

On-shell Approaches for Gravitational Wave Physics

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IHEP High Energy Theory Forum

Outline

Introduction: amplitudes meet GW

Classical amplitudes and EFT matching

Spinning black holes and neutron stars

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Gravitational wave: new window to probe our Universe

New physics!

- ▶ Probe dynamics of black holes
- ▶ Test general relativity
- ▶ Black hole formation
- ▶ Early universe

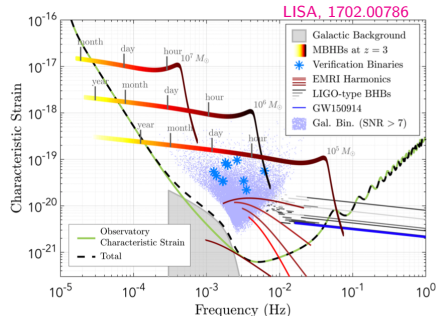
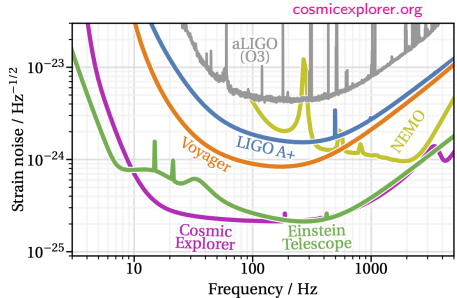
Future ground based observatories

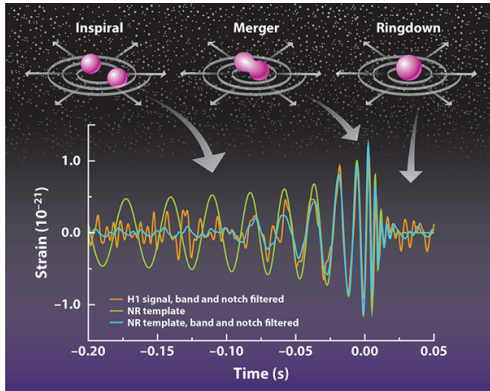
- ▶ Advanced LIGO
- ▶ Einstein Telescope
- ▶ Cosmic Explorer

Future space based observatories

- ▶ LISA
- ▶ Taiji
- ▶ TianQin

Require accurate theoretical prediction





Accurate theoretical prediction of the GW production puts challenges on the understanding of its source

$$H = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - \frac{Gm_1m_2}{|\mathbf{r}|} + \text{(corrections from general relativity)}$$

How to organize perturbations?

- ▶ Post-Newtonian (PN) expansion

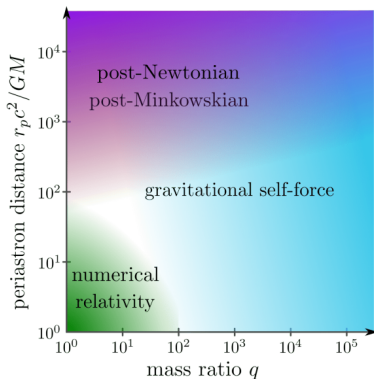
$$v^2 \sim \frac{Gm}{r} \ll 1$$

- ▶ Post-Minkowskian (PM) expansion

$$\frac{Gm}{r} \ll v^2 \sim 1$$

- ▶ Self-force expansion

$$\frac{Gm}{r} \sim v^2 \sim 1, \quad \frac{m_1}{m_2} \ll 1$$



Khalil, Buonanno, Steinhoff, Vines, 2204.05047

Amplitude-based methods naturally lead to PM expansion

PM expansion is relevant to bound orbits with large eccentricity and scattering process

Why don't we just use numerical relativity?

Numerical relativity provides the most accurate observables (waveform, scattering angle, etc) for binaries with mass ratio $q \sim 1$

- ▶ Works for the entire binary merger process
- ▶ Numerical error under very good control (can be considered as the TRUTH)
- ▶ Computationally expensive: need days of computation time on a super-computer to produce one waveform template
- ▶ Matched filtering analysis requires a dense sampling of the parameter space
- ▶ In light of future observatories, we need $\sim 10^6$ waveform templates

We need analytic perturbative computations to efficiently generate observables

Why amplitudeologists are working on GW?

It all starts with an enticing invitation



PHYSICAL REVIEW D **97**, 044038 (2018)

High-energy gravitational scattering and the general relativistic two-body problem

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(Received 29 October 2017; published 26 February 2018)

A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.

DOI: [10.1103/PhysRevD.97.044038](https://doi.org/10.1103/PhysRevD.97.044038)

Latest progress

order	deflection & spin kick					waveform			Integration complexity
	plain	spin-orbit	spin-spin	spin ²	tidal	plain	spin-orbit spin-spin	tidal	
1PM	WQFT WEFT Amps HEFT	WQFT WEFT Amps HEFT	WQFT WEFT Amps HEFT	Amps HEFT	X	trivial	trivial	trivial	~ tree-level
2PM	WQFT WEFT Amps HEFT	WQFT WEFT Amps HEFT	WQFT WEFT Amps HEFT	Amps HEFT	WQFT WEFT Amps	WQFT WEFT Amps HEFT	WQFT WEFT Amps HEFT	WQFT WEFT	~ 1-loop
3PM cons	WQFT WEFT Amps HEFT	WQFT Amps	WQFT (Amps)		WQFT WEFT	Amps HEFT	Amps		~ 2-loop
3PM diss	WQFT WEFT Amps HEFT	WQFT	WQFT		WQFT WEFT				
4PM cons	WQFT WEFT Amps	WQFT			WQFT				~ 3-loop
4PM diss	WQFT WEFT Amps	WQFT			WQFT				
5PM-1SF cons	WQFT								~ 4-loop

r-r: Radiation-reaction (...): partial results

2024 MIAPbP workshop “EFT and Multi-Loop Methods for Advancing Precision in Collider and Gravitational Wave Physics”

Talk by Jan Plefka

Current frontier: partial results at 5PM-2SF for $\mathcal{N} = 8$

Bern, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng, 2509.17412

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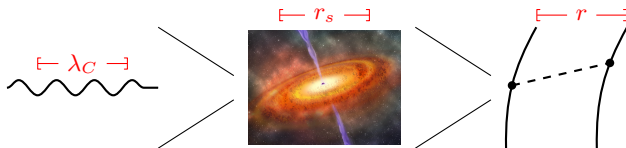
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From quantum to classical two-body problem



$$\lambda_C \ll r_s \implies Gm^2 \gg 1$$

Classical contribution is encoded in every loop order

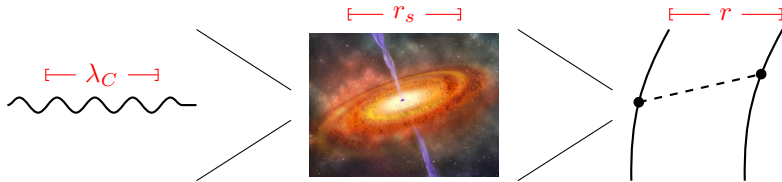
$$r_s \ll r \implies Gmq \ll 1$$

Gmq is the small parameter in the classical perturbative expansion

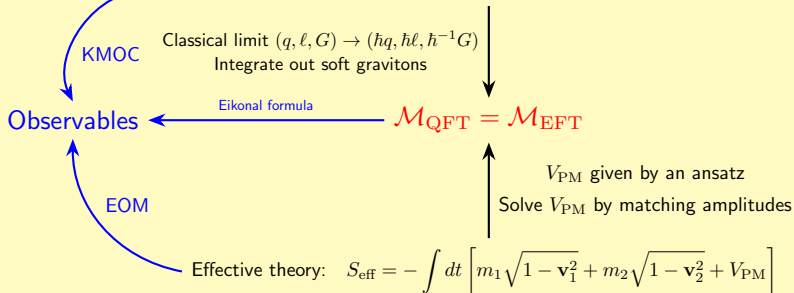
$$\lambda_C \ll r \implies q/m \ll 1$$

Expand in small momentum transfer and keep only the leading order
(Bohr's correspondence principle)

EFT perspective



Full theory:
$$S_{\text{full}} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \frac{1}{2} \int d^4x \sum_{i=1,2} (g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i - m_i^2 \phi_i^2) + \mathcal{O}(R^2 \phi^2)$$



EFT matching Cheung, Rothstein, Solon, 1808.02489

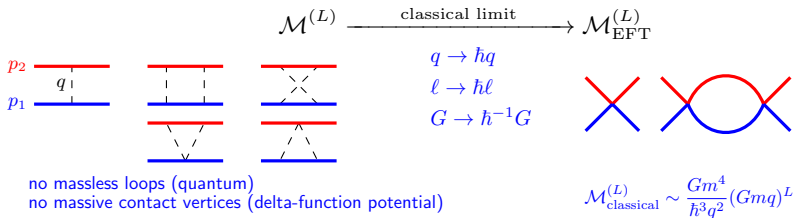
- Full theory: Schwarzschild black hole $\implies$ scalar field ϕ

$$\mathcal{L} = \sqrt{-g} \left[-\frac{1}{16\pi G} R + \frac{1}{2} \sum_{i=1,2} (g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i - m_i^2 \phi_i^2) \right] + \mathcal{O}(R^2 \phi^2)$$

- Effective theory: potential $V(\mathbf{k}, \mathbf{k}')$ given by an ansatz

$$L = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \left[\sum_{i=1,2} a_i^\dagger(-\mathbf{k}) \left(i\partial_t + \sqrt{\mathbf{k}^2 + m_i^2} \right) a_i(\mathbf{k}) - \int \frac{d^3 \mathbf{k}'}{(2\pi)^3} V(\mathbf{k}, \mathbf{k}') a_1^\dagger(\mathbf{k}') a_1(\mathbf{k}) a_2^\dagger(-\mathbf{k}') a_2(\mathbf{k}) \right]$$

- Solve the EFT potential by matching the full theory and EFT amplitudes order-by-order in G in the classical limit



$$E_1 = \sqrt{\mathbf{p}^2 + m_1^2} \quad E_2 = \sqrt{\mathbf{p}^2 + m_2^2}$$

$$\gamma = \frac{E_1 + E_2}{m_1 + m_2} \quad \xi = \frac{E_1 E_2}{(E_1 + E_2)^2}$$

$$\nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \quad \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

Hamiltonian: $H = E_1 + E_2 + \sum_{n=1}^{\infty} \frac{G^n}{|\mathbf{r}|^n} c_{n\text{PM}}(\mathbf{p}^2)$

$$c_{1\text{PM}} = -\frac{\nu^2(m_1 + m_2)^2}{\gamma^2 \xi} (2\sigma^2 - 1) \quad \text{Westpfahl and Goller 1979}$$

$$c_{2\text{PM}} = -\frac{\nu^2(m_1 + m_2)^3}{\gamma^2 \xi} \left[\frac{3(5\sigma^2 - 1)}{4} - \frac{4\nu\sigma(2\sigma^2 - 1)}{\gamma \xi} + \frac{\nu^2(1 - \xi)(2\sigma^2 - 1)^2}{2\gamma^3 \xi^2} \right] \quad \text{Bel, Damour, Deruelle, Ibanez, Martin, 1981}$$

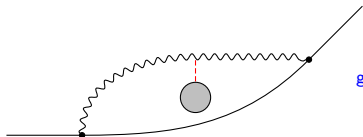
Westpfahl 1985

$$c_{3\text{PM}} = \frac{\nu^2 m^4}{\gamma^2 \xi} \left[\frac{3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3}{12} - \frac{4\nu(3 + 12\sigma^2 - 4\sigma^4) \operatorname{arcsinh} \sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2 - 1}} \right. \\ \left. - \frac{3\nu\gamma(2\sigma^2 - 1)(5\sigma^2 - 1)}{2(1 + \gamma)(1 + \sigma)} + \frac{3\nu\sigma(20\sigma^2 - 7)}{2\gamma \xi} + \frac{\nu^2(3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2)(2\sigma^2 - 1)}{4\gamma^3 \xi^2} \right. \\ \left. + \frac{2\nu^3(3 - 4\xi)\sigma(2\sigma^2 - 1)^2}{\gamma^4 \xi^3} - \frac{\nu^4(1 - 2\xi)(2\sigma^2 - 1)^3}{2\gamma^6 \xi^4} \right] \quad \text{(This is what Damour was asking for)}$$

State-of-the-art: $c_{4\text{PM}}^{\text{hyp}}$ and $c_{5\text{PM}}^{\text{hyp 1SF}}$

- ▶ $c_{3\text{PM}}$ is not known to general relativists before computed this way
- ▶ $c_{5\text{PM}}$ for GR is obtained using the amplitude-worldline hybrid method

Gravitational tail



gravitational tails depend on the trajectory

$$c_4^{\text{hyp}} = \frac{M^7 \nu^2}{4\xi E^2} \left[\mathcal{M}_4^{\text{p}} + \nu \left(4\mathcal{M}_4^{\text{t}} \log\left(\frac{p_\infty}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}} \right) \right] + \mathcal{D}^3 \left[\frac{E^3 \xi^3}{3} c_1^4 \right] + \mathcal{D}^2 \left[\left(\frac{E^3 \xi^3}{p^2} + \frac{E\xi(3\xi-1)}{2} \right) c_1^4 - 2E^2 \xi^2 c_1^2 c_2 \right] \\ + \left(\mathcal{D} + \frac{1}{p^2} \right) \left[E\xi(2c_1 c_3 + c_2^2) + \left(\frac{4\xi-1}{4E} + \frac{2E^3 \xi^3}{p^4} + \frac{E\xi(3\xi-1)}{p^2} \right) c_1^4 + \left((1-3\xi) - \frac{4E^2 \xi^2}{p^2} \right) c_1^2 c_2 \right], \quad (8)$$

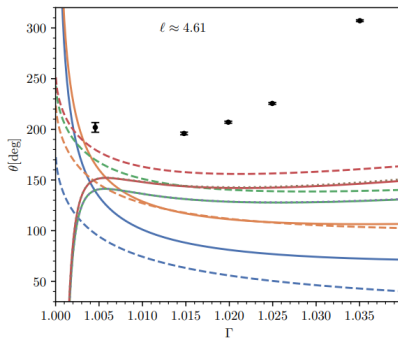
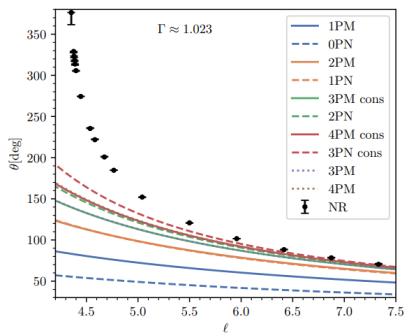
where $\mathcal{D} = \frac{d}{dp^2}$

Bern, Parra-Martinez, Roiban, Ruf, Solon, Shen, Zeng, 2112.10750

Comparison with numerical relativity

Fixed-order perturbative computation only works well for weak couplings

Buonanno, Jakobsen, Mogull, 2402.12342



Effective-one-body (EOB) resummation is necessary for strong couplings

Buonanno, Damour, gr-qc/9811091 Buonanno, Jakobsen, Mogull, 2402.12342

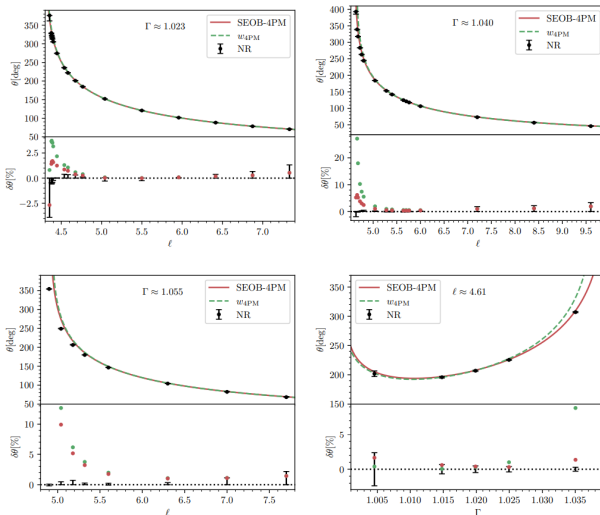


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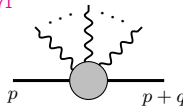
Outlook

Black holes and neutron stars can carry spin

How to incorporate spin into the EFT?

On-shell description of spin

Bern, Luna, Roiban, Shen, Zeng, 2005.03071



- On-shell spin- s states are **symmetric traceless and transverse**

$$\varepsilon_{a_1 a_2 \dots a_s} = \varepsilon_{(a_1 a_2 \dots a_s)} \quad p^{a_1} \varepsilon_{a_1 a_2 \dots a_s} = \eta^{a_1 a_2} \varepsilon_{a_1 a_2 \dots a_s} = 0$$

- Classical limit $\implies$ **spin coherent state** $\varepsilon_{a_1 a_2 \dots a_s}^s = \varepsilon_{a_1}^+ \varepsilon_{a_2}^+ \dots \varepsilon_{a_s}^+$ with large s

$$\begin{aligned} \varepsilon_p^s \cdot M^{ab} \cdot \varepsilon_{p+q}^s &\sim S^{ab} & (M^{ab})_{c(s)}^{d(s)} &= -2is \delta_{(c_1}^{[a} \eta^{b](d_1} \delta_{c_2}^{d_2} \dots \delta_{c_s)}^{d_s}) \\ \varepsilon_p^s \cdot \{M^{ab} M^{cd}\} \cdot \varepsilon_{p+q}^s &\sim S^{ab} S^{cd} & S^{ab} &= (1/m) \varepsilon^{abcd} p_c S_d \end{aligned}$$

- The spin tensor satisfy **covariant spin supplementary condition (SSC)**

$$S^{ab} p_b = 0 \quad (S^{ab} \text{ is boosted from rest frame } S^{ij})$$

- Transversality** and **covariant SSC** are related

- Spin magnitude is conserved: $S^{ab} S_{ab} \sim S^a S_a \sim \mathbf{S}^2 = \text{const}$

Effective field theory for higher-spin particles

Higher spin quantum field theory ($\phi_s \equiv \phi_{a_1 a_2 \dots a_s}$)

$$\begin{aligned}\nabla_\mu \phi_s &= \partial_\mu \phi_s + (i/2) \omega_{\mu ab} M^{ab} \phi_s \\ \mathbb{S}^a &= (-i/2m) \epsilon^{abcd} M_{cd} \nabla_b\end{aligned}$$

$$\begin{aligned}\mathcal{L} = & -\frac{1}{2} \phi_s (\nabla^2 + m^2) \phi_s + \frac{1}{8} R_{abcd} \phi_s M^{ab} M^{cd} \phi_s - \frac{C_2}{2m^2} R_{af_1 b f_2} \nabla^a \phi_s \mathbb{S}^{(f_1} \mathbb{S}^{f_2)} \nabla^b \phi_s \\ & + \frac{D_2}{2m^2} R_{abcd} \nabla_i \phi_s \{M^{ai} M^{cd}\} \phi_s + \frac{E_2 - 2D_2}{2m^4} R_{abcd} \nabla^{(a} \nabla^{i)} \phi_s \{M^b{}_i M^d{}_j\} \nabla^{(c} \nabla^{j)} \phi_s + \mathcal{O}(M_{ab}^3)\end{aligned}$$

We prefer to use a formalism in which the classical and large spin limit is straightforward

- ▶ Contractions of ϕ_s facilitated by M^{ab} only
- ▶ Propagator uniform in s : $i\delta_{a(s)}^{b(s)}/(p^2 - m^2)$
- ▶ There are additional lower spin ($s' < s$) states in the spectrum

Problematic? Not in the classical limit:

- ▶ Ghost nature easily cured by an analytic continuation on classical variables
- ▶ We get a more generic non-rigid spinning object (more internal DOFs)
- ▶ Conventional rigid spinning objects correspond to special Wilson coefficients

Generalized spin coherent state

Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, Vines, 2308.14176

The external state now contains lower spin components. Consider the coherent sum

$$\mathcal{E}_{\mu_1 \dots \mu_s} = \varepsilon_{\mu_1 \dots \mu_s}^{(s)} + u_{(\mu_1} \varepsilon_{\mu_2 \dots \mu_s)}^{(s-1)} + \dots$$

Similar coherent sum was also considered in Aoude, Ochirov, 2108.01649, etc

Classical limit

$$\begin{aligned}\mathcal{E}_p \cdot M^{ab} \cdot \mathcal{E}_{p+q} &\sim S^{ab} & S^{ab} &= S^{ab} + (i/m)(p^a K^b - p^b K^a) \\ \mathcal{E}_p \cdot \{M^{ab} M^{cd}\} \cdot \mathcal{E}_{p+q} &\sim S^{ab} S^{cd}\end{aligned}$$

where K^a is identified as the **boost generator**, and $S^{ab} p_b = K^a p_a = 0$

- ▶ K^a emerges from the transition between spin s and lower spin states
- ▶ Consequently, $S^{ab} S_{ab} \sim \mathbf{S}^2 - \mathbf{K}^2$ is a still constant but $\mathbf{S}^2$ is not

Non-minimal interactions up to $\mathcal{O}(M_{ab}^2)$

Alavverdian, Bern, Kosmopoulos, Luna, Roiban, Scheopner, Roiban, **FT**, 2407.10928, 2503.03739

$$\nabla_\mu \phi_s = \partial_\mu \phi_s + (i/2)\omega_{\mu ab} M^{ab} \phi_s$$

$$\mathbb{S}^a = (-i/2m)\epsilon^{abcd} M_{cd} \nabla_b$$

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}\phi_s(\nabla^2 + m^2)\phi_s + \frac{1}{8}R_{abcd}\phi_s M^{ab} M^{cd}\phi_s - \frac{C_2}{2m^2}R_{af_1bf_2}\nabla^a\phi_s\mathbb{S}^{(f_1}\mathbb{S}^{f_2)}\nabla^b\phi_s \\ & + \frac{D_2}{2m^2}R_{abcd}\nabla_i\phi_s\{M^{ai}M^{cd}\}\phi_s + \frac{E_2 - 2D_2}{2m^4}R_{abcd}\nabla^{(a}\nabla^{i)}\phi_s\{M^b{}_iM^d{}_j\}\nabla^{(c}\nabla^{j)}\phi_s \end{aligned}$$

- The C_2 -operator has an origin in the world-line formalism for neutron stars
Porto, 0511061; Levi, Steinhoff, 1501.04956
- It is the only independent operator assuming that rest frame spin is the only dynamical degree of freedom
- The D_2 and E_2 WCs supports additional K -dependent interactions in the classical limit

$$D_2 = E_2 = 0 \quad \implies \quad \text{Conventional compact object described by } H(\mathbf{r}, \mathbf{p}, \mathbf{S})$$

$$C_2 = D_2 = E_2 = 0 \quad \implies \quad \text{Kerr black hole}$$

Generic values: generic compact object described by $H(\mathbf{r}, \mathbf{p}, \mathbf{S}, \mathbf{K})$

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KMOC formalism Kosower, Maybee, O'Connell, 1811.10950

Observable in a scattering process as an in-in correlator

$$\begin{aligned}\Delta O &= \langle \psi_{\text{out}} | \mathbb{O} | \psi_{\text{out}} \rangle - \langle \psi_{\text{in}} | \mathbb{O} | \psi_{\text{in}} \rangle \\ &= \langle \psi_{\text{in}} | \hat{S}^\dagger \mathbb{O} \hat{S} | \psi_{\text{in}} \rangle - \langle \psi_{\text{in}} | \mathbb{O} | \psi_{\text{in}} \rangle \\ &= i \langle \psi_{\text{in}} | [\mathbb{O}, \hat{T}] | \psi_{\text{in}} \rangle + \langle \psi_{\text{in}} | \hat{T}^\dagger [\mathbb{O}, \hat{T}] | \psi_{\text{in}} \rangle\end{aligned}$$
$$\begin{aligned}\hat{S} &= 1 + i\hat{T} \\ \hat{T} - \hat{T}^\dagger &= i\hat{T}^\dagger \hat{T}\end{aligned}$$

Incoming state: $|\psi_{\text{in}}\rangle = \int d\Phi[p_1] d\Phi[p_2] \phi(p_1) \phi(p_2) e^{ip_1 \cdot b_1 + ip_2 \cdot b_2} |p_1 p_2\rangle$

Single particle phase-space and wavefunction:

$$d\Phi[p] = \frac{d^4 p}{(2\pi)^4} 2\pi \Theta(p^0) \delta(p^2 - m^2) \quad \int d\Phi[p] |\phi(p)|^2 = 1$$

We can compute observables by dressing amplitudes and cuts with the corresponding operators

$$\begin{aligned}\Delta O &= \int d\Phi[p_1] d\Phi[p_2] d\Phi[p'_1] d\Phi[p'_2] \phi(p_1) \phi(p_2) \phi(p'_1) \phi(p'_2) e^{i(p_1 - p'_1) \cdot b_1 + i(p_2 - p'_2) \cdot b_2} \\ &\quad \times \left[i \langle p'_1 p'_2 | [\mathbb{O}, \hat{T}] | p_1 p_2 \rangle + \langle p'_1 p'_2 | \hat{T}^\dagger [\mathbb{O}, \hat{T}] | p_1 p_2 \rangle \right]\end{aligned}$$

Classical observables Kosower, Maybee, O'Connell, 1811.10950

The wave packets are highly localized, while q_i is much smaller than their spread

$$\int d\Phi[p_1]d\Phi[p_2]d\Phi[p'_1]d\Phi[p'_2]\phi(p_1)\phi(p_2)\phi^*(p'_1)\phi^*(p'_2)e^{i(p_1-p'_1)\cdot b_1+i(p_2-p'_2)\cdot b_2}$$

$$\simeq \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q_1) \hat{\delta}(2\bar{p}_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \quad \hat{\delta}(x) = 2\pi\delta(x)$$

Resolution of identity in the two-massive-particle subspace

$$\mathbb{I} = \sum_X \int d\Phi[r_1]d\Phi[r_2] |r_1 r_2 X\rangle \langle r_1 r_2 X|$$

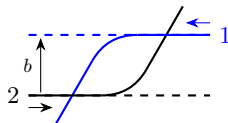
Expansion in the soft region $(q_i, \ell, k, G) \rightarrow (\hbar q_i, \hbar \ell, \hbar k, \hbar^{-1} G)$

Classical observables

$$\Delta O_{\text{cl}} = \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q_1) \hat{\delta}(2\bar{p}_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \left[i \langle p'_1 p'_2 | [\mathbb{O}, \hat{T}] | p_1 p_2 \rangle + \langle p'_1 p'_2 | \hat{T}^\dagger [\mathbb{O}, \hat{T}] | p_1 p_2 \rangle \right]_{\text{cl}}$$

Impulse

Through 3PM: Hermann, Parra-Martinez, Ruf, Zeng, 2104.03957



We compute the expectation value of

$$\mathbb{O} = \mathbb{P}_1^\mu \quad \mathbb{P}_1^\mu |p_1^\mu\rangle = p_1^\mu |p_1\rangle$$

Tree level impulse



$$[\Delta p_1^\mu]_{(0)} = \int \frac{d^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q) e^{iq \cdot b} (-iq^\mu) \left[\underbrace{-\frac{\kappa^2 \bar{m}_1^2 \bar{m}_2^2 (2y^2 - 1)}{2q^2}}_{\text{classical tree amp}} \right] = -\frac{\kappa^2 \bar{m}_1 \bar{m}_2 (2y^2 - 1)}{16\pi \sqrt{y^2 - 1}} \frac{b^\mu}{(-b^2)}$$

One loop impulse (from the S-matrix)



$$\begin{aligned} [\Delta p_1^\mu]_{(1,\perp)} &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q_1) \hat{\delta}(2\bar{p}_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \left[i \langle p'_1 p'_2 | [\mathbb{P}_1^\mu, \hat{T}] | p_1 p_2 \rangle \right] \\ &= \int \frac{d^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q) e^{iq \cdot b} (-iq^\mu) \left[M_{\text{amp-sc}}^{(1)} + M_{\text{amp-cl}}^{(1)} \right] \end{aligned}$$

We have a classically singular and IR divergent contribution $M_{\text{qft-sc}}^{(1)}$, but fortunately we have not done yet

$$M_{\text{amp-sc}}^{(1)} = \frac{i\kappa^4 \bar{m}_1^4 \bar{m}_2^4 (2y^2 - 1)^2}{8\hbar^4} \mathcal{I}_\square \quad M_{\text{amp-cl}}^{(1)} = \frac{3\kappa^4 \bar{m}_1 \bar{m}_2 (\bar{m}_1 + \bar{m}_2) (5y^2 - 1)}{128\hbar^3 \sqrt{-q^2}}$$

Impulse

One loop impulse (from the cut)



$$\check{u}_1 = \frac{y\check{u}_2 - \bar{u}_1}{y^2 - 1} \text{ and } \check{u}_2 = \frac{y\bar{u}_1 - \bar{u}_2}{y^2 - 1}$$

$$\begin{aligned} [\Delta p_1^\mu]_{(1,\parallel)} &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q_1) \hat{\delta}(2\bar{p}_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \left[\langle p'_1 p'_2 | \hat{T}^\dagger [\mathbb{P}_1^\mu, \hat{T}] | p_1 p_2 \rangle \right] \\ &= \int \frac{d^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q) e^{iq \cdot b} \left[(-iq^\mu) M_{\text{cut-sc}}^{(1)} + \left(\frac{\check{u}_2^\mu}{\bar{m}_2} - \frac{\check{u}_1^\mu}{\bar{m}_1} \right) M_{\text{cut-cl}}^{(1)} \right] \end{aligned}$$

The classically singular contribution cancels exactly between the S-matrix and cut

$$M_{\text{amp-sc}}^{(1)} + M_{\text{cut-sc}}^{(1)} = 0 \quad M_{\text{cut-cl}}^{(1)} = \frac{(2y^2 - 1)^2 \bar{m}_1^3 \bar{m}_2^3}{32\pi \hbar^3 \sqrt{y^2 - 1}} \log(-q^2)$$

Full one-loop impulse

$$\begin{aligned} [\Delta p_1^\mu]_{(1,\perp)} + [\Delta p_1^\mu]_{(1,\parallel)} &= \int \frac{d^d q}{(2\pi)^d} \hat{\delta}(2\bar{p}_1 \cdot q) e^{iq \cdot b} \left[(-iq^\mu) M_{\text{amp-cl}}^{(1)} + \left(\frac{\check{u}_2^\mu}{\bar{m}_2} - \frac{\check{u}_1^\mu}{\bar{m}_1} \right) M_{\text{cut-cl}}^{(1)} \right] \\ &= - \underbrace{\frac{3\pi G^2 \bar{m}_1 \bar{m}_2 (\bar{m}_1 + \bar{m}_2) (5y^2 - 1) b^\mu}{4\sqrt{y^2 - 1} (-b^2)^{3/2}}}_{\text{transverse}} - \underbrace{\frac{2G^2 \bar{m}_1^2 \bar{m}_2^2 (2y^2 - 1)^2}{(y^2 - 1) (-b^2)} \left[\frac{\check{u}_2^\mu}{\bar{m}_2} - \frac{\check{u}_1^\mu}{\bar{m}_1} \right]}_{\text{longitudinal}} \end{aligned}$$

Waveform and metric perturbation

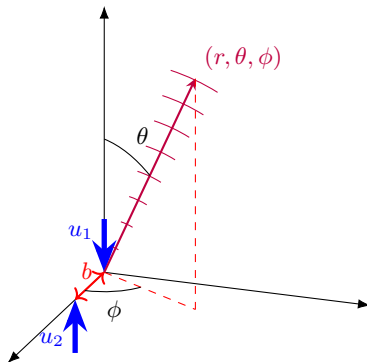
Herderschee, Roiban, **FT**, 2303.06112

Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, **FT**, 2402.06604

De Angelis, Herderschee, Roiban, **FT**, In preparation

Metric perturbation: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

Waveform: $W(T_R, \theta, \phi) = \frac{1}{4G} \lim_{r \rightarrow \infty} r \varepsilon_+^\mu \varepsilon_+^\nu h_{\mu\nu} = \frac{1}{4G} (h_+^\infty - i h_\times^\infty)$



Operator: metric perturbation

$$\mathbb{H} = \varepsilon_+^\mu \varepsilon_+^\nu \hat{h}_{\mu\nu}(x) = \int d\Phi[k] \left[\hat{a}_{--}(k) e^{-ik \cdot x} + \text{c.c.} \right]$$

Since there is no radiation at $t \rightarrow -\infty$,

$$\Delta H(x) = \langle \psi_{\text{in}} | \hat{S}^\dagger \mathbb{H} \hat{S} | \psi_{\text{in}} \rangle = \int d\Phi[k] \left[\tilde{J}(k) e^{-ik \cdot x} - \text{c.c.} \right] \quad \tilde{J}(k) = \langle \psi_{\text{in}} | \hat{S}^\dagger \hat{a}_{--}(k) \hat{S} | \psi_{\text{in}} \rangle$$

Assuming the spatial current $J(y)$ is localized around $y = 0$,

$$\Delta H(x) \Big|_{|\mathbf{x}| \rightarrow \infty} = \frac{1}{4\pi|\mathbf{x}|} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} W(\omega, \mathbf{n}) e^{-i\omega\tau} \quad (\tau = x^0 - |\mathbf{x}| : \text{retarded time})$$

$$\mathcal{W}(\varepsilon, \omega, \mathbf{n}, p_1, p_2, b_1, b_2) = (-i) \langle \psi_{\text{in}} | \hat{S}^\dagger \hat{a}_{--}(k) \hat{S} | \psi_{\text{in}} \rangle \Big|_{k=(\omega, \omega\mathbf{n})}^{\text{IR finite}} \quad (\text{frequency domain waveform})$$

(The IR divergence can be absorbed in the definition of τ)

Relevant matrix elements for waveform

In the space of momentum transfer:

$$(-i)\langle\psi_{\text{in}}|\hat{S}^\dagger\hat{a}(k)\hat{S}|\psi_{\text{in}}\rangle = \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2p_1 \cdot q_1) \hat{\delta}(2p_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \\ \times \left[\langle p'_1 p'_2 k | \hat{T} | p_1 p_2 \rangle - i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \right]$$

The matrix elements can be classified as follows:

$$\langle p'_1 p'_2 k | \hat{T} | p_1 p_2 \rangle \Big|_{\text{conn}} = \mathcal{M}^{\text{tree}} + \mathcal{M}^{1 \text{ loop}} \quad i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{conn}} = \mathcal{S}^{1 \text{ loop}}$$



The full connected contribution to waveform:

$$\mathcal{M}_{\text{conn}} = \mathcal{M}^{\text{tree}} + \mathcal{M}^{1 \text{ loop}} - \mathcal{S}^{1 \text{ loop}}$$

Contribution from disconnected T matrix elements:

$$\langle p'_1 p'_2 k | \hat{T} | p_1 p_2 \rangle \Big|_{\text{disc}} = \mathcal{M}_{\text{const}} \quad i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{disc}} = \mathcal{M}_{\text{disc}}$$



Tree-level amplitudes and LO waveform

From GR: Kovacs, Thorne, Astrophys. J. 224 62, 1978

From world line QFT: Jakobsen, Møglu, Plefka, Steinhoff, 2101.12688

From KMOC: Cristofoli, Gonzo, Kosower, O'Connell, 2107.10193

For convenience, we define the Fourier transform from q -space to b -space,

$$\text{FT}[f(q_1, q_2, k)] = \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \hat{\delta}(2p_1 \cdot q_1) \hat{\delta}(2p_2 \cdot q_2) e^{iq_1 \cdot b_1 + iq_2 \cdot b_2} \hat{\delta}(q_1 + q_2 - k) f(q_1, q_2, k) \Big|_{k=(\omega, \omega \mathbf{n})}$$

LO waveform from KMOC

$$\mathcal{W}(\omega, \mathbf{n}) \Big|_{\text{LO}} = \text{FT}[\mathcal{M}^{\text{tree}}]$$

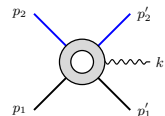
Classical tree-level amplitude Luna, Nicholson, O'Connell, White, 1711.03901

$$\begin{aligned} \mathcal{M}^{\text{tree}} &= -\kappa^2 m_1^2 m_2^2 \left[\frac{(P_{12} \cdot \varepsilon)^2 + \sigma(Q_{12} \cdot \varepsilon)(P_{12} \cdot \varepsilon)}{q_1^2 q_2^2} + \left(\sigma^2 - \frac{1}{d-2} \right) \left(\frac{(Q_{12} \cdot \varepsilon)^2}{4q_1^2 q_2^2} - \frac{(P_{12} \cdot \varepsilon)^2}{4w_1^2 w_2^2} \right) \right] \\ &= \mathcal{M}_{d=4}^{\text{tree}} + \epsilon \mathcal{M}_{\text{extra}}^{\text{tree}} \end{aligned}$$

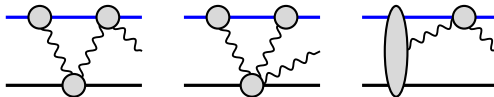
Of course, at LO the $\mathcal{O}(\epsilon)$ term is irrelevant.

$$\begin{aligned} P_{12}^\mu &= w_1 u_2^\mu - w_2 u_1^\mu & u_i &= p_i/m_i & w_i &= u_i \cdot k \\ Q_{12}^\mu &= (q_1 - q_2)^\mu - \frac{q_1^2}{w_1} u_1^\mu + \frac{q_2^2}{w_2} u_2^\mu \end{aligned}$$

One-loop matrix elements Herderschee, Roiban, FT, 2303.06112

$$\langle p'_1 p'_2 k | \hat{T} | p_1 p_2 \rangle \Big|_{1\text{-loop}} = \text{Diagram} = \mathcal{M}^{1\text{ loop}}$$


Use generalized unitarity to construct the integrand



Expand the integrand in the classical limit (method of region, Beneke, Smirnov, hep-ph/9711391)

► Matter propagator

$$\frac{1}{(p + \ell)^2 - m^2 + i0} = \frac{1}{2p \cdot \ell + \ell^2 + i0} = \frac{1}{2p \cdot \ell + i0} \sum_{n=0}^{\infty} \left(-\frac{\ell^2}{2p \cdot \ell + i0} \right)^n$$

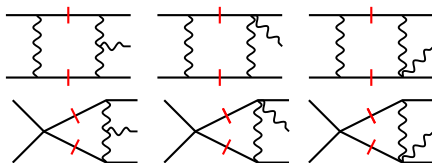
► One matter line per loop

► Matter lines do not touch before IBP

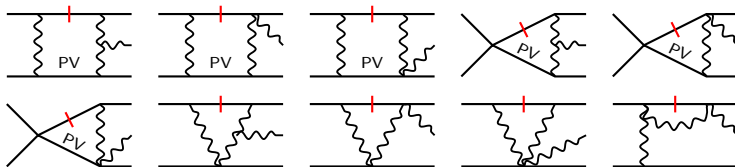
Master integrals: amplitude Herderschee, Roiban, FT, 2303.06112

IBP automated by FIRE6: Smirnov, Chuharev, 1901.07808

Super-classical:
$$\frac{m_1^3 m_2^3}{\hbar^4} \sum_i \alpha_i \mathcal{I}_i^{\text{cut, cut}}$$

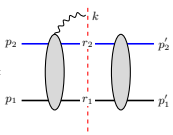


Classical:
$$\frac{1}{\hbar^3} \sum_i (m_1^3 m_2^2 \beta_i \mathcal{I}_i^{\text{cut, pv}} + m_1^2 m_2^3 \gamma_i \mathcal{I}_i^{\text{pv, cut}}) + \frac{1}{\hbar^3} (\text{triangles and bubbles})$$



$$\text{---} \overset{\text{red tick}}{\perp} \text{---} = \hat{\delta}(2u_2 \cdot \ell) \quad \text{---} \overset{\text{PV}}{\text{---}} \text{---} = \frac{1}{2u_1 \cdot \ell + i0} + \frac{1}{2u_1 \cdot \ell - i0}$$

Connected cut contribution $\mathcal{S}^{1 \text{ loop}}$

$$\langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{conn}}^{1 \text{ loop}} = \text{Diagram} = \sum_i \left(\frac{1}{\hbar^4} m_1^3 m_2^3 \alpha_i - \frac{1}{\hbar^3} m_1^3 m_2^2 \beta_i + \frac{1}{\hbar^3} m_1^2 m_2^3 \gamma_i \right) \mathcal{I}_i^{\text{cut, cut}}$$


- ▶ Exactly cancel the super-classical terms in the virtual amplitude
- ▶ Convert the PV to retarded [Caron-Huot, Giroux, Hannesdottir, Mizera, 2308.02125](#)

$$\text{PV} \frac{1}{2u_1 \cdot \ell} \longrightarrow \frac{1}{2u_1 \cdot \ell - i0}$$

- ▶ Classical waveform contains only imaginary IR divergences

Equivalence to world-line formalism proved by Capatti, Zeng, 2412.10864

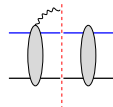
The full connected contribution

Herderschee, Roiban, **FT**, 2303.06112
 Brandhuber, Brown, Chen, De Angelis,
 Gowdy, Travaglini, 2303.06111
 Georgoudis, Heissenberg, Vazquez-Holm, 2303.07006

$$\begin{aligned}
 \mathcal{M}^{1 \text{ loop}} &= -\frac{i \kappa^2}{32\pi} (m_1 w_1 + m_2 w_2) \left[\frac{1}{\epsilon} - \log \frac{w_1 w_2}{\mu^2} \right] \mathcal{M}_{d=4}^{\text{tree}} \\
 &\quad + \kappa^4 \left[A_{\text{rat}}^R + \frac{A_1^R}{\sqrt{w_2^2 - q_1^2}} + \frac{A_2^R}{\sqrt{w_1^2 - q_2^2}} + \frac{A_3^R}{\sqrt{-q_1^2}} + \frac{A_4^R}{\sqrt{-q_2^2}} \right] \\
 &\quad + i \kappa^4 \left[A_{\text{rat}}^I + A_1^I \frac{\text{arcsinh} \frac{w_2}{\sqrt{-q_1^2}}}{\sqrt{w_2^2 - q_1^2}} + A_2^I \frac{\text{arcsinh} \frac{w_1}{\sqrt{-q_2^2}}}{\sqrt{w_1^2 - q_2^2}} + A_3^I \log \frac{q_2^2}{q_1^2} + A_4^I \log \frac{w_1}{w_2} + A_5^I \frac{\text{arccosh} \sigma}{(\sigma^2 - 1)^{3/2}} \right] \\
 &= -\frac{i \kappa^2}{32\pi\epsilon} (m_1 w_1 + m_2 w_2) \mathcal{M}_{d=4}^{\text{tree}} + \widetilde{\mathcal{M}}_{1 \text{ loop}}^{\text{fin}}
 \end{aligned}$$



$$\begin{aligned}
 \mathcal{S}^{1 \text{ loop}} &= \frac{i \kappa^2 \Gamma}{64\pi} (m_1 w_1 + m_2 w_2) \left[\frac{1}{\epsilon} - \log \frac{w_1 w_2}{(\sigma^2 - 1) \mu^2} \right] \mathcal{M}_{d=4}^{\text{tree}} \quad \Gamma = \frac{3\sigma - 2\sigma^3}{(\sigma^2 - 1)^{3/2}} \\
 &\quad + \frac{i \kappa^4}{256\pi} \frac{(2\sigma^2 - 1)^2}{(\sigma^2 - 1)^{3/2}} (m_1 w_1 + m_2 w_2) \left[\frac{1}{\epsilon} - \log \frac{w_1 w_2}{(\sigma^2 - 1) \mu^2} \right] \frac{m_1^2 m_2^2 (u_1 \cdot f \cdot u_2)^2 (w_1^2 + \sigma w_1 w_2 + w_2^2)}{w_1^3 w_2^3} \\
 &\quad + i \kappa^4 \left[A_{\text{rat}}^{\text{cut}} + A_1^{\text{cut}} \log \frac{w_1}{w_2} + A_2^{\text{cut}} \log \frac{w_1 w_2}{-q_1^2} + A_3^{\text{cut}} \log \frac{w_1 w_2}{-q_2^2} + A_4^{\text{cut}} \frac{\text{arccosh} \sigma}{(\sigma^2 - 1)^{3/2}} \right] \\
 &= \frac{i \kappa^2 \Gamma}{64\pi\epsilon} (m_1 w_1 + m_2 w_2) \mathcal{M}_{d=4}^{\text{tree}} + \widetilde{\mathcal{S}}_{1 \text{ loop}}^{\text{fin}} + \mathcal{S}_{1 \text{ loop}}^{\text{local}}
 \end{aligned}$$



The full connected contribution

The IR divergence exponentiates Caron-Huot, Giroux, Hannesdottir, Mizera, 2308.02125

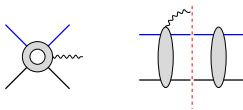
$$\mathcal{M}_{\text{conn}} = \mathcal{M}_{\text{conn}}^{\text{fin}} \exp \left[-i\omega \frac{\kappa^2 E(1 - \Gamma/2)}{32\pi\epsilon} \right] \quad (m_1 w_1 + m_2 w_2 = \omega E)$$

such that it can be absorbed by a re-definition of retarded time

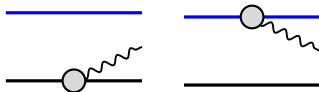
$$\tau \rightarrow \tau - \frac{\kappa^2 E(1 - \Gamma/2)}{32\pi\epsilon}$$

At one-loop level, this leads to

$$\begin{aligned} \mathcal{M}_{\text{conn}}^{\text{fin}} &= \mathcal{M}^{\text{tree}} + \mathcal{M}^{\text{1 loop}} - \mathcal{S}^{\text{1 loop}} + \frac{i\kappa^2(m_1 w_1 + m_2 w_2)(1 - \Gamma/2)}{32\pi\epsilon} \mathcal{M}^{\text{tree}} \\ &= \mathcal{M}^{\text{tree}} + \left[\widetilde{\mathcal{M}}_{\text{1 loop}}^{\text{fin}} + \frac{i\kappa^2(m_1 w_1 + m_2 w_2)\mathcal{M}_{\text{extra}}^{\text{tree}}}{32\pi} \right] - \left[\widetilde{\mathcal{S}}_{\text{1 loop}}^{\text{fin}} + \frac{i\kappa^2\Gamma(m_1 w_1 + m_2 w_2)\mathcal{M}_{\text{extra}}^{\text{tree}}}{64\pi} \right] \\ &= \mathcal{M}^{\text{tree}} + \mathcal{M}_{\text{1 loop}}^{\text{fin}} - \mathcal{S}_{\text{1 loop}}^{\text{fin}} \end{aligned}$$



Disconnected amplitude contribution



They are supported only by zero-energy gravitons

$$\mathcal{W}_{\text{const}}(\omega, \mathbf{n}) = \int d\Phi_{p_1} \mathcal{M}_3(p_1, k) e^{ik \cdot b_1} + \int d\Phi_{p_2} \mathcal{M}_3(p_2, k) e^{ik \cdot b_2} = \hat{\delta}(\omega) \left[\frac{m_1(\varepsilon \cdot u_1)^2}{u_1 \cdot n} + \frac{m_2(\varepsilon \cdot u_2)^2}{u_2 \cdot n} \right]$$

The constant background **agrees with the MPM calculation**

$$\mathcal{W}_{\text{const}}(\tau, \mathbf{n}) = \frac{m_1(\varepsilon \cdot u_1)^2}{u_1 \cdot n} + \frac{m_2(\varepsilon \cdot u_2)^2}{u_2 \cdot n} \quad n = (1, \mathbf{n})$$

It can be removed by a BMS supertranslation [Veneziano, Vilkovisky, 2201.11607](#)

$$\delta_T \left[\frac{h_{ab}^\infty}{r} \right] = \frac{1}{r} (2D_a D_b - \gamma_{ab} \Delta) T(n) - T(n) \partial_\tau \left[\frac{h_{ab}^\infty}{r} \right] \quad T(n) = 2G \left[m_1 \frac{w_1}{\omega} \log \frac{w_1}{\omega} + m_2 \frac{w_2}{\omega} \log \frac{w_2}{\omega} \right]$$

The MPM waveform is in the **intrinsic BMS frame**. If we assume $\mathcal{M}_3 = 0$, then we are in the **canonical BMS frame**. They are related by the above BMS supertranslation.

Disconnected cut contribution [DB, TD, SDA, AG, AH, RR, FT, 2402.06604]

The T matrices in the KMOC cut can be disconnected:

$$i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{disc.}} \supset \int d\Phi[\ell] d\Phi[r_2]$$

- ▶ The cut graviton ℓ only has support on zero energy
- ▶ We only need to consider the soft limit of the left blob

$$\mathcal{M}_6 \simeq \kappa \mathcal{M}_5 \left[\sum_a \frac{\eta_a \varepsilon_{\mu\nu}(\ell) p_a^\mu p_a^\nu}{2\ell \cdot p_a + i0\eta_a} + \frac{\varepsilon_{\mu\nu}(\ell) k^\mu k^\nu}{2\ell \cdot k + i0} \right]$$

- ▶ The sum over massive legs only leads to zero integrals of the form

$$\int d^d \ell \delta(u_2 \cdot \ell) \delta(\ell^2) \Theta(\ell^0) \quad \int d^d \ell \delta(u_1 \cdot \ell) \delta(u_2 \cdot \ell) \delta(\ell^2) \Theta(\ell^0)$$

Disconnected cut contribution [DB, TD, SDA, AG, AH, RR, FT, 2402.06604]

Contribution from the disconnected T matrix elements:

$$i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}_{hh}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{disc.}} = -i\kappa^2 \mathcal{M}^{\text{tree}} \left[m_1^2 w_1^2 I(u_1, k) + m_2^2 w_2^2 I(u_2, k) \right]$$
$$I(u_1, k) = \frac{1}{m_1 \omega} \int \frac{d^d \ell}{(2\pi)^d} \frac{\hat{\delta}(2u_1 \cdot \ell) \hat{\delta}(\ell^2) \Theta(\ell^0)}{2\ell \cdot n}$$

This integral is **not yet properly defined** in $d = 4$; further regularization is needed:

- Need to include the zero-energy graviton contribution
- We should continue u_1 off-shell, $u_1 \rightarrow \tilde{u}_1 - \beta n$, such that

$$2u_1 \cdot \ell = (u_1 + \ell)^2 - 1 \rightarrow (\tilde{u}_1 + \ell)^2 - 1 \simeq 2u_1 \cdot \ell - 2\beta u_1 \cdot n$$

- The integral now becomes

$$I(u_1, k) = \frac{1}{m_1 \omega} \int \frac{d^d \ell}{(2\pi)^d} \frac{\hat{\delta}(2u_1 \cdot \ell - 2\beta u_1 \cdot n) \hat{\delta}(\ell^2) \Theta(\ell^0)}{2\ell \cdot n}$$
$$= \frac{1}{32\pi m_1 w_1} \left[\frac{1}{\epsilon} - \log \frac{w_1^2}{\omega^2} - \log \frac{\beta^2}{\pi} \right]$$

- Divergent terms are absorbed in a further shift of retarded time

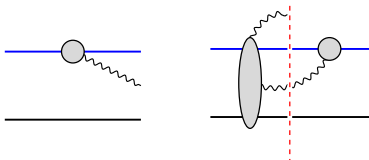
Disconnected cut contribution [DB, TD, SDA, AG, AH, RR, FT, 2402.06604]

The disconnected T matrix elements in the cut lead to

$$\mathcal{M}_{\text{disc}} = i \langle p'_1 p'_2 | \hat{T}^\dagger \hat{a}(k) \hat{T} | p_1 p_2 \rangle \Big|_{\text{disc}} = -2i\omega G \left[m_1 \frac{w_1}{\omega} \log \frac{w_1}{\omega} + m_2 \frac{w_2}{\omega} \log \frac{w_2}{\omega} \right] \mathcal{M}^{\text{tree}} = -i\omega T(n) \mathcal{M}^{\text{tree}}$$

which exactly reproduces the iteration part of the BMS supertranslation

$$\delta_T \left[\frac{h_{ab}^\infty}{r} \right] = \frac{1}{r} (2D_a D_b - \gamma_{ab} \Delta) T(n) - T(n) \partial_\tau \left[\frac{h_{ab}^\infty}{r} \right]$$



Full KMOC waveform at one-loop [DB, TD, SDA, AG, AH, RR, FT, 2402.06604]

$$\mathcal{W}^{\text{KMOC}}(\omega, \mathbf{n}) = \mathcal{W}_{\text{const}} + \mathcal{W}^{\text{tree}} + \text{FT} \left[\mathcal{M}_{1\text{ loop}}^{\text{fin}} - \mathcal{S}_{1\text{ loop}}^{\text{fin}} + \mathcal{M}_{\text{disc}} \right]$$

- ▶ Fourier transform to the impact-parameter space can only be done numerically for generic kinematic setup (see Brunello, De Angelis 2403.08009 for improvements)
- ▶ Analytic computation is available under the soft graviton limit and/or small relative velocity (PN) expansion

Soft expansion (up to the first non-universal coefficient)

$$\mathcal{W}^{\text{KMOC}}(\omega, \mathbf{n}) \sim \frac{\mathcal{A}}{\omega} + \mathcal{B} \log \omega + \mathcal{C} \omega (\log \omega)^2 + \mathcal{D} \omega \log \omega$$

All order conjecture in leading log: Alessio, Di Vecchia, Heissenberg, 2407.04128

PN expansion (up to 2.5PN)

$$\begin{aligned} \mathcal{W}^{\text{KMOC}}(\omega, \mathbf{n}) \sim & 1 + \frac{GM^2\nu}{p_\infty} (1 + p_\infty + p_\infty^2 + p_\infty^3 + p_\infty^4 + p_\infty^5 + \dots) \\ & + \frac{GM}{bp_\infty^2} \frac{GM^2\nu}{p_\infty} (1 + p_\infty + p_\infty^2 + p_\infty^3 + p_\infty^4 + p_\infty^5 + \dots) \end{aligned}$$

$$M = m_1 + m_2$$

$$\nu = m_1 m_2 / M^2$$

$$p_\infty = \sqrt{\sigma^2 - 1}$$

Full KMOC waveform at one-loop [DB, TD, SDA, AG, AH, RR, FT, 2402.06604]

$$\mathcal{W}^{\text{KMOC}}(\omega, \mathbf{n}) = \mathcal{W}_{\text{const}} + \mathcal{W}^{\text{tree}} + \text{FT} \left[\mathcal{M}_{1\text{ loop}}^{\text{fin}} - \mathcal{S}_{1\text{ loop}}^{\text{fin}} + \mathcal{M}_{\text{disc}} \right]$$

- Four
- for
- Anal
- rela

Both in perfect agreement with
the GR based multipolar post-Minkowskian calculation

Bini, Damour, Geralico, 2309.14925

Similar comparison up to $\mathcal{O}(p_\infty^3)$ is done in Georgoudis, Heissenberg, Russo, 2402.06361

rically
all

Soft expansion (up to the first non-universal coefficient)

$$\mathcal{W}^{\text{KMOC}}(\omega, \mathbf{n}) \sim \frac{\mathcal{A}}{\omega} + \mathcal{B} \log \omega + \mathcal{C} \omega (\log \omega)^2 + \mathcal{D} \omega \log \omega$$

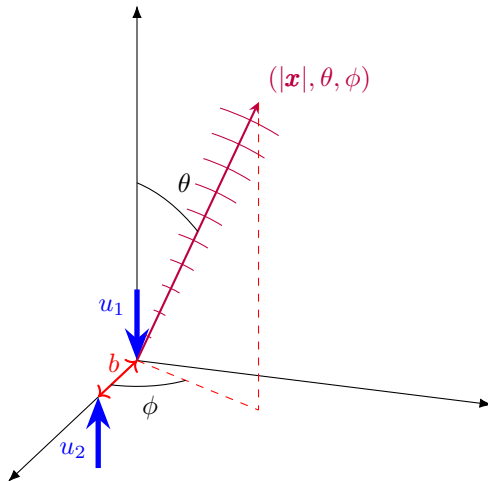
All order conjecture in leading log: Alessio, Di Vecchia, Heissenberg, 2407.04128

PN expansion (up to 2.5PN)

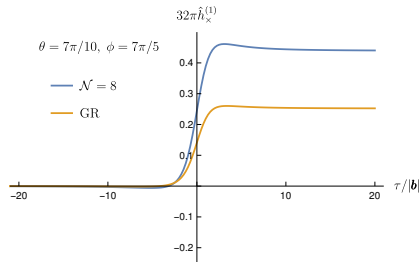
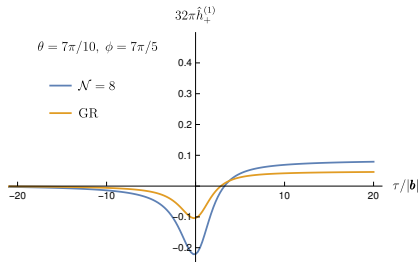
$$\begin{aligned} \mathcal{W}^{\text{KMOC}}(\omega, \mathbf{n}) \sim & 1 + \frac{GM^2 \nu}{p_\infty} (1 + p_\infty + p_\infty^2 + p_\infty^3 + p_\infty^4 + p_\infty^5 + \dots) \\ & + \frac{GM}{bp_\infty^2} \frac{GM^2 \nu}{p_\infty} (1 + p_\infty + p_\infty^2 + p_\infty^3 + p_\infty^4 + p_\infty^5 + \dots) \end{aligned}$$

$$\begin{aligned} M &= m_1 + m_2 \\ \nu &= m_1 m_2 / M^2 \\ p_\infty &= \sqrt{\sigma^2 - 1} \end{aligned}$$

Numerical results

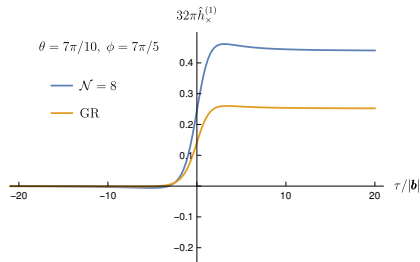
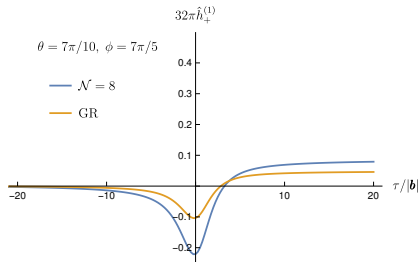


LO waveform



Agree with Jakobsen, Mogull, Plefka, Steinhoff, 2101.12688
From spinning binaries: De Angelis, Novichkov, Gonzo, 2309.17429
Brandhuber, Brown, Chen, Gowdy, Travaglini, 2310.04405
Aoude, Haddad, Heissenberg, Helset, 2310.05832

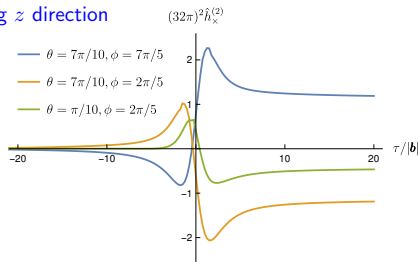
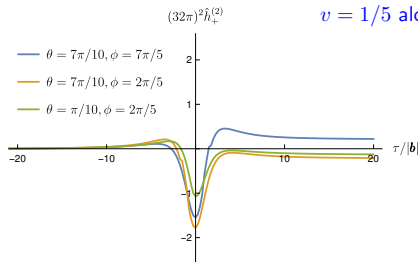
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NLO waveform (GR)

$v = 1/5$ along z direction



$\theta = 7\pi/10$ and $\phi = 7\pi/5$

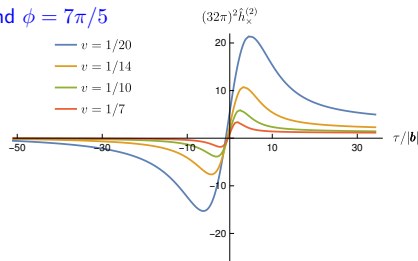
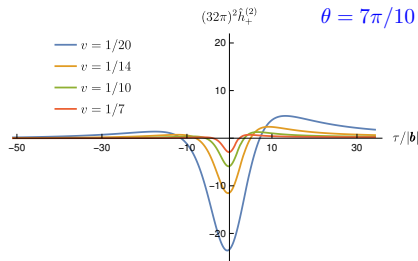


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Spinning black holes and neutron stars

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Synergy between amplitude and GR

Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, **FT**, 2402.06604

One of the important takeaway messages of our new results is that, after having sorted out the subtleties that were hidden in the EFT one-loop waveform, we obtained a remarkable confirmation that the classical limit of an amplitude-based waveform does correctly incorporate the many subtle classical effects that were included in the $O(G^2\eta^5)$ -accurate MPM waveform such as (i) radiation-reaction effects on the worldlines, (ii) high-multipolarity tail effects in the wave-zone, and (iii) cubically nonlinear multipole couplings in the exterior zone. The fact that the road leading to the present successful EFT/MPM comparison had some bumps, which taught us interesting lessons, is another example of the useful synergy between amplitude-based, and classical perturbation-theory-based, approaches to gravitational physics.

Outlook

Explicit higher order spinning and spinless calculation

- ▶ Challenge: IBP reduction and DE for loop integrals; higher rank tensor reduction
- ▶ What do we mean by analytic computation?

Systematic inclusion of tidal operators and absorption

- ▶ Need to understand the running and mixing of WCs

How to describe generic spinning bodies?

- ▶ Perhaps relevant to future higher precision GW observations

Re-summation in observables

- ▶ SCET for gravity
- ▶ Self-force effective theory

How to directly compute bound-state observables?

- ▶ Phenomenological prescription exists and works very well in matching NR
- ▶ Remain as a theoretical challenge

Thanks for listening!