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Witness the High-Dimensional Quantum Steering via Majorization Lattice

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M.-C. Yang, C.-F. Qiao, arXiv:2507.20950

Outline

Outline

A brief history of quantum steering

EPR argument and quantum steering

- EPR argument and origin of quantum steering

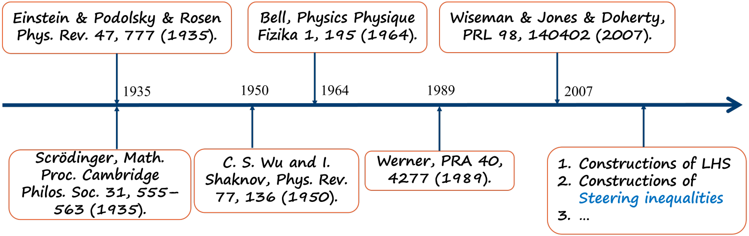


Figure. Origin of quantum steering

- Quantum steering: One party's (Alice) ability to influence another (Bob) remote quantum state by one's choice of local measurements.

$$|\psi\rangle = \frac{|0_{Alice}1_{Bob}\rangle - |1_{Alice}0_{Bob}\rangle}{\sqrt{2}} . \tag{1}$$

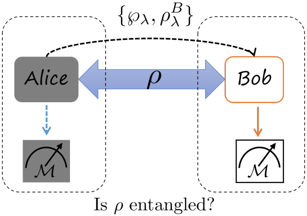
Local hidden state (LHS) model and quantum steering

- Assuming that Alice and Bob share the bipartite quantum state ρ, quantum theory predicts the joint probability

$$p(a_i, b_j|A, B; \rho) = \mathrm{Tr}[(\Pi_i^A \otimes \Pi_j^B)\rho] \ . \tag{2}$$

- LHS model requires the bipartite joint probability

$$\begin{aligned} p(a_i, b_j|A, B; \rho) &= \sum_\lambda \wp(a_i|A, \lambda)\mathrm{Tr}[\Pi_j^B \rho_\lambda^B]\wp_\lambda \\ &= \mathrm{Tr}[\Pi_j^B \rho_{a_i|A}] \ . \end{aligned}$$



- Assemblage $\{\rho_{a_i|A}\}$

$$\rho_{a_i|A} = \sum_\lambda \wp_\lambda \wp(a_i|A, \lambda)\rho_\lambda^B \ . \tag{3}$$

H. M. Wiseman et al., PRL (2007). R. Uola et al., RMP (2020, Review).

A brief history of quantum steering

LHS	Note	Steering Ineq.	Note
R. F. Werner, PRA (1989)	PVM	M. D. Reid, PRA (1989)	CV system
H. M. Wiseman et al., PRL (2007) M. L. Almeida et al., PRL (2007)	PVM PVM	S. J. Jones et al., PRA (2007)	
		E. G. Cavalcanti et al., PRA (2009)	
D. Cavalcanti et al., PRL (2016) P. Hirsch et al., PRL (2016)	General	D. J. Saunders et al., Nat. Phy. (2010)	Qubit, Linear Ineq.
M. Fillettaz et al., PRA (2018)	Qubit	E. G. Cavalcanti et al., PRA (2011) Q. Y. He et al., PRA (2011) S. P. Walborn et al., PRL (2011)	Entropy
H. Chau Nguyen and O. Gühne, PRL (2020)	POVM, dichotomic		
M. J. Renner, PRL (2024) Y. Zhang and E. Chitambar, PRL (2024)	Qubit Werner states, POVM	Q.Y. He and M. D. Reid, PRL (2013) J. Schneeloch et al., PRA (2013)	Multipartite Entropic UR
		D. A. Evans and H. M. Wiseman, PRA (2014) S. Jevtic et al., PRL (2014) T. Moroder et al., PRL (2014) T. Pramanik et al., PRA (2014)	
		I. Kogias et al., PRL (2015) Che-Ming Li et al., PRA(2015) M. Marciniak et al., PRL (2015) P. Skrzypczyk and D. Cavalcanti, PRA(2015)	Probability Combination
		Yi-Zheng Zhen et al., PRA (2016) Huangjun Zhu et al., PRL (2016)	Variance UR
		Yu-Lin Zheng et al., PRA (2017)	
		A. C. S. Costa et al., PRA (2018) T. Kriváchy et al., PRA (2018) A. Riccardi et al., PRA (2018) Bai-Chu Yu et al., PRA (2018) Qiang Zeng et al., PRL (2018)	Tsallis entropy Rényi entropy Multipartite
		Jun-Li Li and Cong-Feng Qiao, Adv. Quantum Technol. (2020) Jun-Li Li and Cong-Feng Qiao, QIP (2021) Gaoyan Zhu et al., PRA(L) (2023)	High dimension, MUB Majoriztion, qubit
		Rui Qu et al., PRL (2022) V. Srivastav er al., PRX (2022)	High dimension, MUB
		Ruo-Chen Wang et al., PRA (2024)	
		Ma-Cheng Yang and Cong-Feng Qiao, arXiv:2507.20950	**Majorization, Qudit, any settings**

How to access the correlation information embedded in high-dimensional systems remains an open and challenging question!

A brief history of quantum steering

Questions

- Bloch representation of a general bipartite state

$$\rho = \frac{1}{d^2}\mathbb{1} \otimes \mathbb{1} + \frac{1}{2d}\vec{u} \cdot \vec{\pi} \otimes \mathbb{1} + \frac{1}{2d}\mathbb{1} \otimes \vec{v} \cdot \vec{\pi} + \frac{1}{4}\sum_{\mu,\nu} \mathcal{T}_{\mu\nu}\pi_\mu \otimes \pi_\nu \,. \qquad (4)$$

- How to construct the LHS model?

- How to directly recognize steering from measurement statistics for general measurement settings and arbitrary dimensional system?

$$\boxed{p(a, b|\mathcal{A}, \mathcal{B}; \rho) = \mathrm{Tr}[(\Pi_a^A \otimes \Pi_a^B)\rho]} \qquad (5)$$

Outline

Probability majorization lattice

- Majorization *partial ordering* (MPO)

  $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$ with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, $\vec{x} \prec \vec{y}$ if and only if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \ , \ k = 1, \cdots, n-1 \, . \tag{6}$$

A. W. Marshall, et al., Inequalities: theory of majorization and its applications (Springer, 1979).

- Probability majorization lattice, a *quadruple* $\langle \mathcal{P}_n, \prec, \vee, \wedge \rangle$ and

$$\mathcal{P}_n \equiv \left\{ (p_1, \cdots, p_n)^{\mathrm{T}} \middle| p_i \geq p_{i+1} \geq 0, \sum_{i=1}^{n} p_i = \text{const.} \right\} , \tag{7}$$

$\forall \vec{p}, \vec{q} \in \mathcal{P}_n$, there is a unique infimum $\vec{p} \wedge \vec{q}$ and a unique supremum $\vec{p} \vee \vec{q}$.

F. Cicalese and U. Vaccaro, IEEE Trans. Inf Theory 48, 933 (2002).

Majorization lattice and quantum steering

Probability majorization lattice and entropy

- Schur-concave function: $\Phi(\cdot)$ is Schur-concave, then

$$\vec{p} \prec \vec{q} \Leftrightarrow \Phi(\vec{p}) \geq \Phi(\vec{q}) \ , \forall \Phi(\cdot) \ . \tag{8}$$

- α-order Rényi entropy

$$H_\alpha(\vec{p}) = \frac{1}{1-\alpha} \log \sum_k p_k^\alpha \ . \tag{9}$$

- Incomparable probability distributions $\Rightarrow$ intercross phenomenon of Lorenz curves $\Rightarrow$ indefinite entropy relations (entanglement catalytic phenomenon, Jonathan and Plenio, PRL, 1999)

- Entropy only partially reflects the MPO structure of probability distributions.

Majorization lattice and quantum steering

Lorenz curve, intercross phenomenon and indefinite entropy ordering

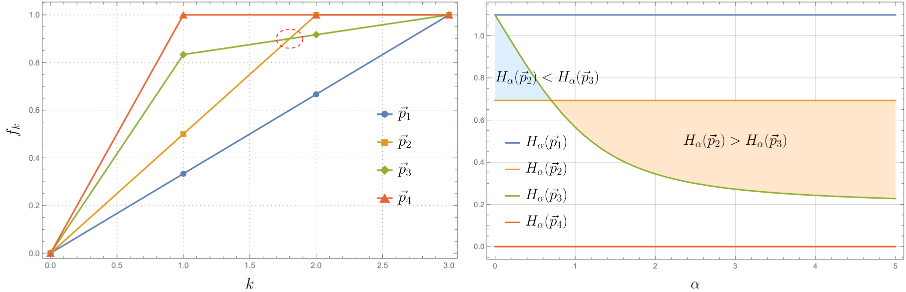


Figure. Lorenz curves of $\vec{p}_1 = (1/3, 1/3, 1/3)^{\mathrm{T}}$, $\vec{p}_2 = (1/2, 1/2, 0)^{\mathrm{T}}$, $\vec{p}_3 = (5/6, 1/12, 1/12)^{\mathrm{T}}$, $\vec{p}_4 = (1, 0, 0)^{\mathrm{T}}$. Majorization relation: $\vec{p}_1 \prec \vec{p}_2 \prec \vec{p}_4$, $\vec{p}_1 \prec \vec{p}_3 \prec \vec{p}_4$, but $\vec{p}_2 \nprec \vec{p}_3$ and $\vec{p}_3 \nprec \vec{p}_2$.

Majorization lattice and quantum steering

Complete lattice and uncertainty relations

- Complete lattice

Lemma

A probability majorization lattice is a complete lattice, meaning that for every subset $S \subseteq \mathcal{P}_n$, both the supremum $\bigvee S \in \mathcal{P}_n$ and the infimum $\bigwedge S \in \mathcal{P}_n$ exist.

Alberti and Uhlmann, Stochasticity and partial order (1982); Bapat, Linear Algebra Appl (1991); Bondar, Linear Algebra Appl (1994).

- Uncertainty and certainty relations via probability majorization lattice

$$\forall \vec{\chi} \in \mathcal{P}_{A,B}\ ,\ \vec{t}(A,B) \prec \vec{\chi} \prec \vec{s}(A,B) \tag{10}$$

where $\mathcal{P}_{A,B} = \{\vec{p}_A \odot \vec{p}_B | \vec{p}_A \in \mathcal{P}_A, \vec{p}_B \in \mathcal{P}_B\}$ with $\odot \in \{\otimes, \oplus, +\}$.

M. Partovi, PRA (2011); S. Friedland, et al., PRL (2013); J. L. Li and C. F. Qiao, Ann Phys (2019); A.-X. Liu, M.-C. Yang, and C.-F. Qiao, QIP (2025).

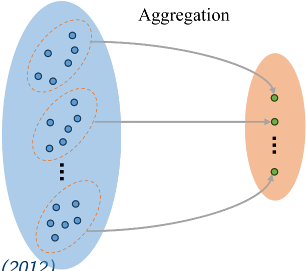
Complete lattice, aggregation operation and quantum steering

- The set of non-steerable states

$$\mathcal{P}_{\mathrm{ns}}^{N} = \left\{ \bigodot_{\mu=1}^{N} \vec{p}(A_\mu \otimes B_\mu)_\rho \middle| \rho \in \mathcal{D}_{\mathrm{ns}} \right\} \subset \mathcal{P}_n \ . \tag{11}$$

- Aggregation operation

$\forall \vec{p} \in \mathcal{P}_n$ and $\vec{q} \in \mathcal{P}_m$, $\vec{q}$ is refereed as an *aggregation* of $\vec{p}$ if there is a partition $\mathcal{I}$ of $\{1, \cdots, n\}$ into disjoint sets $I_1, \cdots, I_m$ such that $q_j = \sum_{i \in I_j} p_i$, for $j = 1, \cdots, m$, denoted as $\vec{q} = \mathcal{I}(\vec{p})$.



M. Vidyasagar, IEEE Transactions on Automatic Control 57, 2464 (2012).

F. Cicalese et al., in 2016 IEEE International Symposium on Information Theory (ISIT) (2016).

F. Cicalese et al., IEEE Trans. Inf Theory 65, 3436 (2019).

Majorization lattice and quantum steering

Theorem

Given orthonormal complete base sets $\mathcal{A} = (A_1, \cdots, A_N)$ *and* $\mathcal{B} = (B_1, \cdots, B_N)$ *for Hilbert spaces* $\mathcal{H}_A$ *and* $\mathcal{H}_B$*, we have the following majorization inequality*

$$\bigodot_{\mu=1}^{N} \mathcal{I}\left(\vec{p}(\mathcal{J}(A_\mu) \otimes \mathcal{E}(B_\mu))_\rho\right) \prec \vec{s}(\mathcal{B}) \ . \tag{12}$$

Here, $\vec{s}(\mathcal{B})$ *is majorization UR bound of* $\mathcal{B} = (B_1, \cdots, B_N)$ *for binary operations* $\odot \in \{\otimes, \oplus, +\}$ *and* $\mathcal{I}\left(\vec{p}(A_\mu \otimes B_\mu)_\rho\right)$ *denotes all aggregations of* $\vec{p}(A_\mu \otimes B_\mu)_\rho$ *with partitions* $\mathcal{I}$ *satisfying* $\mathcal{I}(\vec{p} \otimes \vec{q}) \prec \vec{q}$*;* $\mathcal{E}$ *denotes the local transformations preserving the majorization UR bound* $\vec{s}(\mathcal{B})$*, i.e.* $\vec{s}(\mathcal{E}(\mathcal{B})) = \vec{s}(\mathcal{B})$*;* $\mathcal{J}$ *represents any local transformation of measurement Alice performs.*

Outline

High-dimensional quantum steering

Bipartite steering from aggregation operation

- Aggregation operation of measurement statistics ($Nd^2 \to Nd$)

$$\bigoplus_{\mu=1}^{N} \vec{\Upsilon}(\mathcal{J}(A_\mu), \mathcal{E}(B_\mu); \rho) \prec \vec{s}(\mathcal{B}) \ ,$$

$$\Upsilon_k(A,B;\rho) = \sum_{(i,j)\in I_k} p(a_i, b_j|A,B;\rho) = \frac{1}{d} + \frac{1}{4}\sum_{(i,j)\in I_k} (\vec{a}_i, \mathcal{T}\vec{b}_j)$$
$$I_1 = \{(i,j)|i = j = 1, \cdots, d\} \ ,$$
$$I_k = \{(i,j)|j = (i+k-2) \bmod_1 d, i = 1, \cdots, d\}$$

which yields a family of aggregation $\vec{\Upsilon}$ steering inequalities

$$\mathcal{S}_N(L) \equiv \frac{1}{L}\sum_{k=1}^{L}\left[\bigoplus_{\mu}^{N} \vec{\Upsilon}_\mu\right]_k^{\downarrow} \leq \bar{\Omega}_L \ , \ L = 2, \cdots, Nd \ . \qquad (13)$$

$[\cdot]_k^{\downarrow}$: The k-th largest component; $\bar{\Omega}_L = \frac{\Omega_L}{L}$ with $\Omega_L = \sum_{k=1}^{L} [\vec{s}(\mathcal{B})]_k^{\downarrow}$.

- Steering parameter: $\mathcal{S}_N(L)$; state-independent bound: $\bar{\Omega}_L$ (combinatorial optimization problem, COP).

High-dimensional quantum steering

High-dimensional Werner states and isotropic states

- Steering parameters of isotropic states and Werner states

$$\mathcal{S}_N^{\mathrm{ISO}} = \frac{1+(d-1)w}{d}\,, \ \mathcal{S}_{N(d-1)}^{\mathrm{W}} = \frac{d-1+\eta}{d(d-1)}\,. \tag{14}$$

- Steering thresholds of isotropic states and Werner states

$$w_{\mathrm{steer}} = \left(d\bar{\Omega}_N - 1\right)/(d-1)\,, \tag{15}$$

$$\eta_{\mathrm{steer}} = (d-1)\left(d\bar{\Omega}_{N(d-1)} - 1\right)\,. \tag{16}$$

- Upper bounds of $\bar{\Omega}_N$ for MUBs

Bounds	w_{steer} (Isotropic states)	η_{steer} (Werner states)
$\bar{\Gamma}_N = \frac{N+\sqrt{N}(d-1)}{dN}$	$1/\sqrt{N}$	$(d-1)/\sqrt{N}$
$\bar{\Theta}_N = \frac{1}{N} + \frac{N-1}{N\sqrt{d}}$	$\frac{(N+\sqrt{d})(\sqrt{d}-1)}{N(d-1)}$	$\frac{(\sqrt{d}-1)[(d-1)N+\sqrt{d}]}{N}$
$\bar{\Lambda}_N = \frac{\sqrt{N}+d-1}{d\sqrt{N}}$	$\frac{1}{d-1}\left(\sqrt{d(d+N-1)/N} - 1\right)$	$d\left(\sqrt{(d-1)[(d-1)N+1]/(Nd)} - 1\right) + 1$

Steering thresholds of some special cases

- For qubit system

$$w_{\text{steer}} = 2\bar{\Omega}_N - 1\ . \tag{17}$$

Saunders *et al.*'s inequality: $\frac{1}{N}\sum_{k=1}^{N} |\langle A_k \otimes B_k\rangle| \leq C_N$ with $C_N = \frac{1}{N}\max_{\{a_k\}}\left[\lambda_{\max}\left(\sum_{k=1}^{N} a_k B_k\right)\right]$ and $a_k = \pm 1$. Note: $\bar{\Omega}_N = \frac{1}{2} + C_N$.

D. J. Saunders et al., Nat. Phys. (2010).

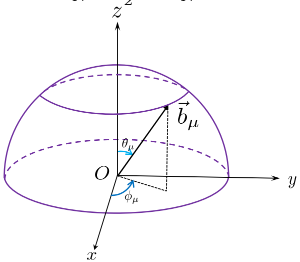
- Infinite measurement settings scenario

The hemisphere:

$$\lim_{N\to\infty} \bar{\Omega}_N = 3/4 \Rightarrow w_{\text{steer}} = 1/2$$

The semi-circle:

$$\lim_{N\to\infty} \bar{\Omega}_N = (\pi + 2)/2\pi \Rightarrow w_{\text{steer}} = 2/\pi$$

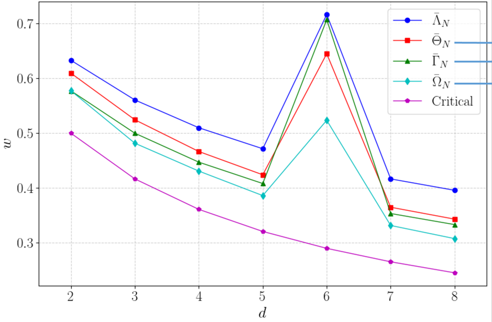


- Two measurement settings for qudit

$\bar{\Omega}_2 = (1+c)/2$ with $c = \max_{i,j} |\langle a_i|b_j\rangle| \Rightarrow w_{\text{steer}} = \frac{d(1+c)-2}{2(d-1)}$.

C.-M. Li et al., PRA (2015); Q. Zeng et al., PRL (2018); Y. Guo et al., PRL (2019).

High-dimensional isotropic states in MUBs scenario



PRA (2015).
PRL (2022).
PRA (2018).
PRA (2017)
entropy
SDP
probability combination

Figure. The steering thresholds of the isotropic states for MUBs.

Y.-Z. Zhen, Y.-L. Zheng, W.-F. Cao, L. Li, Z.-B. Chen, N.-L. Liu, and K. Chen, PRA (2016).
Y.-L. Zheng, Y.-Z. Zhen, Z.-B. Chen, N.-L. Liu, K. Chen, and J.-W. Pan, PRA (2017).
A. C. S. Costa, R. Uola, and O. Gühne, PRA(R) (2018).

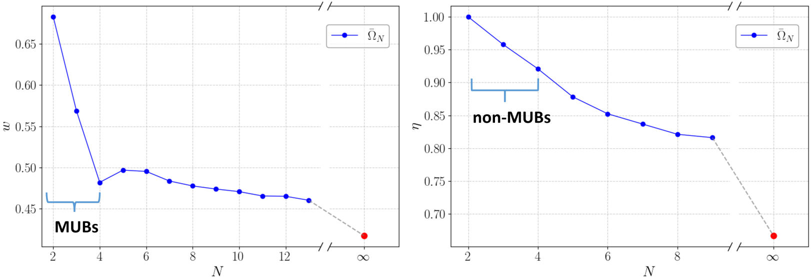
High-dimensional quantum stee ring

Qutrit isotropic and Werner states in general scenario

- Optimization problem of $\bar{\Omega}_N$

$$\begin{aligned} \min \quad & \bar{\Omega}_N \\ \text{s.t.} \quad & \mathcal{B} = (B_1, \cdots, B_N), \forall B_\mu = B_\mu^\dagger . \end{aligned} \tag{18}$$

- Numerical results for N-setting (Cross-Entropy Method)



(a) Qutrit isotropic states **(b)** Qutrit Werner states

Figure. The steering threshold value of the qutrit isotropic and Werner states for N settings.

Outline

Summary and outlook

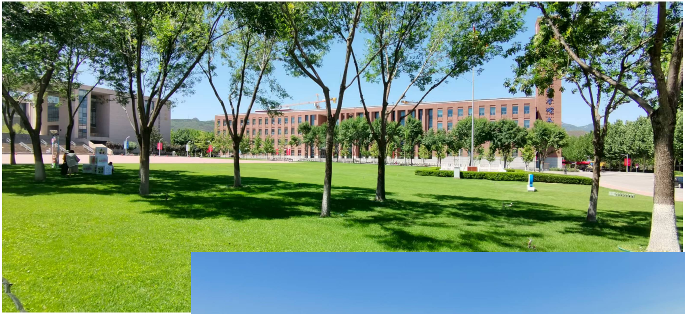
Summary

- Majorization lattice provides a powerful tool for studying quantum steering, particularly in high-dimensional systems.
- Derived steering thresholds for arbitrary-dimensional isotropic states and Werner states in both MUBs and general scenarios.
- Non-MUB measurements are essential in high-dimensional and N-setting contexts, such as high-dimensional Werner states.

Outlook

- Open questions: Is a complete set of MUBs optimal for isotropic states but worst for Werner states? What are the limitations of infinite settings for high-dimensional isotropic and Werner states? POVM or PVM?
- Extension: Explore applications in general quantum resource theory, including quantum coherence, quantum correlations, and multipartite systems.

Acknowledgements



Acknowledgements

Thanks!