Phase Diagram of a Lattice Fermion Model with Symmetric Mass Generation

Sandip Maiti

Central China Normal University

Based on
arXiv:2512.24836, arXiv:2602.18360

with D. Banerjee, S. Chandrasekharan, M. K. Marinkovic

June 18, 2026

Masses in QFTs

Mass terms in QFTs are often an emergent property:

- whether a fermion bilinear condenses
- whether a symmetry is broken – either explicitly or spontaneously
- whether the IR theory is trivial or interacting
- Trivial case: explicit mass

$$\mathcal{L} = \bar{\psi}\, i\partial\!\!\!/ \psi - m\bar{\psi}\psi$$

- Dynamical (continuous) symmetry breaking:

$$\mathcal{L} = \bar{\psi}\, i\partial\!\!\!/ \psi - g(\bar{\psi}\psi)^2$$

At strong coupling: $\langle\bar{\psi}\psi\rangle \neq 0$, giving rise to a fermion mass

Masses in QFTs

- Spontaneous symmetry breaking (SSB) with scalar fields:

$$\mathcal{L} = |D_\mu \phi|^2 - V(\phi) + y\phi\,\bar{\psi}_L \psi_R$$

After SSB,

$$\langle \phi \rangle \neq 0,$$

fermions acquire mass

$$m_\psi = y\langle \phi \rangle$$

together with massless Goldstone bosons

Symmetric Mass Generation (SMG)

Can fermions become massive without spontaneous symmetry breaking?

Symmetric Mass Generation provides such a mechanism

- Requires strong interactions, typically multi-fermion interactions
  Ayyar & Chandrasekharan, PRD **91**, 065035 (2015); Catterall, JHEP **01**, 121 (2016); Martin & Grover, arXiv:2507.23032.
- All fermion bilinears are forbidden by symmetry
- IR is a featureless symmetric gapped phase
- No ’t Hooft anomalies: anomaly cancellation is essential
- Originally proposed in lattice field theory in the 1980s, but long regarded as lattice artefacts
- Recent work on pure fermionic theories, fermions coupled to gauge fields Butt, Catterall & Hasenfratz, PRL **134** (2025). , and anomaly cancellation has renewed interest
- Which microscopic models realize this phase?

Lattice model

- Model in $(2+1)$-d Euclidean space-time system, $n_f = 2$ massless staggered fermions, denoted u and d
- Four Grassmann-valued staggered fermion fields at each site i, namely $\bar{u}_i$, u_i, $\bar{d}_i$, and d_i

$$S = \sum_{\langle ij \rangle} \eta_{ij} \Big(\bar{u}_i u_j - \bar{u}_j u_i + \bar{d}_i d_j - \bar{d}_j d_i \Big)$$

$$- U_I \sum_i \left(\bar{u}_i u_i \, \bar{d}_i d_i \right) - U_B \sum_{\langle ij \rangle} \left(\bar{u}_i u_i \, \bar{u}_j u_j + \bar{d}_i d_i \, \bar{d}_j d_j \right)$$

- Phase factors η_{ij} implement a π-flux through each plaquette:

$$\eta_{\hat{t}} = 1, \qquad \eta_{\hat{x}} = (-1)^t, \qquad \eta_{\hat{y}} = (-1)^{t+x}$$

Symmetries I

- $$S_F = \sum_{\langle ij \rangle} \eta_{ij} \left(\bar{u}_i u_j - \bar{u}_j u_i + \bar{d}_i d_j - \bar{d}_j d_i \right) = \sum_{\langle ij \rangle} \left(\bar{u}_i M_{ij} u_j + \bar{d}_i M_{ij} d_j \right)$$
- Free fermion hopping; invariant under internal $SU(4) \times U(1)$

$$\begin{pmatrix} u_i \\ \bar{u}_i \\ d_i \\ \bar{d}_i \end{pmatrix} \rightarrow V\, e^{i\theta} \begin{pmatrix} u_i \\ \bar{u}_i \\ d_i \\ \bar{d}_i \end{pmatrix}, \qquad \begin{pmatrix} \bar{u}_j \\ u_j \\ \bar{d}_j \\ d_j \end{pmatrix} \rightarrow V^* e^{-i\theta} \begin{pmatrix} \bar{u}_j \\ u_j \\ \bar{d}_j \\ d_j \end{pmatrix}$$

- Site i is even and j odd, V is an element of $SU(4)$
- $e^{i\theta}$ represents a $U(1)$ phase → axial symmetry of staggered fermions.
- $$S_I = -U_I \sum_i \left(\bar{u}_i u_i \, \bar{d}_i d_i \right)$$
- Invariant under $SU(4)$ but explicitly breaks the $U(1)$ symmetry
- Behaves like a ’t Hooft vertex, introducing instanton-like effects

Symmetries II

- $$S_B = -U_B \sum_{\langle ij \rangle} \left(\bar{u}_i u_i \, \bar{u}_j u_j + \bar{d}_i d_i \, \bar{d}_j d_j \right)$$
- Invariant under $SU(2) \times U(1)_{\text{u-quark}}$ and $SU(2) \times U(1)_{\text{d-quark}}$
- Thus, the transformation on the u-quarks is, with $V_u \in SU(2)$:

$$\begin{pmatrix} u_i \\ \bar{u}_i \end{pmatrix} \to V_u e^{i\theta_u} \begin{pmatrix} u_i \\ \bar{u}_i \end{pmatrix}, \qquad \begin{pmatrix} \bar{u}_j \\ u_j \end{pmatrix} \to V_u^* e^{-i\theta_u} \begin{pmatrix} \bar{u}_j \\ u_j \end{pmatrix}$$

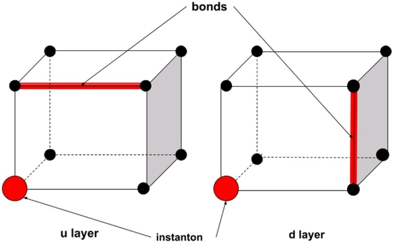
- Action on d-quarks is similar, with $V_u \to V_d$ and $e^{i\theta_u} \to e^{i\theta_d}$
- For $\theta_u = \theta_d = \theta$, one obtains the $U(1)$ symmetry of S_I
- Another independent $U_\chi(1)$ symmetry of S_B is obtained for

$$\chi = \theta_u = -\theta_d$$

- The $U_\chi(1)$ is a subgroup of the $SU(4)$ symmetry of S_I
- For $U_B = 0$, $U_I \neq 0$, the full $SU(4)$ is intact, but $U_B \neq 0, \quad U_I \neq 0$ gives $SU(2) \times SU(2) \times U_\chi(1)$

Computation Approach

- In order to study our GN model using Monte Carlo methods we use the **fermion bag approach**
- Idea: Group fermion worldlines inside regions called fermion bags (Eur.Phys.J.A49,90(2013)
- The partition function is then written as a sum over configurations of fermion bags
- One possible way to identify fermion bags is to introduce new variables, the **dimers or bonds** $b_{x,y}$ for nearest neighbor interactions and **instantons** i_x for single site interactions (Phys.Rev.D97,054501)



Strong Coupling Approach

$$Z = \int [\mathcal{D}\bar{u}\mathcal{D}u\mathcal{D}\bar{d}\mathcal{D}d] e^{-\sum_{x,y}(\bar{u}_x M_{x,y} u_y + \bar{d}_x M_{x,y} d_y) + U_B \sum_{x,y}(\bar{u}_x u_x \bar{u}_y u_y + \bar{d}_x d_x \bar{d}_y d_y) + U_I \sum_x \bar{u}_x u_x \bar{d}_x d_x}$$

$$\begin{aligned} e^{U_B \bar{u}_x u_y \bar{u}_y u_x} &= (1 + U_B \bar{u}_x u_x \bar{u}_y u_y) = \sum_{b_{x,y}=0,1} (U_B \bar{u}_x u_x \bar{u}_y u_y)^{b_{x,y}} \\ e^{U_I \bar{u}_x u_x \bar{d}_x d_x} &= (1 + U_I \bar{u}_x u_x \bar{d}_x d_x) = \sum_{i_x=0,1} (U_I \bar{u}_x u_x \bar{d}_x d_x)^{i_x} \end{aligned}$$

The partition function then becomes the sum over all configurations of [i] and [b] given by

$$Z = \sum_{[b,i]} U_I^{N_i} U_B^{N_u+N_d} det(W_u) det(W_d)$$

Weak Coupling Approach

An alternate way of constructing the fermion bag is to consider each configuration $[b,i]$ as a term in the perturbative expansion:

$$Z = \sum_{[b,i]} U_I^{N_i} U_B^{N_u+N_d} \left\{ \int [\mathcal{D}\bar{u}\mathcal{D}u] e^{-\sum\limits_{x,y}(\bar{u}_x M_{x,y} u_y)} \bar{u}_{z_1} u_{z_1} \ldots \bar{u}_{z_k} u_{z_k} \right\}$$

$$\times \left\{ \int [\mathcal{D}\bar{d}\mathcal{D}d] e^{-\sum\limits_{x,y}(\bar{d}_x M_{x,y} d_y)} \bar{d}_{w_1} d_{w_1} \ldots \bar{d}_{w_\ell} d_{w_\ell} \right\}$$

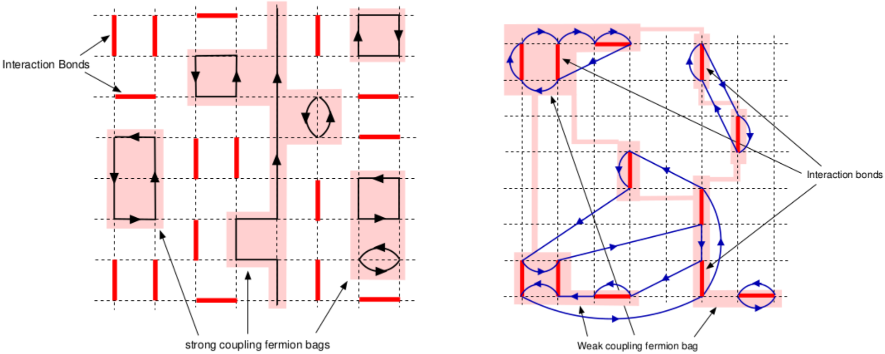
In this expression $k = 2N_u + N_i$ and $\ell = 2N_d + N_i$

$$Z = [\det(M)]^2 \sum_{[b,i]} U_I{}^{N_i} U^{N_u+N_d} \det(G_u) \ \det(G_d)$$

where G_u and G_d are $k \times k$ and $\ell \times \ell$ propagator matrix between the sites $z_1, \ldots z_k$ and $w_1, \ldots, w_\ell$ respectively

- To optimize the efficiency of the fermion bag algorithm, it is often beneficial to switch between different fermion bag formulations

An illustration of fermion bag



Credit: Chandrasekharan, Li, PhysRevLett.108.140404

Condensates and Scalings

- Bilinear condensates $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$ spontaneously break $U_\chi(1)$
- Detect by measuring the following susceptibilities:

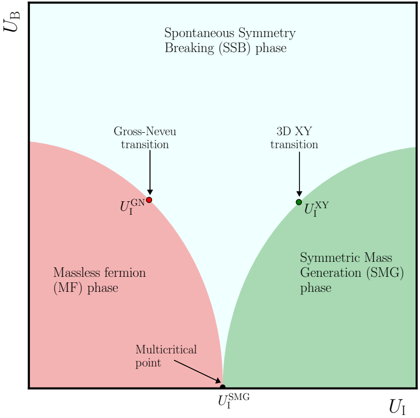
$$\chi_{ud} = \frac{1}{2L^3} \sum_{i,j} \left\langle \bar{u}_i u_i \, \bar{d}_j d_j \right\rangle$$

$$\chi_{uu} = \frac{1}{2L^3} \sum_{i,j} \left\langle \bar{u}_i u_i \, \bar{u}_j u_j \right\rangle$$

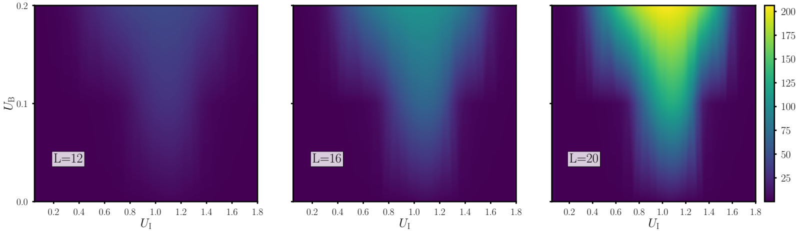
$$\chi_{dd} = \frac{1}{2L^3} \sum_{i,j} \left\langle \bar{d}_i d_i \, \bar{d}_j d_j \right\rangle$$

- Note: $\chi_{uu} = \chi_{dd}$ (symmetry of action), but $\chi_{ud} \approx \chi_{uu}$ on large lattices, except when U_I is small
- In a phase with SSB, $\chi \sim L^3$, while in both the MF phase and the SMG phase we expect $\chi \sim$ const.
- At a critical point, in contrast, we expect the finite-size scaling behavior $\chi \sim L^{2-\eta}$

Phase diagram

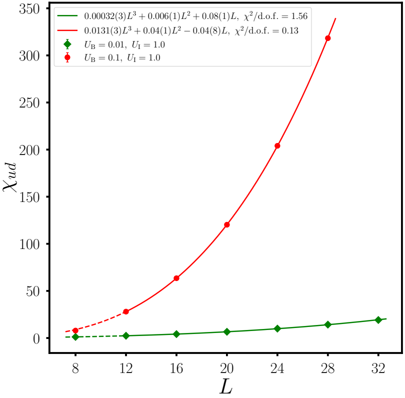


Evidence from susceptibility



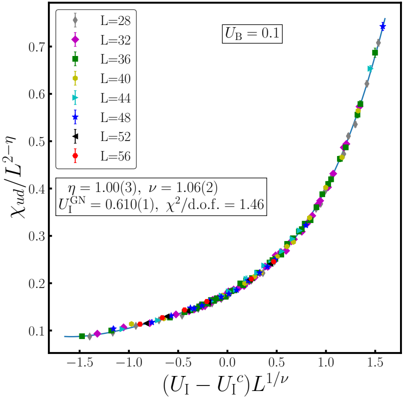
- χ_{ud} heat maps for $L = 12, 16, 20$, illustrating the structure of the phase diagram
- Enhanced susceptibilities indicative of a SSB phase are visible for U_B values as small as $U_B = 0.01$ at $U_I = 1.0$

$U_I(c)$ at $U_B = 0$: multicritical point?



- Even $U_B = 0.01$ causes SSB at $U_I(c)$, detect by χ-PT inspired fits
- SSB breaks the $U_\chi(1)$ subgroup in $SU(4) = SU(2) \times SU(2) \times U_\chi(1)$

MF to SSB transition



Exponents extracted by fitting χ_{uu} and χ_{dd} using the finite-size scaling relation $\chi/L^{2-\eta} = f\left((U_I - U_I^c)L^{1/\nu}\right)$ at the critical point

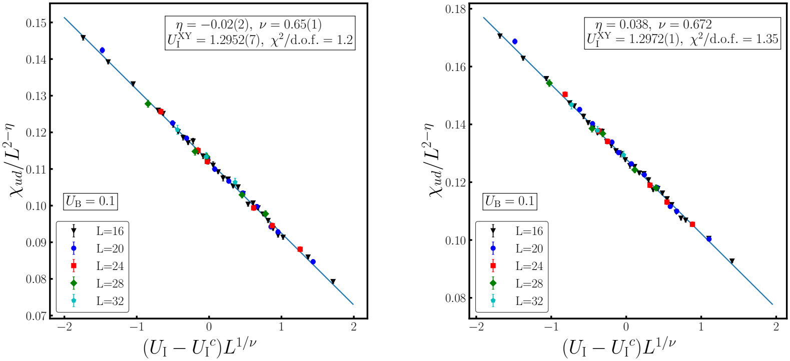
MF to SSB: Gross-Neveu universality

- Staggered $n_f = 1$ in $d = 3$ corresponds to $N_f = 2$ of four-component Dirac fermions at long distances
- We should expect $N_f = 4$ four-component Dirac fermions and mean-field critical exponents
- Exponents checked using various stability fits

ν	η	Study
1.06(2)	1.00(3)	This work
1.13	0.93(9)	MC (Bilayer Graphene)
1.06(9)	0.99(5)	ϵ-expansion (4-loop)
1.092(6)	0.926(13)	ϵ-expansion (interpolation)

Critical exponents reported in various studies for the Gross–Neveu phase transition in three Euclidean dimensions with four flavors of four-component Dirac fermions. At this transition, massless fermions acquire a mass through the formation of a fermion-bilinear condensate that spontaneously breaks a $U(1)$ symmetry

SSB to SMG: 3d XY universality



Different analysis: (left) fits critical exponents, while (right) fixes the critical exponents to the 3d XY values

Summary and Outlook

- Three phases identified: MF, SSB, and SMG
- The SSB phase exhibits spontaneous $U_\chi(1)$ breaking, massive fermions
- The SMG phase remains fully symmetric with $\langle\bar{\psi}\psi\rangle = 0$
- The previously observed direct MF ↔ SMG transition is reinterpreted as an $SU(4)$-symmetric SMG multicritical point
- The multicritical point connects the Gross–Neveu and 3D-XY universality classes
- Anomaly cancellation appears to be a necessary ingredient for realizing symmetric mass generation

---

$$S_{\mathrm{I}} = -U_I \sum_i \left(\bar{u}_i u_i \bar{d}_i d_i \right)$$

- $SU(4)$ symmetry:

$\chi_1 = \bar{u}$, $\chi_2 = u$, $\chi_3 = \bar{d}$ and $\chi_4 = d$

$\chi_1 = V_{1\alpha}\chi_\alpha$, $\chi_2 = V_{2\beta}\chi_\beta$, $\chi_3 = V_{2\gamma}\chi_\gamma$ and $\chi_4 = V_{4\delta}\chi_\delta$

If $\alpha \neq \beta \neq \gamma \neq \delta$

$$\chi_1\chi_2\chi_3\chi_4 \longrightarrow SU(4)$$

- Because of the on-site term, S_I breaks $U(1)$ symmetry explicitly