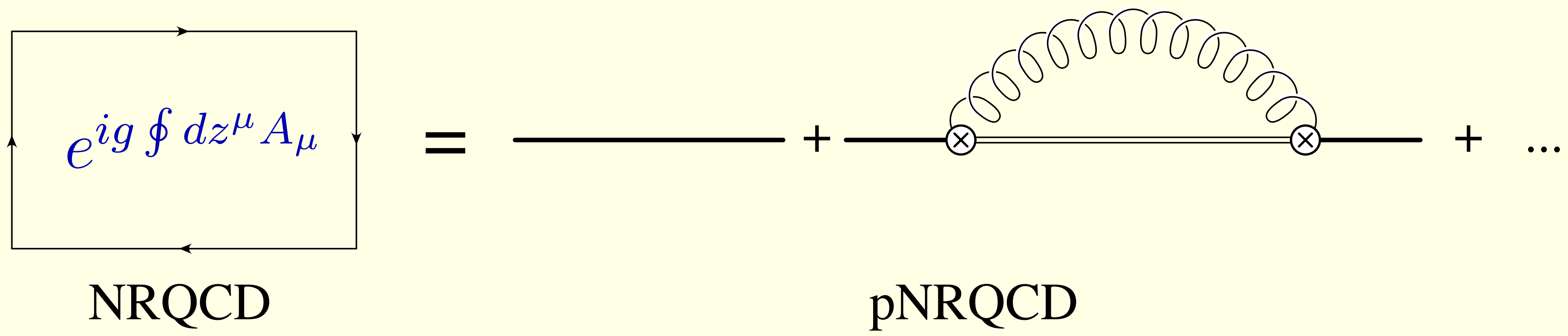
Potentials from pNRQCD $(r < \Lambda^{-1})$





$$\mathcal{L}_{\rm pNRQCD} = \sum_k \sum_n \frac{1}{m^k} c_k(\alpha_{\rm s}(m/\mu)) \times V(r\mu', r\mu) \times O_n(\mu', \lambda)\, r^n$$

The static singlet QQbar potential and energy$(r < \Lambda^{-1}$



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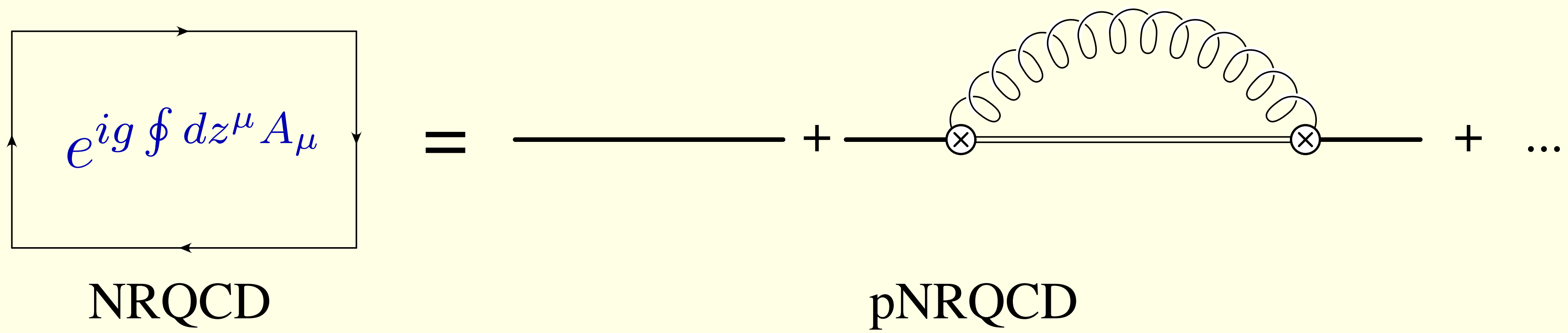


potential

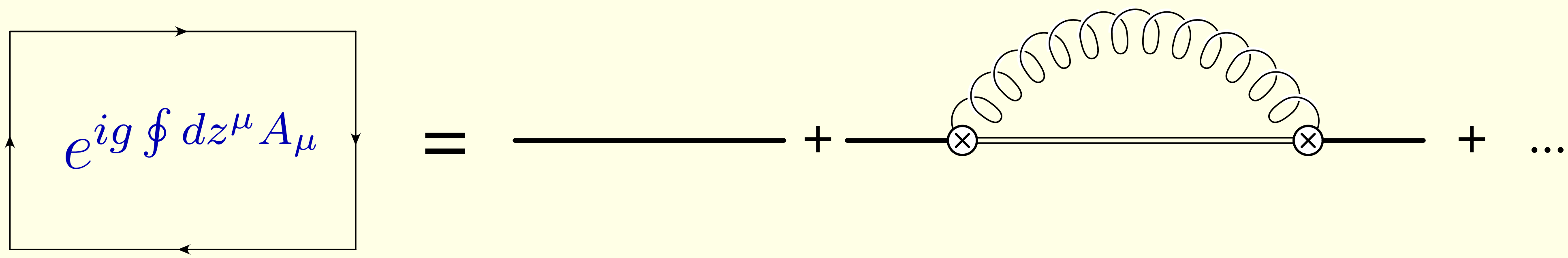
$$\lim_{T\to\infty} \frac{i}{T} \ln \langle \square \rangle = V_s(r,\mu) - i \frac{g^2}{N_c} V_A^2 \int_0^\infty dt\, e^{-it(V_o - V_s)} \langle \mathrm{Tr}(r \cdot E(t)\, r \cdot E(0)) \rangle(\mu) + \ldots$$

static energy

ultrasoft contribution

contributes from 3 loops

The static singlet QQbar potential and energy$(r < \Lambda^{-1}$



NRQCD

pNRQCD

potential

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* The μ dependence cancels between the two terms in the right-hand side:

$V_s \sim \ln r\mu, \ln^2 r\mu$, ...

ultrasoft contribution $\sim \ln(V_o - V_s)/\mu, \ln^2(V_o - V_s)/\mu, \ldots \ln r\mu, \ln^2 r\mu, \ldots$

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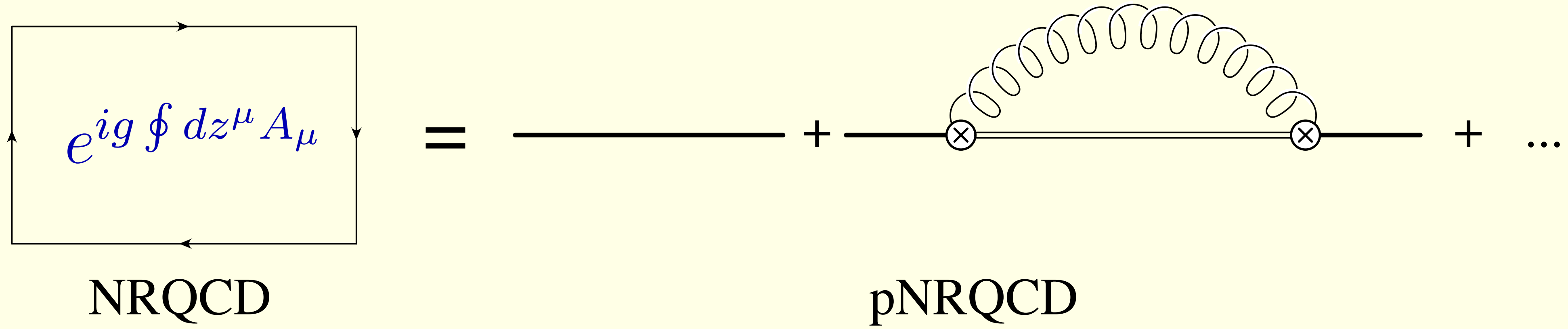

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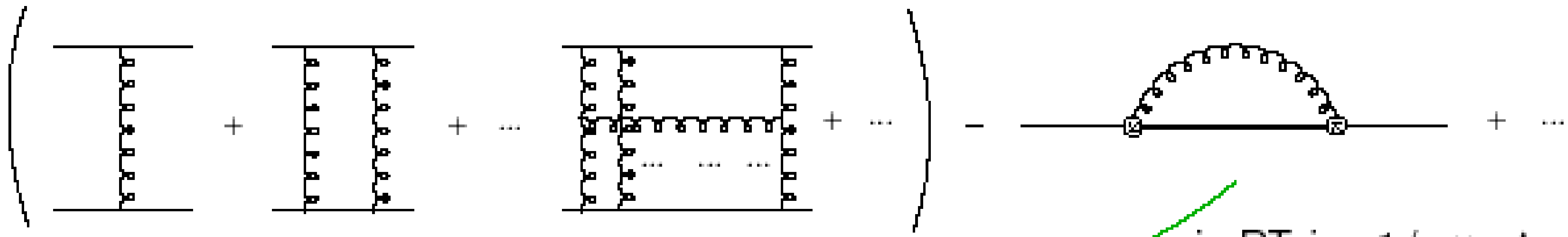
The static energy is a physical quantity and does not depend on the ultrasoft cutoff

Static singlet potential at N^4LO

$V =$



in PT, i.e. $1/r \gg \Lambda_{\rm QCD}$

$$= -\frac{4}{3}\frac{\alpha_s(1/r)}{r}\left[1 + a_1\frac{\alpha_s(1/r)}{4\pi} + a_2\left(\frac{\alpha_s(1/r)}{4\pi}\right)^2 \right.$$
$$+ \left(144\,\pi^2\,\ln r\mu + a_3\right)\left(\frac{\alpha_s(1/r)}{4\pi}\right)^3$$
$$\left. + \left(a_4^{L2}\ln^2 r\mu + \left(a_4^L + 48\pi^2\,\beta_0(-5 + 6\ln 2)\right)\ln r\mu + a_4\right)\left(\frac{\alpha_s(1/r)}{4\pi}\right)^4 + \dots\right]$$

$$a_4^{L2} = -144\pi^2\,\beta_0$$

$$a_4^L = 432\pi^2\left[a_1 + 2\gamma_E\beta_0 + n_f\left(-\frac{20}{27} + \frac{4}{9}\ln 2\right) + \frac{149}{9} - \frac{22}{3}\ln 2 + \frac{4}{3}\pi^2\right]$$

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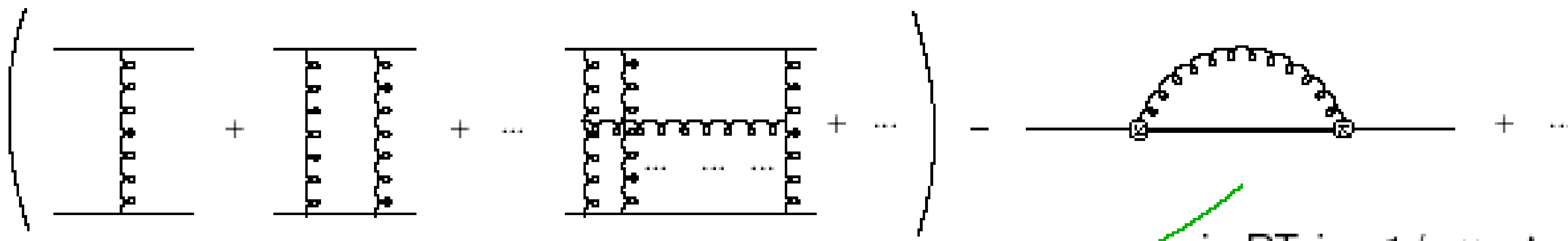
Static singlet potential at N^4LO

$V =$ 

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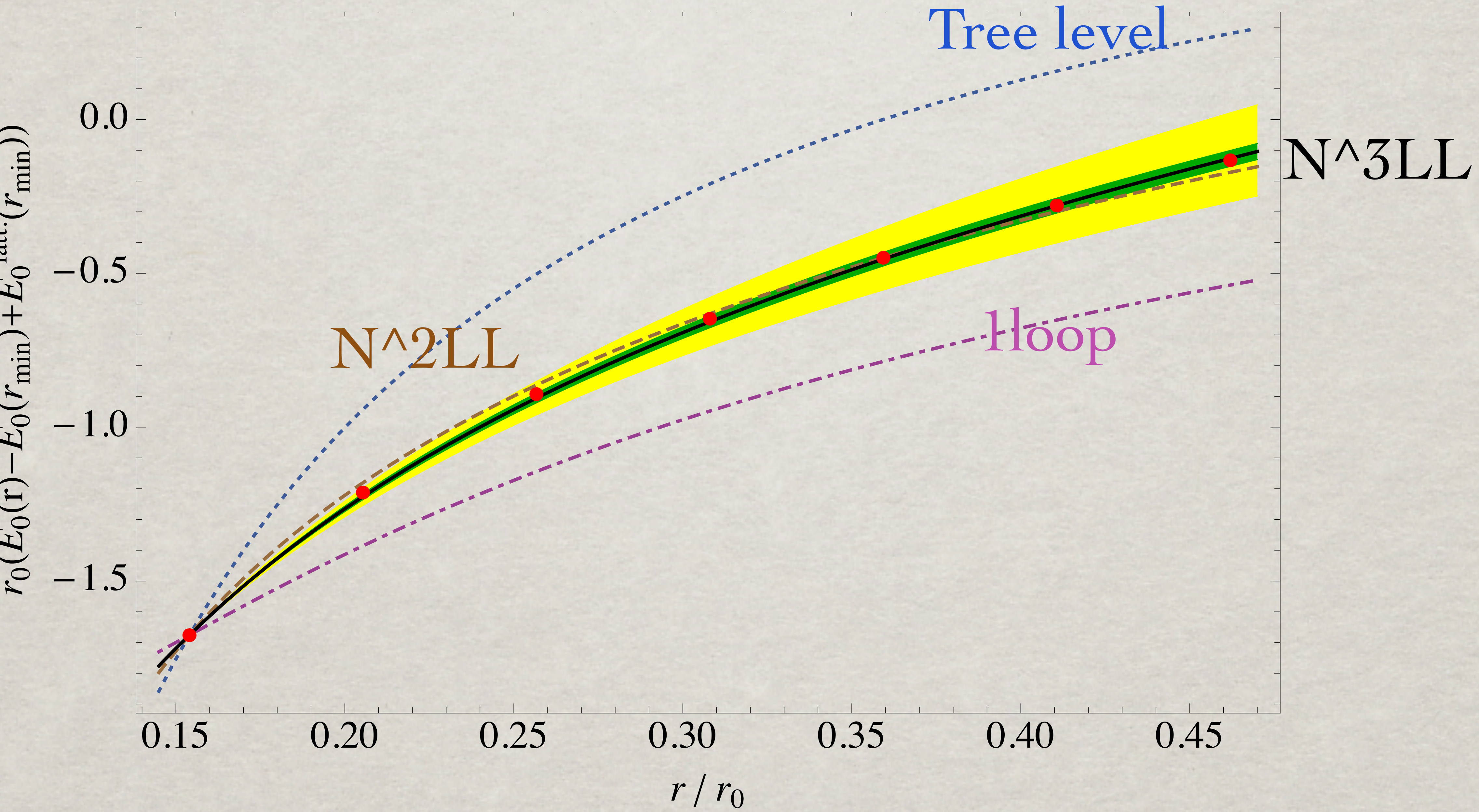
- The logarithmic contribution at N^3LO may be extracted from the one-loop calculation of the ultrasoft contribution;
- the single logarithmic contribution at N^4LO may be extracted from the two-loop calculation of the ultrasoft contribution.

Singlet static energy at N^3LL in comparison to lattice data (red points Necco sommer 2002)

Obtain the static energy: 1) subtract the renormalon 2) resum the logs in the energy scales ratio



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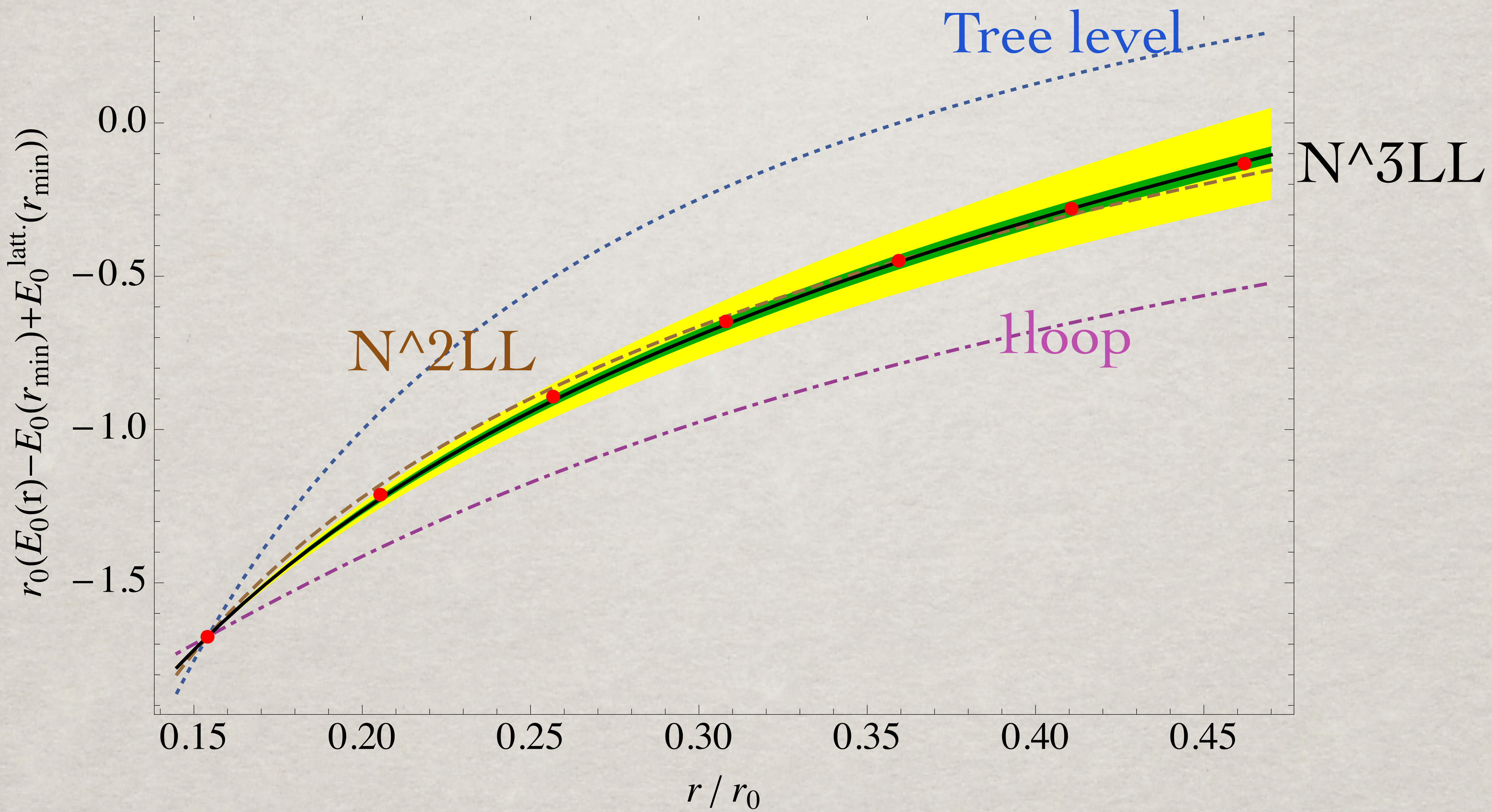

- The lattice data are perfectly described from perturbation theory up to more than 0.2 fm
- Allows precise extraction of fundamental parameters of QCD

What is known about the QQQ potential/static energy and why it is interesting?

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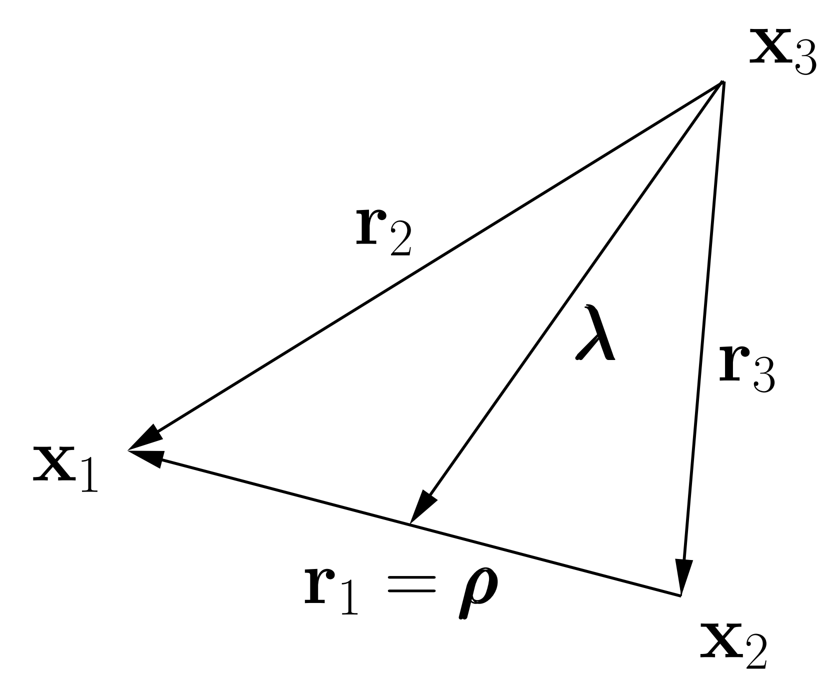

- Two independent relative distances

$$\mathbf{r}_1 = \mathbf{x}_1 - \mathbf{x}_2\,, \qquad \mathbf{r}_2 = \mathbf{x}_1 - \mathbf{x}_3\,, \qquad \mathbf{r}_3 = \mathbf{x}_2 - \mathbf{x}_3\,,$$

$$\boldsymbol{\rho} = \mathbf{r}_1\,, \qquad \boldsymbol{\lambda} = \frac{1}{2}(\mathbf{r}_2 + \mathbf{r}_3)\,.$$

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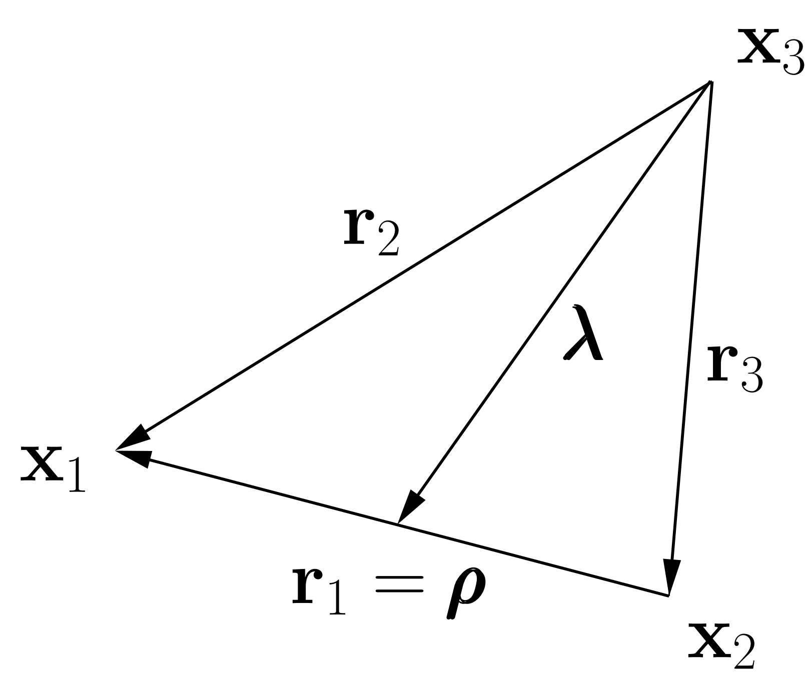

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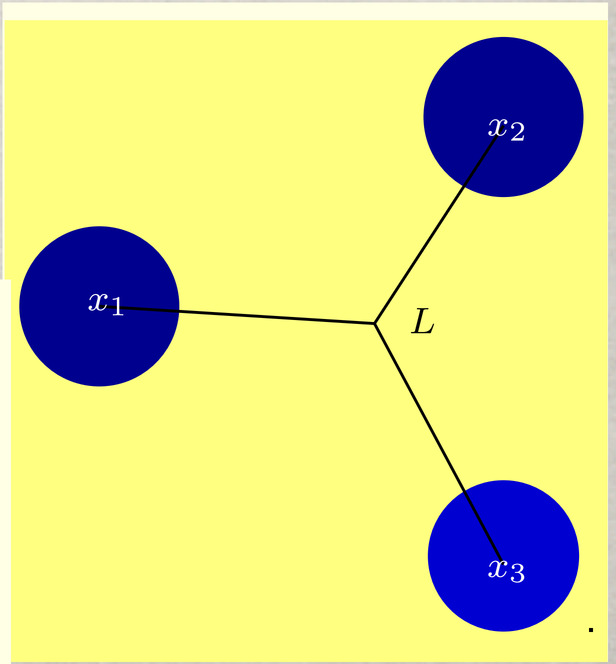
- We define L the sum of the distances of the three quarks from the Torricelli point, which has minimum distance from the quarks.



A richer dynamical structure

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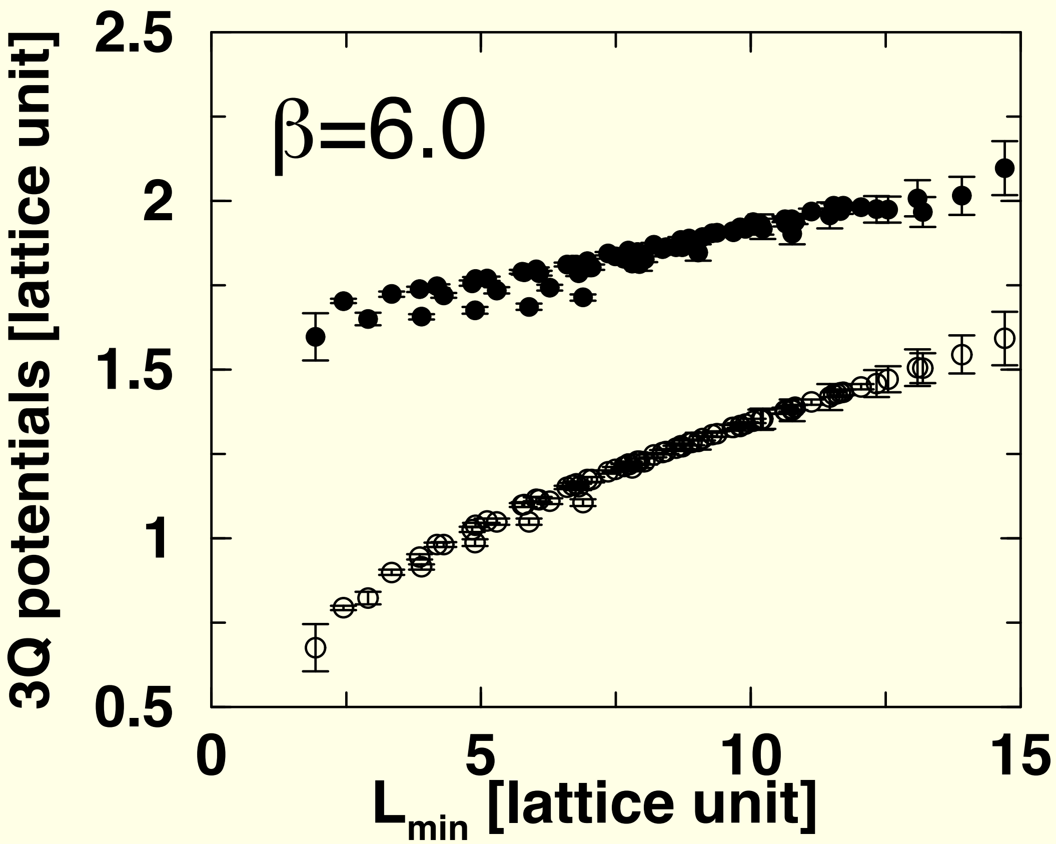
QQQ static energies on the lattice



- Ground state and first gluonic excitation.
- In the short range, Coulomb-like behavior.
- In the long range, linearly raising potential and three-body interaction depending on one length L.

- Takahashi Matsufuru Nemoto Suganuma PRL 86 (2001) 18
PRD 65 (2002) 114509, Takahashi Suganuma PRD 70 (2004) 074506

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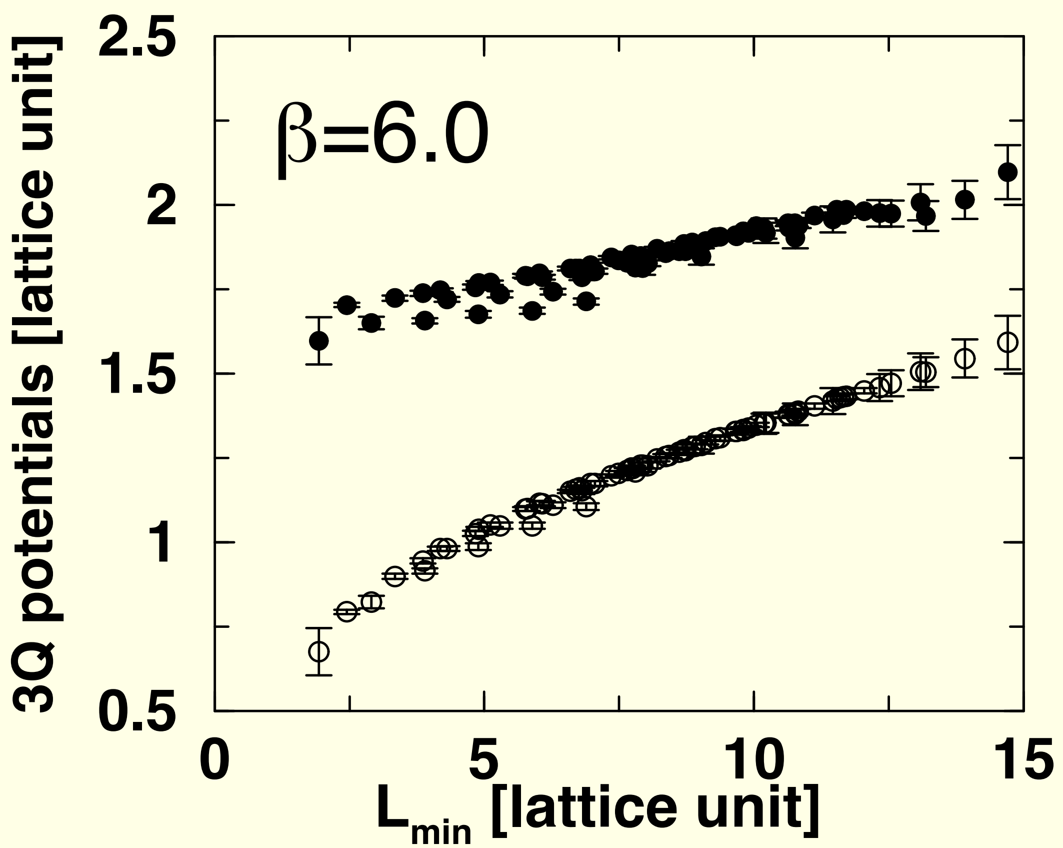

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the transition region spectacularly leads from a two body Coulomb interaction to a three body one, depending on one length only

EFT for static-static-static quarks

N. B. , T. Roesch, A. Vairo 2005, N.B. J. Ghiglieri, A. Vairo 2010

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- Consider $r_q \ll \Lambda_{\rm QCD}^{-1}$

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> - *The (weakly coupled) EFT for* QQQ *baryons contains:*
> q, *gluons*, $(QQQ)_1 = S$, $(QQQ)_8 = (O^{A1}, \ldots, O^{A8})$, $(QQQ)_8 = (O^{S1}, \ldots, O^{S8})$ *and* $(QQQ)_{10} = (\Delta^1, \ldots, \Delta^{10})$.

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In our choice, $O^{\rm S}$ and $O^{\rm A}$ are respectively symmetric and antisymmetric for exchanges of the quarks located in $\mathbf{x}_1$ and $\mathbf{x}_2$.

Since octets mix already at LO, it is useful to define: $O^a = \begin{pmatrix} O^{Aa} \\ O^{Sa} \end{pmatrix}$.

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- The EFT Lagrangian reads $\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \sum_{f=1}^{n_f} \bar{q}_f\, i\not{D} q_f + \delta\mathcal{L}^{(0)} + \cdots$

dots stand for h.o. terms in the multipole expansion.

N. B. , T. Roesch, A. Vairo 2005, N.B. J. Ghiglieri, A. Vairo 2010

pNRQCD Lagrangian

order in ($\frac{1}{m}$, multipole) expansion

$$\downarrow\downarrow$$

$$\mathcal{L}^{QQQ}_{\text{pNRQCD}} = \mathcal{L}^{(0,0)}_{\text{pNRQCD}} + \mathcal{L}^{(0,1)}_{\text{pNRQCD}} + \ldots,$$

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$$\begin{aligned}\mathcal{L}^{(0,0)}_{\text{pNRQCD}} = \int \mathrm{d}^3\rho\, \mathrm{d}^3\lambda \Big\{ & S^\dagger \left(\mathrm{i}\partial_0 - V^s\right) S + \Delta^\dagger \left(\mathrm{i}D_0 - V^\Delta\right) \Delta \\ & + O^{A\dagger} \left(\mathrm{i}D_0 - V^o_A\right) O^A + O^{S\dagger} \left(\mathrm{i}D_0 - V^o_S\right) O^S \\ & + O^{A\dagger} \left(-V^o_{AS}\right) O^S + O^{S\dagger} \left(-V^o_{AS}\right) O^A \Big\},\end{aligned}$$

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$$\begin{aligned}\mathcal{L}^{(0,1)}_{\rm pNRQCD} = \int d^3\rho\, d^3\lambda \Big\{ & V^{(0,1)}_{S\boldsymbol{\rho}\cdot\mathbf{E}O^S} \frac{g}{2\sqrt{2}} \left[S^\dagger \boldsymbol{\rho}\cdot\mathbf{E}^a O^{Sa} + O^{Sa\dagger} \boldsymbol{\rho}\cdot\mathbf{E}^a S \right] \\ & - V^{(0,1)}_{S\boldsymbol{\lambda}\cdot\mathbf{E}O^A} \frac{g}{\sqrt{6}} \left[S^\dagger \boldsymbol{\lambda}\cdot\mathbf{E}^a O^{Aa} + O^{Aa\dagger} \boldsymbol{\lambda}\cdot\mathbf{E}^a S \right] + \dots \Big\}.\end{aligned}$$

V_S V_O^A V_O^S V_Δ potentials (Wilson coefficients) to be calculated in the matching

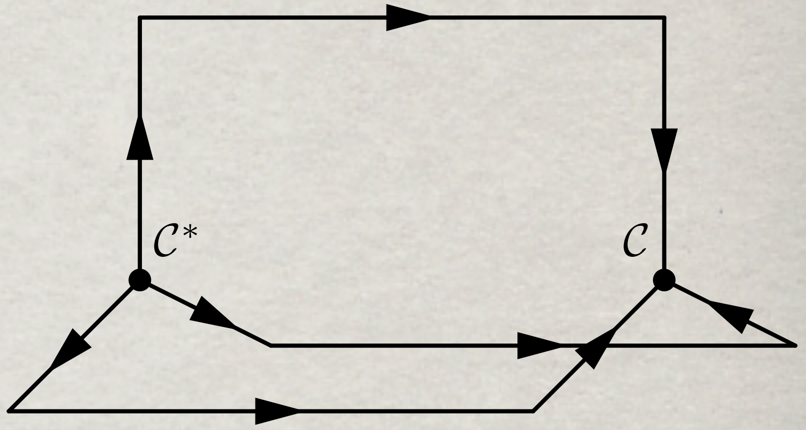
Matching the QQQ potential

up to two loops:

$$V_{\mathcal{C}}(\mathfrak{r}) = \lim_{T_W \to \infty} \frac{i}{T_W} \ln \frac{\langle 0 | \mathcal{C}^u \, W \, \mathcal{C}^{v\dagger} | 0 \rangle}{\mathcal{C}^u_{mno} \mathcal{C}^{v\dagger}_{mno}},$$

$\mathfrak{r} = \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$

representation



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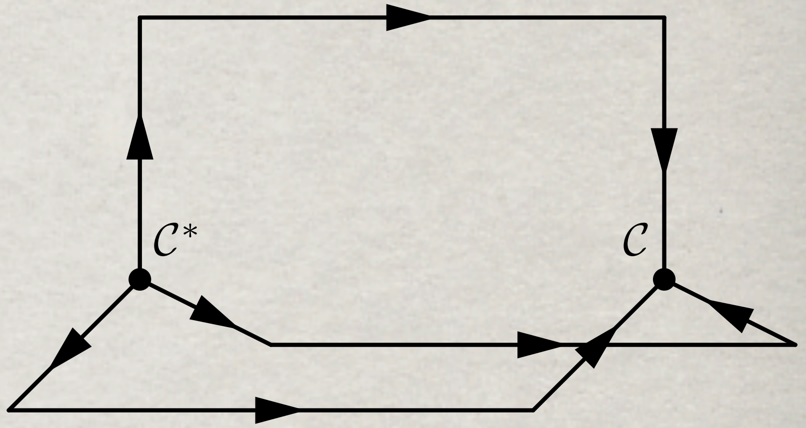

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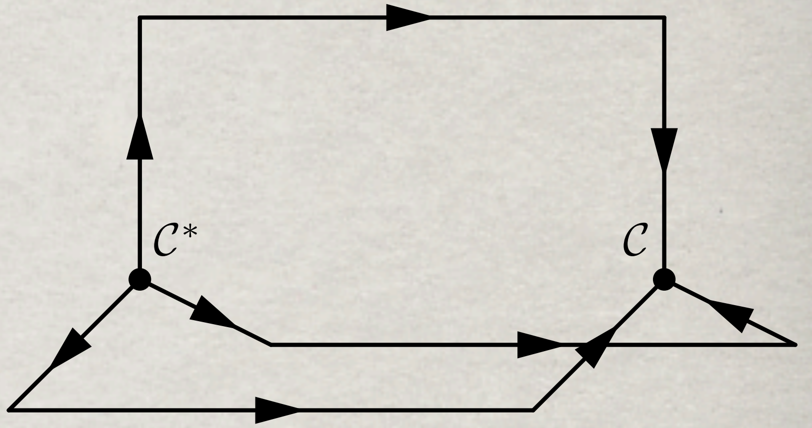

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the three body part is the part that vanishes when putting one of the quarks at infinite distance from the other two

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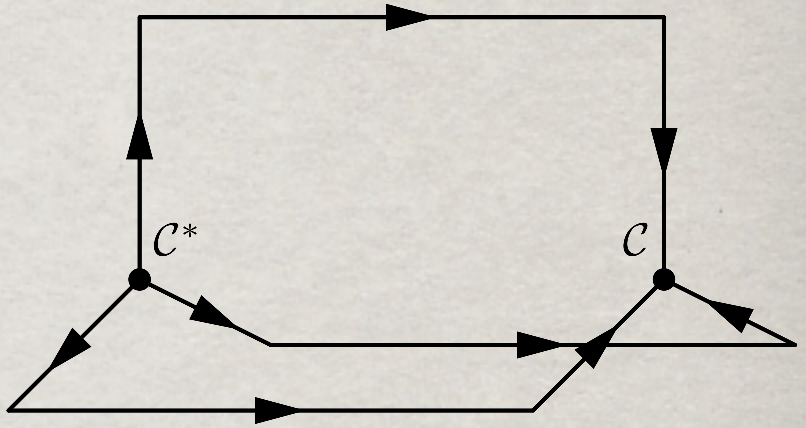

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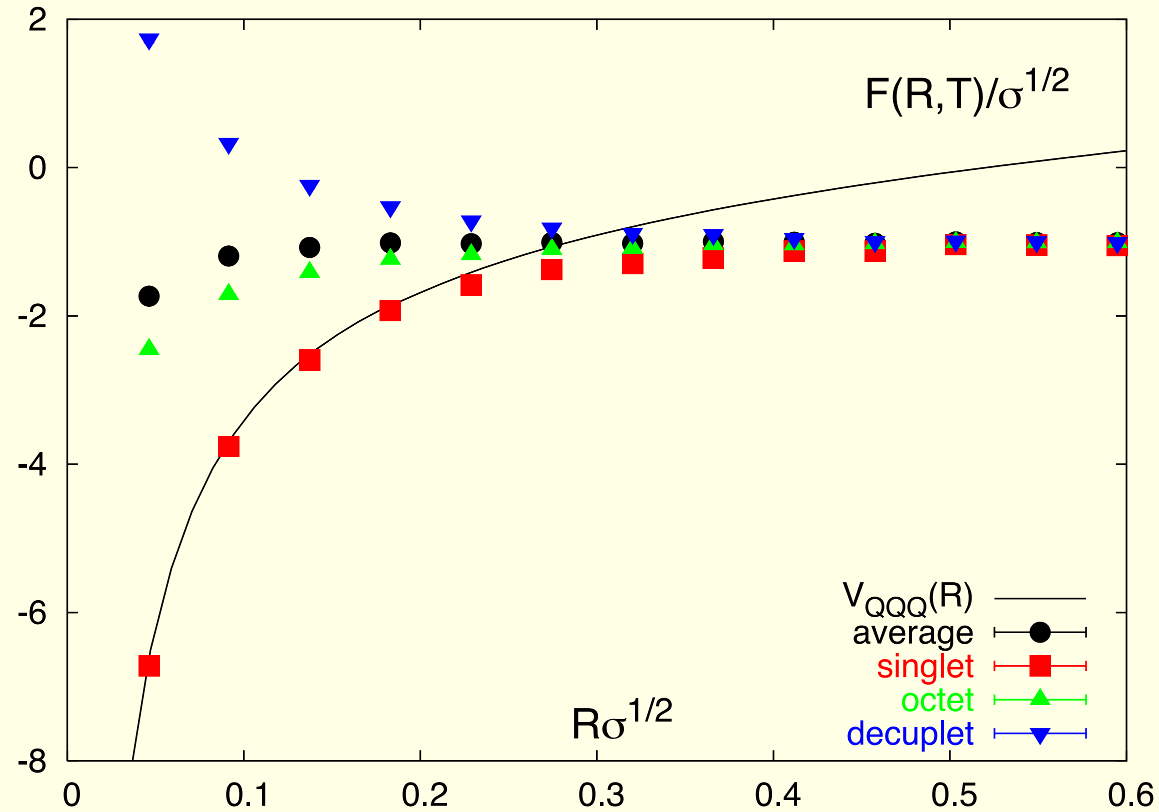
At leading order:

$$V_S^{(0)} = -\frac{2}{3}\,\alpha_s \left(\frac{1}{|\boldsymbol{r}_1|} + \frac{1}{|\boldsymbol{r}_2|} + \frac{1}{|\boldsymbol{r}_3|} \right) + \dots$$

$$V_O^{(0)} = \alpha_s \left[\frac{1}{|\boldsymbol{r}_1|} \begin{pmatrix} -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} + \frac{1}{|\boldsymbol{r}_2|} \begin{pmatrix} \frac{1}{12} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{5}{12} \end{pmatrix} + \frac{1}{|\boldsymbol{r}_3|} \begin{pmatrix} \frac{1}{12} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{5}{12} \end{pmatrix} \right] + \dots$$

$$V_\Delta^{(0)} = \frac{\alpha_s}{3} \left(\frac{1}{|\boldsymbol{r}_1|} + \frac{1}{|\boldsymbol{r}_2|} + \frac{1}{|\boldsymbol{r}_3|} \right) + \dots$$

QQQ lattice potentials in different color representations

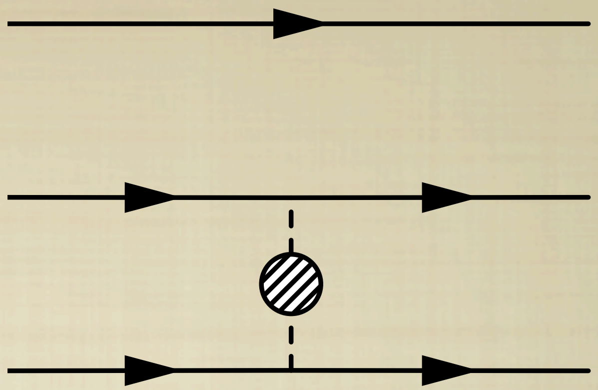


○ Hübner Karsch Kaczmarek Vogt PRD 77 (2008) 074504

- At short distances, one recovers the zero temperature potentials.
- Singlet, octet and decuplet potentials in an equilateral configuration:

$$V_S^{(0)} = -2\frac{\alpha_s}{r} + \dots, \quad V_O^{(0)} = -\frac{\alpha_s}{2r} + \dots, \quad V_\Delta^{(0)} = \frac{\alpha_s}{r} + \dots$$

QQQ potential at NLO



$$V^1_{\mathcal{C}}(\mathfrak{r}) = \sum_{i=1}^{3} f^0_q(\mathcal{C}) \frac{\alpha_{\overline{MS}}(\mathbf{r}_q)}{|\mathbf{r}_q|} \left[1 + \frac{\alpha_{\overline{MS}}(\mathbf{r}_q)}{4\pi} (2\beta_0\gamma + a_1)\right]$$

same colour factor as the LO one

$$a_1 = \frac{31}{9} C_A - \frac{20}{9} T_F n_f$$

C = singlet, octet, decuplet

at NLO QQbar and QQQ potential only differ for the overall colour representation but the effective coupling of the potential is the same

$$\alpha_V(1/|\mathbf{r}_q|) = \alpha_{\rm s}(1/|\mathbf{r}_q|) \left[1 + \frac{\alpha_{\rm s}}{4\pi} (2\beta_0\gamma_E + a_1)\right],$$

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QQQ singlet static potential at NNLO

$$
\begin{aligned}
V_S^{(0)} \;=\; & -\frac{2}{3}\sum_{q=1}^{3}\frac{\alpha_s(1/|\boldsymbol{r}_q|)}{|\boldsymbol{r}_q|}\left\{1+\frac{\alpha_s(1/|\boldsymbol{r}_q|)}{4\pi}\left[\frac{31}{3}+22\gamma_E-\left(\frac{10}{9}+\frac{4}{3}\gamma_E\right)n_f\right]\right. \\
& +\left(\frac{\alpha_s}{4\pi}\right)^2\left[66\zeta(3)+484\gamma_E^2+\frac{1976}{3}\gamma_E+\frac{3}{4}\pi^4+\frac{121}{3}\pi^2+\frac{4343}{18}\right. \\
& -\left(\frac{52}{3}\zeta(3)+\frac{176}{3}\gamma_E^2+\frac{916}{9}\gamma_E+\frac{44}{9}\pi^2+\frac{1229}{27}\right)n_f \\
& \left.\left.+\left(\frac{16}{9}\gamma_E^2+\frac{80}{27}\gamma_E+\frac{4}{27}\pi^2+\frac{100}{81}\right)n_f^2\right]\right\} \\
& +V_S^{\mathrm{3body}}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)
\end{aligned}
$$

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QQQ singlet static potential at NNLO

$$V_S^{(0)} = -\frac{2}{3}\sum_{q=1}^{3}\frac{\alpha_s(1/|\boldsymbol{r}_q|)}{|\boldsymbol{r}_q|}\left\{1+\frac{\alpha_s(1/|\boldsymbol{r}_q|)}{4\pi}\left[\frac{31}{3}+22\gamma_E-\left(\frac{10}{9}+\frac{4}{3}\gamma_E\right)n_f\right]\right.$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^2\left[66\zeta(3)+484\gamma_E^2+\frac{1976}{3}\gamma_E+\frac{3}{4}\pi^4+\frac{121}{3}\pi^2+\frac{4343}{18}\right.$$
$$-\left(\frac{52}{3}\zeta(3)+\frac{176}{3}\gamma_E^2+\frac{916}{9}\gamma_E+\frac{44}{9}\pi^2+\frac{1229}{27}\right)n_f$$
$$\left.\left.+\left(\frac{16}{9}\gamma_E^2+\frac{80}{27}\gamma_E+\frac{4}{27}\pi^2+\frac{100}{81}\right)n_f^2\right]\right\}$$
$$+V_S^{3\text{body}}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)$$

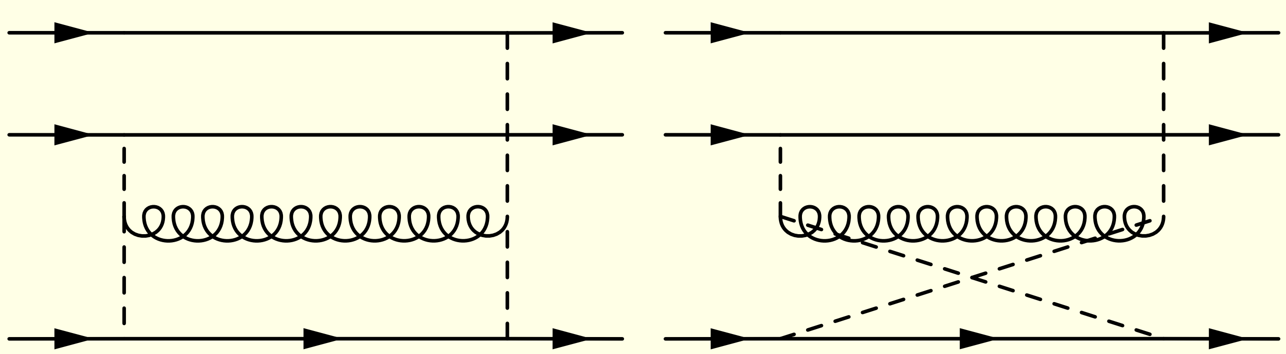
$$V_S^{3\text{body}} = -\alpha_s\left(\frac{\alpha_s}{4\pi}\right)^2\left[v(\boldsymbol{r}_2,\boldsymbol{r}_3)+v(\boldsymbol{r}_1,-\boldsymbol{r}_3)+v(-\boldsymbol{r}_2,-\boldsymbol{r}_1)\right]$$

where

$$v(\boldsymbol{\rho},\boldsymbol{\lambda}) = 16\pi\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\lambda}}\int_0^1 dx\int_0^1 dy\,\frac{1}{R}\left[\left(1-\frac{M^2}{R^2}\right)\arctan\frac{R}{M}+\frac{M}{R}\right]$$
$$+16\pi\hat{\boldsymbol{\rho}}^i\hat{\boldsymbol{\lambda}}^j\int_0^1 dx\int_0^1 dy\,\frac{\hat{\boldsymbol{R}}^i\hat{\boldsymbol{R}}^j}{R}\left[\left(1+3\frac{M^2}{R^2}\right)\arctan\frac{R}{M}-3\frac{M}{R}\right]$$

with $\boldsymbol{R}=x\boldsymbol{\rho}-y\boldsymbol{\lambda}$, $R=|\boldsymbol{R}|$ and $M=|\boldsymbol{\rho}|\sqrt{x(1-x)}+|\boldsymbol{\lambda}|\sqrt{y(1-y)}$

Relevant diagrams in Coulomb gauge:



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QQQ singlet static energy at order $O(\alpha_s^4 \ln \alpha_s)$

QQQ singlet static potential at order $O(\alpha_s^4 \ln \mu)$

$$E^s(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = V^s(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3; \mu) + \delta^s_{\rm US}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3; \mu),$$

it is sufficient to calculate the leading divergence in the ultrasoft correction: a one loop calculation in the EFT

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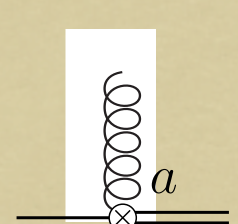
it is sufficient to calculate the leading divergence in the ultrasoft correction: a one loop calculation in the EFT

$= \theta(T)\mathrm{e}^{-iV^sT}$ (singlet propagator),

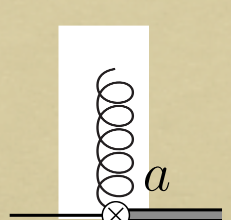
b a $= \theta(T)\mathrm{e}^{-iV^o_ST}\delta_{ab}$ (symmetric octet propagator),

b a $= \theta(T)\mathrm{e}^{-iV^o_AT}\delta_{ab}$ (antisymmetric octet propagator),

b a $=$ b a $= -iV^o_{AS}\delta_{ab}$ (octet mixing potential),



$$= ig\,\frac{1}{2\sqrt{2}}\,\boldsymbol{\rho}\cdot\mathbf{E}^a,$$



$$= -ig\,\frac{1}{\sqrt{6}}\,\boldsymbol{\lambda}\cdot\mathbf{E}^a.$$

singlet couples differently to symmetric or antisymmetric octets

QQQ singlet static potential at order $O(\alpha_s^4 \ln \mu)$

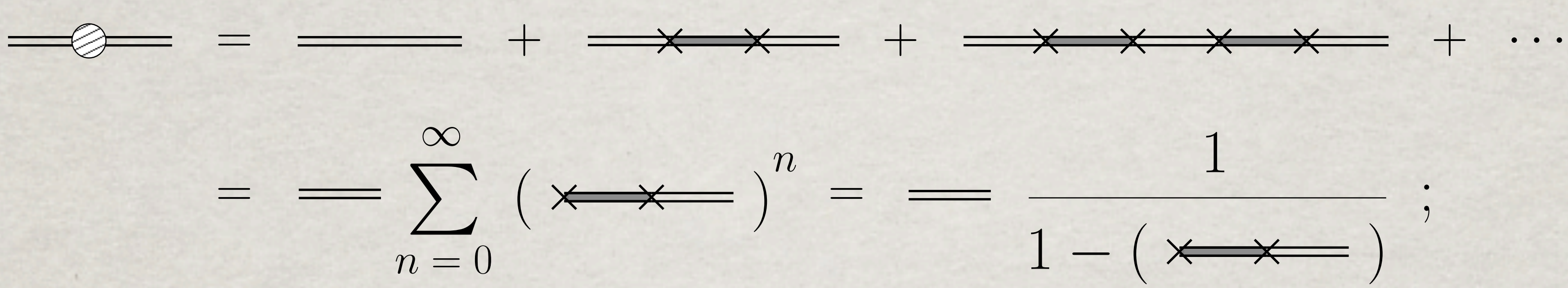
The biggest difference with respect to QQbar is that the singlet couples to two distinct octet fields and that octet fields mix

the mixing of the octet fields is of the same order of the octet energies : it must be considered to all order when computing physical octet to octet propagators

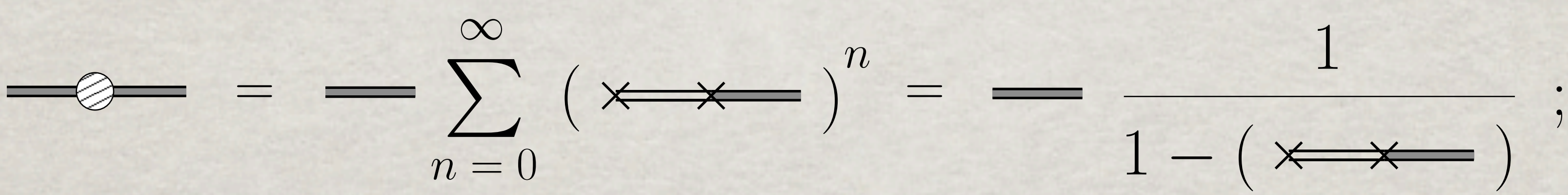
the resummation of the octet mixing potential gives rise to three different sets of resummed octet propagators

Resummed octet propagators

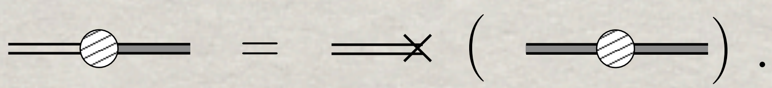
(1) a resummed octet propagator, G^o_S, that describes the propagation from a symmetric initial state to a symmetric final state:



(2) a resummed octet propagator, G^o_A, that describes the propagation from an antisymmetric initial state to an antisymmetric final state:



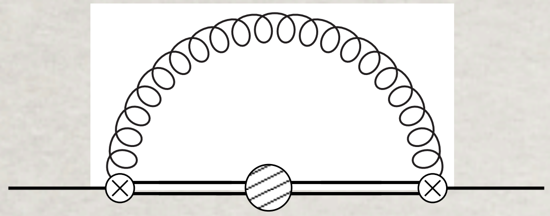
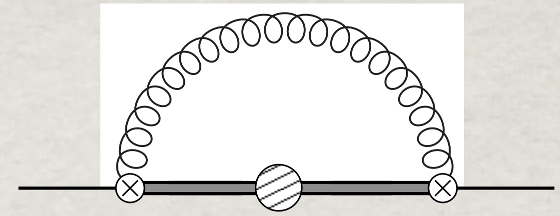
(3) a resummed octet propagator, G^o_{AS}, that describes the propagation from a symmetric initial state to an antisymmetric final state or vice versa:



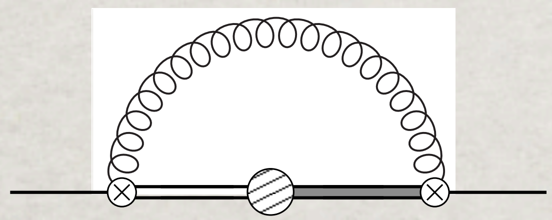
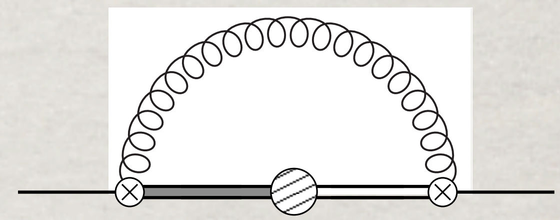
$$-i\,[G^o_S(E)]_{ab} = \frac{i\delta_{ab}(E - V^o_A)}{(E - V^o_S + i\epsilon)(E - V^o_A + i\epsilon) - (V^o_{AS})^2}\,, \quad -i\,[G^o_A(E)]_{ab} = \frac{i\delta_{ab}(E - V^o_S)}{(E - V^o_S + i\epsilon)(E - V^o_A + i\epsilon) - (V^o_{AS})^2}\,,$$

$$-i\,[G^o_{AS}(E)]_{ab} = \frac{i\delta_{ab}V^o_{AS}}{(E - V^o_S + i\epsilon)(E - V^o_A + i\epsilon) - (V^o_{AS})^2}\,,$$

Calculation of the ultrasoft contribution up to α_s^4

$\delta^s_{\rm US} =$  $+$ 

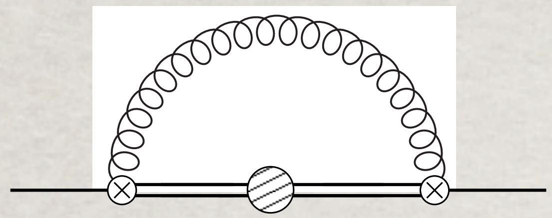
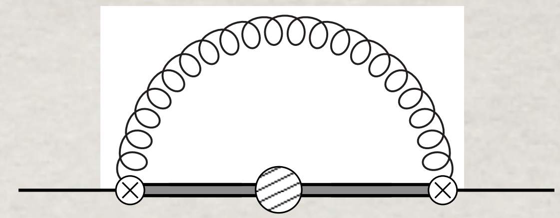
no decuplet contribution

$+$  $+$ 

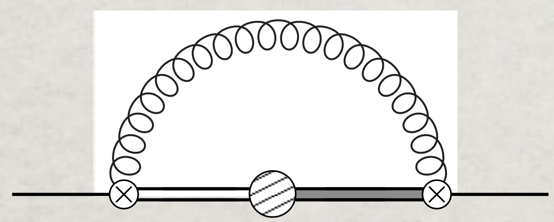
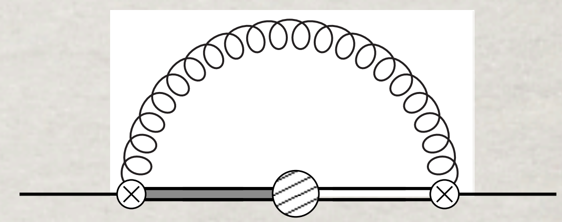
$$\delta^s_{\rm US} = -ig^2\left(\frac{1}{2\sqrt{2}}\right)^2\int_0^\infty {\rm d}t\,\frac{1}{E_1-E_2}\left[(E_1-V_A^o){\rm e}^{-it(E_1-V^s)} - (E_2-V_A^o){\rm e}^{-it(E_2-V^s)}\right]\langle \boldsymbol{\rho}\cdot\mathbf{E}^a(t)\boldsymbol{\rho}\cdot\mathbf{E}^a(0)\rangle$$
$$-ig^2\left(\frac{1}{\sqrt{6}}\right)^2\int_0^\infty {\rm d}t\,\frac{1}{E_1-E_2}\left[(E_1-V_S^o){\rm e}^{-it(E_1-V^s)} - (E_2-V_S^o){\rm e}^{-it(E_2-V^s)}\right]\langle \boldsymbol{\lambda}\cdot\mathbf{E}^a(t)\boldsymbol{\lambda}\cdot\mathbf{E}^a(0)\rangle$$
$$+2ig^2\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{6}}\int_0^\infty {\rm d}t\,\frac{V_{AS}^o}{E_1-E_2}\left[{\rm e}^{-it(E_1-V^s)} - {\rm e}^{-it(E_2-V^s)}\right]\langle \boldsymbol{\rho}\cdot\mathbf{E}^a(t)\boldsymbol{\lambda}\cdot\mathbf{E}^a(0)\rangle,$$

$$E_{1,2} = \frac{V_A^o+V_S^o}{2} \pm \sqrt{\left(\frac{V_A^o-V_S^o}{2}\right)^2 + (V_{AS}^o)^2 - i\epsilon}\,.$$

Calculation of the ultrasoft contribution up to α_s^4

$\delta^s_{\rm US} =$  $+$ 

no decuplet contribution

$+$  $+$ 

$$\delta^s_{\rm US} = \frac{4}{3}\frac{\alpha_{\rm s}}{\pi}\frac{1}{E_1 - E_2}\left[\left(\frac{|\boldsymbol{\rho}|^2}{4}(E_1 - V^o_A) + \frac{|\boldsymbol{\lambda}|^2}{3}(E_1 - V^o_S) - \frac{\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\sqrt{3}}V^o_{AS}\right)(E_1 - V^s)^3 \right.$$
$$\times\left(\frac{1}{\varepsilon} - \gamma_E - \ln\frac{(E_1 - V^s)^2}{\pi\mu^2} + \frac{5}{3}\right)$$
$$-\left(\frac{|\boldsymbol{\rho}|^2}{4}(E_2 - V^o_A) + \frac{|\boldsymbol{\lambda}|^2}{3}(E_2 - V^o_S) - \frac{\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\sqrt{3}}V^o_{AS}\right)(E_2 - V^s)^3$$
$$\left.\times\left(\frac{1}{\varepsilon} - \gamma_E - \ln\frac{(E_2 - V^s)^2}{\pi\mu^2} + \frac{5}{3}\right)\right],$$

QQQ singlet static potential at order $O(\alpha_s^4 \ln \mu)$

$$E^s(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = V^s(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3; \mu) + \delta^s_{\mathrm{US}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3; \mu),$$

The divergence and the $\alpha_s^4 \ln \mu$ in δ^s_{US} must cancel against a divergence and a term $\alpha_s^4 \ln \mu$ in the singlet static potential

QQQ singlet static potential at order $O(\alpha_s^4 \ln \mu)$

$$
\begin{aligned}
V^s(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3; \mu) = & \; V^s_{\mathrm{NNLO}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\
& -\frac{\alpha_s^4}{3\pi} \ln \mu \left[\left(\mathbf{r}_1^2 + \frac{(\mathbf{r}_2 + \mathbf{r}_3)^2}{3} \right) \left(\frac{1}{|\mathbf{r}_1|^2} + \frac{1}{|\mathbf{r}_2|^2} + \frac{1}{|\mathbf{r}_3|^2} - \frac{1}{4} \frac{|\mathbf{r}_1| + |\mathbf{r}_2| + |\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|} \right) \right. \\
& \qquad \times \left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|} + \frac{1}{|\mathbf{r}_3|} \right) \\
& + \left(\mathbf{r}_1^2 - \frac{(\mathbf{r}_2 + \mathbf{r}_3)^2}{3} \right) \left(\frac{1}{|\mathbf{r}_1|^2} + \frac{1}{|\mathbf{r}_2|^2} + \frac{1}{|\mathbf{r}_3|^2} + \frac{5}{4} \frac{|\mathbf{r}_1| + |\mathbf{r}_2| + |\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|} \right) \\
& \qquad \times \left(\frac{1}{|\mathbf{r}_1|} - \frac{1}{2|\mathbf{r}_2|} - \frac{1}{2|\mathbf{r}_3|} \right) \\
& + \mathbf{r}_1 \cdot (\mathbf{r}_2 + \mathbf{r}_3) \left(\frac{1}{|\mathbf{r}_1|^2} + \frac{1}{|\mathbf{r}_2|^2} + \frac{1}{|\mathbf{r}_3|^2} + \frac{5}{4} \frac{|\mathbf{r}_1| + |\mathbf{r}_2| + |\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|} \right) \\
& \qquad \left. \times \left(\frac{1}{|\mathbf{r}_2|} - \frac{1}{|\mathbf{r}_3|} \right) \right] .
\end{aligned}
$$

the new term proportional to $\alpha_s^4 \ln \mu$ that we have added is a genuine three body potential

QQQ singlet static energy at order $O(\alpha_s^4 \ln \alpha_s)$

summing the potential and the US contribution

$$
\begin{aligned}
E^s(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) \;=\; & V^s_{\rm NNLO}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) \\
& -\frac{\alpha_s^4}{3\pi}\ln\alpha_s \left[\left(\mathbf{r}_1^2+\frac{(\mathbf{r}_2+\mathbf{r}_3)^2}{3}\right)\left(\frac{1}{|\mathbf{r}_1|^2}+\frac{1}{|\mathbf{r}_2|^2}+\frac{1}{|\mathbf{r}_3|^2}-\frac{1}{4}\frac{|\mathbf{r}_1|+|\mathbf{r}_2|+|\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|}\right.\right. \\
& \qquad \left.\times\left(\frac{1}{|\mathbf{r}_1|}+\frac{1}{|\mathbf{r}_2|}+\frac{1}{|\mathbf{r}_3|}\right)\right) \\
& \qquad +\left(\mathbf{r}_1^2-\frac{(\mathbf{r}_2+\mathbf{r}_3)^2}{3}\right)\left(\frac{1}{|\mathbf{r}_1|^2}+\frac{1}{|\mathbf{r}_2|^2}+\frac{1}{|\mathbf{r}_3|^2}+\frac{5}{4}\frac{|\mathbf{r}_1|+|\mathbf{r}_2|+|\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|}\right. \\
& \qquad \left.\times\left(\frac{1}{|\mathbf{r}_1|}-\frac{1}{2|\mathbf{r}_2|}-\frac{1}{2|\mathbf{r}_3|}\right)\right) \\
& \qquad +\mathbf{r}_1\cdot(\mathbf{r}_2+\mathbf{r}_3)\left(\frac{1}{|\mathbf{r}_1|^2}+\frac{1}{|\mathbf{r}_2|^2}+\frac{1}{|\mathbf{r}_3|^2}+\frac{5}{4}\frac{|\mathbf{r}_1|+|\mathbf{r}_2|+|\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|}\right. \\
& \qquad \left.\left.\times\left(\frac{1}{|\mathbf{r}_2|}-\frac{1}{|\mathbf{r}_3|}\right)\right)\right]
\end{aligned}
$$

The logarithm of α_s signals that an ultraviolet divergence from the US scale has canceled against an infrared divergence from the soft scale.

Renormalization Group improvement of the singlet static potential in an equilateral geometry

The US logs that start appearing in the potential at N^3LO may be resummed using RG equation. These are a set of eqs. that describe the scale dependence of the static potentials in different color representations: they follow by requiring that the static energies are independent of the renormalization scheme

The potentials in different color representations mix under renormalization

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$$\mu\, \mathrm{d}V^s/\mathrm{d}\mu = -\mu\, \mathrm{d}\delta^s_{\mathrm{US}}/\mathrm{d}\mu$$

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$$\mu\, \mathrm{d}V^s/\mathrm{d}\mu = -\mu\, \mathrm{d}\delta^s_{\mathrm{US}}/\mathrm{d}\mu$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}V^s = -\frac{8}{3}\frac{\alpha_s}{\pi}\left\{\left[\frac{V_S^o - V_A^o}{2}\left(\frac{|\boldsymbol{\rho}|^2}{4} - \frac{|\boldsymbol{\lambda}|^2}{3}\right) - V_{AS}^o\frac{\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\sqrt{3}}\right]\right.$$
$$\times\left[3\left(\frac{V_S^o+V_A^o}{2} - V^s\right)^2 + \frac{(V_S^o - V_A^o)^2}{4} + (V_{AS}^o)^2\right]$$
$$\left. + \left(\frac{V_S^o+V_A^o}{2} - V^s\right)\left(\frac{|\boldsymbol{\rho}|^2}{4} + \frac{|\boldsymbol{\lambda}|^2}{3}\right) \times\left[\left(\frac{V_S^o+V_A^o}{2} - V^s\right)^2 + 3\frac{(V_S^o - V_A^o)^2}{4} + 3(V_{AS}^o)^2\right]\right\}$$

Renormalization Group improvement of the singlet static potential in an equilateral geometry

We can solve singlet, octet, decuplet potential RG coupled equations. There is however a difference with respect to the QQbar case. In the QQbar case there is only one length r, in the QQQ case we have more than one length. For a general three-body geometry logs corrections in the US scale can be as important as finite logs involving ratios among the different lengths. The calculation of this finite terms requires the calculation of the QQQ Wilson loop. however these logs are not important if the length scales are similar.

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We work in the equilateral geometry $|\mathbf{r}_1| = |\mathbf{r}_2| = |\mathbf{r}_3| = r.$

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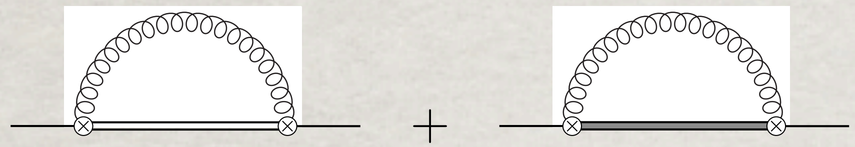
We calculate US contribution for the decuplet and the octet and obtain the corresponding RG equations

Renormalization Group improvement of the singlet static potential in an equilateral geometry

$\delta^s_{\mathrm{US}} =$



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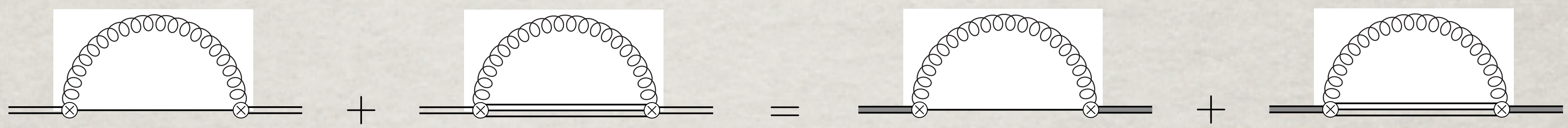

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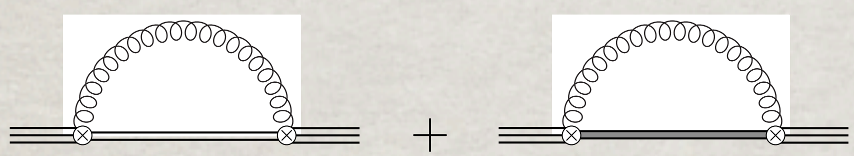
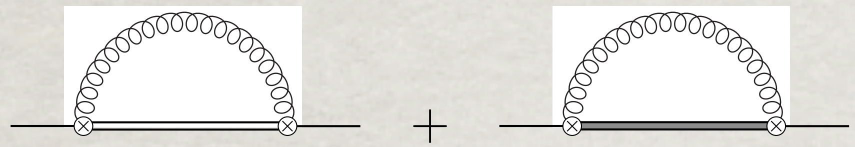
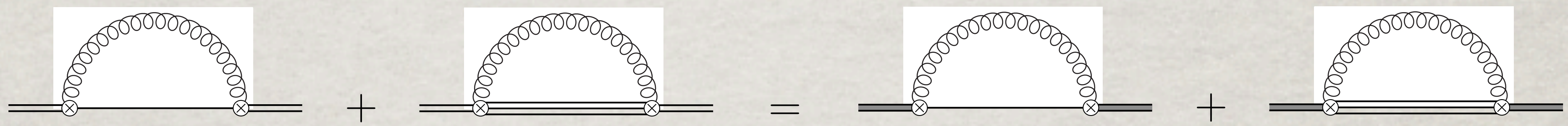
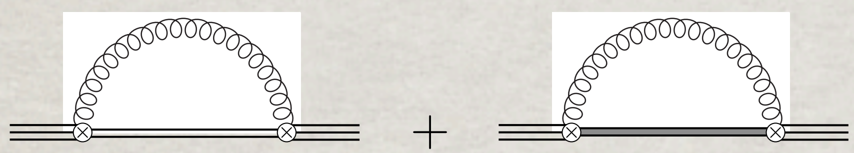


Renormalization Group improvement of the singlet static potential in an equilateral geometry

$\delta^s_{\rm US} =$ 

$\delta^o_{\rm US} =$ 

$\delta^d_{\rm US} =$ 

$$\begin{cases} \mu \dfrac{\rm d}{{\rm d}\mu} V^s = -\dfrac{4}{3\pi}\alpha_{\rm s} r^2 (V^o - V^s)^3 + \mathcal{O}(\alpha_{\rm s}^5) \\ \mu \dfrac{\rm d}{{\rm d}\mu} V^o = \dfrac{1}{12\pi}\alpha_{\rm s} r^2 \left[(V^o - V^s)^3 + 5(V^o - V^d)^3\right] + \mathcal{O}(\alpha_{\rm s}^5) \\ \mu \dfrac{\rm d}{{\rm d}\mu} V^d = -\dfrac{2}{3\pi}\alpha_{\rm s} r^2 (V^o - V^d)^3 + \mathcal{O}(\alpha_{\rm s}^5) \\ \mu \dfrac{\rm d}{{\rm d}\mu} \alpha_{\rm s} = \alpha_{\rm s}\beta(\alpha_{\rm s}) \end{cases} .$$

RG coupled eqs

Renormalization Group improvement of the singlet static potential in an equilateral geometry

solutions of the equations

$$V^s(r;\mu) = V^s_{\rm NNLO}(r) - 9\frac{\alpha_s^3(1/r)}{\beta_0 r}\ln\frac{\alpha_s(1/r)}{\alpha_s(\mu)}\,,$$

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provides the singlet static potentials at NNLL accuracy in the equilateral geometry

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RG coupled eqs

Conclusions

We have computed the QQQ singlet static potential at order $\alpha_s^4 \ln \mu$ and the singlet static energy at order $\alpha_s^4 \ln \alpha_s$

These are the most accurate determinations of the QQQ singlet static and energy in perturbative QCD

The new contribution to the potential is a three body interaction and together with the three body interaction at two loop order may provide new insight on the emergence of a long range three body interaction governed by only one fundamental length

In the special situation where the quarks are located at the corners of an equilateral triangle we have solved the RG eqs at NNLL accuracy obtaining the expression for the QQQ singlet static potential at NNLL accuracy

Backup

Let us consider some simple geometries

Isosceles geometry in a plane $|\mathbf{r}_2| = |\mathbf{r}_3| = r$ and $\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{r}}_3 = \cos\theta$.

$$V^{\rm tot}_{\mathcal{H}\mathcal{C}}(r,\theta) = f_{\mathcal{H}}(\mathcal{C})\alpha_{\rm s}^3 \frac{c_{\mathcal{H}}(\theta)}{r}.$$

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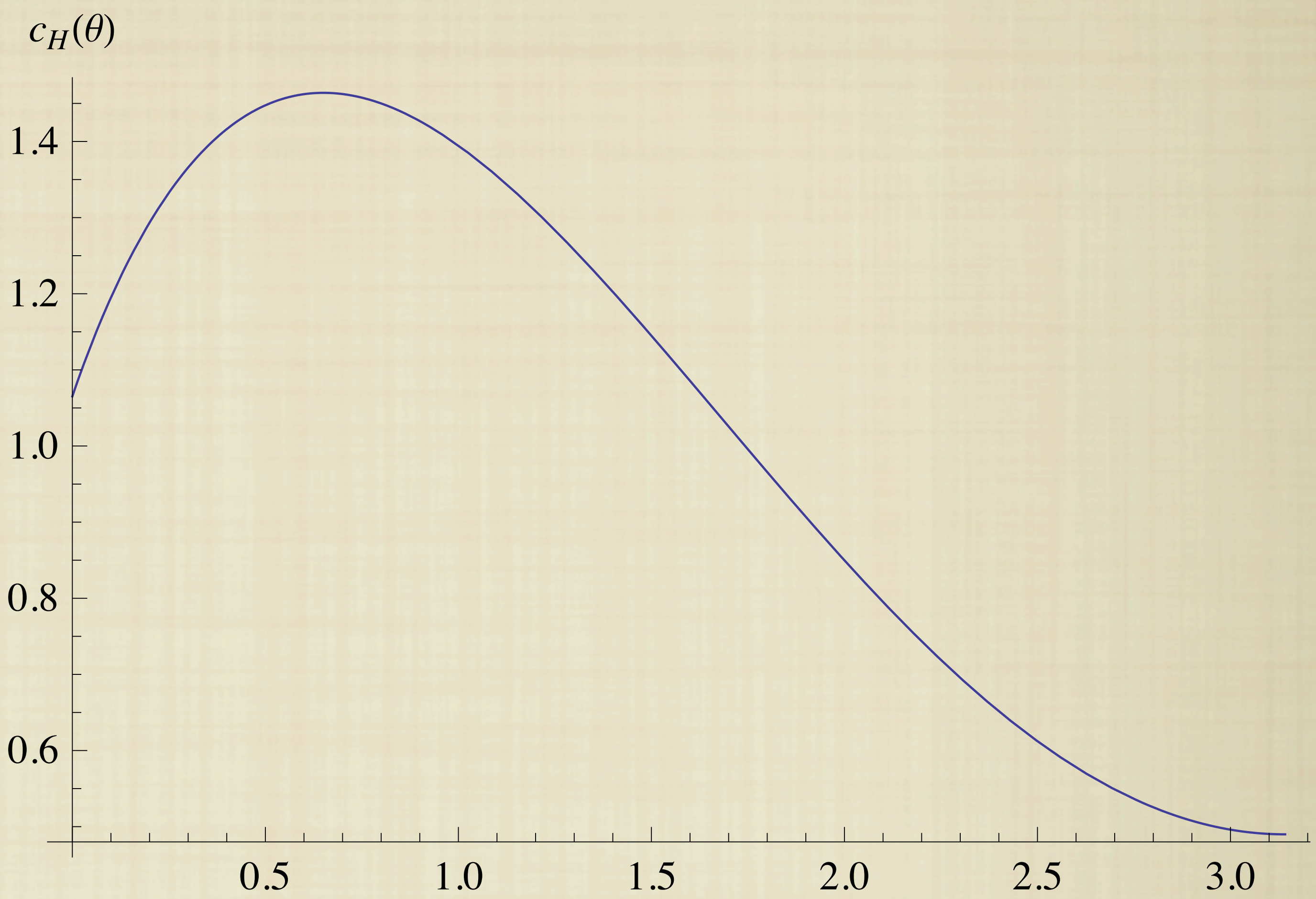

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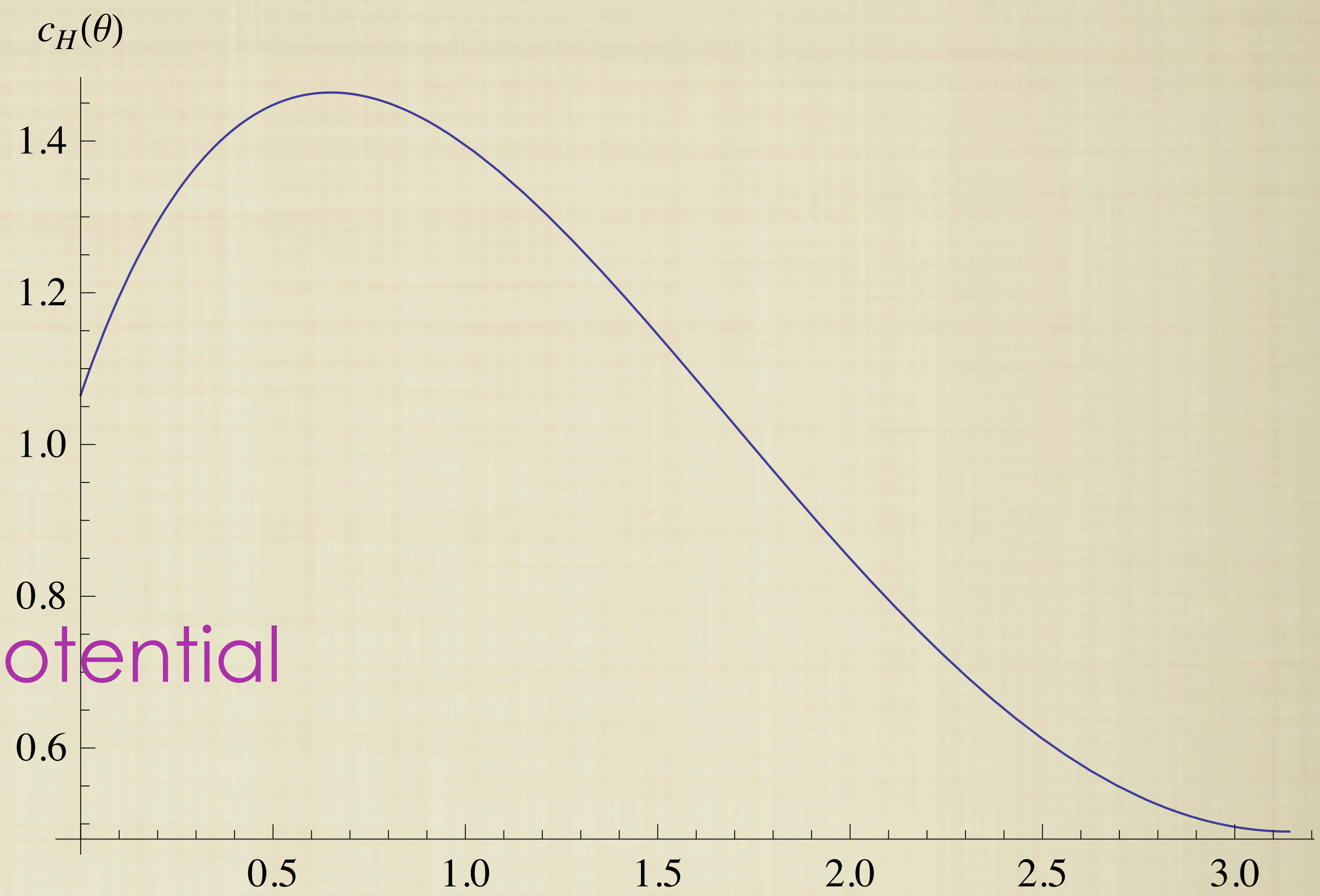

attractive contribution to the potential

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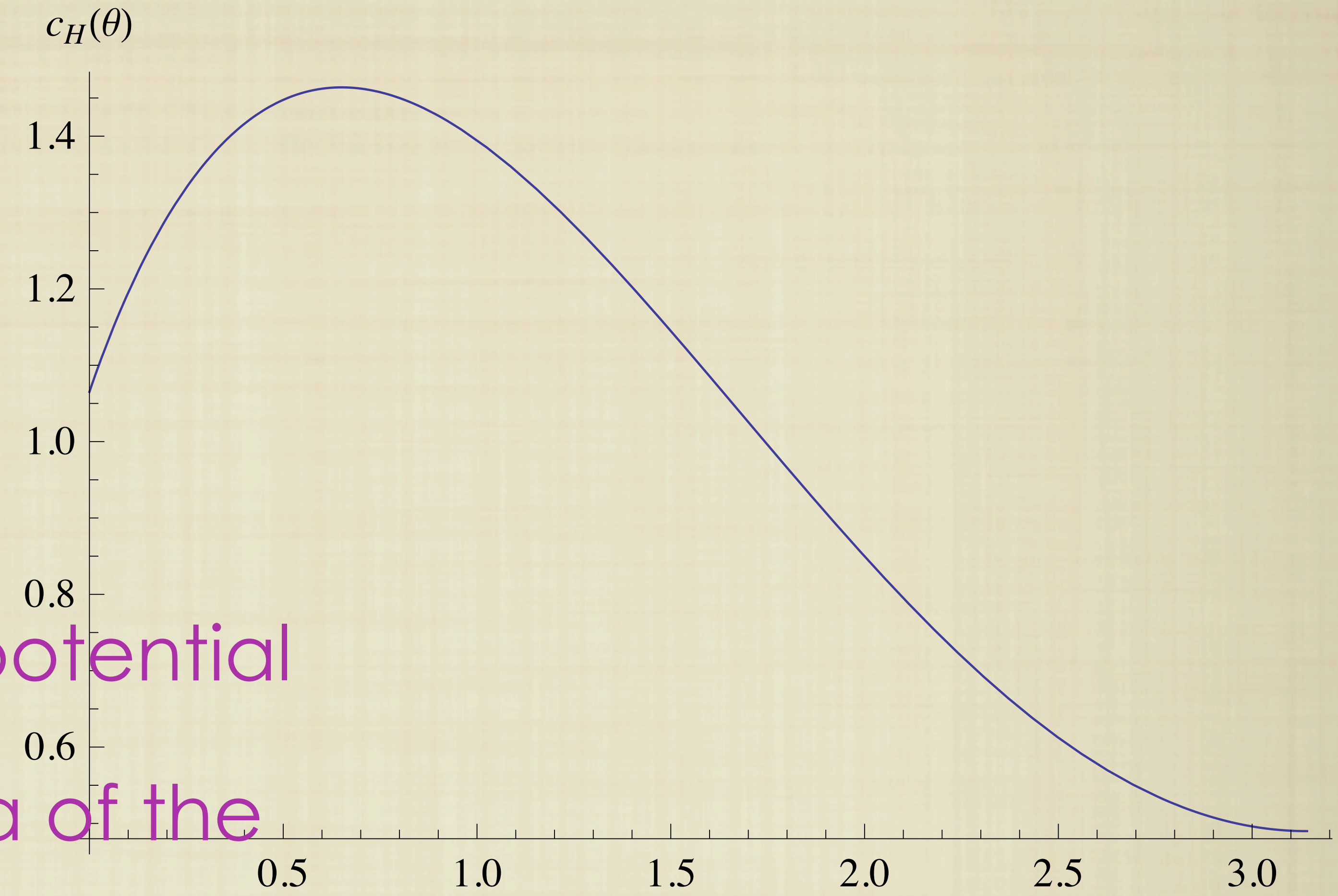

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weak dependence on theta of the 3body potential

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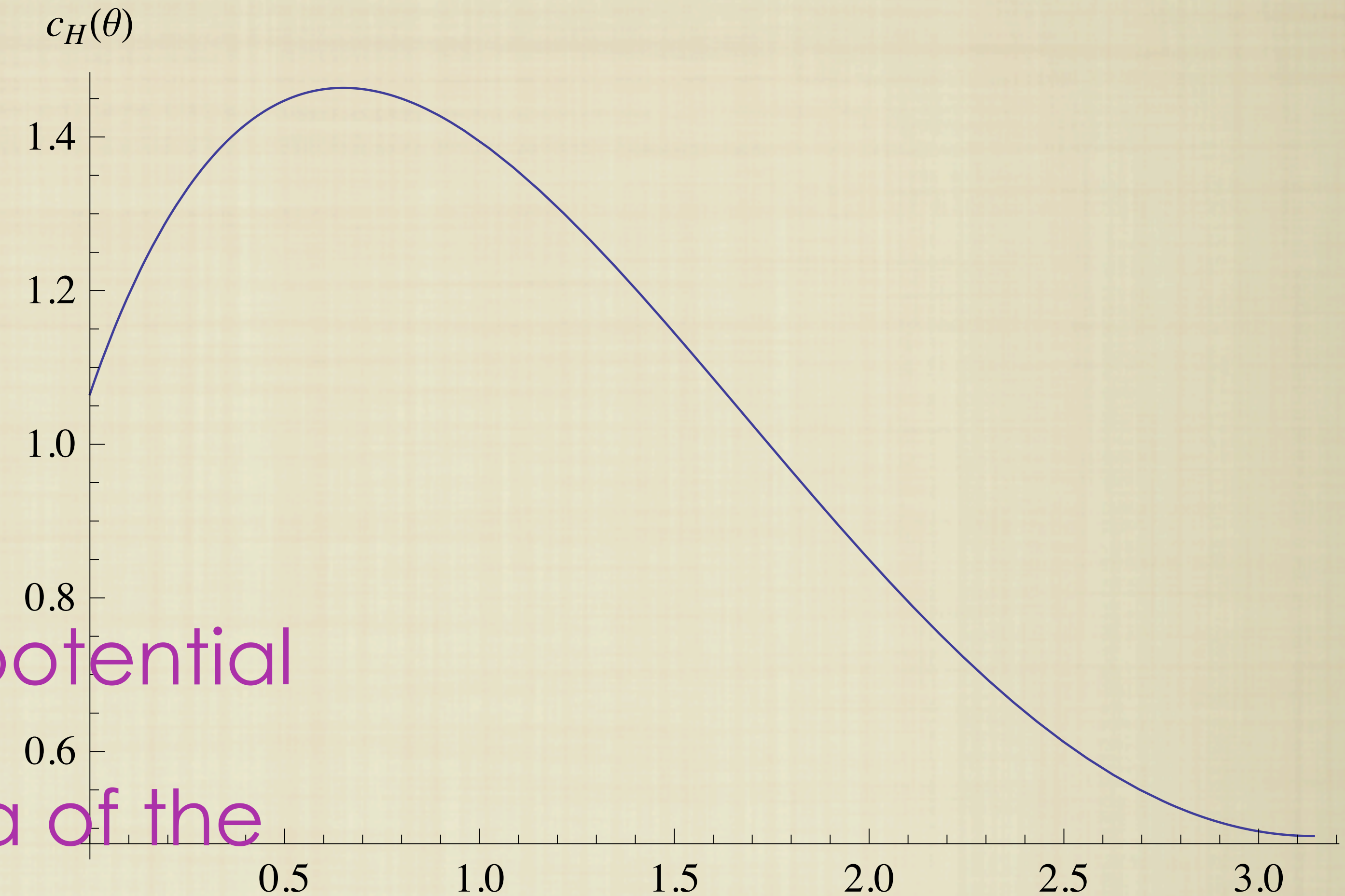

attractive contribution to the potential

weak dependence on theta of the 3body potential

may indicate the onset of a smooth transition towards the long distance Y shaped three body potential seen in the lattice data?

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$\theta = \pi/3$: *planar equilateral geometry*

In the equilateral case, we have $c_{\mathcal{H}}(\pi/3) \approx 1.377$.

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We can compare the relative magnitude of the three-body contribution to the tree level potential. For the singlet

$$\frac{V^{\rm tot}_{\mathcal{H}\,s}(r)}{V^{(0)}_s(r)} = \frac{c_{\mathcal{H}}(\pi/3)}{4}\alpha_{\rm s}^2(1/r) \approx \frac{\alpha_{\rm s}^2(1/r)}{2.90},$$

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using α_{s} at one loop, $V_{\mathcal{H}\,s}^{\mathrm{tot}}(r)$ may become as large as one sixth of the tree-level Coulomb potential in the region around 0.3 fm, where, at least in the $Q\overline{Q}$ case, perturbation theory

still holds

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Generic geometry

In the most general geometry the three body potential depends on two coordinates, we may choose one of them to be L_min, leaving the other not specified

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(B.1) Planar lattice geometry with two fixed quarks

In Fig 10, we plot the three-body potential obtained by placing the three quarks in a plane (x, y), fixing the position of the first quark in $(0, 0)$, the second one in $(1, 0)$ and moving the third one in the lattice $(0.5+0.125\, n_x, 0.125\, n_y)$ with $n_x \in \{0, 1, ..., 20\}$ and $n_y \in \{0, 1, ..., 24\}$. The plot clearly shows the dependence on the geometry at fixed L, however, the dependence is weaker than in the two-body case.

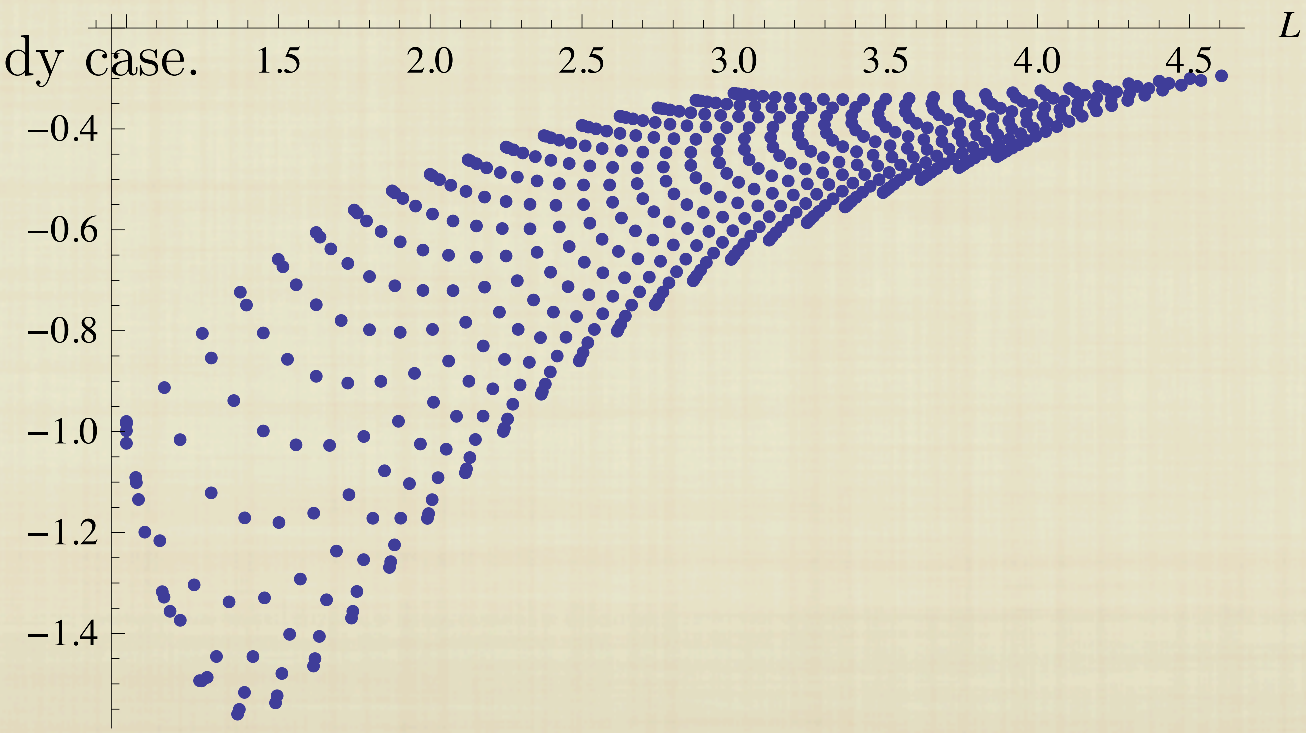

FIG. 10: The normalized three-body potential, $V^{\rm tot}_{\mathcal{HC}}(L, ...)/(-f_{\mathcal{H}}(\mathcal{C})\alpha_{\rm s}^3)$, plotted as function of L

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Three-dimensional lattice geometry with the three quarks moving along the axes

[28] T. T. Takahashi and H. Suganuma, Phys. Rev. **D70**, 074506 (2004), hep-lat/0409105.

In the lattice calculation of Ref. [28], the three quarks were located along the axes of a three-dimensional lattice, namely at $(n_x, 0, 0)$, $(0, n_y, 0)$ and $(0, 0, n_z)$, with $n_x \in \{0, 1, ..., 6\}$ and $n_y, n_z \in \{1, ..., 6\}$. For the sake of comparison, we consider the same geometry and plot the corresponding three-body potential in Fig. 11. The plot shows a weak dependence on the geometry: much weaker than in the two-body case, but also somewhat weaker than in the geometry considered in (B.1).

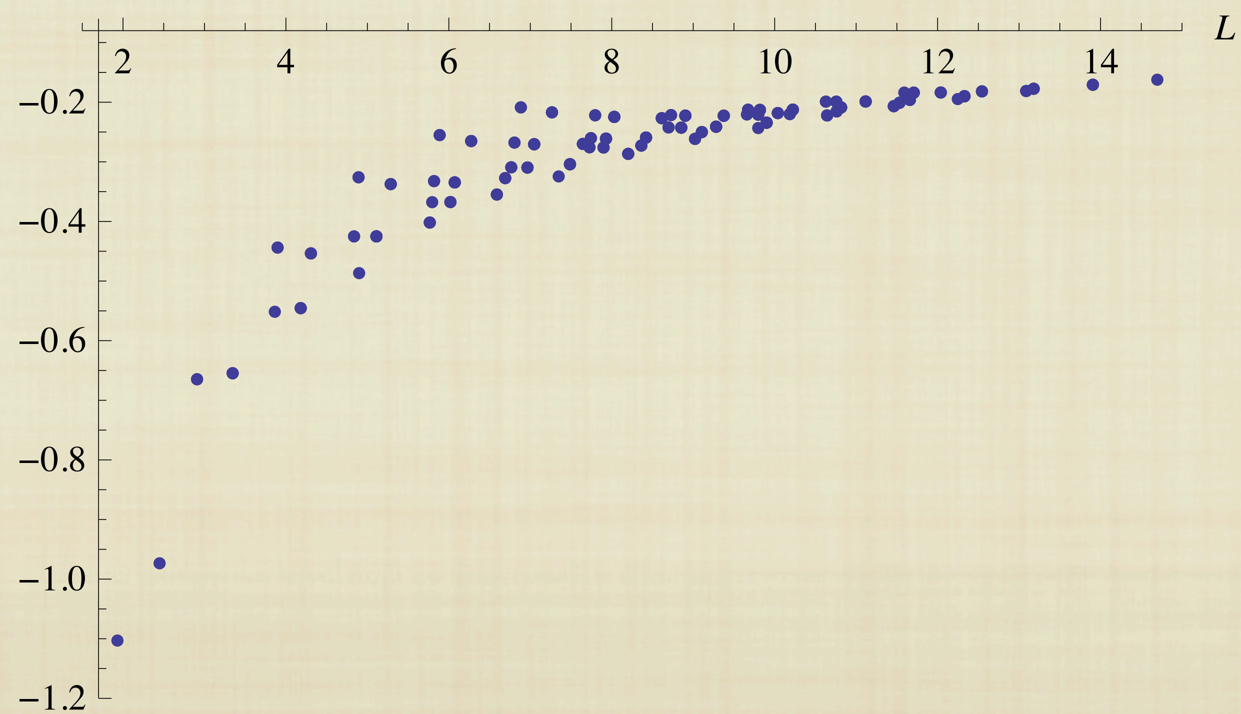
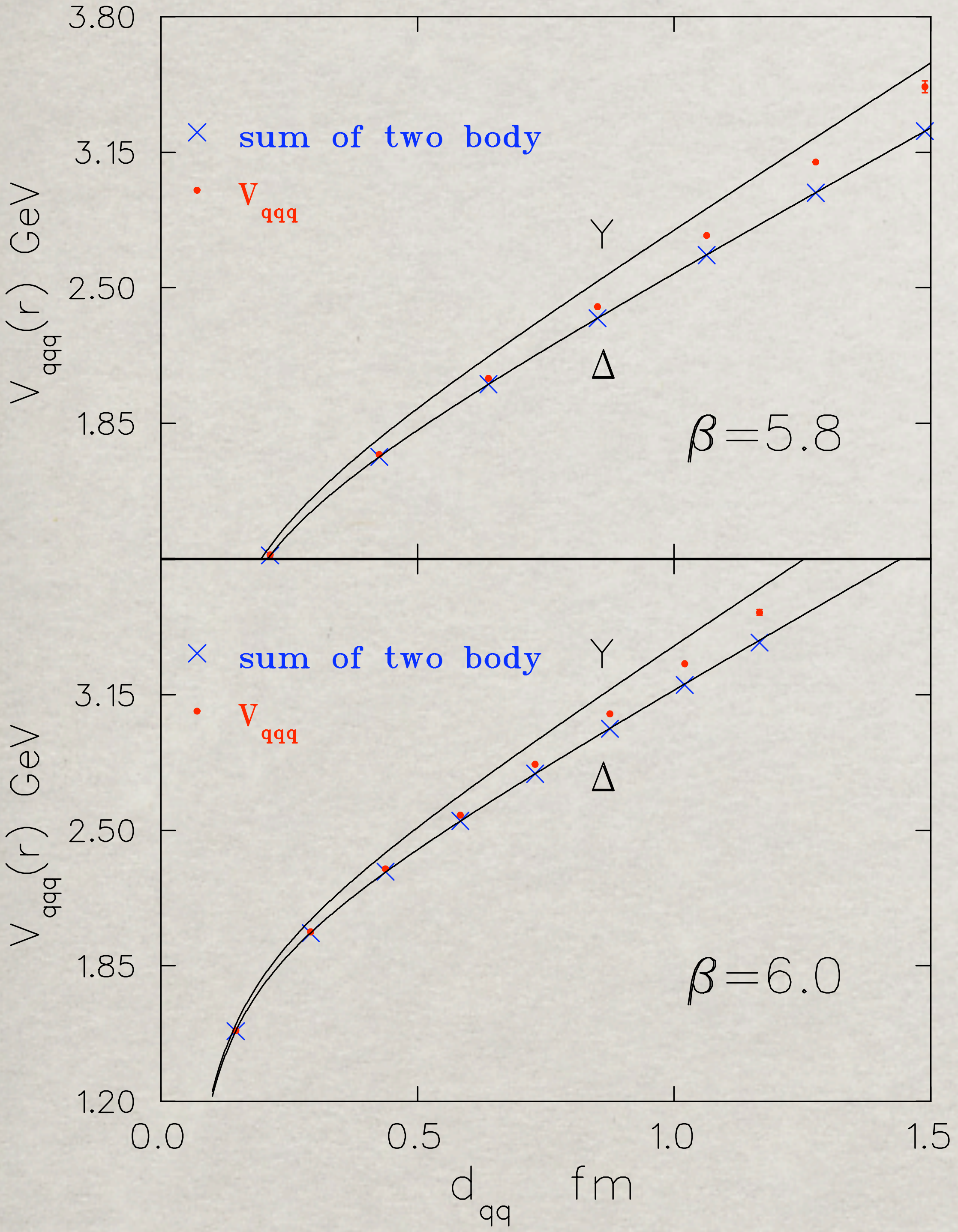


FIG. 11: The normalized three-body potential, $V^{\rm tot}_{\mathcal{HC}}(L, ...)/(-f_{\mathcal{H}}(\mathcal{C})\alpha_s^3)$, plotted as function of L

The precise behaviour of the QQQ potential is still object of investigation on the lattice



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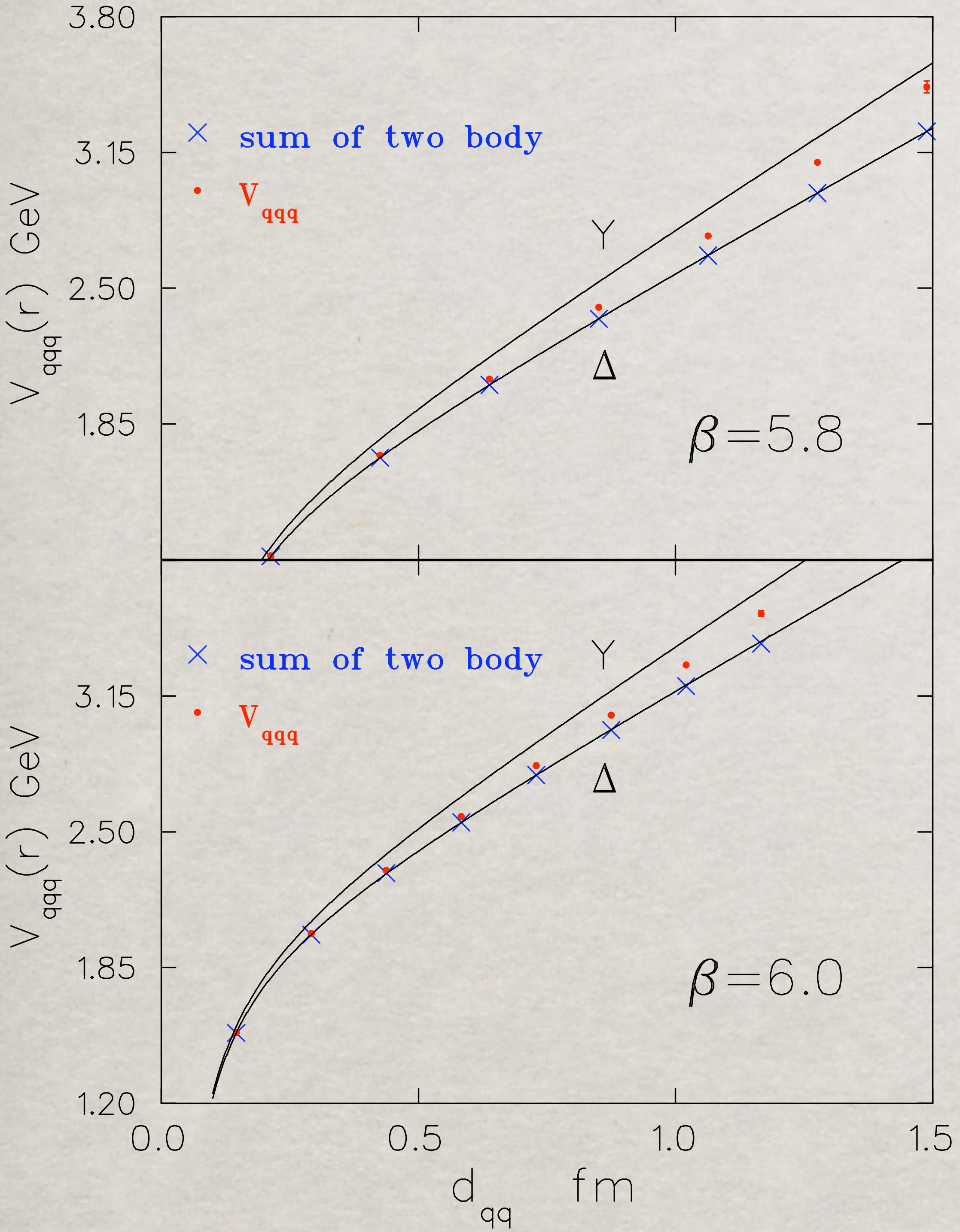

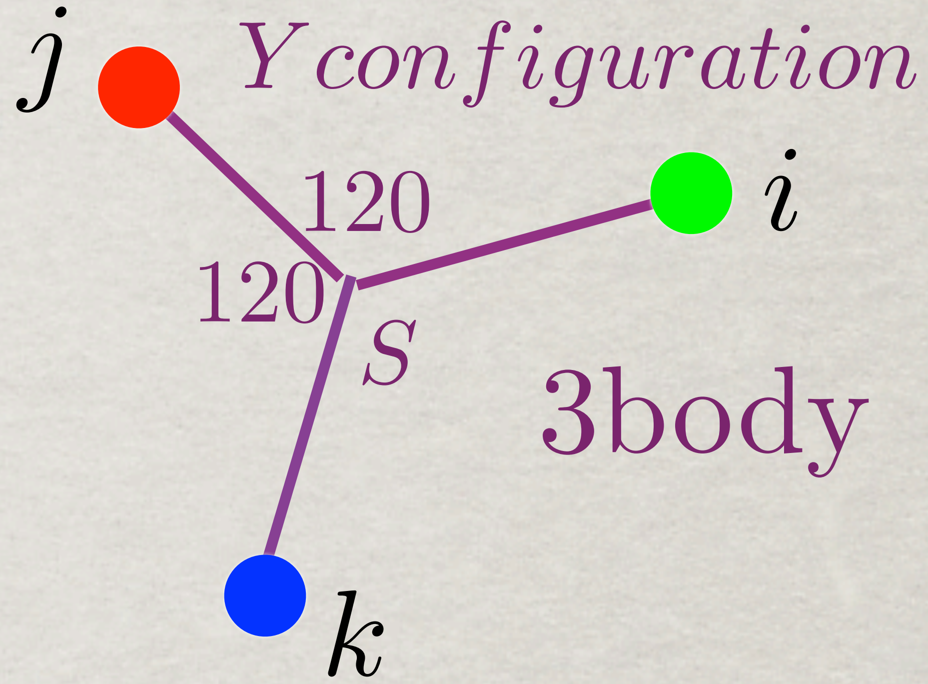

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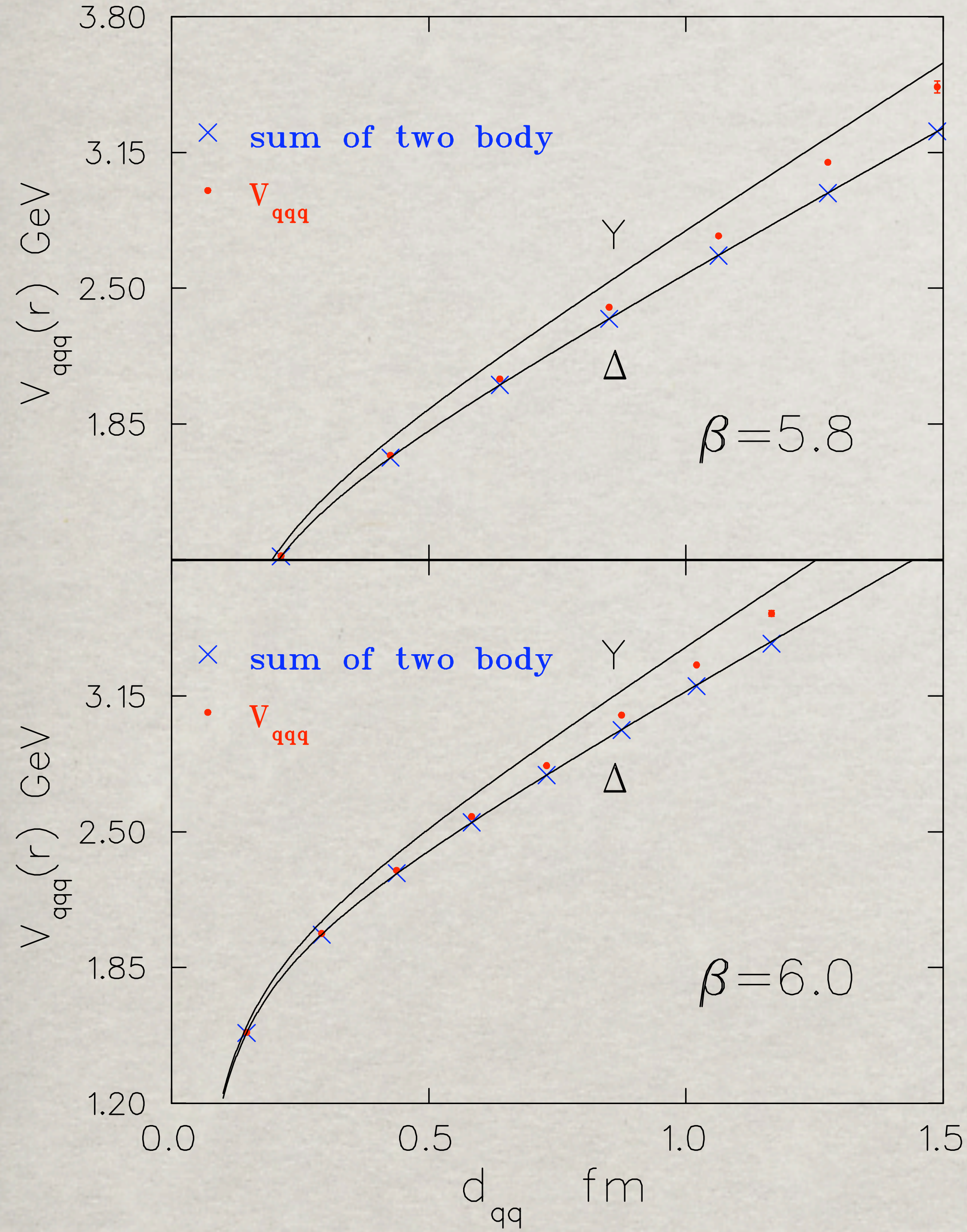

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