Hadron structure functions in the Hamiltonian framework

Manuel Schneider
`manuel.schneider@nycu.edu.tw`

NATIONAL
YANG MING CHIAO TUNG
UNIVERSITY

理論與計算物理中心
Center for Theoretical and Computational Physics

Quantum Information Science in High Energy Nuclear Physics
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QIS - HENP

Outline

1. Introduction: Hadron Structure Functions
2. Progress: Hamiltonian approaches
3. Example: Schwinger Model
4. Summary & Outlook

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Hadron Structure Functions: PDF

[EIC]

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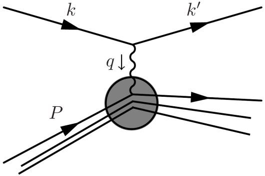
- parton distribution functions (PDF):
  probability of constituent with momentum fraction ξ

[EIC]



Deep Inelastic Scattering
[Schwartz 2014]

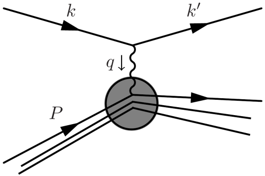
Hadron Structure Functions: PDF

- parton distribution functions (PDF):
  probability of constituent with momentum fraction ξ
- factorization at large energy $Q^2 = -q^2$, e.g. DIS:

$$\sigma(\xi, Q^2) = \hat{\sigma}(\xi, Q^2) \otimes f(\xi)$$

experiment perturbative non-perturbative

[EIC]



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$$\boxed{f_\psi(\xi) = \int dz^- e^{-i\xi P^+ z^-} \langle P | \bar{\psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\psi(0) | P \rangle}$$

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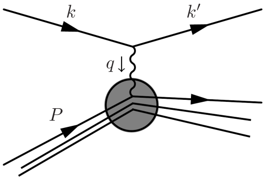

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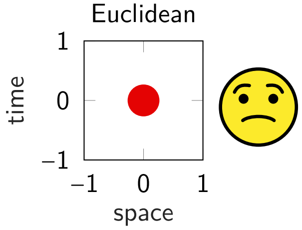
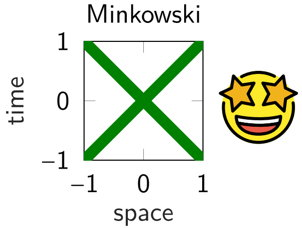
- z^-: lightcone coordinate
- lattice QCD in Euclidean space: lightcone → point
- Hamiltonian formalism: lightcone in Minkowski space
  → tensor network states / quantum devices



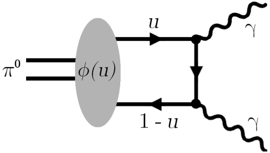


Hadron Structure Functions: generalizations

- lightcone distribution amplitude (LCDA): decay or hadronization

$$\pi^0 \to q(u)\bar{q}(1-u) \to \gamma\gamma$$

$$iF_M\phi(u) = \int dz^- e^{iuP^+z^-} \langle 0 | \bar{\psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\psi(0) | P \rangle$$



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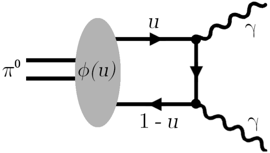
- general hadron structure functions:

$$\boxed{\int dz^- e^{i\xi P^+ z^-} \langle F \,|\, J(z^-)\gamma W(z^- \leftarrow 0) J(0) \,|\, I \rangle}$$



- form factors: photon interaction
  $F \propto \langle P' | \bar{\psi}(0)\gamma\psi(0) | P \rangle$
- hadronic tensor: DIS
  $W^{\mu,\nu} \propto \int dz^- e^{iqP^+z^-} \langle P \,|\, J^\nu(z^-) J^\mu(0) \,|\, P \rangle;\ J^\mu = \bar{\psi}\gamma^\mu\psi$
- fragmentation functions: hadronization
  $D \propto \int dz^- e^{ik^+z^-} \langle 0 \,|\, \psi(z) \,|\, P \rangle \gamma \left\langle P \middle| \bar{\psi}(0) \middle| I \right\rangle$
- generalized parton distributions (GPD): full kinematics
  $H(\xi, \Delta P) = \int dz^- e^{-i\xi P^+ z^-} \left\langle P' \middle| \bar{\psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\psi(0) \middle| P \right\rangle$

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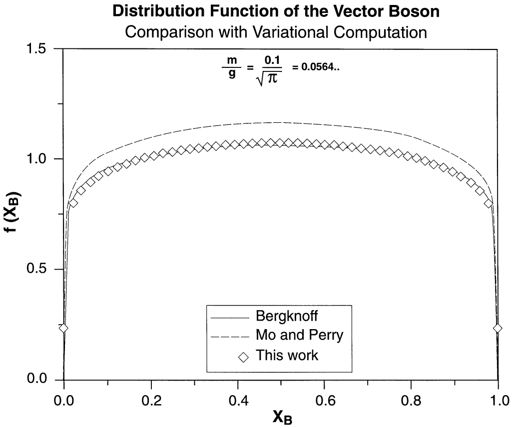
Progress on hadron structure functions in the Hamiltonian approach

- Hamiltonian calculation with exact diagonalization (ED)
  [Bergknoff 1977; Eller et al. 1987; Mo and Perry 1993; Kröger and Scheu 1998; Jirari et al. 1999]



Schwinger model
[Kröger and Scheu 1998]

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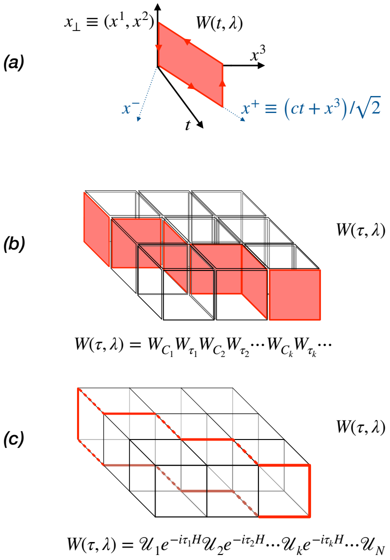
  [Pedernales et al. 2014; Echevarria et al. 2021; Pisarski 2022]



$$W(\tau, \lambda) = W_{C_1} W_{\tau_1} W_{C_2} W_{\tau_2} \cdots W_{C_k} W_{\tau_k} \cdots$$

$$W(\tau, \lambda) = \mathcal{U}_1 e^{-i\tau_1 H} \mathcal{U}_2 e^{-i\tau_2 H} \ldots \mathcal{U}_k e^{-i\tau_k H} \ldots \mathcal{U}_N$$

[M.G. Echevarria, I.L. Egusquiza, E. Rico, G. Schnell]

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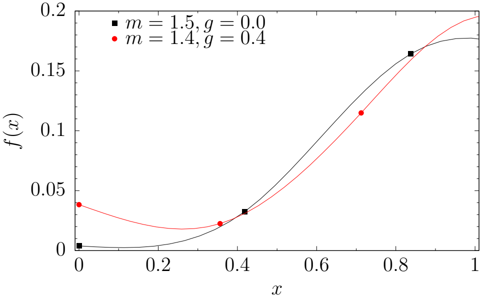
- Hadronic tensor in Hamiltonian formalism
  [Lamm et al. 2020]



PDF from ED
Thirring model
[Lamm et al. 2020]

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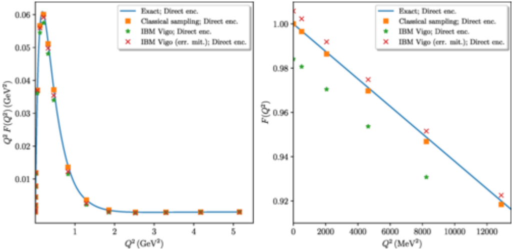

Form factors in basis light-front quantization Nambu–Jona-Lasinio model (NJL)
[Kreshchuk et al. 2021b]

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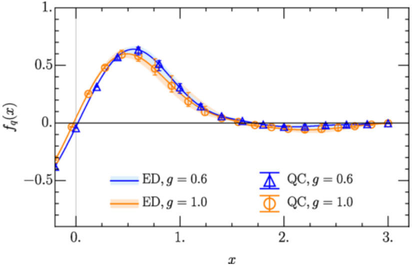
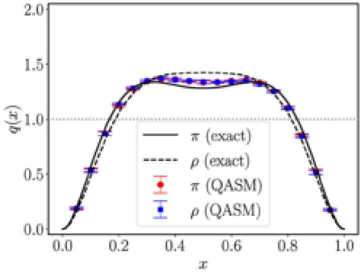
  [Li et al. 2022; Qian et al. 2022]



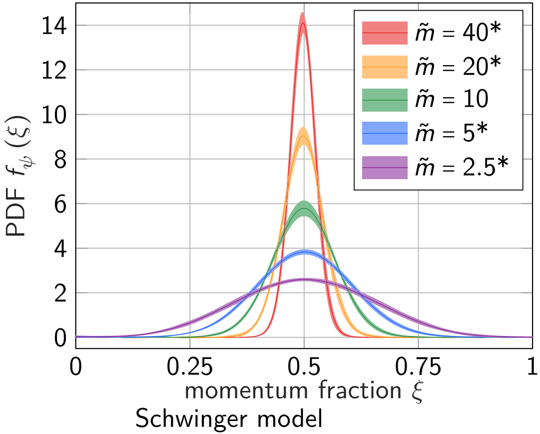
NJL model [Li et al. 2022]



Effective model [Qian et al. 2022]

Progress on hadron structure functions in the Hamiltonian approach (2)

- PDF in gauge theory with tensor networks
  [Schneider et al. 2025; Bañuls et al. 2026]



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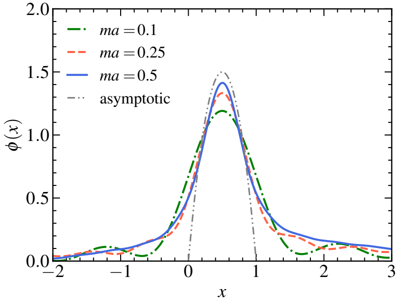

NJL model
[Z.-B. Kang, N. Moran, P. Nguyen, W. Qian]

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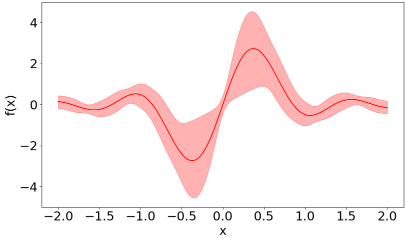

Schwinger model [Chen et al. 2025]

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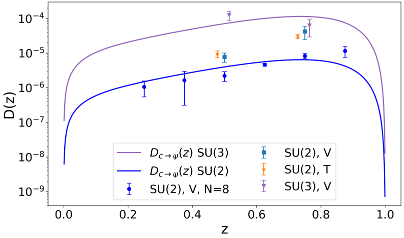

truncated, light-front quantized QCD
[Gálvez-Viruet et al. 2025]

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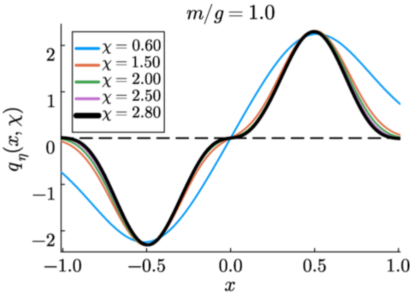
- Quasi- PDFs, GPDs, and fragmentation functions with tensor networks
  [Grieninger et al. 2024; Grieninger and Zahed 2024; Grieninger et al. 2026]



Schwinger model [Grieninger et al. 2026]

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Schwinger model [Hamer et al. 1997]

- quantum electrodynamics in 1+1 dimensions, $U(1)$ symmetry
- fermion couples to gauge boson → partons
- bound states → hadrons [Bañuls et al. 2013]
- scattering → PDF [Dai et al. 1994, 1995]
- Lagrange density:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - g\not{A} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_0\rho$$

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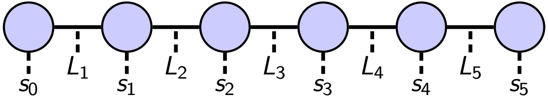

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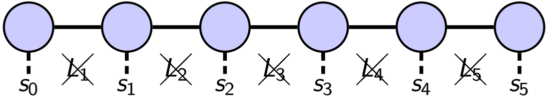


$$\boxed{H = x\sum_{n=0}^{N-2}\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\right] + \frac{\mu}{2}\sum_{n=0}^{N-1}\left[1 + (-1)^n\sigma_n^z\right] + \sum_{n=0}^{N-2}\left[\frac{1}{2}\sum_{k=0}^{n}\left((-1)^k + \sigma_k^z + 2q_k\right)\right]^2}$$

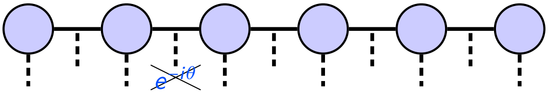
$$\left(x = \frac{1}{a^2g^2}, \mu = \frac{2m}{ag^2}\right)$$

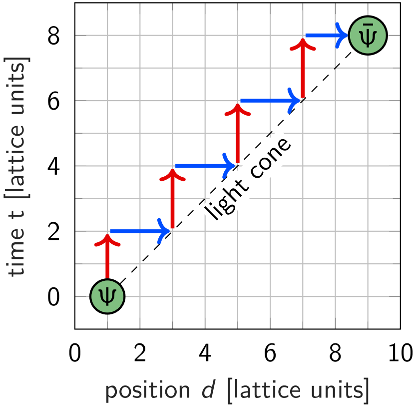
PDF in the Schwinger model

- matrix elements:

$$\mathcal{M} = \left\langle P \middle| \bar{\Psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\Psi(0) \middle| P \right\rangle$$

- lightcone
  → small time- and space-like steps
- spatial evolution:
  change electric field along the path
- trotterized time evolution





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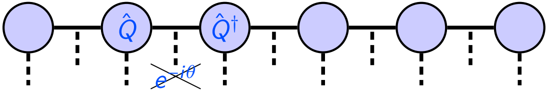

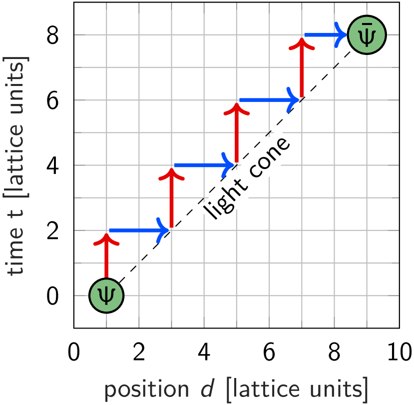

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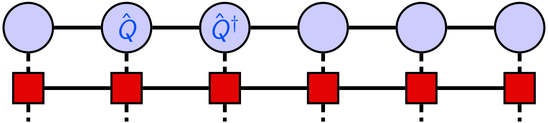

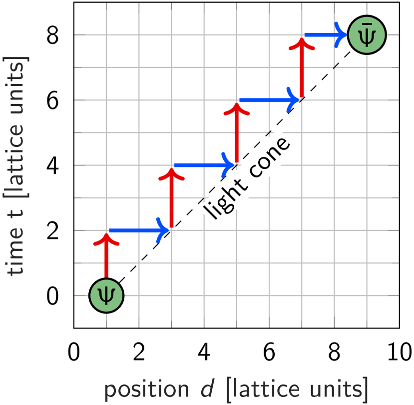

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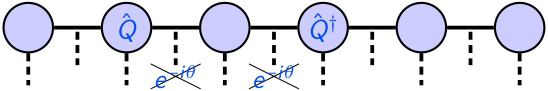

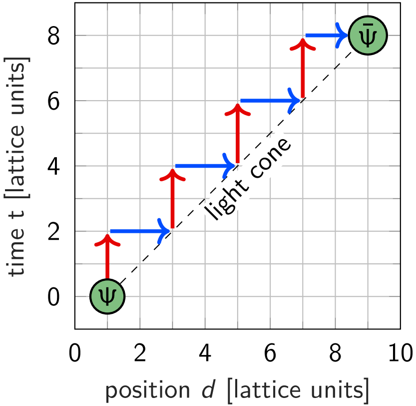

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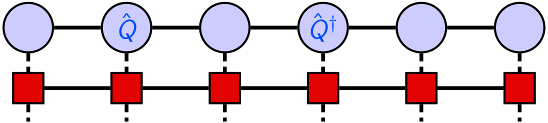

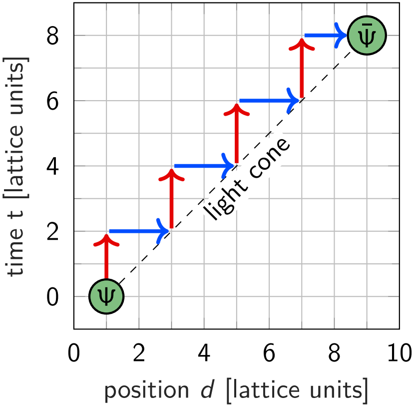

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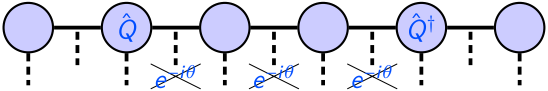

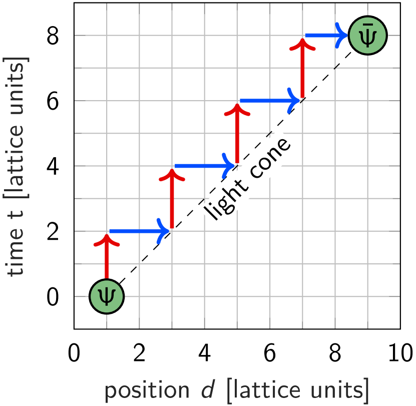

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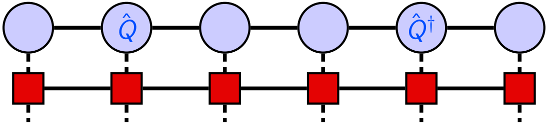
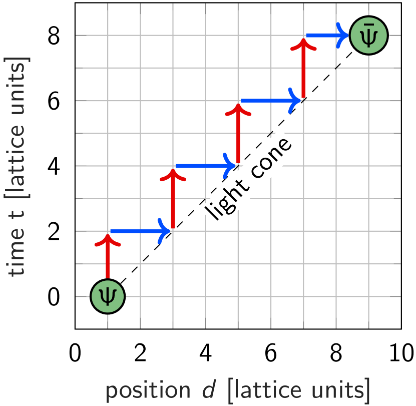
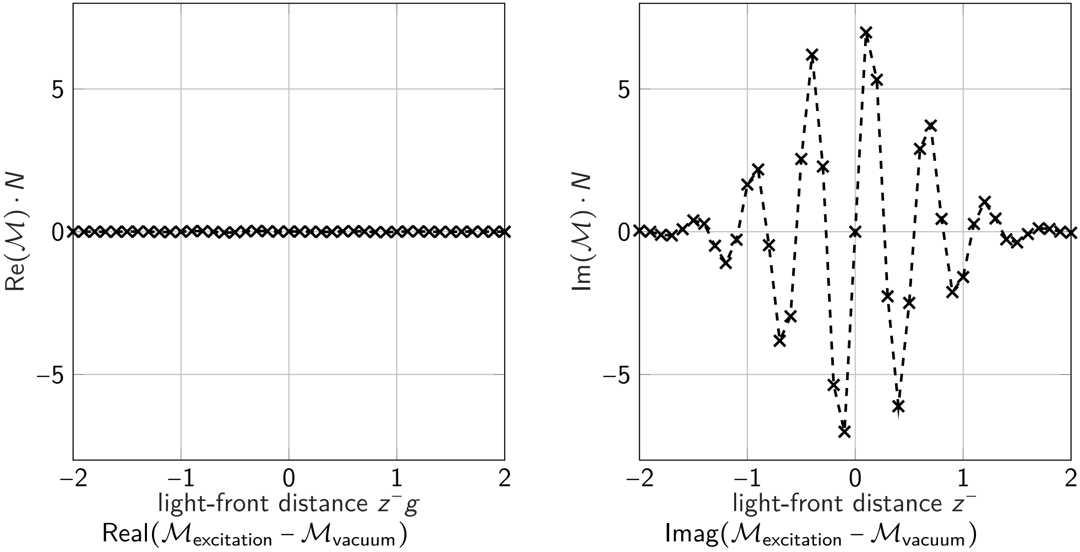


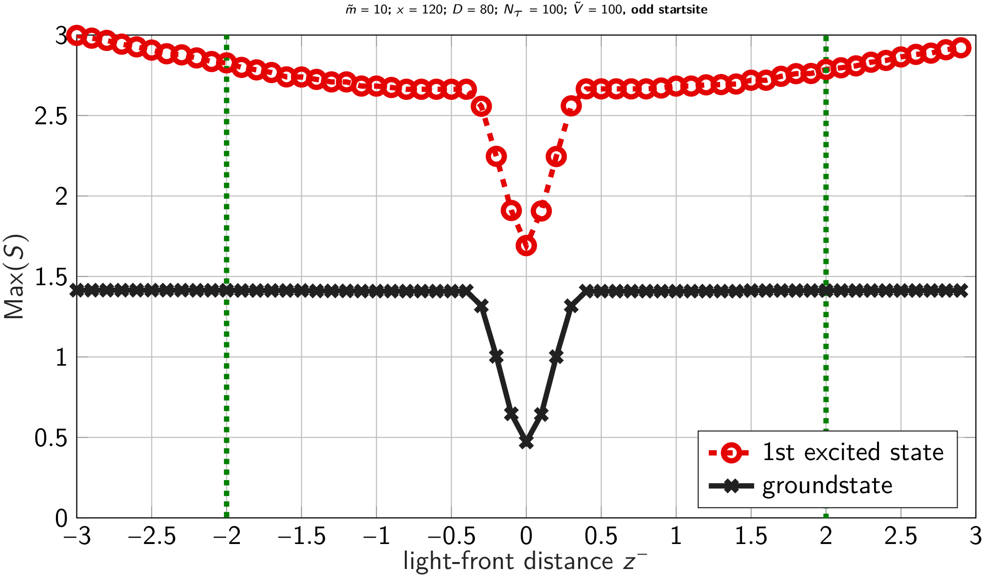


Matrix elements

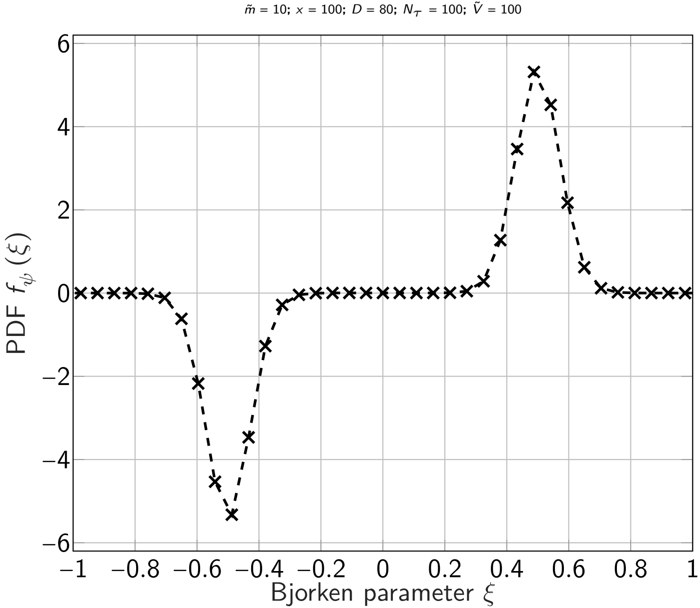
$\tilde{m} = 10;\ x = 100;\ D = 80;\ N_{\mathcal{T}} = 100;\ \tilde{V} = 100$



Entanglement entropy



PDF



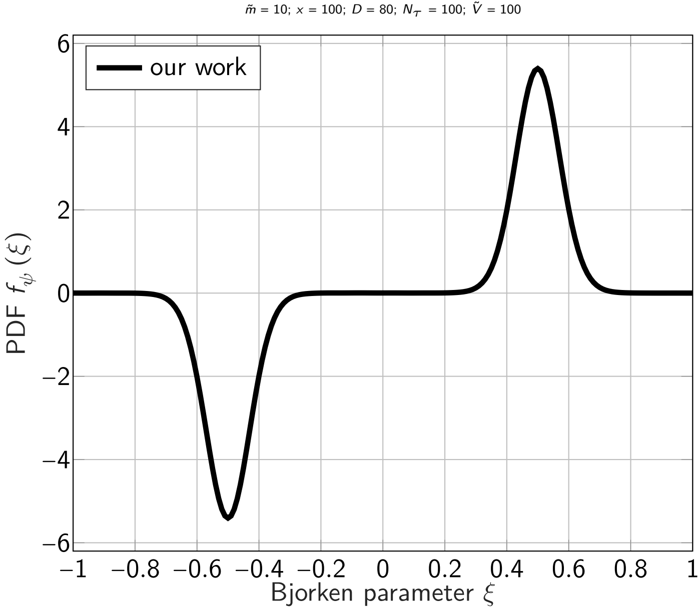
observations:

- $\xi > 0$: $f_{\psi} \approx$ symmetric around $\xi = 0.5$
- antifermion PDF from negative ξ:

$$f_{\overline{\psi}}(\xi) = -f_{\psi}(-\xi)$$

- observed symmetry
  $\rightarrow f_{\overline{\psi}}(\xi) = f_{\psi}(\xi)$
  $\Rightarrow$ meson ✓

PDF



observations:

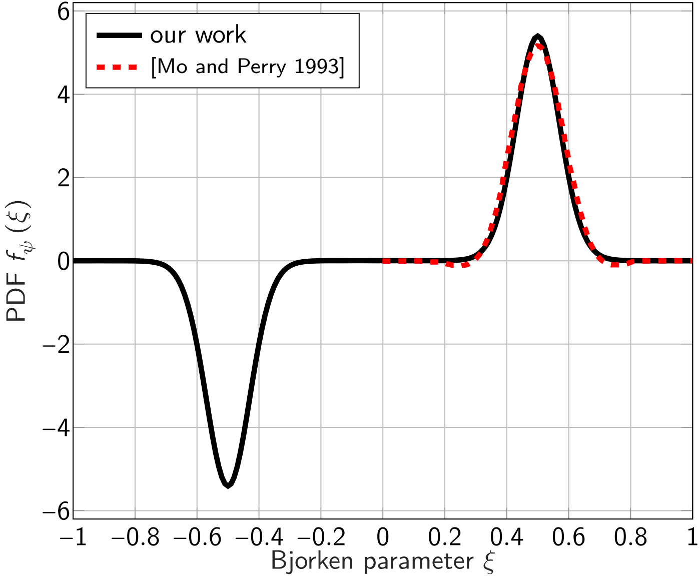
- $\xi > 0$: $f_\psi \approx$ symmetric around $\xi = 0.5$
- antifermion PDF from negative ξ:

$$f_{\overline{\psi}}(\xi) = -f_\psi(-\xi)$$

- observed symmetry
  $\rightarrow f_{\overline{\psi}}(\xi) = f_\psi(\xi)$
  $\Rightarrow$ meson ✓

PDF

$\tilde{m} = 10;\ x = 100;\ D = 80;\ N_{\mathcal{T}} = 100;\ \tilde{V} = 100$



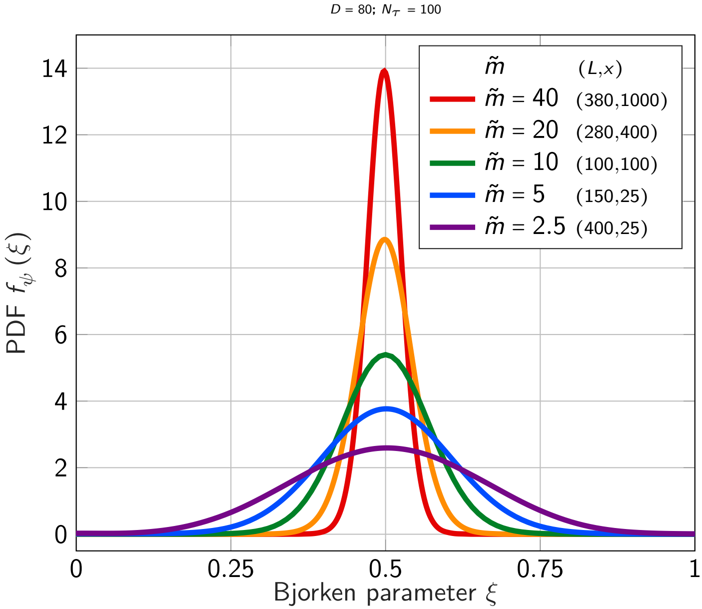
observations:

- $\xi > 0$: $f_\psi \approx$ symmetric around $\xi = 0.5$
- antifermion PDF from negative ξ:

$$f_{\overline{\psi}}(\xi) = -f_\psi(-\xi)$$

- observed symmetry
  $\rightarrow f_{\overline{\psi}}(\xi) = f_\psi(\xi)$
  $\Rightarrow$ meson ✓

PDF



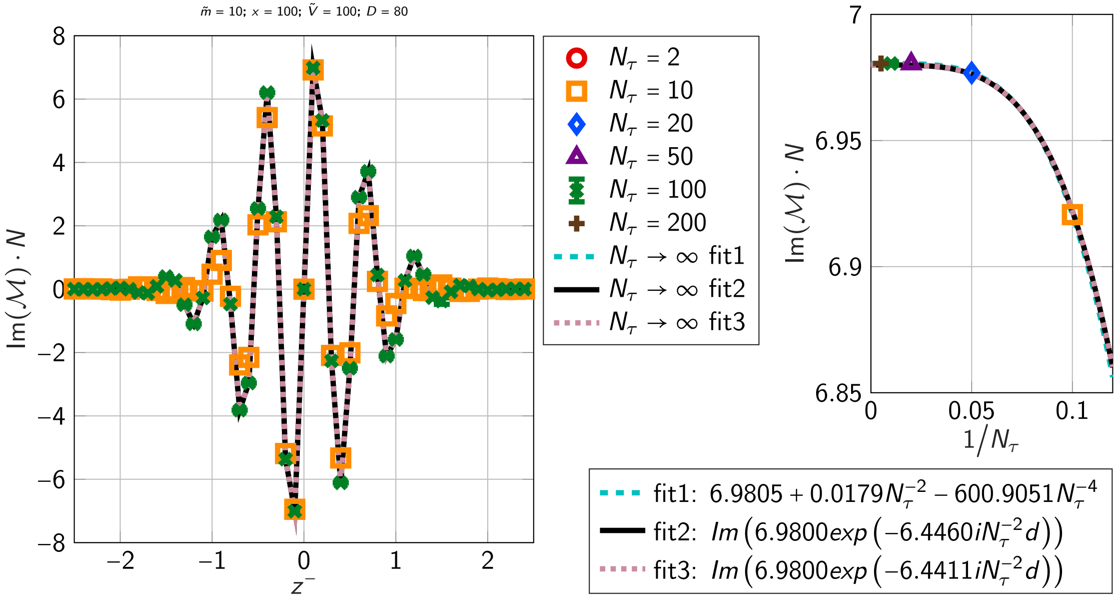
observations:

- $\xi > 0$: $f_\psi \approx$ symmetric around $\xi = 0.5$
- antifermion PDF from negative ξ:

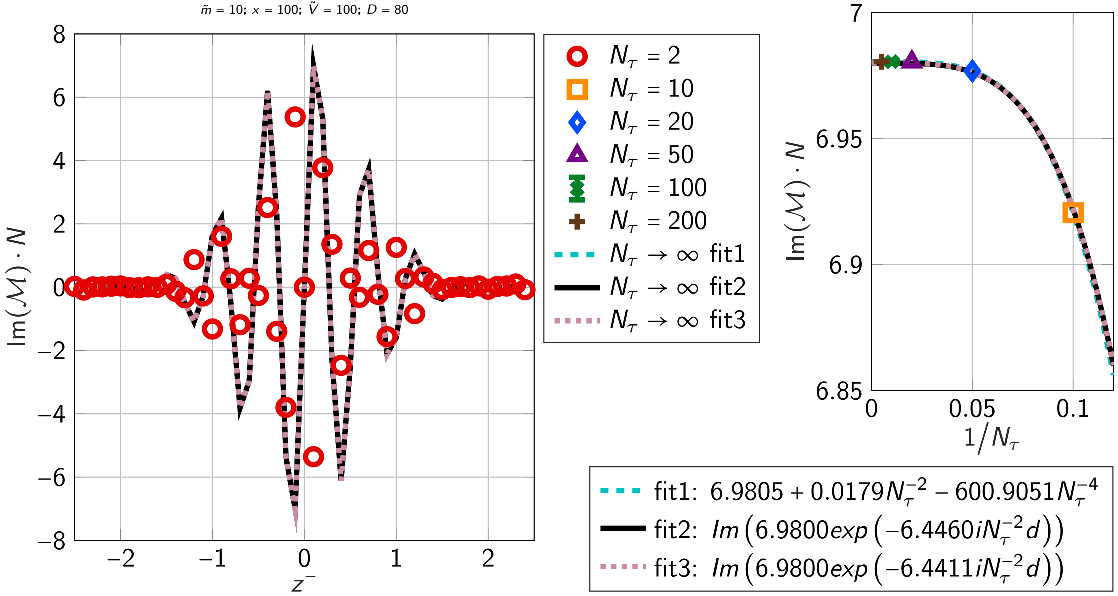
$$f_{\overline{\psi}}(\xi) = -f_\psi(-\xi)$$

- observed symmetry
  $\rightarrow f_{\overline{\psi}}(\xi) = f_\psi(\xi)$
  $\Rightarrow$ meson ✓
- peak broadens with decreasing fermion mass ✓

N_τ-dependence



N_τ-dependence – too small

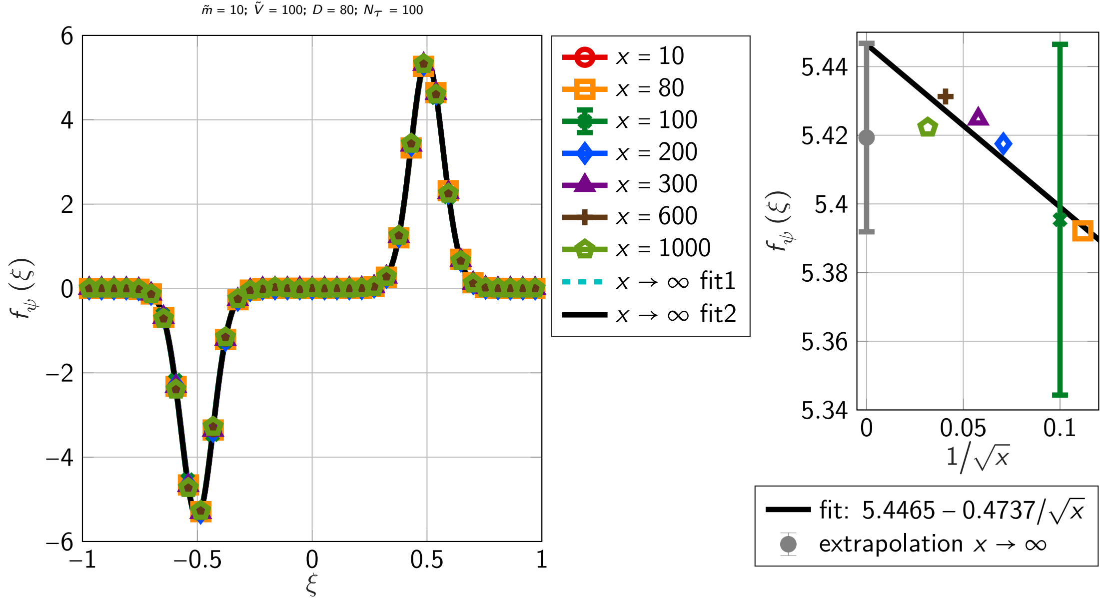


fit1: $6.9805 + 0.0179 N_\tau^{-2} - 600.9051 N_\tau^{-4}$

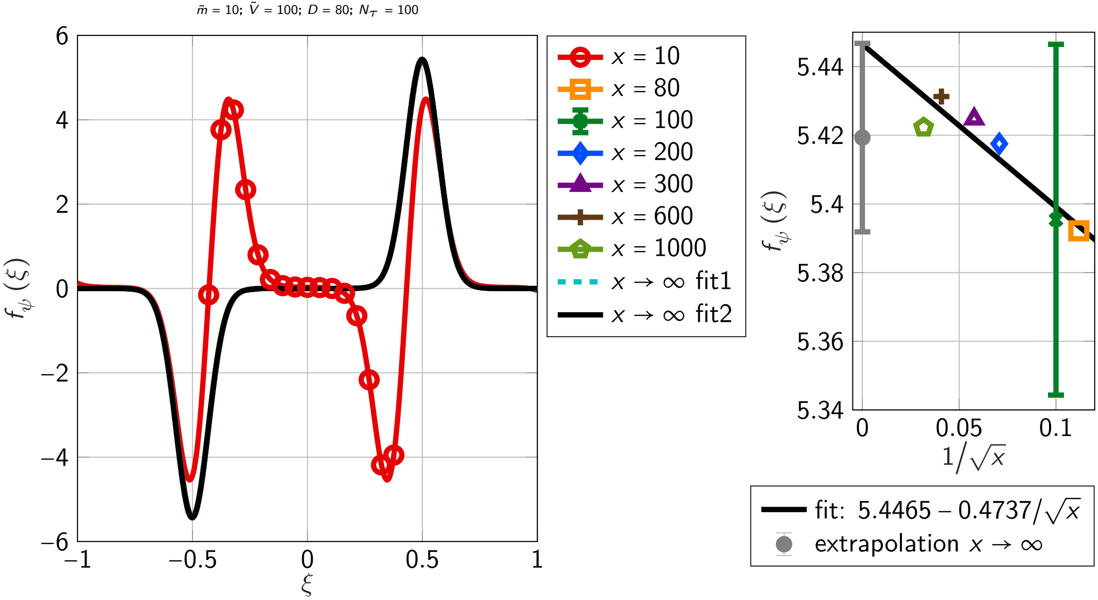
fit2: $Im\left(6.9800 exp\left(-6.4460 i N_\tau^{-2} d\right)\right)$

fit3: $Im\left(6.9800 exp\left(-6.4411 i N_\tau^{-2} d\right)\right)$

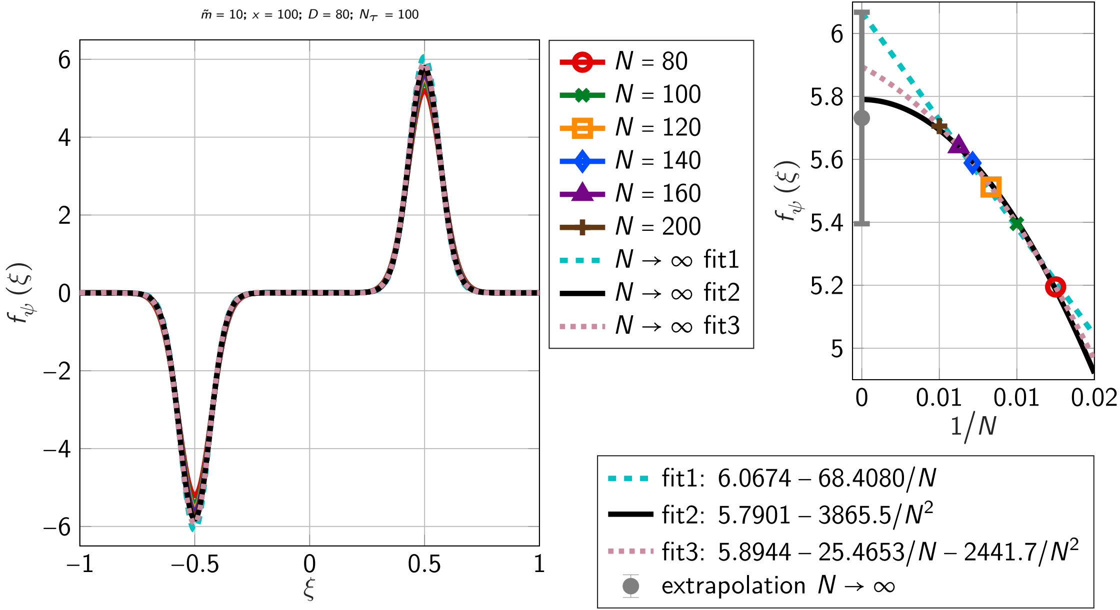
x-dependence



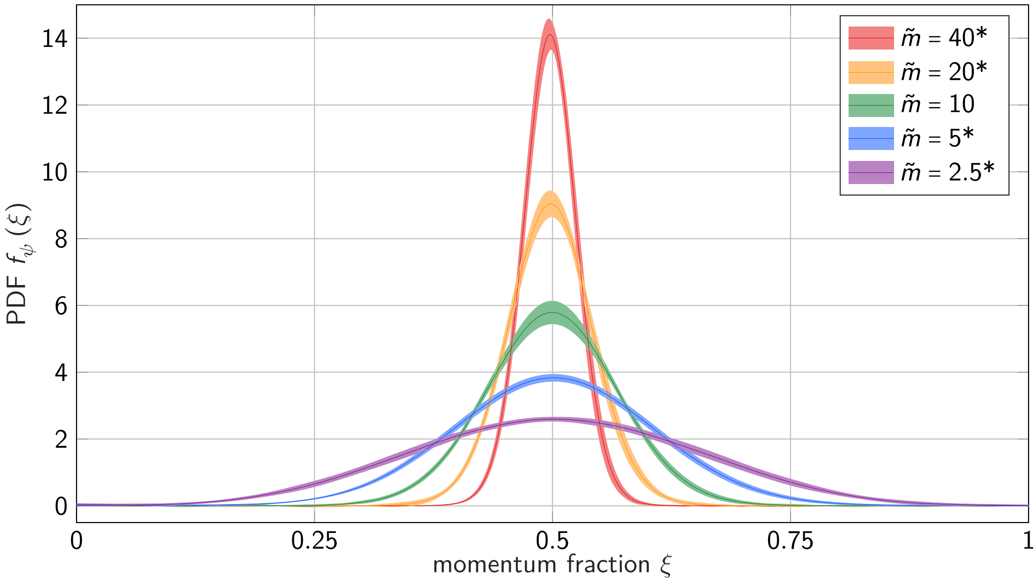
x-dependence – too small



N-dependence

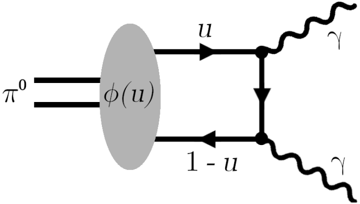


PDF [*preliminary]

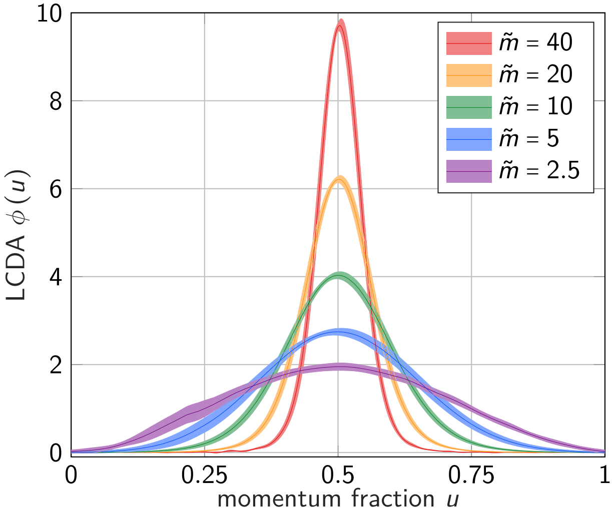


Lightcone Distribution Amplitude (LCDA) [preliminary]

LCDA: decay or hadronization



$$\pi^0 \to q(u)\bar{q}(1-u) \to \gamma\gamma$$



$$\boxed{iF_M\phi(u) = \int dz^- e^{iuP^+z^-}\left\langle 0\middle|\bar{\psi}(z^-)\gamma^+ W(z^- \leftarrow 0)\psi(0)\middle|P\right\rangle}$$

Outline

1 Introduction: Hadron Structure Functions

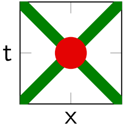
2 Progress: Hamiltonian approaches

3 Example: Schwinger Model

4 Summary & Outlook

Summary

- hadron structure functions
  → universal properties of hadrons
  → important for experiments
- Euclidean space: lightcone → point
- ⇒ use tensor network states / quantum devices
- 1+1D: established (many works)
- Schwinger model:
  fermion- and anti-fermion- PDF and LCDA of vector meson ✓



Collaborators:

Mari
Carmen
Bañuls

Krzysztof
Cichy

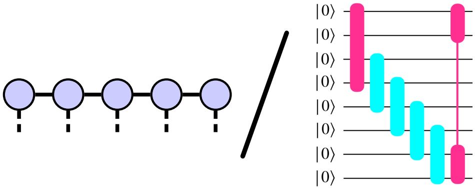
C.-J. David
Lin

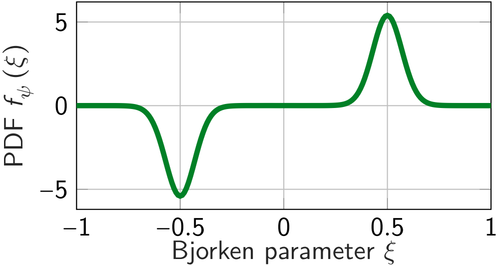
[arXiv:2504.07508]
[arXiv:2409.16996]

Outlook

- quantum computing
    - enforcing Gauss law
    - efficient state preparation and time evolution
    - scaling
- tensor networks
    - classical computation complementary to Monte Carlo ✓
    - verification, parameter scans ✓
    - entanglement barriers (?)
- quantum simulation
- theory
    - renormalization effects
    - truncating gauge degrees of freedom
- physics program
    - higher dimensions
    - non-Abelian
    - further parton structure functions

⇒ QCD ☺





[1] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, (Cambridge University Press, Mar. 2014), ISBN: 978-1-107-03473-0, 978-1-107-03473-0, doi:10.1017/9781139540940.

[2] H. Bergknoff, "Physical particles of the massive schwinger model," Nuclear Physics B **122**, 215–236, ISSN: 0550-3213 (1977) doi:https://doi.org/10.1016/0550-3213(77)90204-8.

[3] T. Eller, H. C. Pauli, and S. J. Brodsky, "Discretized Light Cone Quantization: The Massless and the Massive Schwinger Model," Phys. Rev. D **35**, 1493 (1987) doi:10.1103/PhysRevD.35.1493.

[4] Y. Mo and R. J. Perry, "Basis function calculations for the massive schwinger model in the light-front tamm-dancoff approximation," Journal of Computational Physics **108**, 159–174 (1993) doi:10.1006/jcph.1993.1171.

[5] H. Kröger and N. Scheu, "The massive schwinger model - a hamiltonian lattice study in a fast moving frame," Physics Letters B **429**, 58–63, ISSN: 0370-2693 (1998) doi:https://doi.org/10.1016/S0370-2693(98)00449-3.

[6] H. Jirari, H. Kröger, K. Moriarty, and N. Scheu, "The massive schwinger model in a fast moving frame," Nuclear Physics B - Proceedings Supplements **73**, 733–735, ISSN: 0920-5632 (1999) doi:https://doi.org/10.1016/S0920-5632(99)85188-3.

[7] J. S. Pedernales, R. Di Candia, I. L. Egusquiza, J. Casanova, and E. Solano, "Efficient quantum algorithm for computing *n*-time correlation functions," Phys. Rev. Lett. **113**, 020505 (2014) doi:10.1103/PhysRevLett.113.020505.

[8] M. G. Echevarria, I. L. Egusquiza, E. Rico, and G. Schnell, “Quantum simulation of light-front parton correlators,” Phys. Rev. D **104**, 014512 (2021) `doi:10.1103/PhysRevD.104.014512`.

[9] R. D. Pisarski, “Wilson loops in the hamiltonian formalism,” Phys. Rev. D **105**, L111501 (2022) `doi:10.1103/PhysRevD.105.L111501`.

[10] H. Lamm, S. Lawrence, and Y. Yamauchi, “Parton physics on a quantum computer,” Phys. Rev. Res. **2**, 013272 (2020) `doi:10.1103/PhysRevResearch.2.013272`.

[11] M. Kreshchuk, S. Jia, W. M. Kirby, G. Goldstein, J. P. Vary, and P. J. Love, “Simulating hadronic physics on noisy intermediate-scale quantum devices using basis light-front quantization,” Phys. Rev. A **103**, 062601 (2021) `doi:10.1103/PhysRevA.103.062601`.

[12] M. Kreshchuk, S. Jia, W. M. Kirby, G. Goldstein, J. P. Vary, and P. J. Love, “Light-front field theory on current quantum computers,” Entropy **23**, ISSN: 1099-4300 (2021) `doi:10.3390/e23050597`.

[13] M. Kreshchuk, W. M. Kirby, G. Goldstein, H. Beauchemin, and P. J. Love, “Quantum simulation of quantum field theory in the light-front formulation,” Phys. Rev. A **105**, 032418 (2022) `doi:10.1103/PhysRevA.105.032418`.

[14] T. Li, X. Guo, W. K. Lai, X. Liu, E. Wang, H. Xing, D.-B. Zhang, and S.-L. Zhu, “Partonic collinear structure by quantum computing,” Phys. Rev. D **105**, L111502 (2022) `doi:10.1103/PhysRevD.105.L111502`.

[15] W. Qian, R. Basili, S. Pal, G. Luecke, and J. P. Vary, “Solving hadron structures using the basis light-front quantization approach on quantum computers,” Phys. Rev. Res. **4**, 043193 (2022) `doi:10.1103/PhysRevResearch.4.043193`.

[16] M. Schneider, M. C. Bañuls, K. Cichy, and C.-J. D. Lin, “Parton Distribution Functions in the Schwinger Model with Tensor Networks,” in Proceedings of the 41st international symposium on lattice field theory — pos(lattice2024), Vol. 466 (2025), p. 024, `doi:10.22323/1.466.0024`.

[17] M. C. Bañuls, K. Cichy, C.-J. D. Lin, and M. Schneider, “Parton distribution functions in the schwinger model from tensor network states,” Phys. Rev. D **113**, L011502 (2026) `doi:10.1103/lcyh-rxf3`.

[18] Z.-B. Kang, N. Moran, P. Nguyen, and W. Qian, “Partonic distribution functions and amplitudes using tensor network methods,” Journal of High Energy Physics **2025**, 176, ISSN: 1029-8479 (2025) `doi:10.1007/JHEP09(2025)176`.

[19] J.-W. Chen, Y.-T. Chen, and G. Meher, “Parton distributions on a quantum computer,” (2025).

[20] J. J. Gálvez-Viruet, F. J. Llanes-Estrada, N. M. de Arenaza, M. Gómez-Rocha, and T. J. Hobbs, “Quantum computing hadron fragmentation functions in light-front chromodynamics,” (2025).

[21] J. J. Gálvez-Viruet, F. J. Llanes-Estrada, and M. Gómez-Rocha, “Preparations for quantum computing in hadron physics,” International Journal of Modern Physics A **41**, 2630003 (2026) `doi:10.1142/S0217751X26300036`.

[22] S. Grieninger, K. Ikeda, and I. Zahed, “Quasiparton distributions in massive qed2: toward quantum computation,” Phys. Rev. D **110**, 076008 (2024) doi:10.1103/PhysRevD.110.076008.

[23] S. Grieninger and I. Zahed, “Quasifragmentation functions in the massive schwinger model,” Phys. Rev. D **110**, 116009 (2024) doi:10.1103/PhysRevD.110.116009.

[24] S. Grieninger, J. Montgomery, F. Ringer, and I. Zahed, “Tensor network simulations of quasi-gpds in the massive schwinger model,” Phys. Rev. D **113**, 074509 (2026) doi:10.1103/sz3z-8cyx.

[25] M. C. Bañuls, K. Cichy, J. I. Cirac, and K. Jansen, “The mass spectrum of the schwinger model with matrix product states,” JHEP **2013**, 158 (2013) doi:10.1007/JHEP11(2013)158.

[26] J. Dai, J. Hughes, and J. Liu, “Next-to-leading order parton model calculations in the massless schwinger model,” Physics Letters B **327**, 347–353, ISSN: 0370-2693 (1994) doi:https://doi.org/10.1016/0370-2693(94)90740-4.

[27] J. Dai, J. Hughes, and J. Liu, “Perturbative analysis of the massless schwinger model,” Phys. Rev. D **51**, 5209–5215 (1995) doi:10.1103/PhysRevD.51.5209.

[28] C. J. Hamer, Z. Weihong, and J. Oitmaa, “Series expansions for the massive schwinger model in hamiltonian lattice theory,” Phys. Rev. D **56**, 55–67 (1997) doi:10.1103/PhysRevD.56.55.

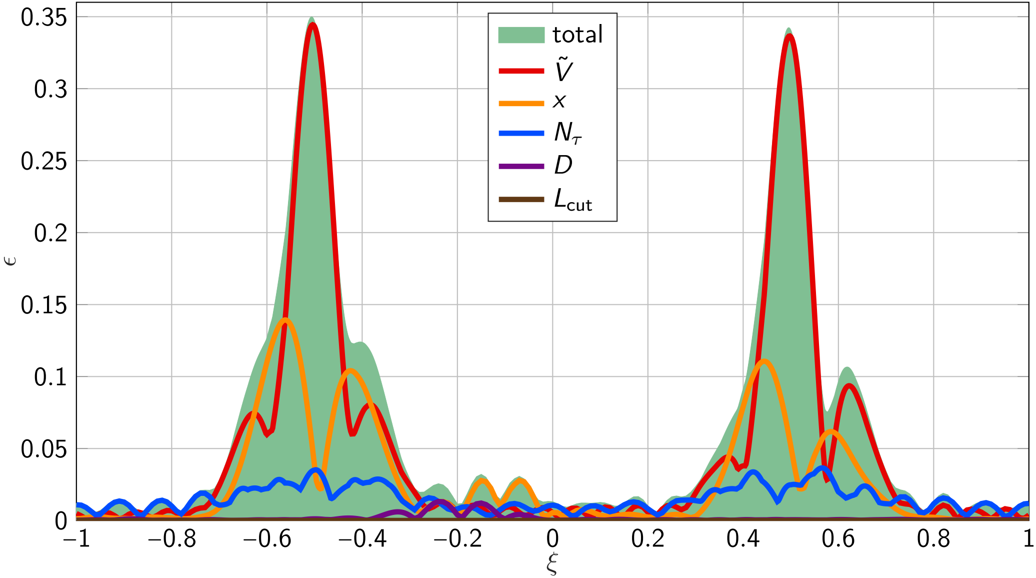
[29] Further image sources, EIC, www.computerhistory.org/timeline/1981, https://openmoji.org, www.flaticon.com/free-icons/search, arXiv:2510.07125.

[30] M. B. Hastings, "An area law for one-dimensional quantum systems," Journal of Statistical Mechanics: Theory & Exp. **2007**, 08024 (2007) `doi:10.1088/1742-5468/2007/08/P08024`.

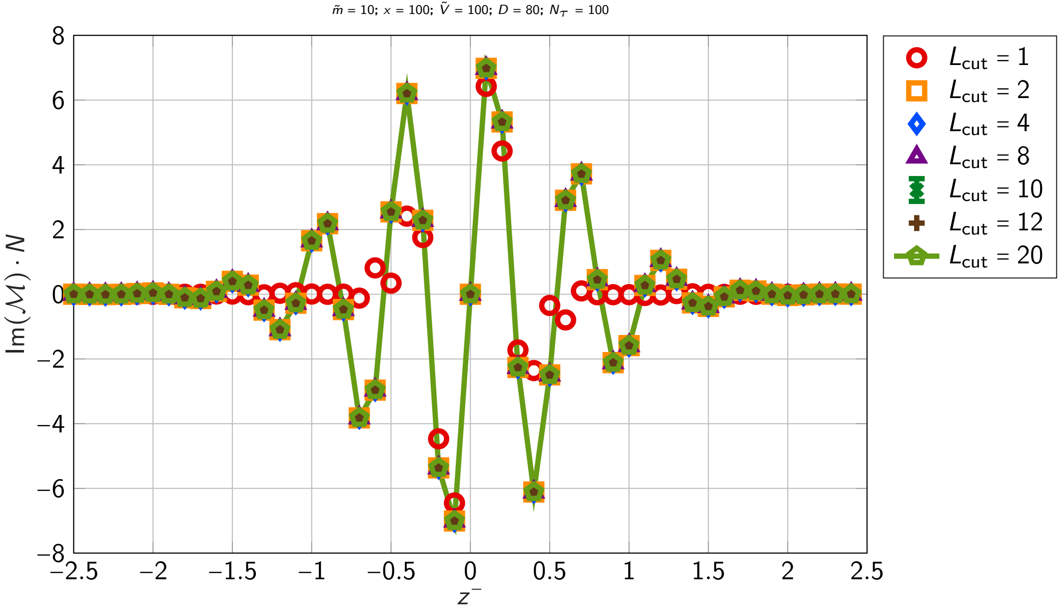
Outline

5 Backup

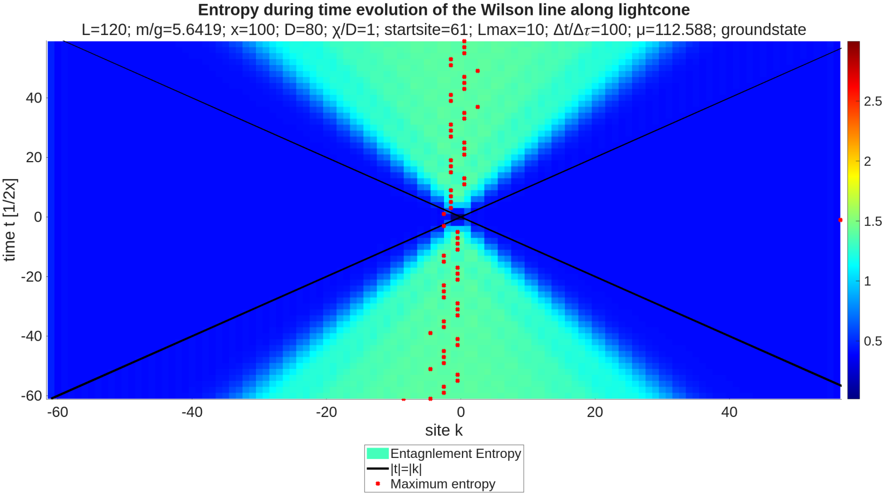
Contributions to error



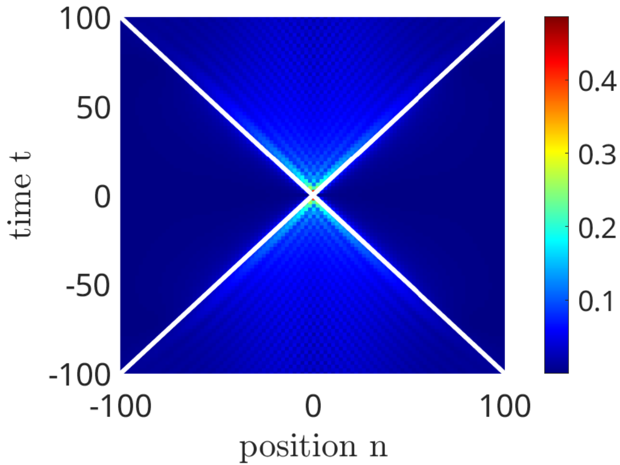
L_{cut}-dependence: truncation of electric field



Entanglement entropy



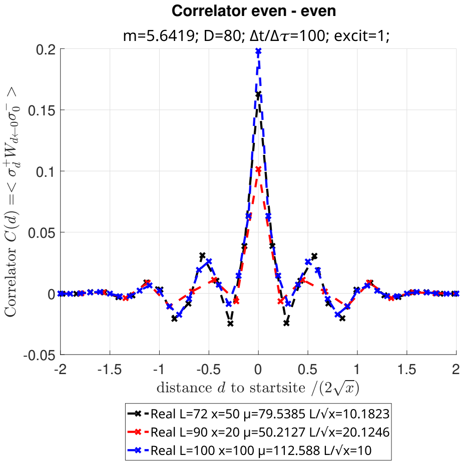
Light-cone structure



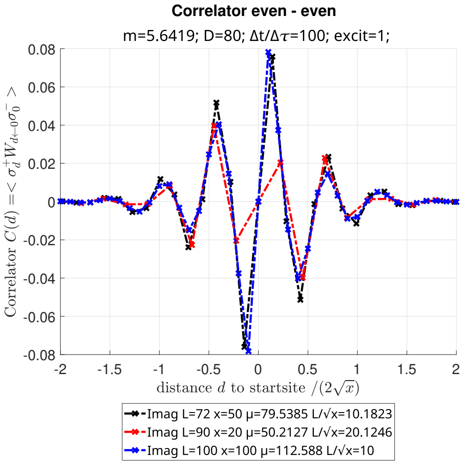
$$\langle P|\, e^{iHt} \prod_{k<n} (i\sigma_k^z)\, \sigma_n^+ e^{-iH_0 t} \prod_{k'<0} (-i\sigma_{k'}^z)\, \sigma_0^- \,|P\rangle$$

- even-to-even matrix element
- calculated to each site at each timeslice
- static charge fixed at orign

Even-even matrix elements



$\mathrm{Re}(C_{ee})$



$\mathrm{Im}(C_{ee})$

Tensor Network States

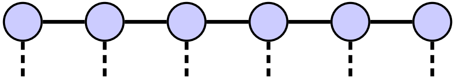
- generic state scales exponentially

$$|\psi\rangle = \sum_{s_1,s_2,\dots,s_N} \Psi^{s_1 s_2 \dots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \dots \otimes |s_N\rangle$$

- tensor network state as ansatz

$$\Psi^{s_1 s_2 \dots s_N} = \sum_{\{i_x\}} A^{1,s_1}_{i_1} \cdot A^{2,s_2}_{i_1,i_2} \cdot A^{3,s_3}_{i_2,i_3} \dots A^{N,s_N}_{i_{N-1}}$$

=



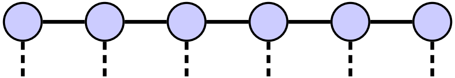
Tensor Network States

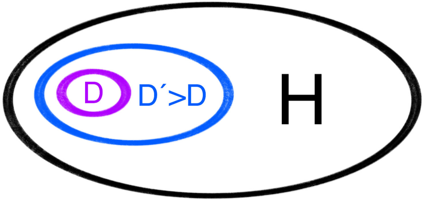
- generic state scales exponentially
- tensor network state as ansatz
- truncation to bond dimenion D
- polynomial resource scaling

$$|\psi\rangle = \sum_{s_1,s_2,\ldots,s_N} \Psi^{s_1 s_2 \ldots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \ldots \otimes |s_N\rangle$$

$$\Psi^{s_1 s_2 \ldots s_N} \approx \sum_{\{i_x\}=1}^{D} A^{1,s_1}_{i_1} \cdot A^{2,s_2}_{i_1,i_2} \cdot A^{3,s_3}_{i_2,i_3} \ldots A^{N,s_N}_{i_{N-1}}$$

=





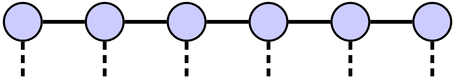
Tensor Network States

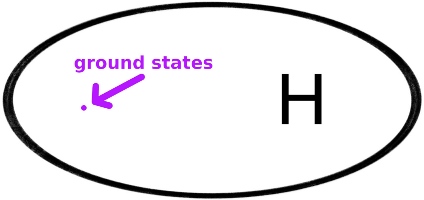
- generic state scales exponentially
- tensor network state as ansatz
- truncation to bond dimenion D
- polynomial resource scaling
- good for ground states and low excited states [Hastings 2007]
- no sampling, fundamentally different systematics compared to Monte Carlo

$$|\psi\rangle = \sum_{s_1,s_2,\dots,s_N} \Psi^{s_1 s_2 \dots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \dots \otimes |s_N\rangle$$

$$\Psi^{s_1 s_2 \dots s_N} \approx \sum_{\{i_x\}=1}^{D} A^{1,s_1}_{i_1} \cdot A^{2,s_2}_{i_1,i_2} \cdot A^{3,s_3}_{i_2,i_3} \dots A^{N,s_N}_{i_{N-1}}$$

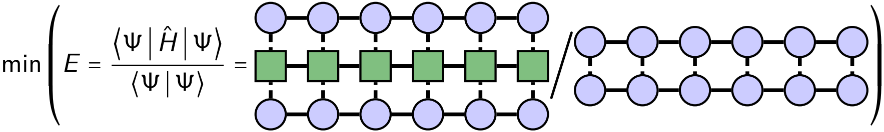
$=$





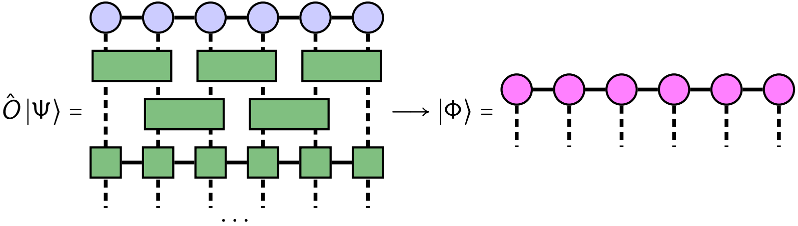
Efficient Tensor Network operations

- Find groundstate and excited states



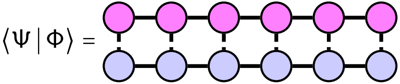
$$\min \left(E = \frac{\langle \Psi | \hat{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \right)$$

- Apply operators / time evolution



$$\hat{O} |\Psi\rangle = \quad \cdots \quad \longrightarrow |\Phi\rangle =$$

- Calculate overlap



$$\langle \Psi | \Phi \rangle =$$

Time evolution with MPS

$$H = x\sum_{n=0}^{N-2}\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\right] + \frac{\mu}{2}\sum_{n=0}^{N-1}\left[1+(-1)^n\sigma_n^z\right] + \sum_{n=0}^{N-2}\left[\frac{1}{2}\sum_{k=0}^{n}\left((-1)^k+\sigma_k^z+2q_k\right)\right]^2$$

Suzuki-Trotter decomposition:

$$e^{-i\tau H} \approx \left(e^{-i\delta\tau H_{eo}}e^{-i\delta\tau H_{oe}}e^{-i\delta\tau H_L}\right)^{N_\tau}$$

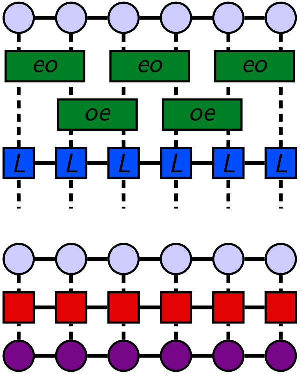
$$e^{-i\delta\tau H_{eo}} = \prod_{n=0,2,\dots} e^{-i\delta\tau x\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\right]}$$

$$e^{-i\delta\tau H_{oe}} = \prod_{n=1,3,\dots} e^{-i\delta\tau x\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+\right]}$$

$e^{-i\delta\tau H_L}$: MPO with indices $L_n \in [-n, n]$

→ truncate to L_{cut}

Optimize to get new MPS



Factorization

cross section:

$$\sigma \propto L^{\mu\nu}(k,q)\,W_{\mu\nu}(q,P)$$

hadronic Tensor:

$$W_{\mu\nu}(\xi,P) = \sum_i \int_x^1 \frac{dz}{z} f_i(z)\,\hat{W}_{\mu\nu}\left(\frac{\xi}{z},Q\right)$$

leading order with $\hat{W} \propto \delta\left(1-\frac{\xi}{z}\right)$:

$$W_{\mu\nu}(q,P) = 4\pi\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)F_1+\frac{8\pi x}{Q^2}\left(P_\mu-\frac{P\cdot q}{q^2}q_\mu\right)\left(P^\nu-\frac{P\cdot q}{q^2}q^\nu\right)F_2$$

factorization (leading order):

$$F_1(\xi) = \frac{1}{2}\sum_i e_i^2 f_i(\xi)$$
$$F_2(\xi) = 2xF_1(\xi)$$