# Hadron Structure by Quantum Computing

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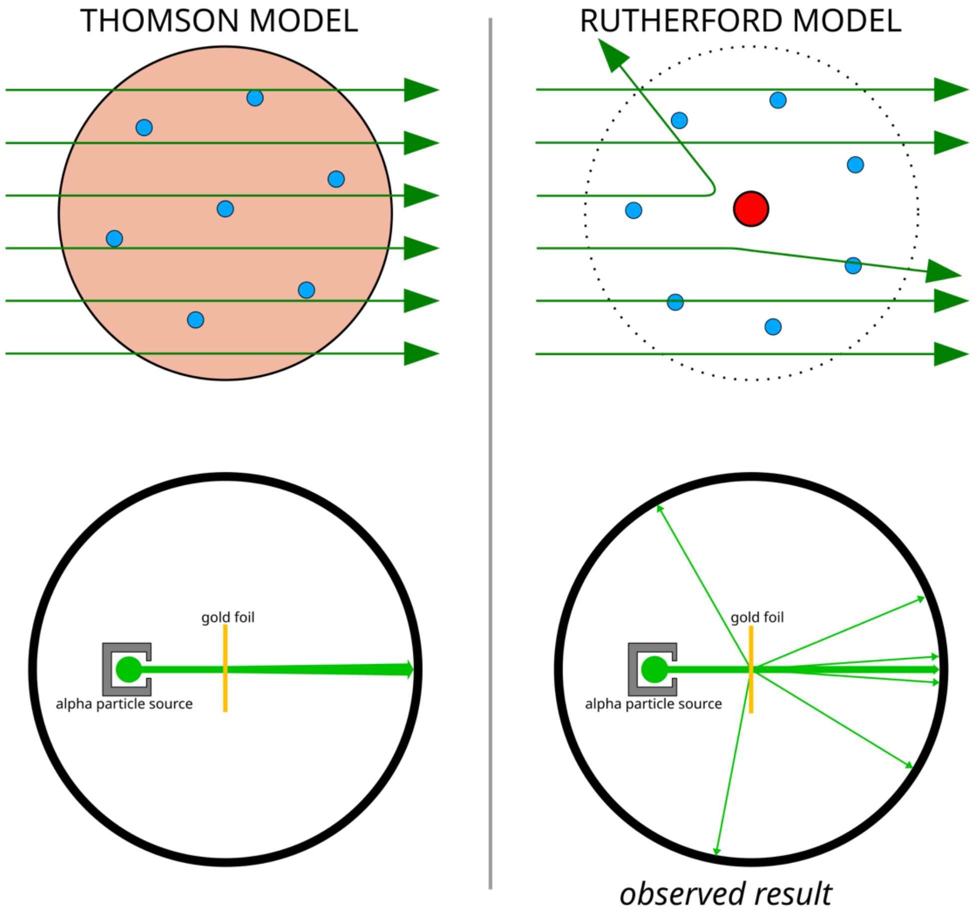
From RIKEN iTHEMS

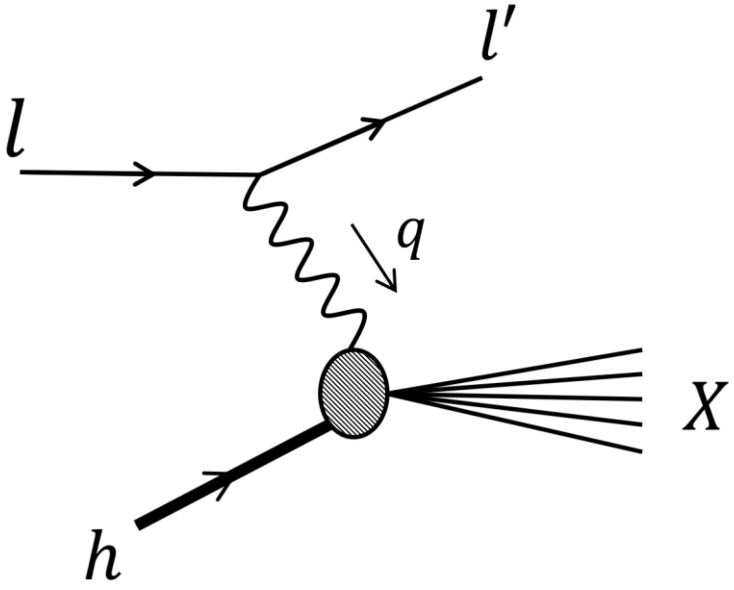
C3NT workshop

# From Rutherford scattering to deep inelastic scattering

Rutherford scattering: prob the structure of the atom



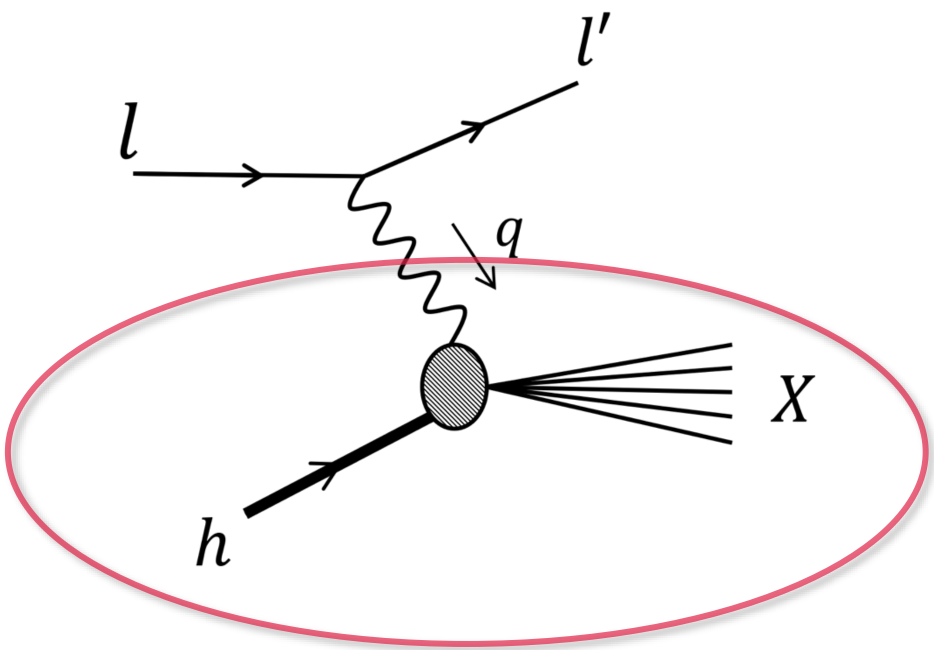

Deep inelastic scattering: prob the structure of the proton (hadron)




$$l(p) + h(k) \to l(p') + X$$

$$-q^2 \gg m_h^2$$

# Hadronic tensor




The cross-section of deep inelastic scattering:

$$\sigma \sim L^{\mu\nu} W_{\mu\nu}$$

- $L^{\mu\nu}$: leptonic tensor, can be calculated analytically.
- $W^{\mu\nu}$: hadronic tensor, can be used to extract hadron structure.

Hadronic tensor is a **time-dependent** **non-perturbative** quantity:

$$W^{\mu\nu}(q) = \int d^{\mathrm{d}}z\, \mathrm{e}^{\mathrm{i}qz} \boxed{\left\langle \boldsymbol{h} \right| \boldsymbol{e^{iHz^0}} J^{\mu}(0,\vec{z})\, \boldsymbol{e^{-iHz^0}} J^{\nu}(0) \left| \boldsymbol{h} \right\rangle}$$

**Quantum computing is a non-perturbative method without sign problems!**

$J^{\mu}(z) = \bar{\psi}(z)\gamma^{\mu}\psi(z)$: The electric current operator.

# Procedures for simulating hadronic tensor on quantum computers

"... and if you want to make a simulation of nature, you'd better make it quantum mechanical, ..." – R. Feynman

$$W^{\mu\nu}(q) = \int d^{\mathrm{d}}z\, \mathrm{e}^{\mathrm{i}qz} \left\langle \boldsymbol{h} \right| \boldsymbol{e}^{\boldsymbol{iHz^0}} J^{\mu}(0, \vec{z})\, \boldsymbol{e}^{\boldsymbol{-iHz^0}} J^{\nu}(0) \left| \boldsymbol{h} \right\rangle$$

Four steps for simulating $W^{\mu\nu}(q)$ on quantum computers:

1. Put quantum fields on lattice.
2. Map lattice fields to qubits.
3. Preparation of the hadron state $|\boldsymbol{h}\rangle$.
4. Evaluate the two-point correlation function.

The procedure is clear, let's move on!

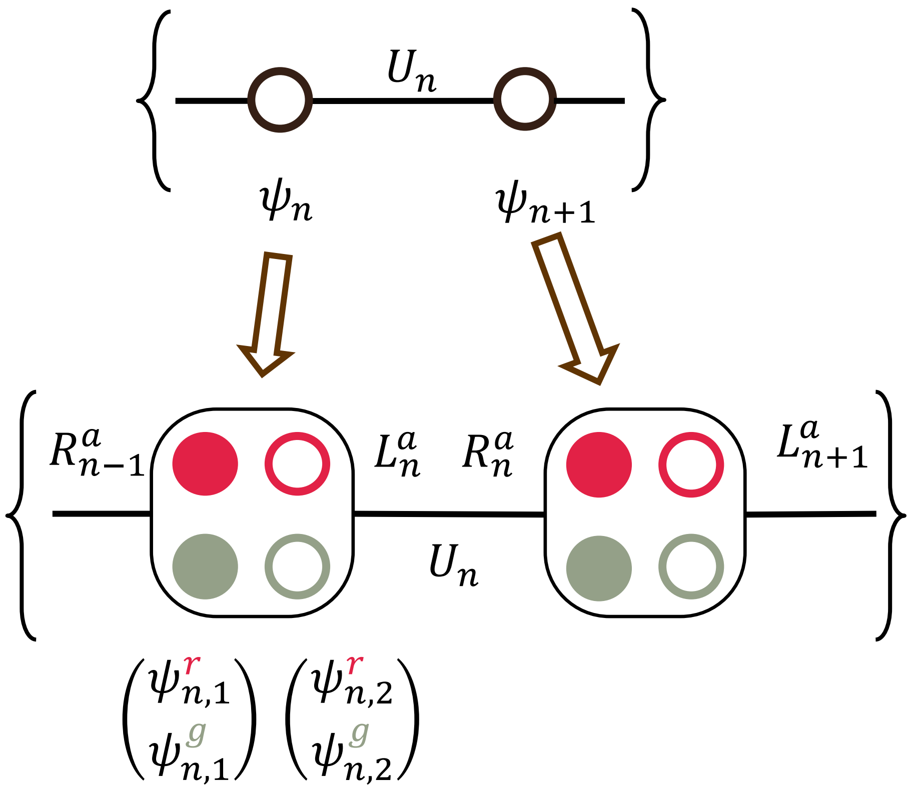
# U(1) and SU(2) lattice gauge theories

Hamiltonian on lattice:

$$H = -\sum_n \frac{i}{2}\bar{\psi}_n\gamma^1[U_n(r+i\gamma^1)\psi_{n+1} - h.c.]$$
$$+(m+r)\sum_n \bar{\psi}_n\psi_n + \frac{g^2}{2}\sum_n L_n^2 = H_{kin} + H_m + H_E$$

- We use **Wilson fermion** to discretize the fermion field because it preserves both **charge conjugation (C) and parity (P) symmetry**.
- We fix the axial gauge $A_1^a = 0$ to eliminate the gauge field:
  - $U_n = \exp(igA_1^a(n)t_a) = I$
  - Gauss's law $L_n^a - L_{n-1}^a = \bar{\psi}_n\gamma^0 t^a\psi_n$
  - Map to qubits by Jordan-Wigner transformation.
- For U(1) case, we set number of color as 1 and $t^a = 1$.

- Example: two space points



# Hadron state of U(1) and SU(2) gauge theories

A hadron state is labeled by the quantum numbers:

Quantum number of U(1) gauge theory with periodic boundary condition (PBC):

- Momentum, defined by the translation operator $T$:

$$T|p,Q\rangle = e^{-ip}|p,Q\rangle$$

- Electric charge $Q$:

$$Q|p,Q\rangle = Q|p,Q\rangle$$

- Meson state at rest $|h_M(p=0)\rangle$:
  - $T|h_M(p=0)\rangle = |h_M(p=0)\rangle$
  - $Q|h_M(p=0)\rangle = 0$

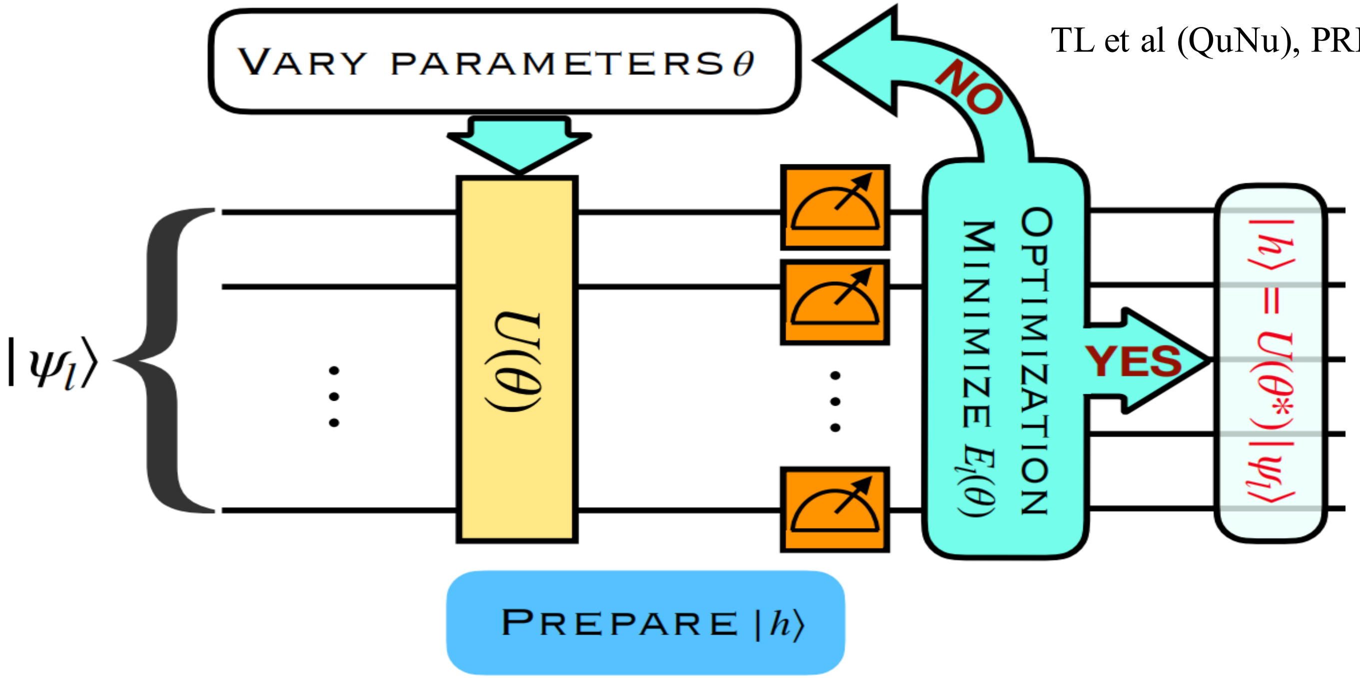
Quantum number of SU(2) gauge theory with PBC:

- Momentum
- Baryon number $B$
- Non-abelian charge $Q^a = \sum_n \bar{\psi}_n \gamma^0 t^a \psi_n$:
  - $Q^3|Phys\rangle = 0.$
  - $\sum_a (Q^a)^2\,|Phys\rangle = 0.$
- Meson state: $B|h_M(p)\rangle = 0.$
- Baryon state: $B|h_B(p)\rangle = +1|h_B(p)\rangle.$

# Preparation of a hadron state by VQE

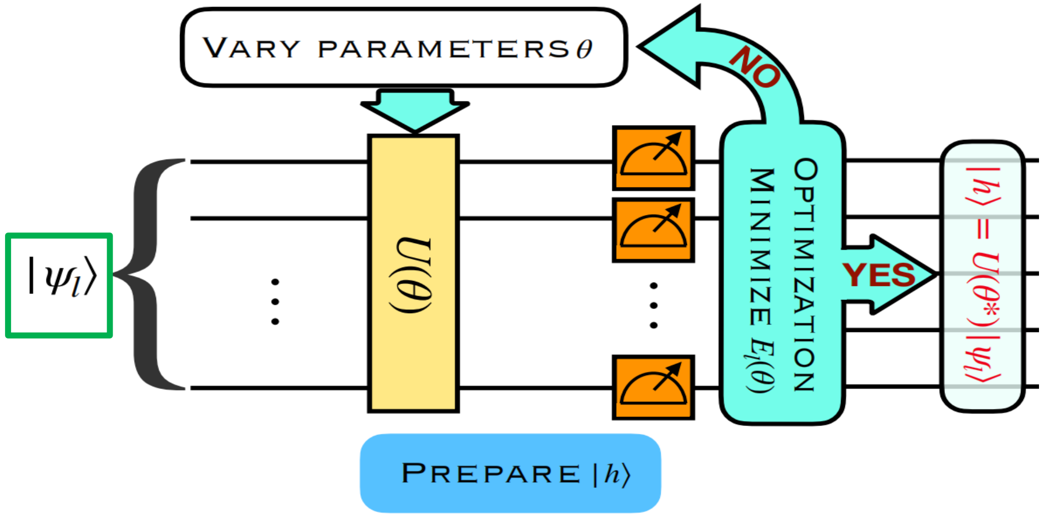
We prepare the hadron state by quantum number resolving VQE

TL et al (QuNu), PRD (letter, 2022)




- $|\psi_l\rangle$ : initial reference states, have the same quantum number with hadron state $|h\rangle$.
- $U(\theta)$: VQE ansatz, preserves the quantum number.
- $E_l(\theta)$: loss function, need to be minimized.

# Preparation of a hadron state by VQE




Example: reference states of vacuum and the lightest meson in SU(2).

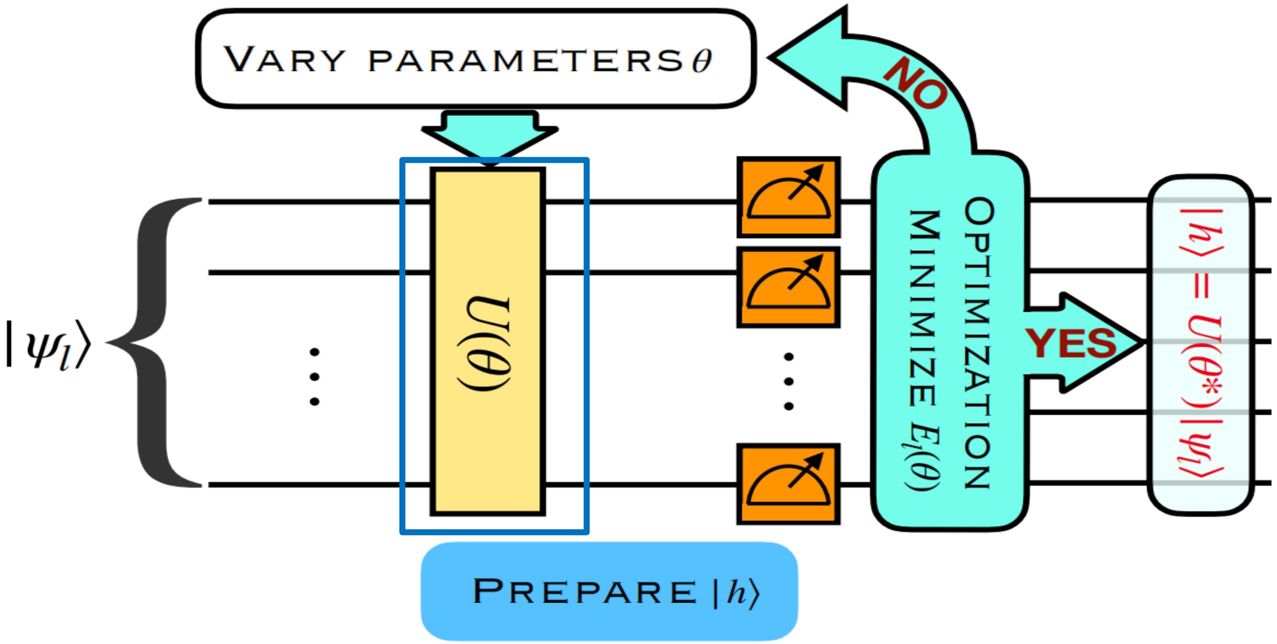
- ➢ We need two reference states.
- ➢ Quantum numbers: $Q = 0, p = 0, Q^3 = 0, (Q)^2 = 0, B = 0.$

- ➢ 4-qubits local reference states on lattice point $n$ $|\psi_{B,i}\rangle_n$:
  - Color singlet: $Q_n^3|\psi_{B,i}\rangle_n = 0, (Q)^2|\psi_{B,i}\rangle_n = 0$
  - $|\psi_{B=0,i=0}\rangle_n = \frac{1}{2}(|0011\rangle - |0110\rangle + |1001\rangle + |1100\rangle)$
  - $|\psi_{B=0,i=1}\rangle_n = \frac{1}{\sqrt{2}}(|0011\rangle - |1100\rangle)$
- ➢ Zero momentum reference states
  - Vacuum reference state $\otimes_n |\psi_{B=0,i=0}\rangle_n$
  - Meson reference state :

$$\sum_i \left(\otimes_{i=0}^{n-1} |\psi_{B=0,i=0}\rangle_i \otimes |\psi_{B=0,i=1}\rangle_n \otimes_{j=n+1}^{N-1} |\psi_{B=0,i=0}\rangle_j\right)$$

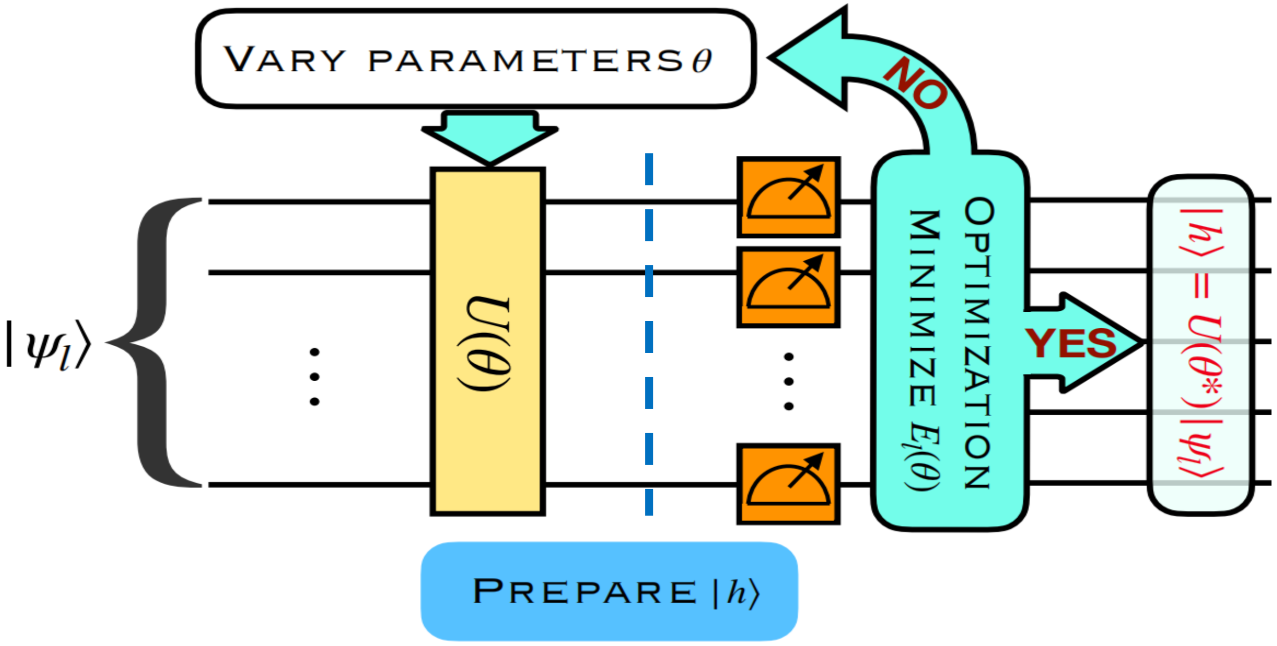
- ➢ Reference states of baryons can be constructed similarly.

# Preparation of a hadron state by VQE




- For given quantum numbers $l$, we use $k$ reference states to obtain the first $k$ excited states.
- Divide $H = H_1 + H_2 + \cdots + H_n$:
  - $[H_i, H_{i+1}] \neq 0$.
  - Every $H_i$ preserve all symmetry of the full Hamiltonian $H$.
  - $U(\theta) = \prod_{i=1}^{p}(\prod_{j=1}^{n} \exp(i\theta_{ij}H_j))$.

# Preparation of a hadron state by VQE

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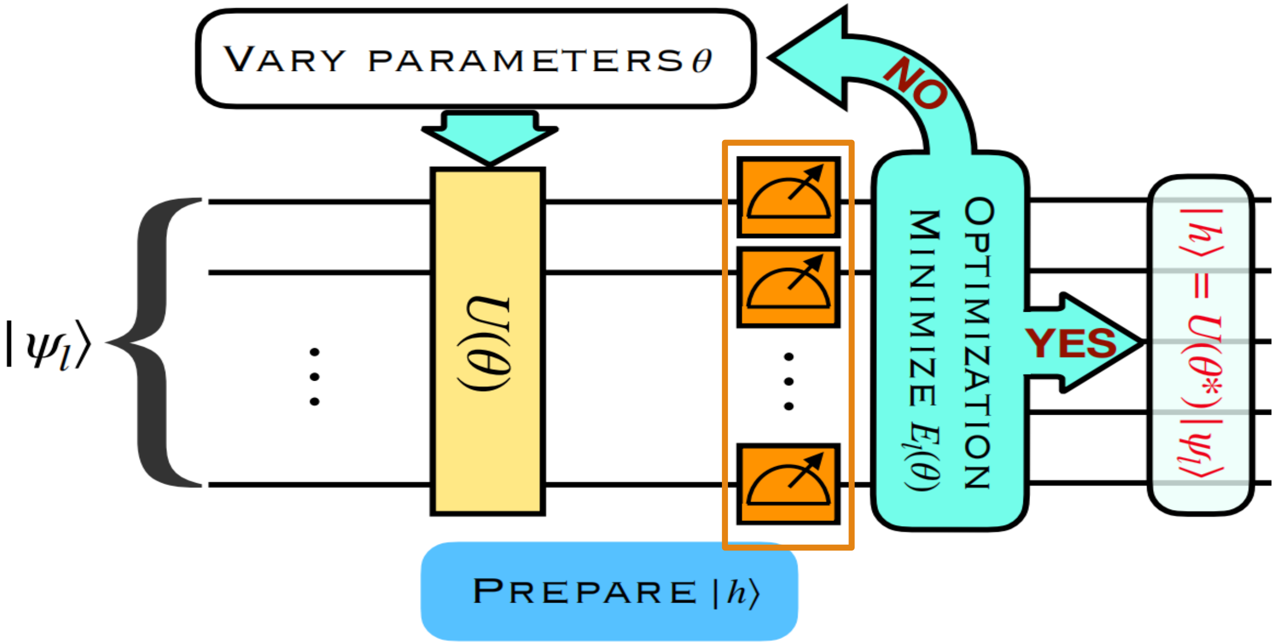


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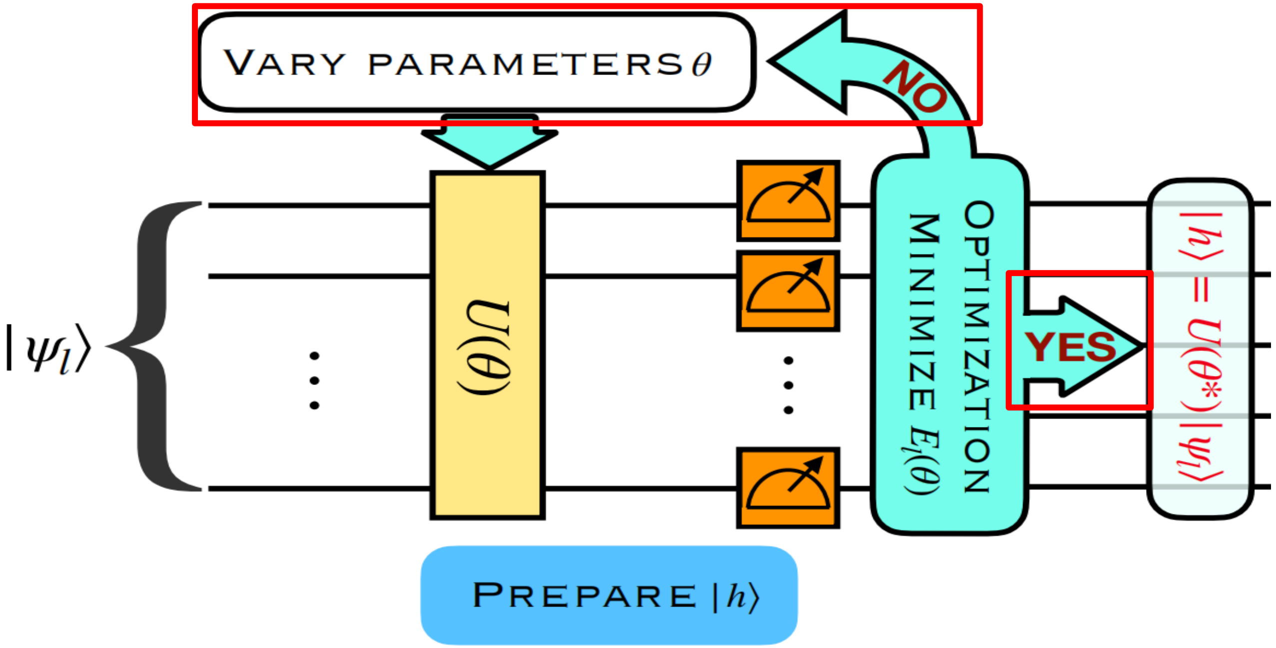
- Measure the loss function $E_l(\theta) = \sum_i^k w_{li}\langle\psi_{li}(\theta)|H|\psi_{li}(\theta)\rangle$ on quantum computers.

# Preparation of a hadron state by VQE

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- Trial wave function: $|\psi_{li}(\theta)\rangle = U(\theta)|\psi_{li}\rangle_{ref}$
- Measure the loss function $E_l(\theta) = \sum_i^k w_{li}\langle\psi_{li}(\theta)|H|\psi_{li}(\theta)\rangle$ on quantum computers.
- Optimize loss function on classical computers

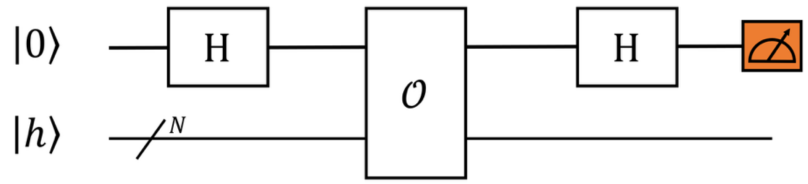
# Evaluation of the current-current correlation functions

- The current-current correlation functions

$$W^{\mu\nu}(q) = \int d^{\mathrm{d}}z\, \mathrm{e}^{\mathrm{i}qz} \left\langle \boldsymbol{h} \right| \boldsymbol{e^{iHz^0}} J^{\mu}(0,\vec{z})\, \boldsymbol{e^{-iHz^0}} J^{\nu}(0) \left| \boldsymbol{h} \right\rangle$$

$$\mathcal{O}_{nm}(t) = \langle h|\, \mathrm{e}^{\mathrm{i}Ht}\, J_n^{\mu}\, \mathrm{e}^{-\mathrm{i}Ht}\, J_m^{\nu}\, |h\rangle$$

Pedernales et al, PRL (2014)




U(1) :

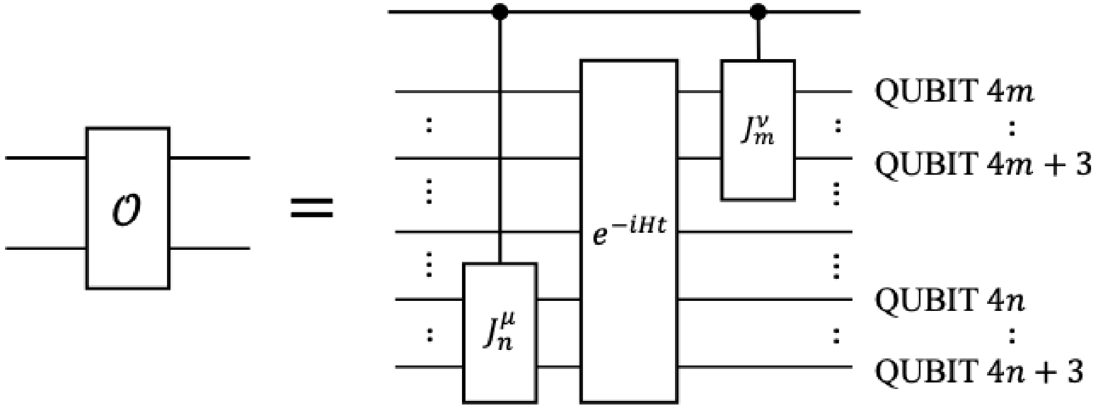
$$J_n^0 = \frac{1}{2}[(-1)^n + \sigma_n^3]$$

$$J_n^1 = \sigma_n^+\, \sigma_{n+1}^- + \sigma_{n+1}^+ \sigma_n^-$$

SU(2) :

$$J_n^0 = \frac{1}{2}(\sigma_{2n}^3 + \sigma_{2n+1}^3)$$

$$J_n^1 = \sigma_{2n}^+\, \sigma_{2n+1}^- + \sigma_{2n+1}^+ \sigma_{2n}^-$$

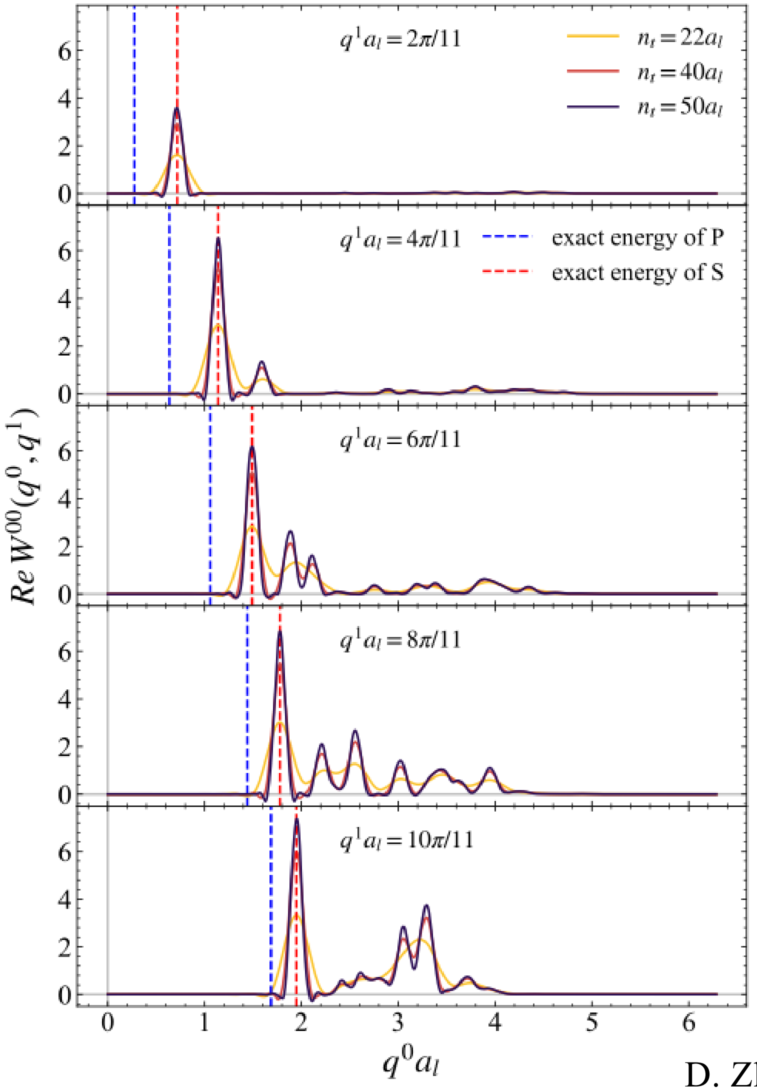


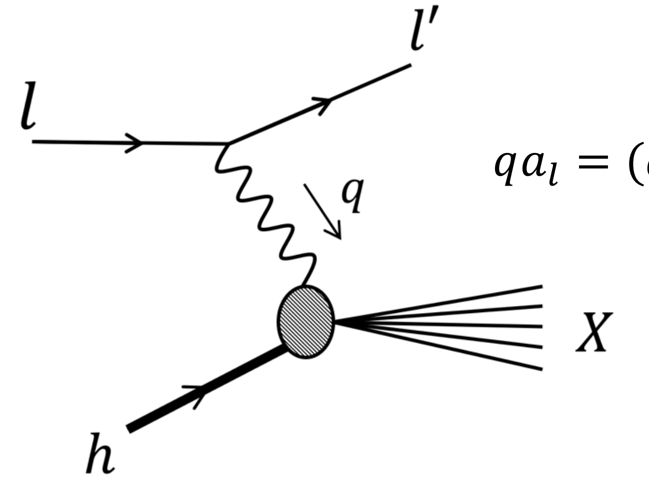

- The Fourier transformation

$$W^{\mu\nu}(q^0, q^1) = \int \mathrm{d}t \sum_{\vec{z}} \mathrm{e}^{\mathrm{i}q^0 t}\, \mathrm{e}^{-\mathrm{i}q^1 z}\, \mathcal{O}_{nm}(t)$$

D. Zhou et al (QuNu), arXiv:2606.17003

# Results of U(1) meson



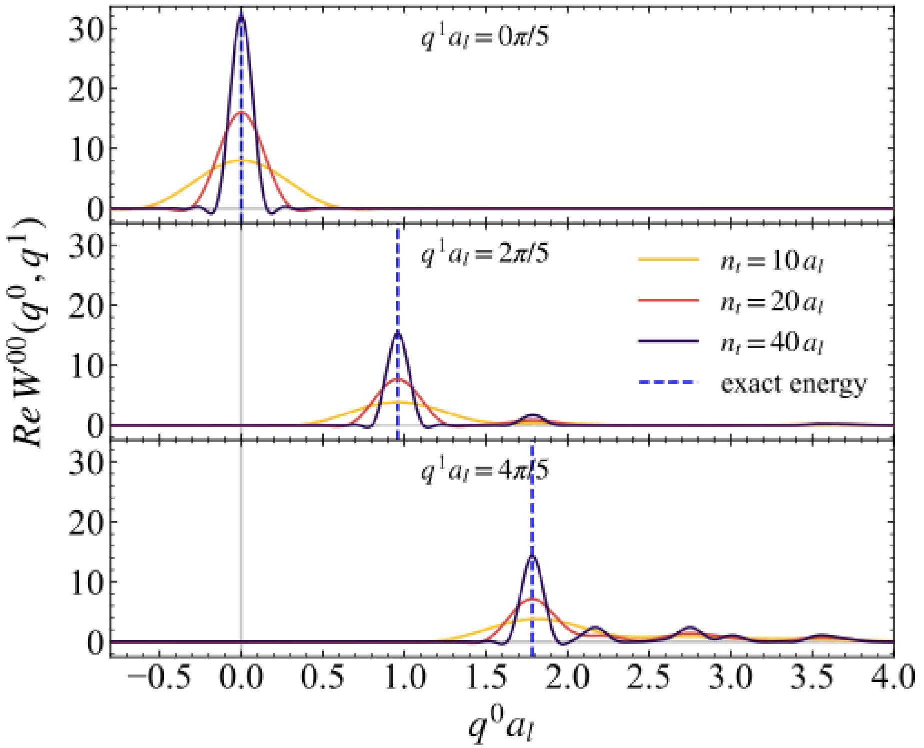



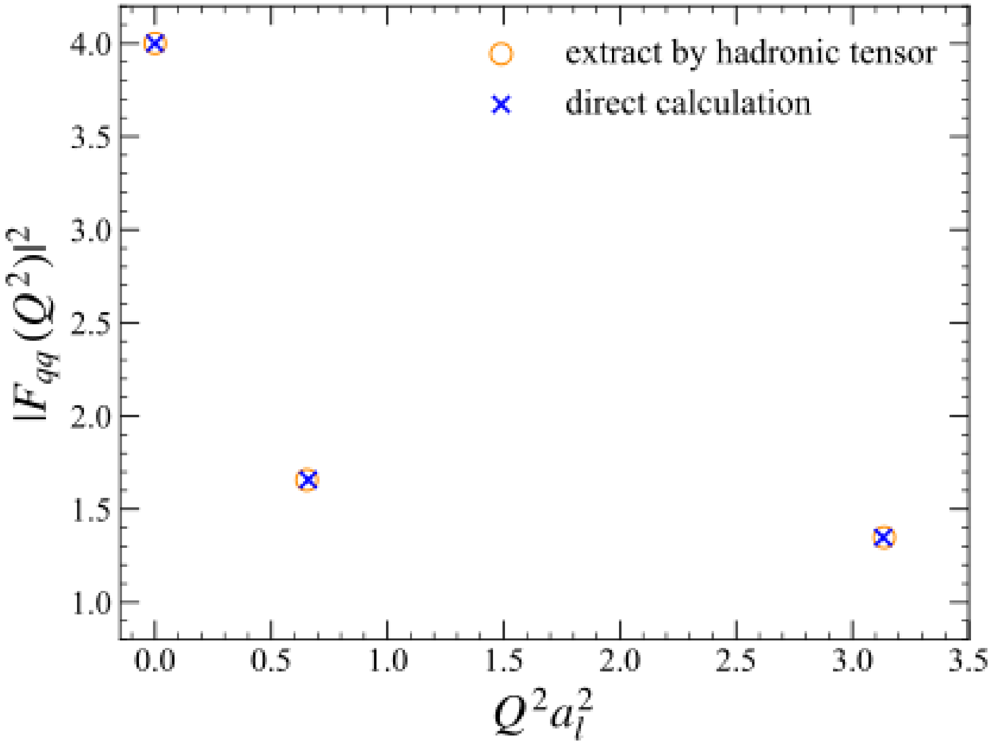

$qa_l = (q^0 a_l, q^1 a_l),\ a_l = 1$

- Blue line: Form factor $|\langle h(p)|J^0|h(p=0)\rangle|^2$.
  - Killed by charge conjugation symmetry.
  - Advantage of Wilson fermion.
- Red line: Inelastic process $|\langle h'(p')|J^0|h(p)\rangle|^2$ .
- Other peaks: contributed by higher excited state and multi-particle final state $|X\rangle$.

D. Zhou et al (QuNu), arXiv:2606.17003

# Results of SU(2) baryon




- Blue line: Form factor $|\langle h(p')|J^0|h(p=0)\rangle|^2$.
- Other peaks: contributed by higher excited state and multi-particle final state $|X\rangle$.

SU(2) baryon form factor:

- Blue cross: from hadronic tensor.
- Orange circle: from exact diagonalization.

D. Zhou et al (QuNu), arXiv:2606.17003

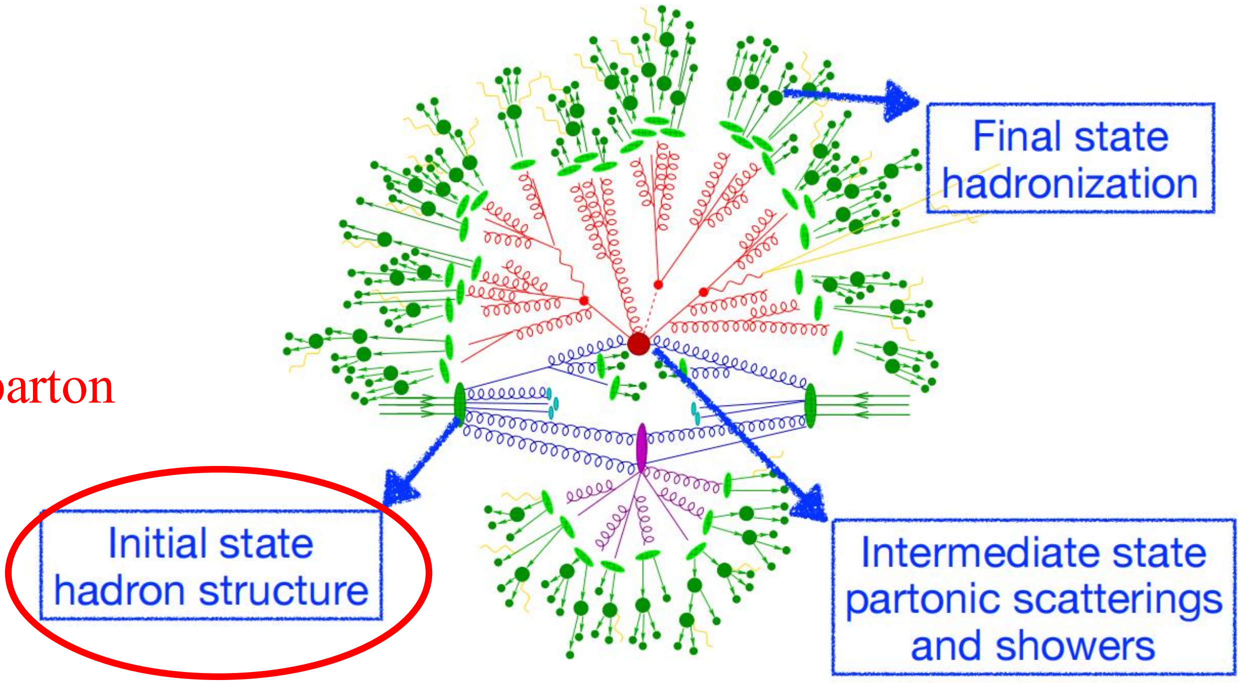
# From hadronic tensor to parton distribution functions (PDFs)

We do an electron-hadron scattering experiment on quantum computers. How to obtain the hadron structure from the experimental data $\sigma$?

Using QCD factorization:

$$\sigma = f \otimes \hat{\sigma} \otimes D$$

$f$: Non-perturbative parton distribution function




On one hand, we can extract PDFs from hadronic tensor $W_{\mu\nu}$. **On the other hand, we can simulate PDFs on quantum computers directly.**

# Procedure of simulating PDFs on quantum computers

Operator definition of the PDF:

$$n = (1,1)$$

$$f_{q/h}(x) = \int \frac{dz}{4\pi} e^{-ix(P\cdot n)z} \langle h(P) | \bar{\psi}(zn)(n\cdot\gamma)\mathcal{W}(zn,0)\psi(0) | h(P) \rangle$$

$$= \int \frac{dz}{4\pi} e^{-ixm_h z} \langle h(\vec{p}=0) | e^{iHz}\bar{\psi}(0,z)e^{-iHz}\gamma^{-}\mathcal{W}(zn,0)\psi(0) | h(\vec{p}=0) \rangle$$

$$\equiv \int \frac{dz}{4\pi} e^{-ixm_h z} \boxed{\tilde{f}_{q/h}(z)}$$

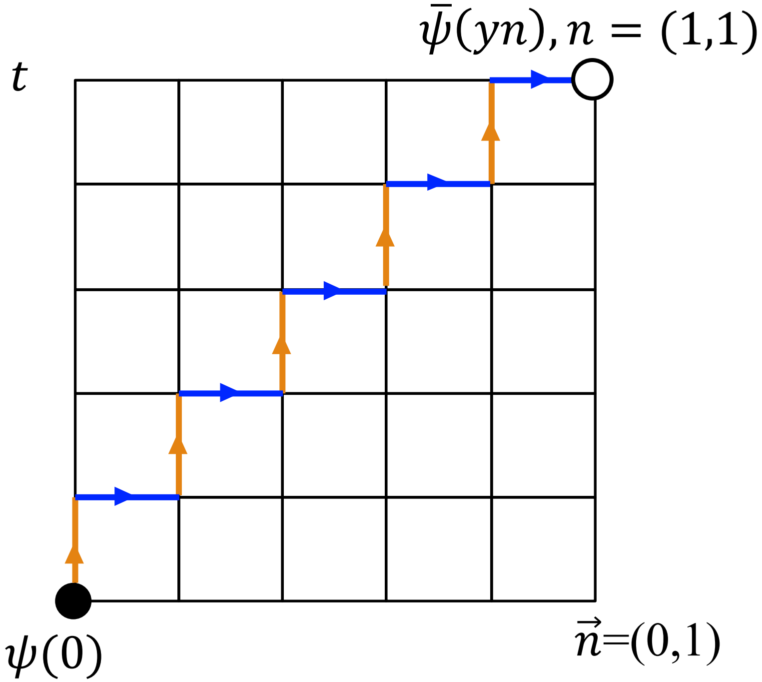
Need to be simulated on quantum computers

Four steps to simulate PDFs on a quantum computer.

- Map the fermion field to qubits. (known)
- Prepare the hadron state $|h\rangle$. (known)
- Deal with the Wilson line.
- Evaluate the dynamical correlation function. (known)

# Two methods for dealing with Wilson line $\mathcal{W}(y^-,0)$

Method 1: Trotterization of the Wilson line




$= \exp[-igaA_0^a(t, y\vec{n})T_a]$

$= \exp[igaA_1^a(t, y\vec{n})T_a]$

Method 2: Map Wilson line to local correlators by introducing auxiliary field

- Introduce an auxiliary fermion field $Q$:

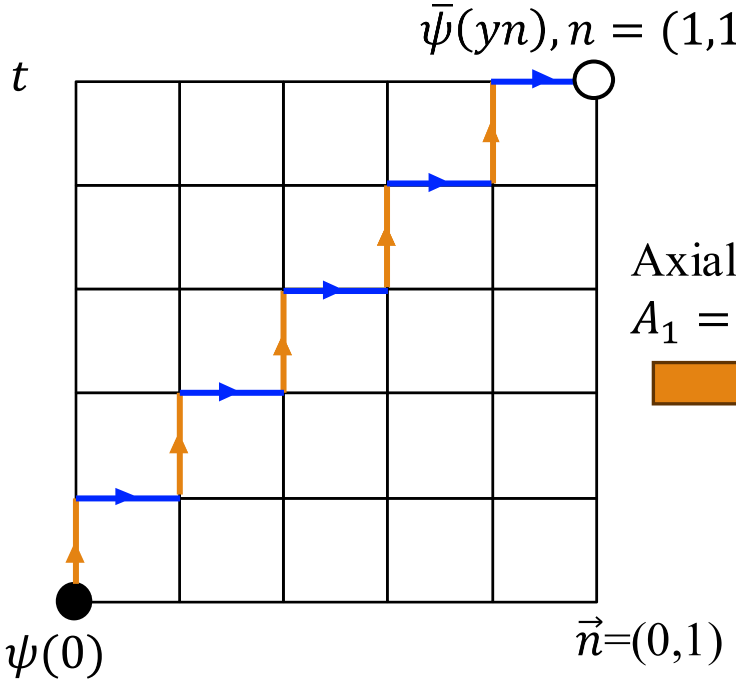
$$S = S_{QCD} + \int d^4x \, i\bar{Q}(n \cdot D)Q$$

- $\tilde{f}_{q/P}(zn) = \langle P|\bar{\psi}_q(zn)\gamma^+ Q(zn)\bar{Q}(0)\psi_q(0)|P\rangle$

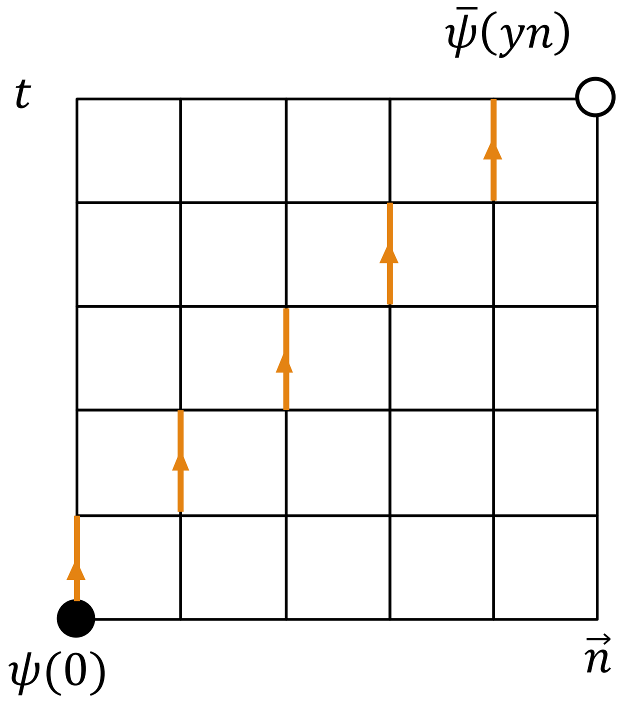
Method 1: Deeper quantum circuit, fewer qubits.

Method 2: Fewer gate cost, more qubits.

# Wilson line of Schwinger model




Axial gauge:
$A_1 = 0$




Gauss law: $E_n - E_{n-1} = Q_n$

Gauge field$A_0$: $A_0(n+1) - A_0(n) = E_n$

Differential equation for $A_0$:

$$\boldsymbol{\partial_1^2 A_0(n) = Q_n}$$

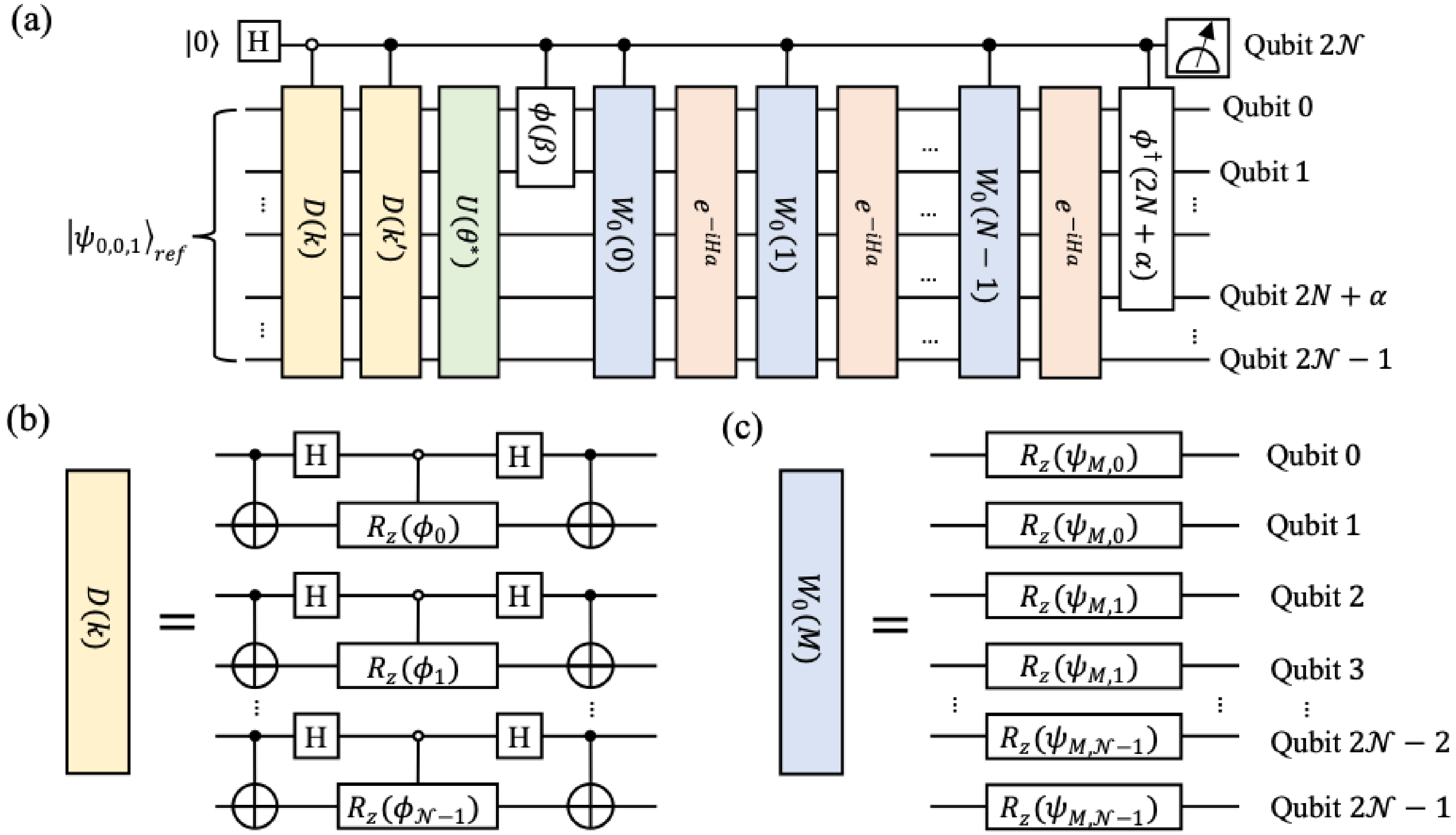
$$\boldsymbol{A_0(n) = \sum_{n'} [V(n-n') - V(0)] Q_{n'}}$$

 $= \exp[-igaA_0(y, y\vec{n})]$

 $= \exp[igaA_1(y, y\vec{n})]$

 $= \exp[-igaA_0(y, y\vec{n})]$

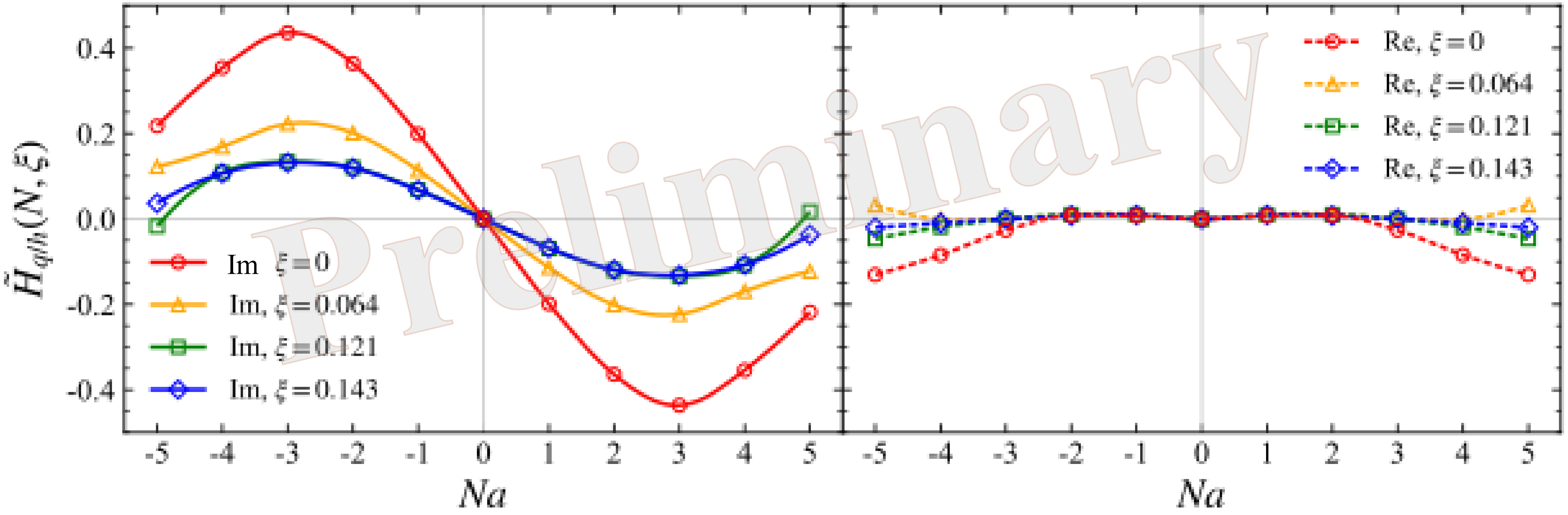
# Quantum circuit for simulating PDFs and GPDs




- Our quantum circuit can be used to simulate generalized parton distributions (GPDs).
- When $k = k'$, the out put is PDFs.
- The Wilson line is nothing but a layer of $R_z$ gate.

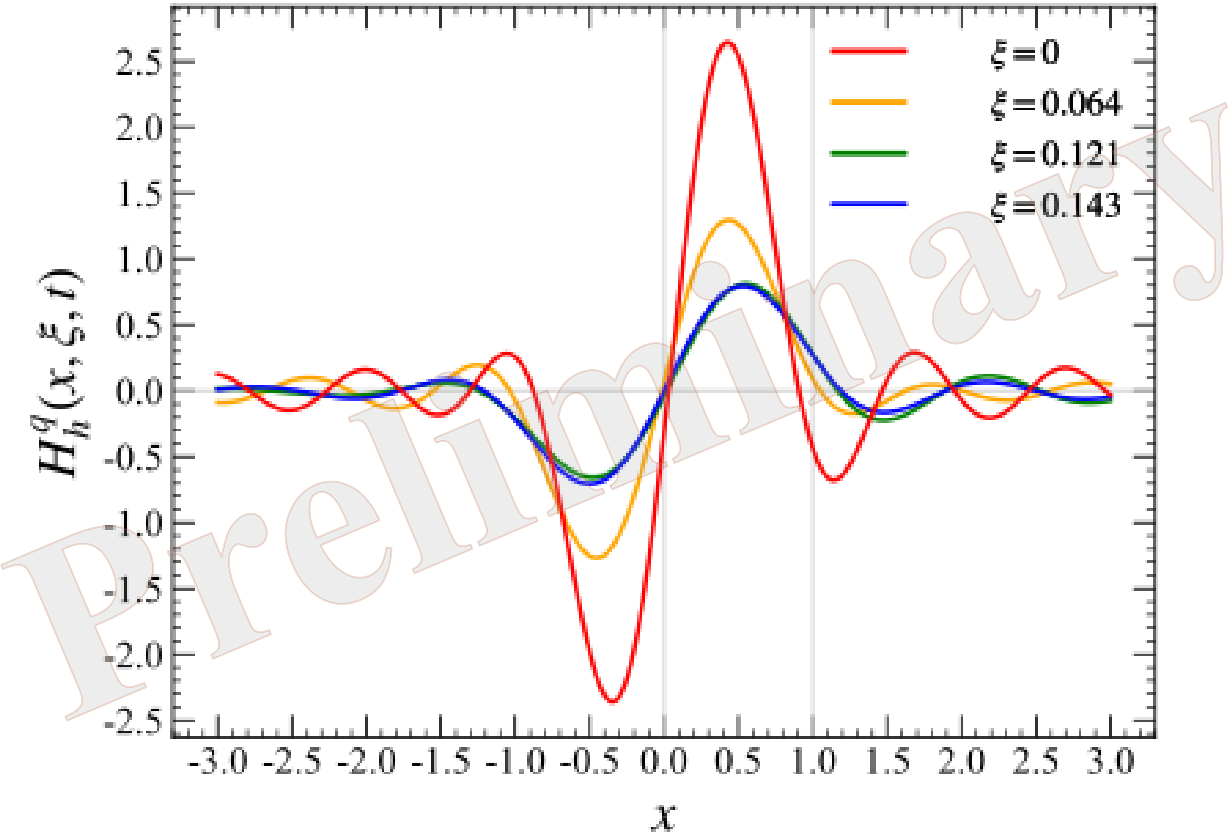
# Results of PDFs and GPDs

Gauge invariance light-cone two-point function




- $\xi = 0$ corresponds to PDFs $\widetilde{f}_{q/h}(x)$.
- $\xi \neq 0$ corresponds to the GPDs.
- We can see the bound state behavior of meson states of Schwinger model.

# Results of PDFs and GPDs




- Momentum transfer $\xi$ and $t$ are not independent in (1+1)-D.
- Both PDFs and GPDs are odd functions due to C symmetry.
- Meson electric form factor:

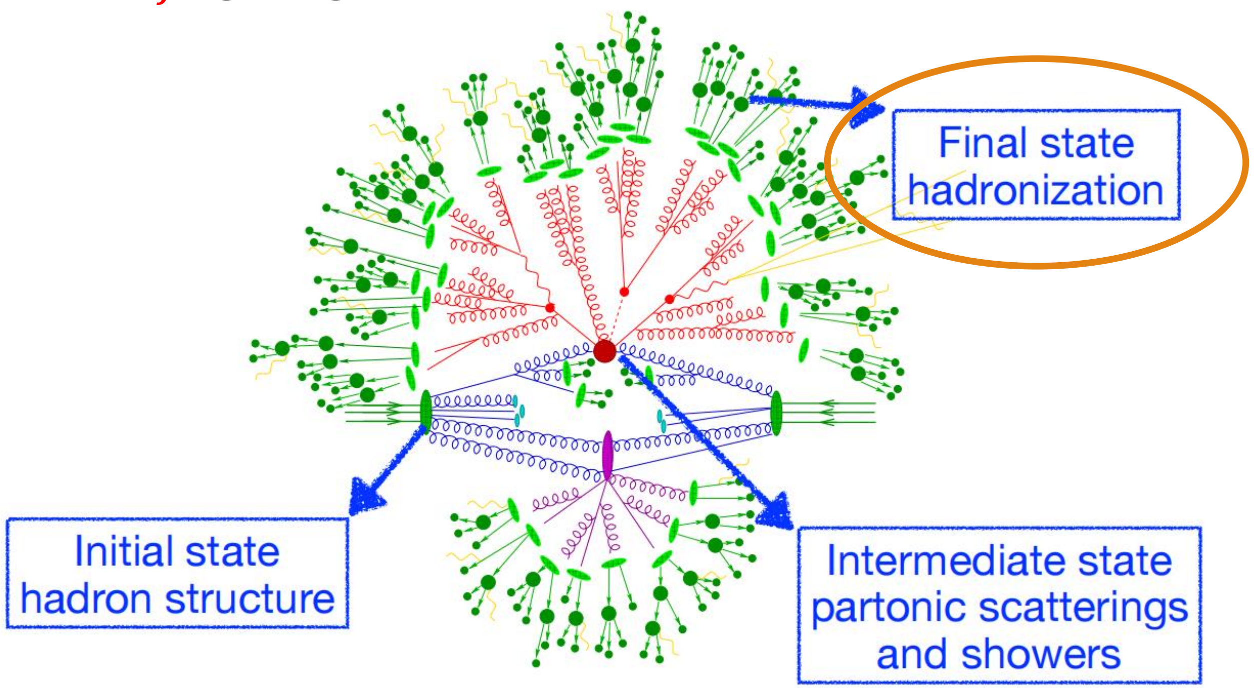
$$F_M(t) = \int dx\, H_h^q(x,\xi,t) = 0.$$

# Final state hadronization: fragmentation functions (FFs)

QCD factorization:

$$\sigma = f \otimes \hat{\sigma} \otimes D$$

$D$: Non-perturbative FFs for semi-inclusive process



# FFs on quantum computers

Operator definition of quark FF:

$$D_i^h(z) = z\int_{-\infty}^{+\infty}\frac{dy}{4\pi}e^{-iym_h/z}\underbrace{\mathrm{Tr}\left\{\langle\Omega|\psi(yn)\sum_X|h,X;out\rangle\langle h,X;out|\,\gamma^+\,\bar{\psi}(0)|\Omega\rangle\right\}}_{\widetilde{D}_i^h(y)}$$

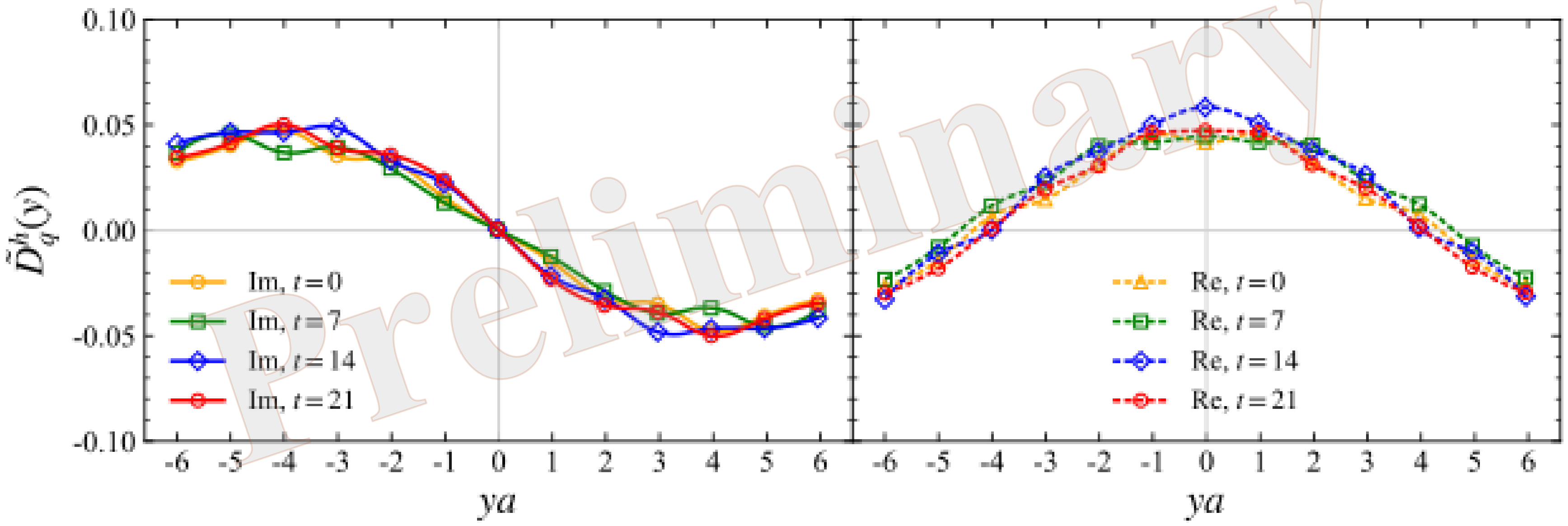
Map to qubits

$$\widetilde{D}_i^h(y) = \lim_{t\to\infty}\sum_{n,n'=0}^{N-1}\sum_{\alpha,\beta}\langle\Omega|\psi_\alpha(yn)e^{iHt}U(\theta^*)(I-\sigma_{2n}^z)(I+\sigma_{2n+1}^z)U^\dagger(\theta^*)e^{-iHt}\,\psi_\beta^\dagger(0,n')|\Omega\rangle$$

- $t\to\infty$ because $|h,X;out\rangle$ are out states.
- $U(\theta^*)$ can prepare all stable one-hadron state from initial reference states.
- $\sum_X$ becomes identity operator on a subspace.

TL et al (QuNu), arXiv:2406.05683

# Results of FFs of Schwinger model




- If we prepare $|h, X; out\rangle$ state on quantum computers exactly, FFs should not depend on $t$.
- FFs has large finite lattice volume effect because of the multi-particle out states.

# Looking forward to QCD: work in axial gauge

For a given number of lattice sites $V = L^3$, lattice spacing $a$, energy scale $E$, bare coupling $g$, and desired precision $\epsilon_s$, the qubit cost for simulating QCD is



$$N_q \approx \boxed{V \log_2\left[\epsilon_s^{-1}\left(V^{\frac{4}{3}}(Ea) + g^2 V^3\right)\right]} \equiv VK$$

Qubit cost for encoding the gauge field on a given lattice site.

Time evolution: Block encoding + QSVT:

$$M_U \sim \mathrm{poly}(\epsilon_s, V, g, E, |t|)$$

A typical parameter choice:

- $L = 10^2$ , $a \approx 0.1\ GeV^{-1}$, $E \approx 1\ GeV$, $\epsilon_s = 10^{-8}$.
- $V = L^3 = 10^6$,  $K = \log_2\left[\epsilon_s^{-1}\left(V^{\frac{4}{3}}(Ea) + g^2 V^3\right)\right] = 50$
- The number of qubits is $N_q \approx 5 \times 10^7$

# Summary

- We proposed a quantum algorithm for simulating the hadronic tensor of U(1) and SU(2) gauge theories on quantum computers.
- We also simulated the PDFs and GPDs of the Schwinger model on quantum computers.
- We also discussed the simulations of FFs on quantum computers.
- Finally, we are looking forward to QCD.