



中国科学院大学

University of Chinese Academy of Sciences

Field Theory of Non-Equilibrium Systems

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Reference

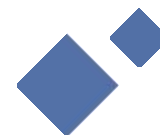
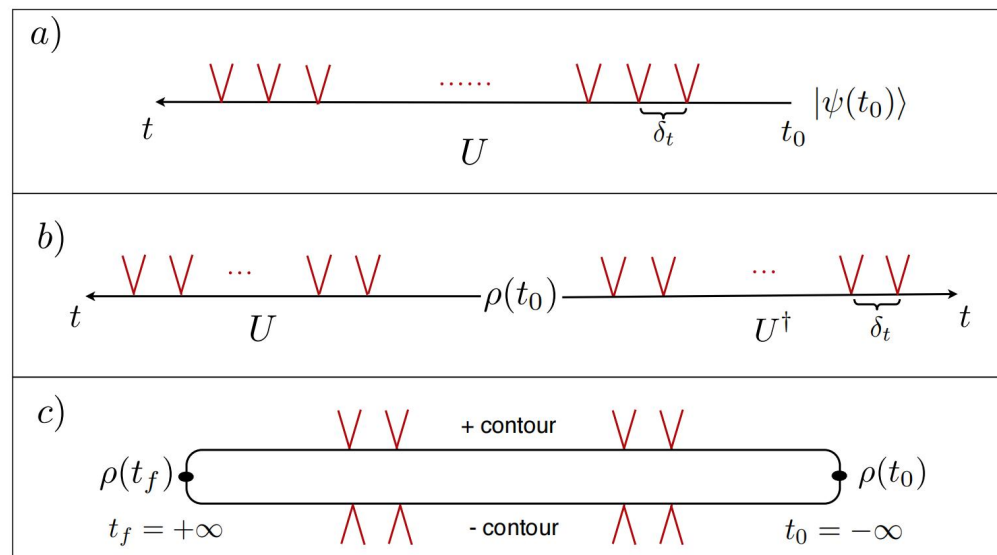
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Review of Schwinger-Keldysh Formalism

- “ Partition Function ”

$$\hat{\mathcal{U}}_{t,t'} = \lim_{N \rightarrow \infty} e^{-i\hat{H}(t-\delta_t)\delta_t} e^{-i\hat{H}(t-2\delta_t)\delta_t} \dots e^{-i\hat{H}(t-N\delta_t)\delta_t} e^{-i\hat{H}(t')\delta_t} = \mathbb{T} \exp \left(-i \int_{t'}^t \hat{H}(t) dt \right)$$

$$Z[V] \equiv \frac{\text{Tr} \left\{ \hat{\mathcal{U}}_{+\infty,-\infty} \hat{\rho}(-\infty) \hat{\mathcal{U}}_{+\infty,-\infty}^\dagger \right\}}{\text{Tr} \{ \hat{\rho}(-\infty) \}} = \frac{\text{Tr} \left\{ \hat{\mathcal{U}}_{+\infty,-\infty} \hat{\rho}(-\infty) \hat{\mathcal{U}}_{-\infty,+\infty} \right\}}{\text{Tr} \{ \hat{\rho}(-\infty) \}} = \frac{\text{Tr} \left\{ \hat{\mathcal{U}}_c[V] \hat{\rho}(-\infty) \right\}}{\text{Tr} \{ \hat{\rho}(-\infty) \}}$$



Review of Schwinger-Keldysh Formalism

- Properties of Coherent States

- Over-Complete Coherent State Basis

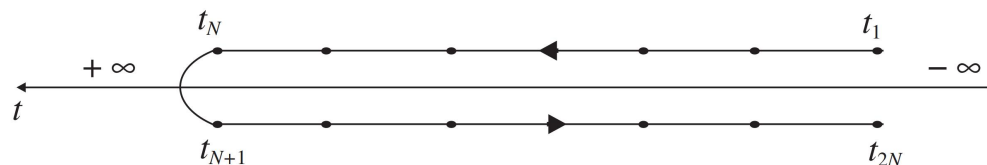
$$\hat{1} = \int d[\bar{\phi}_j, \phi_j] e^{-|\phi_j|^2} |\phi_j\rangle \langle \phi_j|$$

- Identity

$$f(\rho) \equiv \langle \phi | \rho^{\hat{b}^\dagger \hat{b}} | \phi' \rangle = e^{\bar{\phi} \phi' \rho}$$

- Trace of Operators in Number and Coherent Basis

$$\begin{aligned} \text{Tr}\{\hat{O}\} &\equiv \sum_{n=0}^{\infty} \langle n | \hat{O} | n \rangle = \sum_{n=0}^{\infty} \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \langle n | \hat{O} | \phi \rangle \langle \phi | n \rangle \\ &= \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \sum_{n=0}^{\infty} \langle \phi | n \rangle \langle n | \hat{O} | \phi \rangle = \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \langle \phi | \hat{O} | \phi \rangle \end{aligned}$$



Review of Schwinger-Keldysh Formalism

$$H(\bar{\phi}_j, \phi_{j-1}) = \omega_0 \bar{\phi}_j \phi_{j-1}$$

$$\langle \phi_1 | e^{-\beta(\omega_0 - \mu)b^\dagger b} | \phi_{2N} \rangle = \exp \left\{ \bar{\phi}_1 \phi_{2N} \rho(\omega_0) \right\}$$

- In the 3×3 Discrete Case

$$\begin{aligned} \text{Tr} \left\{ \hat{U}_c \hat{\rho}_0 \right\} &\stackrel{(2.10)}{=} \int d[\bar{\phi}_6, \phi_6] e^{-|\phi_6|^2} \langle \phi_6 | \mathbb{T} \exp \left[- \int_t^{t'} dt \hat{H}(t) \right] \mathbb{T} \exp \left[- \int_{t'}^t dt \hat{H}(t) \right] \hat{\rho}_0 | \phi_6 \rangle \quad t' \leq t, N=3, t = t' + 2\delta_t \\ &\approx \int d[\bar{\phi}_6, \phi_6] e^{-|\phi_6|^2} \langle \phi_6 | \left\{ \exp \left[-\hat{H}(t)(-\delta_t) \right] \exp \left[-\hat{H}(t-\delta_t)(-\delta_t) \right] \right\} \left\{ \exp \left[-\hat{H}(t-\delta_t)(\delta_t) \right] \exp \left[-\hat{H}(t)(\delta_t) \right] \right\} \hat{\rho}_0 | \phi_6 \rangle \\ &= \int d[\bar{\phi}_6, \phi_6] e^{-|\phi_6|^2} \langle \phi_6 | \hat{U}_{-\delta_t}(t_5) \hat{U}_{-\delta_t}(t_4) \hat{U}_{+\delta_t}(t_3) \hat{U}_{+\delta_t}(t_2) | \phi_6 \rangle \\ &= \int d[\bar{\phi}_6, \phi_6] e^{-|\phi_6|^2} \langle \phi_6 | \hat{U}_{-\delta_t}(t_5) \left(\int d[\bar{\phi}_5, \phi_5] e^{-|\phi_5|^2} | \phi_5 \rangle \langle \phi_5 | \right) \hat{U}_{-\delta_t}(t_4) \left(\int d[\bar{\phi}_4, \phi_4] e^{-|\phi_4|^2} | \phi_4 \rangle \langle \phi_4 | \right) \hat{U}_{+\delta_t}(t_3) \\ &\quad \left(\int d[\bar{\phi}_3, \phi_3] e^{-|\phi_3|^2} | \phi_3 \rangle \langle \phi_3 | \right) \hat{U}_{+\delta_t}(t_2) \left(\int d[\bar{\phi}_2, \phi_2] e^{-|\phi_2|^2} | \phi_2 \rangle \langle \phi_2 | \right) \left(\int d[\bar{\phi}_1, \phi_1] e^{-|\phi_1|^2} | \phi_1 \rangle \langle \phi_1 | \right) | \phi_6 \rangle \\ &= \int \prod_{j=1}^6 d[\bar{\phi}_j, \phi_j] e^{-\sum_{i=1}^6 |\phi_i|^2} \langle \phi_6 | \hat{U}_{-\delta_t} | \phi_5 \rangle \langle \phi_5 | \hat{U}_{-\delta_t} | \phi_4 \rangle \langle \phi_4 | \hat{1} | \phi_3 \rangle \langle \phi_3 | \hat{U}_{+\delta_t} | \phi_2 \rangle \langle \phi_2 | \hat{U}_{+\delta_t} | \phi_1 \rangle \langle \phi_1 | \hat{\rho}_0 | \phi_6 \rangle \\ &\quad \langle \phi_j | \hat{U}_{\pm \delta_t} | \phi_{j-1} \rangle \equiv \langle \phi_j | e^{\mp i \hat{H}(b^\dagger, b) \delta_t} | \phi_{j-1} \rangle \approx \langle \phi_j | \left(1 \mp i \hat{H}(b^\dagger, b) \delta_t \right) | \phi_{j-1} \rangle \\ &\quad = \langle \phi_j | \phi_{j-1} \rangle \left(1 \mp i H(\bar{\phi}_j, \phi_{j-1}) \delta_t \right) \approx e^{\bar{\phi}_j \phi_{j-1}} e^{\mp i H(\bar{\phi}_j, \phi_{j-1}) \delta_t} \end{aligned}$$



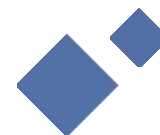
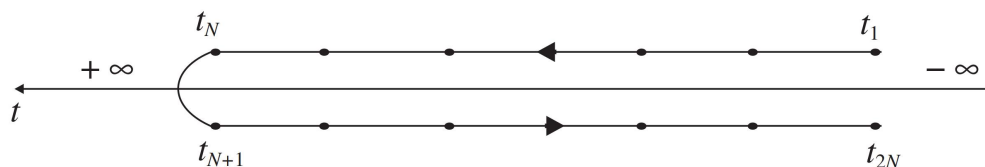
Review of Schwinger-Keldysh Formalism

- Partition Function in Discrete Coherent Basis

$$Z = \frac{1}{\text{Tr} \{ \hat{\rho}_0 \}} \int \prod_{j=1}^{2N} d[\bar{\phi}_j, \phi_j] \exp \left(i \sum_{j,j'=1}^{2N} \bar{\phi}_j G_{jj'}^{-1} \phi_{j'} \right)$$

$$iG_{jj}^{-1} \equiv \left(\begin{array}{ccc|ccc} -1 & & & & & \\ h_- & -1 & & & & \\ & h_- & -1 & & & \\ \hline & & 1 & -1 & & \\ & & & h_+ & -1 & \\ & & & & h_+ & -1 \end{array} \right) \quad \rho(\omega_0) \quad h_{\mp} \equiv 1 \mp i\omega_0 \delta_t$$

- Initial Density Matrix : Proper Inverse / Green Function



Review of Schwinger-Keldysh Formalism

- Normalization and Continuum Notation

- Determinant

$$\det \left[-i\hat{G}^{-1} \right] = 1 - \rho(\omega_0)(h_- h_+)^{N-1} = 1 - \rho(\omega_0)(1 + \omega_0^2 \delta_t^2)^{N-1}$$

$$\approx 1 - \rho(\omega_0) e^{\omega_0^2 \delta_t^2 (N-1)} \xrightarrow{N \rightarrow \infty} 1 - \rho(\omega_0)$$

- Normalization

$$\text{Tr} \{ \hat{\rho}_0 \} = \sum_{n=0}^{\infty} e^{-\beta(\omega_0 - \mu)n} = [1 - \rho(\omega_0)]^{-1} \Rightarrow Z = \frac{1}{\text{Tr} \{ \hat{\rho}_0 \}} \frac{1}{\det \left[-i\hat{G}^{-1} \right]} = 1$$

- Continuum Notation

$$Z = \int \mathbf{D}[\bar{\phi}(t), \phi(t)] e^{iS[\bar{\phi}, \phi]} \quad \mathbf{D}[\bar{\phi}(t), \phi(t)] = \prod_{j=1}^{2N} d[\bar{\phi}_j, \phi_j] / \text{Tr} \{ \hat{\rho}_0 \}$$

$$S[\bar{\phi}, \phi] = \sum_{j=2}^{2N} \delta t_j \left[i\bar{\phi}_j \frac{\phi_j - \phi_{j-1}}{\delta t_j} - \omega_0 \bar{\phi}_j \phi_{j-1} \right] + i\bar{\phi}_1 [\phi_1 - i\rho(\omega_0)\phi_{2N}] \quad \delta t_j \equiv t_j - t_{j-1} = \pm \delta t$$

$$S[\bar{\phi}, \phi] = \int_c dt \bar{\phi}(t) \hat{G}^{-1} \phi(t) \quad \hat{G}^{-1} = i\partial_t - \omega_0$$



Review of Schwinger-Keldysh Formalism

- Normalization and Continuum Notation

- Continuum Notation

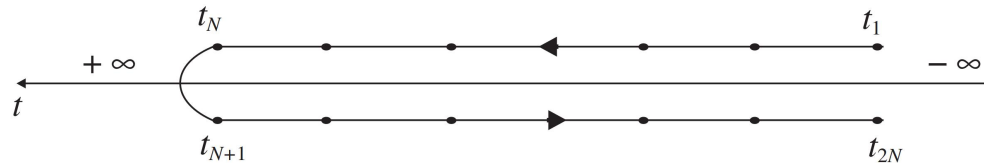
$$S[\bar{\phi}, \phi] = \int_{\mathcal{C}} dt \bar{\phi}(t) \hat{G}^{-1} \phi(t) \quad \hat{G}^{-1} = i\partial_t - \omega_0$$

- Zero Mode and Boundary Terms (To Uniquely Determine Inverse Operator $\hat{G}$)

$$\hat{G}^{-1} e^{-i\omega_0 t} = (i\partial_t - \omega_0) e^{-i\omega_0 t} = 0 \quad \Rightarrow \quad \hat{G}^{-1} \psi(t) = x = \hat{G}^{-1} (\psi(t) + e^{-i\omega_0 t})$$

- To Avoid Integration along the Closed Time Contour

$$S[\bar{\phi}, \phi] = \int_{-\infty}^{+\infty} dt \left[\bar{\phi}^+(t) (i\partial_t - \omega_0) \phi^+(t) - \bar{\phi}^-(t) (i\partial_t - \omega_0) \phi^-(t) \right]$$



Review of Schwinger-Keldysh Formalism

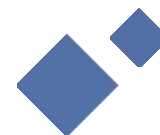
- Green Functions
 - Correlator of Two Bosonic Fields

$$\langle \phi_j \bar{\phi}_{j'} \rangle \equiv \int \mathbf{D}[\bar{\phi}, \phi] \phi_j \bar{\phi}_{j'} \exp \left(i \sum_{k,k'=1}^{2N} \bar{\phi}_k G_{kk'}^{-1} \phi_{k'} \right) = iG_{jj'}$$

- Inverse $i\hat{G}^{-1}$ to Derive $i\hat{G}$

$$iG_{jj'}^{-1} = \left(\begin{array}{cc|cc} -1 & & & \rho(\omega_0) \\ h_- & -1 & & \\ & h_- & -1 & \\ \hline & & 1 & -1 \\ & & & h_+ & -1 \\ & & & h_+ & -1 \end{array} \right) \rightarrow iG_{jj'} = \frac{1}{\det[-i\hat{G}^{-1}]} \left(\begin{array}{ccc|ccc} 1 & \rho h_+^2 h_- & \rho h_+^2 & \rho h_+^2 & \rho h_+ & \rho \\ h_- & 1 & \rho h_+^2 h_- & \rho h_+^2 h_- & \rho h_+ h_- & \rho h_- \\ h_-^2 & h_- & 1 & \rho h_+^2 h_-^2 & \rho h_+ h_-^2 & \rho h_-^2 \\ \hline h_-^2 & h_- & 1 & 1 & \rho h_-^2 h_+ & \rho h_-^2 \\ h_-^2 h_+ & h_- h_+ & h_+ & h_+ & 1 & \rho h_-^2 h_+ \\ h_-^2 h_+^2 & h_- h_+^2 & h_+^2 & h_+^2 & h_+ & 1 \end{array} \right)$$

$$iG = \begin{pmatrix} \langle \phi_j^+ \bar{\phi}_{j'}^+ \rangle & \langle \phi_j^+ \bar{\phi}_{j'}^- \rangle \\ \langle \phi_j^- \bar{\phi}_{j'}^+ \rangle & \langle \phi_j^- \bar{\phi}_{j'}^- \rangle \end{pmatrix} \equiv \begin{pmatrix} iG_{jj'}^{++} & iG_{jj'}^{+-} \\ iG_{jj'}^{-+} & iG_{jj'}^{--} \end{pmatrix} \equiv \begin{pmatrix} iG_{jj'}^{\mathbb{T}} & iG_{jj'}^{<} \\ iG_{jj'}^{>} & iG_{jj'}^{\tilde{\mathbb{T}}} \end{pmatrix}$$



Review of Schwinger-Keldysh Formalism

- Green Functions

- Matrix Elements

$$\langle \phi_j^+ \bar{\phi}_{j'}^- \rangle \equiv iG_{jj'}^< = \frac{\rho h_+^{j'-1} h_-^{j-1}}{\det [-i\hat{G}^{-1}]} \quad \langle \phi_j^- \bar{\phi}_{j'}^+ \rangle \equiv iG_{jj'}^> = \frac{h_+^{N-j} h_-^{N-j'}}{\det [-i\hat{G}^{-1}]} = \frac{(h_+ h_-)^{N-1} h_+^{1-j} h_-^{1-j'}}{\det [-i\hat{G}^{-1}]}$$

$$\langle \phi_j^+ \bar{\phi}_{j'}^+ \rangle \equiv iG_{jj'}^{\mathbb{T}} = \frac{h_-^{j-j'}}{\det [-i\hat{G}^{-1}]} \times \begin{cases} 1, & j \geq j' \\ \rho(h_+ h_-)^{N-1}, & j < j' \end{cases} \quad \langle \phi_j^- \bar{\phi}_{j'}^- \rangle \equiv iG_{jj'}^{\mathbb{T}} = \frac{h_+^{j'-j}}{\det [-i\hat{G}^{-1}]} \times \begin{cases} \rho(h_+ h_-)^{N-1}, & j > j' \\ 1, & j \leq j' \end{cases}$$

- Continuum Limit

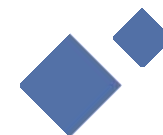
$$\langle \phi^+(t) \bar{\phi}^-(t') \rangle = iG^<(t, t') = n_B e^{-i\omega_0(t-t')} \quad \langle \phi^-(t) \bar{\phi}^+(t') \rangle = iG^>(t, t') = (n_B + 1) e^{-i\omega_0(t-t')}$$

$$\langle \phi^+(t) \bar{\phi}^+(t') \rangle = iG^{\mathbb{T}}(t, t') = \theta(t - t') iG^>(t, t') + \theta(t' - t) iG^<(t, t')$$

$$\langle \phi^-(t) \bar{\phi}^-(t') \rangle = iG^{\tilde{\mathbb{T}}}(t, t') = \theta(t' - t) iG^>(t, t') + \theta(t - t') iG^<(t, t')$$

Bosonic Occupation Number

$$n_B(\omega_0) = \frac{\rho(\omega_0)}{1 - \rho(\omega_0)}$$



Review of Schwinger-Keldysh Formalism

- Green Functions

- Four Green Functions are not Independent

$$G^{\mathbb{T}}(t, t') + G^{\tilde{\mathbb{T}}}(t, t') - G^{>}(t, t') - G^{<}(t, t') = 0$$

- Keldysh Rotation

$$\phi^{\text{cl}}(t) = \frac{1}{\sqrt{2}}(\phi^+(t) + \phi^-(t)) \quad \phi^{\text{q}}(t) = \frac{1}{\sqrt{2}}(\phi^+(t) - \phi^-(t))$$

- Retarded, Advanced and Keldysh Components of Green Functions

$$\begin{aligned} \langle \phi^\alpha(t) \bar{\phi}^\beta(t') \rangle &\equiv iG^{\alpha\beta}(t, t') = \begin{pmatrix} iG^K(t, t') & iG^R(t, t') \\ iG^A(t, t') & 0 \end{pmatrix} = \begin{pmatrix} iG^{\text{cl,cl}}(t, t') & iG^{\text{cl,q}}(t, t') \\ iG^{\text{q,cl}}(t, t') & 0 \end{pmatrix} = \begin{pmatrix} \langle \phi^{\text{cl}}(t) \bar{\phi}^{\text{cl}}(t') \rangle & \langle \phi^{\text{cl}}(t) \bar{\phi}^{\text{q}}(t') \rangle \\ \langle \phi^{\text{q}}(t) \bar{\phi}^{\text{cl}}(t') \rangle & \langle \phi^{\text{q}}(t) \bar{\phi}^{\text{q}}(t') \rangle \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \langle (\phi^+(t) + \phi^-(t)) (\bar{\phi}^+(t') + \bar{\phi}^-(t')) \rangle & \langle (\phi^+(t) + \phi^-(t)) (\bar{\phi}^+(t') - \bar{\phi}^-(t')) \rangle \\ \langle (\phi^+(t) - \phi^-(t)) (\bar{\phi}^+(t') + \bar{\phi}^-(t')) \rangle & \langle (\phi^+(t) - \phi^-(t)) (\bar{\phi}^+(t') - \bar{\phi}^-(t')) \rangle \end{pmatrix} \\ &= \frac{i}{2} \begin{pmatrix} G^{\mathbb{T}}(t, t') + G^{\tilde{\mathbb{T}}}(t, t') + G^{>}(t, t') + G^{<}(t, t') & G^{\mathbb{T}}(t, t') - G^{\tilde{\mathbb{T}}}(t, t') + G^{>}(t, t') - G^{<}(t, t') \\ G^{\mathbb{T}}(t, t') - G^{\tilde{\mathbb{T}}}(t, t') - G^{>}(t, t') + G^{<}(t, t') & G^{\mathbb{T}}(t, t') + G^{\tilde{\mathbb{T}}}(t, t') - G^{>}(t, t') - G^{<}(t, t') \end{pmatrix} \end{aligned}$$



Review of Schwinger-Keldysh Formalism

- Keldysh Rotation
 - Retarded, Advanced and Keldysh Components of Green Functions

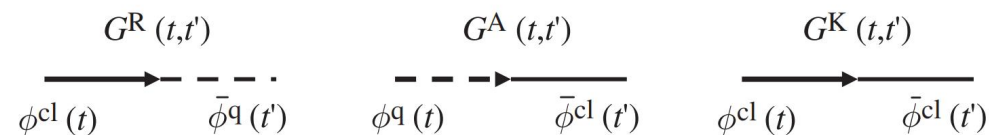
$$\begin{aligned}\langle \phi^\alpha(t) \bar{\phi}^\beta(t') \rangle &\equiv iG^{\alpha\beta}(t, t') = \begin{pmatrix} iG^K(t, t') & iG^R(t, t') \\ iG^A(t, t') & 0 \end{pmatrix} \\ &= \frac{i}{2} \begin{pmatrix} G^\mathbb{T}(t, t') + G^{\tilde{\mathbb{T}}}(t, t') + G^>(t, t') + G^<(t, t') & G^\mathbb{T}(t, t') - G^{\tilde{\mathbb{T}}}(t, t') + G^>(t, t') - G^<(t, t') \\ G^\mathbb{T}(t, t') - G^{\tilde{\mathbb{T}}}(t, t') - G^>(t, t') + G^<(t, t') & G^\mathbb{T}(t, t') + G^{\tilde{\mathbb{T}}}(t, t') - G^>(t, t') - G^<(t, t') \end{pmatrix} \\ &= i \begin{pmatrix} G^>(t, t') + G^<(t, t') & \theta(t - t') [G^>(t, t') - G^<(t, t')] \\ \theta(t' - t) [G^<(t, t') - G^>(t, t')] & 0 \end{pmatrix}\end{aligned}$$



Review of Schwinger-Keldysh Formalism

- Keldysh Rotation

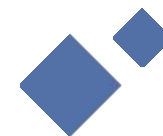
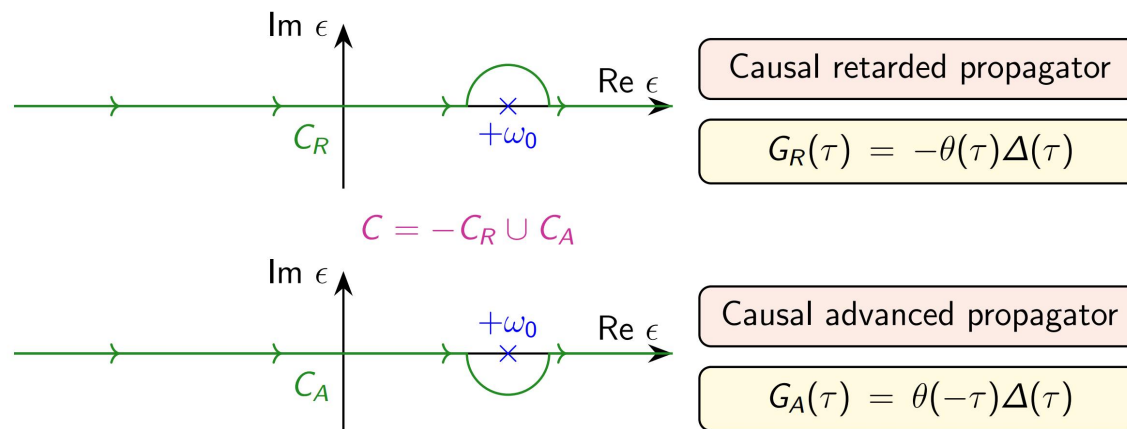
- Toy Example of Single Boson Level $\hat{H}(\hat{b}^\dagger, \hat{b}) = \omega_0 \hat{b}^\dagger \hat{b}$



$$G^R = -i\theta(t - t')e^{-i\omega_0(t-t')} \xrightarrow{\text{FT}} (\epsilon - \omega_0 + i0)^{-1}$$

$$G^A = i\theta(t' - t)e^{-i\omega_0(t-t')} \xrightarrow{\text{FT}} (\epsilon - \omega_0 - i0)^{-1}$$

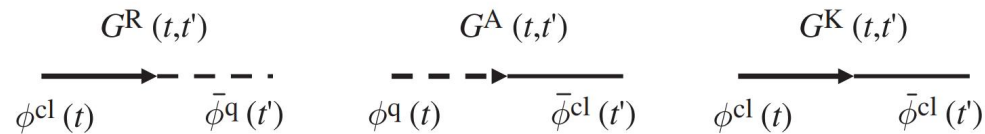
$$G^K = -i[2n_B(\omega_0) + 1]e^{-i\omega_0(t-t')} \xrightarrow{\text{FT}} -2\pi i[2n_B(\epsilon) + 1]\delta(\epsilon - \omega_0)$$



Review of Schwinger-Keldysh Formalism

- Keldysh Rotation

- Toy Example of Single Boson Level $\hat{H}(\hat{b}^\dagger, \hat{b}) = \omega_0 \hat{b}^\dagger \hat{b}$
- Graphic Representation



- Fluctuation-Dissipation Theorem

$$\rho = e^{-(\omega_0 - \mu)/T} \quad n_B = \frac{1}{e^{(\omega_0 - \mu)/T} - 1}$$

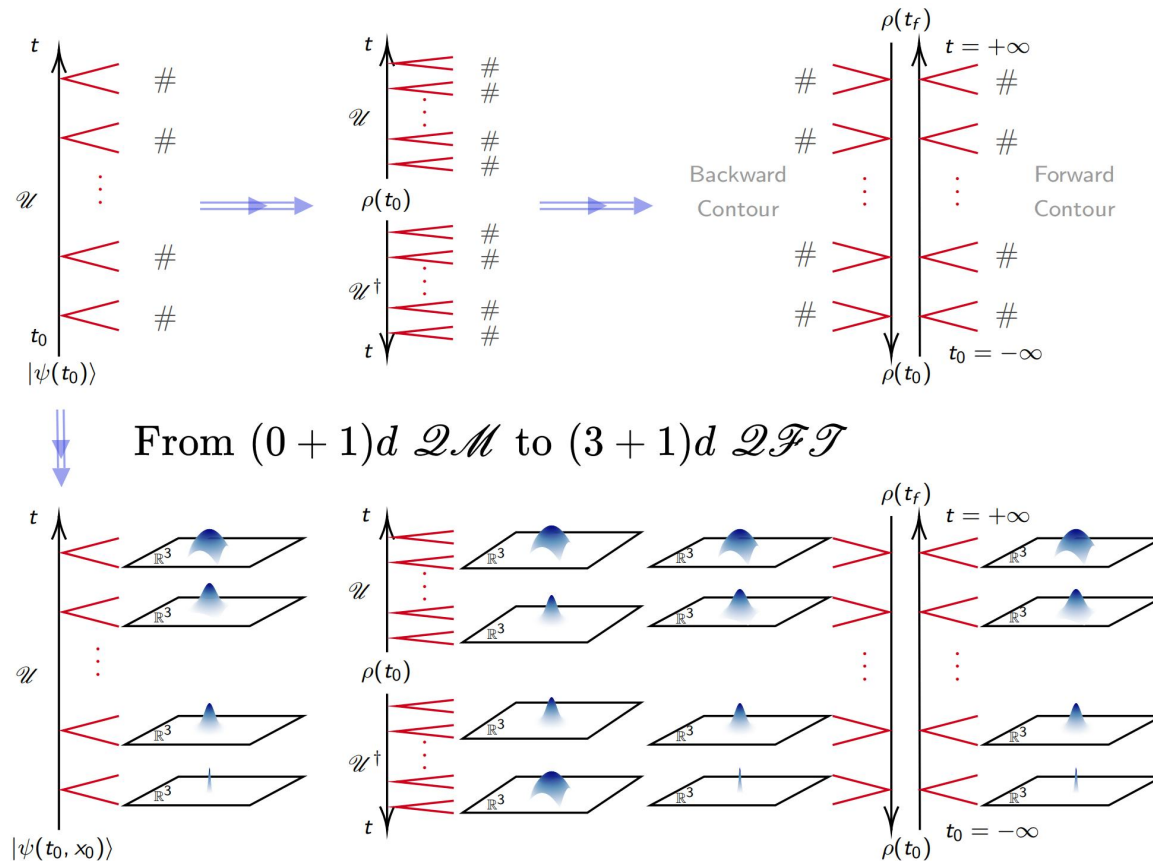
$\Downarrow$

$$G^K(\epsilon) = \coth \frac{\epsilon - \mu}{2T} [G^R(\epsilon) - G^A(\epsilon)]$$



Review of Schwinger-Keldysh Formalism

- From $(0+1)$ d **QM** to $(3+1)$ d **QFT**



Single-Particle Quantum Mechanics

- Parametrize the Action by its Real and Imaginary Parts

$$\hat{b}(t) = \frac{1}{\sqrt{2\omega_0}} (\omega_0 \hat{X}(t) + i\hat{P}(t)) \quad \hat{b}^\dagger(t) = \frac{1}{\sqrt{2\omega_0}} (\omega_0 \hat{X}(t) - i\hat{P}(t))$$

$\Downarrow$

$$\phi(t) = \frac{1}{\sqrt{2\omega_0}} (\omega_0 X(t) + iP(t)) \quad \bar{\phi}(t) = \frac{1}{\sqrt{2\omega_0}} (\omega_0 X(t) - iP(t))$$

$$S[\bar{\phi}, \phi] = \int_c dt \bar{\phi}(t) \hat{G}^{-1} \phi(t) = \int_{-\infty}^{+\infty} dt \left[\bar{\phi}^+(t) (i\partial_t - \omega_0) \phi^+(t) - \bar{\phi}^-(t) (i\partial_t - \omega_0) \phi^-(t) \right] \quad \hat{G}^{-1} = i\partial_t - \omega_0$$

$\Downarrow$

$$\begin{aligned} S[\bar{\phi}, \phi] &= \int_c dt \frac{1}{\sqrt{2\omega_0}} (\omega_0 X(t) - iP(t)) (i\partial_t - \omega_0) \frac{1}{\sqrt{2\omega_0}} (\omega_0 X(t) - iP(t)) \\ &= \int_c dt \left[P\dot{X} - \frac{1}{2}P^2 - \frac{\omega_0^2}{2}X^2 \right] = \int_c dt \left[\frac{i\omega_0}{2} \dot{X}X + \frac{i}{2\omega_0} \dot{P}P - \frac{1}{2}\dot{P}X + \frac{1}{2}P\dot{X} - \frac{\omega_0^2}{2}X^2 - \frac{1}{2}P^2 \right] \\ &= \left[\frac{i}{4\omega_0} (P^2 + \omega_0^2 X^2) - \frac{1}{2}PX \right]_c + \int_c dt \left[P\dot{X} - \frac{1}{2}P^2 - \frac{\omega_0^2}{2}X^2 \right] \end{aligned}$$



Single-Particle Quantum Mechanics

Saddle Point

$$P = \dot{X}$$

- Indeed the Action of the Quantum Harmonic Oscillator

$$S[X, P] = \int_c dt \left[P\dot{X} - \frac{1}{2}P^2 - \frac{\omega_0^2}{2}X^2 \right]$$

- Integrate out Momentum to Derive the Lagrangian Form

$$S[X] = \int_c dt \left[\frac{1}{2}\dot{X}^2 - \frac{\omega_0^2}{2}X^2 \right]$$

- Feynman Lagrangian Action of the Harmonic Oscillator **on CTP**

$$S[X] = \int_c dt \left[\frac{1}{2}\dot{X}^2 - V(X) \right]$$

- Keldysh Rotation

$$X^{\text{cl}}(t) = \frac{1}{2} [X^+(t) + X^-(t)] \quad X^{\text{q}}(t) = \frac{1}{2} [X^+(t) - X^-(t)]$$



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Single-Particle Quantum Mechanics

Saddle Point

$$P = \dot{X}$$

- Keldysh Rotation

$$\begin{aligned} S[X] &= \int_C dt \left[\frac{1}{2} \dot{X}^2 - V(X) \right] = \int_{-\infty}^{+\infty} dt \left\{ \left[\frac{1}{2} (\dot{X}^+)^2 - V(X^+) \right] - \left[\frac{1}{2} (\dot{X}^-)^2 - V(X^-) \right] \right\} \\ &= \int_{-\infty}^{+\infty} dt \left[\frac{1}{2} (\dot{X}^+ + \dot{X}^-) (\dot{X}^+ - \dot{X}^-) - (V(X^+) - V(X^-)) \right] \\ &= \int_{-\infty}^{+\infty} dt \left[2\dot{X}^q \dot{X}^{cl} - V(X^{cl} + X^q) + V(X^{cl} - X^q) \right] \\ &= \left\{ \left\{ \frac{i}{2\omega_0} \left[(\dot{X}^+)^2 + V(X^+) \right] - \frac{1}{2} X^+ \dot{X}^+ \right\} - \left\{ \frac{i}{2\omega_0} \left[(\dot{X}^-)^2 + V(X^-) \right] - \frac{1}{2} X^- \dot{X}^- \right\} + 2X^q \dot{X}^{cl} \right\}_{-\infty}^{+\infty} \\ &\quad + \int_{-\infty}^{+\infty} dt \left[-2X^q \ddot{X}^{cl} - V(X^{cl} + X^q) + V(X^{cl} - X^q) \right] \\ &= \left\{ \frac{i}{2\omega_0} \left[4\dot{X}^{cl} \dot{X}^q + V(X^{cl} + X^q) - V(X^{cl} - X^q) \right] + X^q \dot{X}^{cl} - \dot{X}^q X^{cl} \right\}_{-\infty}^{+\infty} \\ &\quad + \int_{-\infty}^{+\infty} dt \left[-2X^q \ddot{X}^{cl} - V(X^{cl} + X^q) + V(X^{cl} - X^q) \right] \\ &\equiv S[X^{cl}, X^q] = \int_{-\infty}^{+\infty} dt \left[-2X^q \ddot{X}^{cl} - V(X^{cl} + X^q) + V(X^{cl} - X^q) \right] \end{aligned}$$



Single-Particle Quantum Mechanics

- Small Fluctuation of Quantum Components

$$S[X^{\text{cl}}, X^{\text{q}}] = - \int_{-\infty}^{+\infty} dt \left[2X^{\text{q}} \left(\ddot{X}^{\text{cl}} + V'(X^{\text{cl}}) \right) + O[(X^{\text{q}})^3] \right]$$

$\Downarrow$

$$Z = \int \mathbf{D}[X^{\text{cl}}, X^{\text{q}}] e^{iS[X^{\text{cl}}, X^{\text{q}}]} = \int \mathbf{D}[X^{\text{cl}}] 2\pi \delta \left(2 \left(\ddot{X}^{\text{cl}} + V'(X^{\text{cl}}) \right) \right)$$

- Classical Newtonian Dynamics of

$$\ddot{X}^{\text{cl}} = -V'(X^{\text{cl}})$$

- Higher-Order Terms : Both are Subject to Quantum Fluctuations

- Harmonic Oscillator $V(X) = \omega_0^2 X^2/2$

$$S[\vec{X}] = \frac{1}{2} \int_{-\infty}^{+\infty} dt \vec{X}^{\text{T}} \hat{D}^{-1} \vec{X}$$

$$\vec{X}(t) = \begin{pmatrix} X^{\text{cl}}(t) \\ X^{\text{q}}(t) \end{pmatrix} \quad \hat{D}^{-1} = \begin{pmatrix} 0 & [D^{-1}]^{\text{A}} \\ [D^{-1}]^{\text{R}} & [D^{-1}]^{\text{K}} \end{pmatrix}$$



Single-Particle Quantum Mechanics

- Quantum Particle in Contact with an Environment

$$S = S_p + S_{\text{bath}} + S_{\text{int}}$$

$$S_p[X] = \int_{-\infty}^{+\infty} dt \left[-2X^q \ddot{X}^{\text{cl}} - V(X^{\text{cl}} + X^q) + V(X^{\text{cl}} - X^q) \right]$$

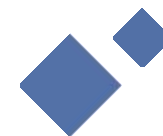
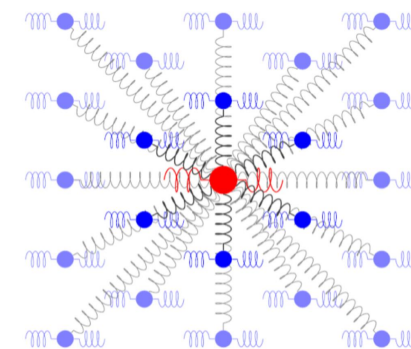
$$S_{\text{bath}}[\varphi_s] = \frac{1}{2} \sum_s \int_{-\infty}^{+\infty} dt \vec{\varphi}_s^T \hat{D}_s^{-1} \vec{\varphi}_s \quad S_{\text{int}}[X, \varphi_s] = \sum_s g_s \int_{-\infty}^{+\infty} dt \vec{X}^T \hat{\sigma}_1 \vec{\varphi}_s$$

- Interaction between the Particle and Bath Oscillators

$$\sum_s g_s \int_{\mathcal{C}} dt X(t) \varphi_s(t) = \sum_s g_s \int dt (X^+ \varphi_s^+ - X^- \varphi_s^-)$$

- Dissipative Action (Integrate out the Bath)

Saddle Point $\frac{\delta(S_{\text{bath}} + S_{\text{int}})}{\delta \vec{\varphi}_s} = 0 \Rightarrow \vec{\varphi}_s = -g_s \hat{D}_s \vec{X}^T \hat{\sigma}_1 = -g_s \hat{D}_s \hat{\sigma}_1 \vec{X}$



Single-Particle Quantum Mechanics

- Dissipative Action (Integrate out the Bath)

Saddle Point $\vec{\varphi}_s = -g_s \hat{D}_s \vec{X}^T \hat{\sigma}_1 = -g_s \hat{D}_s \hat{\sigma}_1 \vec{X}$

$$S_p + S_{\text{bath}} + S_{\text{int}} \rightarrow S_p + S_{\text{diss}}$$

$$S_{\text{diss}}[X] = \frac{1}{2} \sum_s \int_{-\infty}^{+\infty} dt \left(-g_s \hat{D}_s \hat{\sigma}_1 \vec{X} \right)^T \hat{D}_s^{-1} \left(-g_s \hat{D}_s \hat{\sigma}_1 \vec{X} \right) + \sum_s g_s \int_{-\infty}^{+\infty} dt \vec{X}^T \hat{\sigma}_1 \left(-g_s \hat{D}_s \hat{\sigma}_1 \vec{X} \right)$$

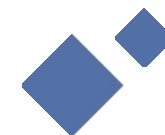
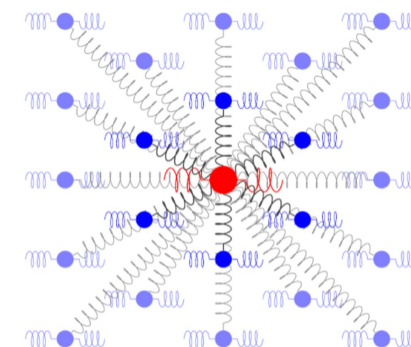
$\Downarrow$

$$S_{\text{diss}} = \frac{1}{2} \iint_{-\infty}^{+\infty} dt dt' \vec{X}^T(t) \hat{\mathfrak{D}}^{-1}(t-t') \vec{X}(t') \quad \hat{\mathfrak{D}}^{-1}(t-t') = -\hat{\sigma}_1 \left[\sum_s g_s^2 \hat{D}_s(t-t') \right] \hat{\sigma}_1$$

$$[\hat{\mathfrak{D}}^{-1}(\epsilon)]^{\text{R(A)}} = -\frac{1}{2} \sum_s \frac{g_s^2}{(\epsilon \pm i0)^2 - \omega_s^2} = \int \frac{d\omega}{2\pi} \frac{\omega J(\omega)}{\omega^2 - (\epsilon \pm i0)^2}$$

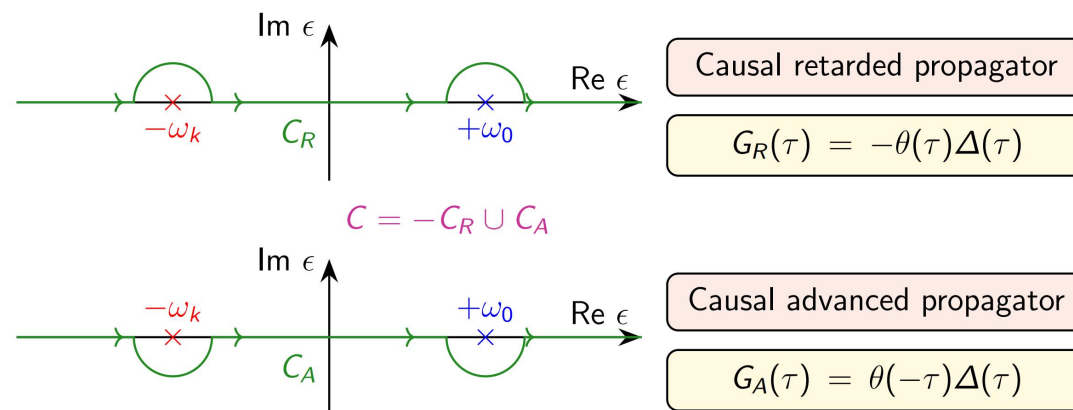
- Bath Spectral Density

$$J(\omega) = \pi \sum_s (g_s^2 / \omega_s) \delta(\omega - \omega_s)$$



Single-Particle Quantum Mechanics

- Dissipative Action (Integrate out the Bath)
 - Causality Structure

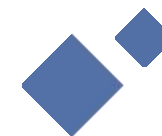
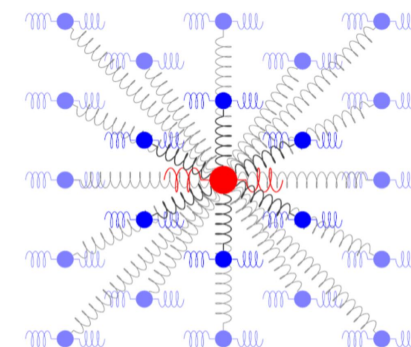


- Ohmic Bath (More Realistic Model of the Environment) $J(\omega) = 4\gamma\omega$

$$[\mathcal{D}^{-1}(\epsilon)]^{\text{R(A)}} = 4\gamma \int \frac{d\omega}{2\pi} \frac{\omega^2}{\omega^2 - (\epsilon \pm i0)^2} = \text{const} \pm 2i\gamma\epsilon$$

- Keldysh Component (FDT)

$$[\mathcal{D}^{-1}(\epsilon)]^{\text{K}} = \left([\mathcal{D}^{\text{R}}]^{-1} - [\mathcal{D}^{\text{A}}]^{-1} \right) \coth \frac{\epsilon}{2T} = 4i\gamma\epsilon \coth \frac{\epsilon}{2T}$$



Single-Particle Quantum Mechanics

- Dissipative Action (Integrate out the Bath)

- Ohmic Bath

- Time Domain (R/A : Local K : Non-Local)

$$\left[\mathcal{D}^{R(A)}\right]^{-1} = \mp 2\gamma\delta(t-t')\partial_{t'}$$

$$\left[\mathcal{D}^{-1}(t-t')\right]^K = 4i\gamma\left[(2T+C)\delta(t-t') - \frac{\pi T^2}{\sinh^2[\pi T(t-t')]} \right]$$

$$C = \pi T^2 \int dt / \sinh^2(\pi T t) \quad \int dt [\mathcal{D}^{-1}(t)]^K = [\mathcal{D}^{-1}(\epsilon=0)]^K = 8i\gamma T$$

- Keldysh Action of the Particle Connected to the Ohmic Bath

$$S[\vec{X}] = \int_{-\infty}^{+\infty} dt \left[-2X^q \left(\ddot{X}^{cl} + \gamma \dot{X}^{cl} \right) - V(X^{cl} + X^q) + V(X^{cl} - X^q) \right] \\ + 2i\gamma \int_{-\infty}^{+\infty} dt \left[2T(X^q(t))^2 + \frac{\pi T^2}{2} \int_{-\infty}^{+\infty} dt' \frac{(X^q(t) - X^q(t'))^2}{\sinh^2[\pi T(t-t')]} \right]$$



Single-Particle Quantum Mechanics

- Dissipative Action (Integrate out the Bath)
 - Keldysh Action of the Particle Connected to the Ohmic Bath

$$S[\vec{X}] = \int_{-\infty}^{+\infty} dt \left[-2X^q \left(\ddot{X}^{\text{cl}} + \gamma \dot{X}^{\text{cl}} \right) - V(X^{\text{cl}} + X^q) + V(X^{\text{cl}} - X^q) \right] \\ + 2i\gamma \int_{-\infty}^{+\infty} dt \left[2T(X^q(t))^2 + \frac{\pi T^2}{2} \int_{-\infty}^{+\infty} dt' \frac{(X^q(t) - X^q(t'))^2}{\sinh^2[\pi T(t - t')]} \right]$$

- q-q Component : Finite instead of Higher Order
- Broken Time-Reversal Symmetry / Unitarity after Integrating out

$$S[X^{\text{cl}}, -X^q] = -S[X^{\text{cl}}, X^q]$$

- Linear Term $\Rightarrow$ Newtonian Equation with the Viscous Friction Force

$$\ddot{X}^{\text{cl}} = -V'(X^{\text{cl}}) - \gamma \dot{X}^{\text{cl}}$$

- Hubbard-Stratonovich Transformation



Rotating-Wave Approximation

- Self-Energy Related to Quantum Fluctuation

$$[\Sigma(\epsilon)]^K = -iJ(\epsilon) \coth \frac{\epsilon}{2T}$$

- Predicament : Non-Locality at Small Temperatures

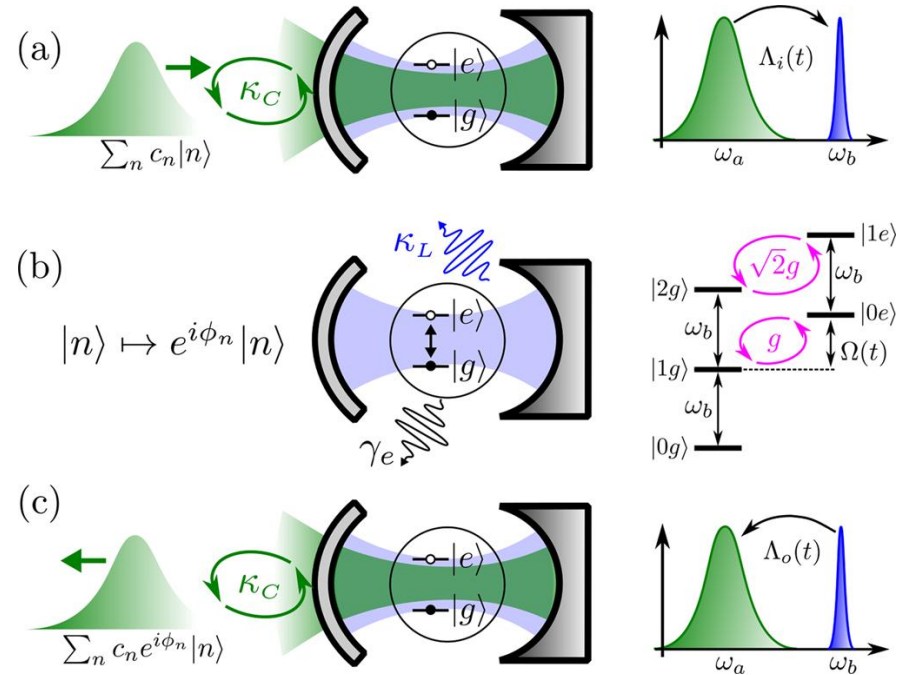
$$[\Sigma(\epsilon)]^K = -iJ(\epsilon) \coth \frac{\epsilon}{2T} \longrightarrow [\Sigma(\epsilon)]^K = -iJ(\epsilon) \operatorname{sign} \epsilon$$

$$J(\epsilon) \sim \epsilon \quad [\Sigma(\epsilon)]^K \sim |\epsilon| \quad [\Sigma(\epsilon)]^K \sim (t - t')^{-2}$$



Rotating-Wave Approximation

- Particular Mode of a Resonant Cavity
 - Filter out Low-Frequency Part of the Bath Spectrum
 - Self-Select Relatively Narrow High-Frequency Band
 - Weakly Nonlinear Bosonic Oscillator



Rotating-Wave Approximation

- Particular Mode of a Resonant Cavity
 - Weakly Nonlinear Bosonic Oscillator
 - External Monochromatic Drive

$$H = H_p + H_{\text{bath}} + H_{\text{int}}$$

$$\hat{H}_p = \omega_0 \hat{b}^\dagger \hat{b} + \frac{g}{12} (\hat{b}^\dagger + \hat{b})^4 + 2i\eta (\hat{b}^\dagger - \hat{b}) \cos \omega_p t$$

$$\hat{H}_{\text{bath}} = \sum_s \omega_s \hat{a}_s^\dagger \hat{a}_s \quad \hat{H}_{\text{int}} = \sum_s \frac{g_s}{\sqrt{2\omega_s}} (\hat{a}_s^\dagger \hat{b} + \hat{b}^\dagger \hat{a}_s)$$

- Unitary Transformation to the Rotating Frame $\hat{U}(t) = e^{i\omega_p t \hat{b}^\dagger \hat{b}}$

$$\hat{\tilde{b}}^\dagger = \hat{U}(t) \hat{b}^\dagger \hat{U}^\dagger(t) = \hat{b}^\dagger + i\omega_p t [\hat{N}, \hat{b}^\dagger] + \frac{(i\omega_p t)^2}{2!} [\hat{N}, [\hat{N}, \hat{b}^\dagger]] + \dots = \hat{b}^\dagger e^{i\omega_p t}$$

$$\hat{\tilde{b}} = \hat{U}(t) \hat{b} \hat{U}^\dagger(t) = \hat{b} + i\omega_p t [\hat{N}, \hat{b}] + \frac{(i\omega_p t)^2}{2!} [\hat{N}, [\hat{N}, \hat{b}]] + \dots = \hat{b} e^{-i\omega_p t}$$



Rotating-Wave Approximation

- Particular Mode of a Resonant Cavity
 - Corresponding Schrodinger Equation

$$\begin{aligned}i\partial_t|\Psi(t)\rangle &= \hat{H}(t)|\Psi(t)\rangle \\ \Downarrow \\ \hat{U}(t)i\partial_t|\Psi(t)\rangle &= \left(\hat{U}(t)\hat{H}(t)\hat{U}^\dagger(t)\right)|\tilde{\Psi}(t)\rangle \\ \Downarrow \\ i\partial_t|\tilde{\Psi}(t)\rangle &= \left[\hat{U}(t)\hat{H}(t)\hat{U}^\dagger(t) + i\left(\partial_t\hat{U}(t)\right)\hat{U}^\dagger(t)\right]|\tilde{\Psi}(t)\rangle = \hat{\tilde{H}}(t)|\tilde{\Psi}(t)\rangle\end{aligned}$$

- Hamiltonian under this Frame

$$\hat{\tilde{H}}(t) \equiv \hat{U}(t)\hat{H}(t)\hat{U}^\dagger(t) + i\left(\partial_t\hat{U}(t)\right)\hat{U}^\dagger(t) \quad |\tilde{\Psi}(t)\rangle \equiv \hat{U}(t)|\Psi(t)\rangle$$



Rotating-Wave Approximation

- Rotating-Wave Approximation (RWA)

- Hamiltonian

$$\begin{aligned}\hat{H}_p &= \omega_0 \hat{b}^\dagger \hat{b} + \frac{g}{2} \left[\hat{U}(\hat{b} + \hat{b}^\dagger) \hat{U}^\dagger \right]^2 + i\eta \hat{U}(\hat{b}^\dagger - \hat{b}) \hat{U}^\dagger (e^{i\omega_p t} + e^{-i\omega_p t}) + i(\partial_t \hat{U}) \hat{U}^\dagger \\ &= \omega_0 \hat{b}^\dagger \hat{b} + \frac{g}{2} (\hat{b}^\dagger e^{i\omega_p t} + \hat{b} e^{-i\omega_p t})^2 + i\eta (\hat{b}^\dagger e^{i\omega_p t} - \hat{b} e^{-i\omega_p t}) (e^{i\omega_p t} + e^{-i\omega_p t}) - \omega_p \hat{b}^\dagger \hat{b} \\ &\stackrel{\text{RWA}}{\approx} (\omega_0 - \omega_p) \hat{b}^\dagger \hat{b} + \frac{g}{12} (\hat{b}^\dagger \hat{b}^\dagger + \hat{b} \hat{b} + 12 \hat{b}^\dagger \hat{b} + 3) + i\eta (\hat{b}^\dagger - \hat{b})\end{aligned}$$

$\Downarrow$

$$\hat{H}_p = \Delta \hat{b}^\dagger \hat{b} + \frac{g}{2} \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} + i\eta (\hat{b}^\dagger - \hat{b})$$

$$\hat{H}_{\text{bath}} = \sum_s \omega_s \hat{a}_s^\dagger \hat{a}_s \quad \hat{H}_{\text{int}} = \sum_s \frac{g_s}{\sqrt{2\omega_s}} \left(\hat{a}_s^\dagger \hat{b} e^{-i\omega_p t} + \hat{b}^\dagger \hat{a}_s e^{i\omega_p t} \right)$$

- Detuning Frequency

$$\Delta \equiv \omega_0 - \omega_p + g \ll \omega_p$$



Rotating-Wave Approximation

Sokhotski-Plemelj Theorem

$$\frac{1}{x \pm i0} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

- Dissipative Action for the Oscillator

- Fourier Transformation

$$\hat{\Sigma}(t-t') = \sum_s \frac{g_s^2}{2\omega_s} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_s^R & G_s^K \\ 0 & G_s^A \end{pmatrix} (t-t') \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{i\omega_p(t-t')} = \sum_s \frac{g_s^2}{2\omega_s} \begin{pmatrix} 0 & G_s^A \\ G_s^R & G_s^K \end{pmatrix} (t-t') e^{i\omega_p(t-t')}$$

$$\begin{aligned} \hat{\Sigma}(\epsilon) &= \sum_s \frac{g_s^2}{2\omega_s} \int \frac{d(t-t')}{\sqrt{2\pi}} \hat{\Sigma}(t-t') e^{i\epsilon(t-t')} = \sum_s \frac{g_s^2}{2\omega_s} \int \frac{d\tau}{\sqrt{2\pi}} \begin{pmatrix} 0 & G_s^A \\ G_s^R & G_s^K \end{pmatrix} (\tau) e^{i(\epsilon+\omega_p)\tau} \\ &= \sum_s \frac{g_s^2}{2\omega_s} \begin{pmatrix} 0 & G_s^A \\ G_s^R & G_s^K \end{pmatrix} (\epsilon + \omega_p) = \sum_s \frac{g_s^2}{2\omega_s} \begin{pmatrix} 0 & G^A \\ G^R & G^K \end{pmatrix} (\epsilon + \omega_p, \omega_s) \\ &= \int \frac{d\omega}{2\pi} J(\omega) \begin{pmatrix} 0 & G^A \\ G^R & G^K \end{pmatrix} (\epsilon + \omega_p, \omega) = \int \frac{d\omega}{2\pi} J(\omega) \begin{pmatrix} 0 & [(\epsilon + \omega_p) - \omega + i0]^{-1} \\ [(\epsilon + \omega_p) - \omega - i0]^{-1} & -2\pi i [2n_B(\epsilon) + 1] \delta(\epsilon + \omega_p - \omega) \end{pmatrix} \\ &= \int \frac{d\omega}{2\pi} J(\omega + \omega_p) \begin{pmatrix} 0 & (\epsilon - \omega + i0)^{-1} \\ (\epsilon - \omega - i0)^{-1} & -2\pi i [2n_B(\epsilon) + 1] \delta(\epsilon - \omega) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \int \frac{d\omega}{2\pi} J(\omega + \omega_p) [\mathcal{P}(\frac{1}{\epsilon - \omega}) - i\pi\delta(\epsilon - \omega)] \\ \int \frac{d\omega}{2\pi} J(\omega + \omega_p) [\mathcal{P}(\frac{1}{\epsilon - \omega}) + i\pi\delta(\epsilon - \omega)] & -iJ(\omega + \omega_p) [2n_B(\epsilon) + 1] \end{pmatrix} \end{aligned}$$



Rotating-Wave Approximation

- Dissipative Action for the Oscillator

- Dissipative Self-Energy

$$\text{Im}[\Sigma(\epsilon)]^{\text{R(A)}} = \mp \frac{i}{2} J(\epsilon + \omega_p) \quad [\Sigma(\epsilon)]^{\text{K}} = -iJ(\epsilon + \omega_p) \coth \frac{\epsilon + \omega_p}{2T}$$

- Narrow Band around $\epsilon \approx \Delta$

$$\epsilon + \omega_p \approx \Delta + \omega_p \approx \omega_0$$

$\Downarrow$

$$\text{Im}[\Sigma(\epsilon)]^{\text{R(A)}} \approx \mp i\kappa \quad [\Sigma(\epsilon)]^{\text{K}} \approx -2i\kappa(2n_B + 1) = -2i\kappa_1$$

- Locality

- Spectral Density of the Bath at the Resonance Frequency

$$\kappa = \frac{J(\omega_0)}{2} \quad \frac{\kappa_1 + \kappa}{\kappa_1 - \kappa} = \frac{n_B + 1}{n_B} = e^{\omega_0/T}$$

$$n_B = n_B(\omega_0) = \left(e^{\omega_0/T} - 1\right)^{-1}$$



Rotating-Wave Approximation

- Dissipative Action for the Oscillator
 - ϵ -Independent Dissipative Self-Energy $\Rightarrow$ Time-Local Effective Action

$$S[\phi^{\text{cl}}, \phi^{\text{q}}] = \int_{-\infty}^{+\infty} dt \left(\bar{\phi}^{\text{cl}}, \bar{\phi}^{\text{q}} \right) \begin{pmatrix} 0 & i\partial_t - \Delta - \frac{g}{2} |\phi^{\text{cl}}|^2 - i\kappa \\ i\partial_t - \Delta - \frac{g}{2} |\phi^{\text{cl}}|^2 + i\kappa & 2i\kappa_1 \end{pmatrix} \begin{pmatrix} \phi^{\text{cl}} \\ \phi^{\text{q}} \end{pmatrix} \\ - \int_{-\infty}^{+\infty} dt \left[\frac{g}{2} \left(\bar{\phi}^{\text{q}} \bar{\phi}^{\text{q}} \phi^{\text{q}} \phi^{\text{cl}} + \bar{\phi}^{\text{cl}} \bar{\phi}^{\text{q}} \phi^{\text{q}} \phi^{\text{q}} \right) + \sqrt{2}i\eta \left(\bar{\phi}^{\text{q}} - \phi^{\text{q}} \right) \right]$$

- Shift the Classical Field to Eliminate the Driving Term $\sim \eta$

$$\phi^{\text{cl}} \rightarrow \phi^{\text{cl}} - \phi_{\eta}$$

$$S \rightarrow S + \int_{-\infty}^{+\infty} dt \left\{ \bar{\phi}^{\text{q}} \left[\left(0 - \Delta - \frac{g}{2} |\phi_{\eta}|^2 + i\kappa \right) (-\phi_{\eta}) - \sqrt{2}i\eta \right] + \phi^{\text{q}} \left[\left(0 - \Delta - \frac{g}{2} |\phi_{\eta}|^2 - i\kappa \right) (-\bar{\phi}_{\eta}) + \sqrt{2}i\eta \right] \right\} \\ + \int_{-\infty}^{+\infty} dt \left\{ \bar{\phi}^{\text{q}} \left\{ \left[i\partial_t - \Delta - \frac{g}{2} (\phi^{\text{cl}} - \phi_{\eta}) (\bar{\phi}^{\text{cl}} - \bar{\phi}_{\eta} + i\kappa) \right] \phi^{\text{cl}} + \left[\Delta + \frac{g}{2} (|\phi^{\text{cl}}|^2 - \phi_{\eta} \bar{\phi}^{\text{cl}} - \bar{\phi}_{\eta} \phi^{\text{cl}}) - i\kappa \right] \phi_{\eta} + 2i\kappa \phi^{\text{q}} - \frac{g}{2} |\phi^{\text{q}}|^2 (\phi^{\text{cl}} - \phi_{\eta}) \right\} \right. \\ \left. + \phi^{\text{q}} \left\{ \left[-i\partial_t - \Delta - \frac{g}{2} (\phi^{\text{cl}} - \phi_{\eta}) (\bar{\phi}^{\text{cl}} - \bar{\phi}_{\eta}) - i\kappa \right] \bar{\phi}^{\text{cl}} + \left[\Delta + \frac{g}{2} (|\phi^{\text{cl}}|^2 - \phi_{\eta} \bar{\phi}^{\text{cl}} - \bar{\phi}_{\eta} \phi^{\text{cl}}) + i\kappa \right] - \frac{g}{2} |\phi^{\text{q}}|^2 (\bar{\phi}^{\text{cl}} - \bar{\phi}_{\eta}) \right\} \right\}$$



Rotating-Wave Approximation

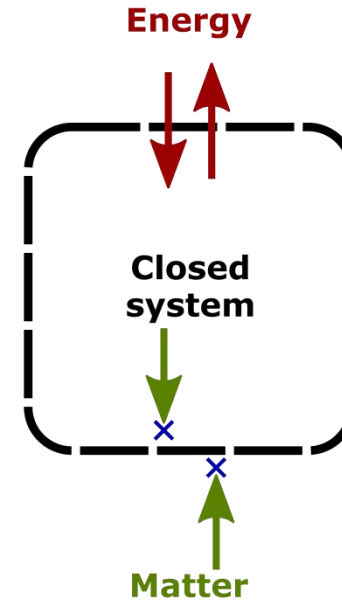
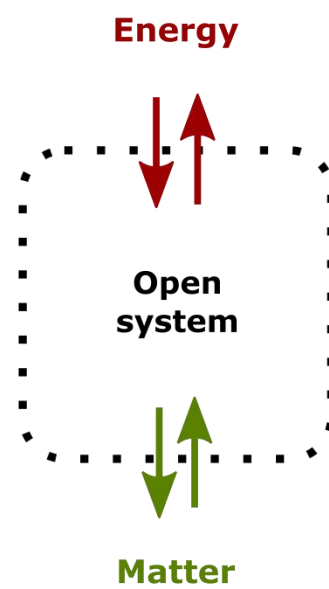
- Dissipative Action for the Oscillator
 - Shift the Classical Field to Eliminate the Driving Term $\sim \eta$
 - **Non-Linear Equation** $(\Delta - i\kappa)\phi_\eta + \frac{g}{2}|\phi_\eta|^2\phi_\eta = \sqrt{2}i\eta$
 - **Bogoliubov & Non-Linear Terms** $\bar{\phi}^q\bar{\phi}^{cl} \quad \phi^q\phi^{cl} \quad \bar{\phi}^q\phi^{cl}\phi^{cl} \quad \bar{\phi}^q\phi^q\phi^q.$
 - Higher Order (Cubic) Terms in Quantum Component
 - Contain Full Quantum Information
 - von-Neumann Equation : Unitary Evolution $\partial_t\hat{\rho}(t) = -i[\hat{H}(t), \hat{\rho}(t)]$
 - Time-Local Driven-Dissipative Action : Non-Unitary Evolution
 - Bath Degrees of Freedom were Integrated out
 - Non-Unitary Evolution Equation for the Reduced Density Matrix (Lindblad Equation)



Lindblad Equation

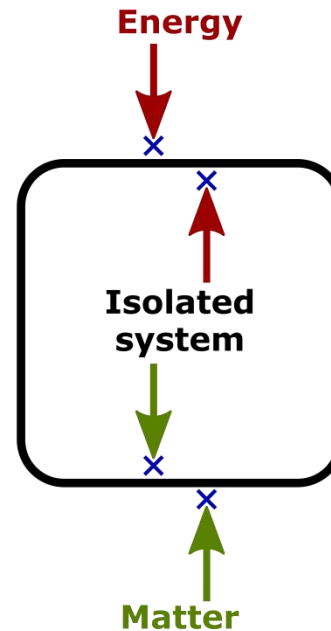
- Non-Unitary Evolution
- Particle Loss; Decoherence
- Lindblad Master Equation

$$\frac{d}{dt}\hat{\rho}(t) = \mathcal{L}\hat{\rho}(t)$$



- Unitary Evolution

$$\frac{d}{dt}|\Psi(t)\rangle = (\hat{H}_S + \hat{V})|\Psi(t)\rangle$$



- Unitary Evolution

$$\frac{d}{dt}|\Psi(t)\rangle = \hat{H}_S|\Psi(t)\rangle$$



Lindblad Equation

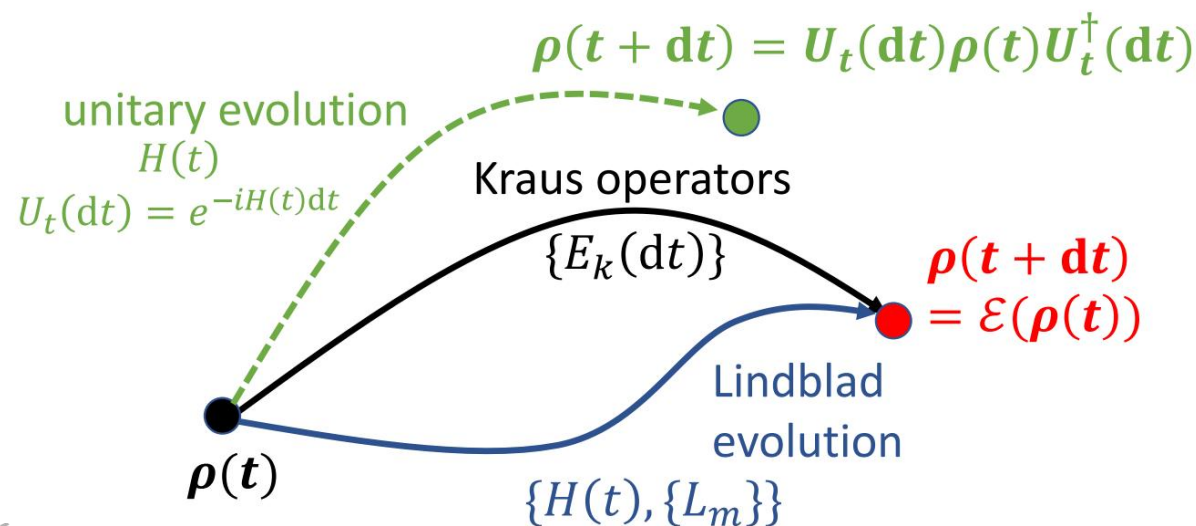
- Reduced Density Matrix / Trace out

$$\hat{\rho}_p = \text{Tr}_{\text{bath}} \{\hat{\rho}\} \quad \text{Tr}\{\dots\} = \text{Tr}_p \{\text{Tr}_{\text{bath}} \{\dots\}\} \quad \text{Tr}\{\hat{\rho}\} = 1$$

- von-Neuman Equation Lindblad Equation

$$\partial_t \hat{\rho}_p(t) = -i \left[\hat{H}'_p, \hat{\rho}_p(t) \right] + \sum_a \gamma_a \left(\hat{L}_a \hat{\rho}_p(t) \hat{L}_a^\dagger - \frac{1}{2} \left\{ \hat{L}_a^\dagger \hat{L}_a, \hat{\rho}_p(t) \right\} \right)$$

- Non-Hermitian Operators $\hat{L}_a$: Lindblad / Quantum Jump Operators



Lindblad Equation

- GKSL Equation (A Brief History of the GKLS Equation : arXiv:1710.05993)

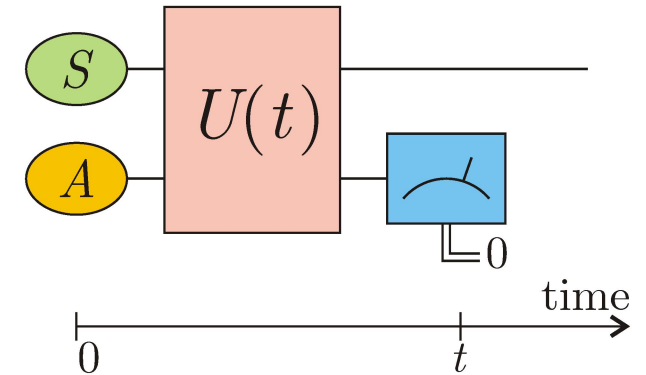
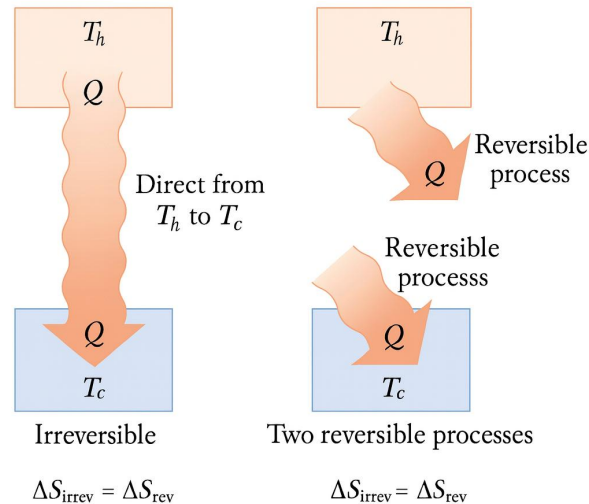
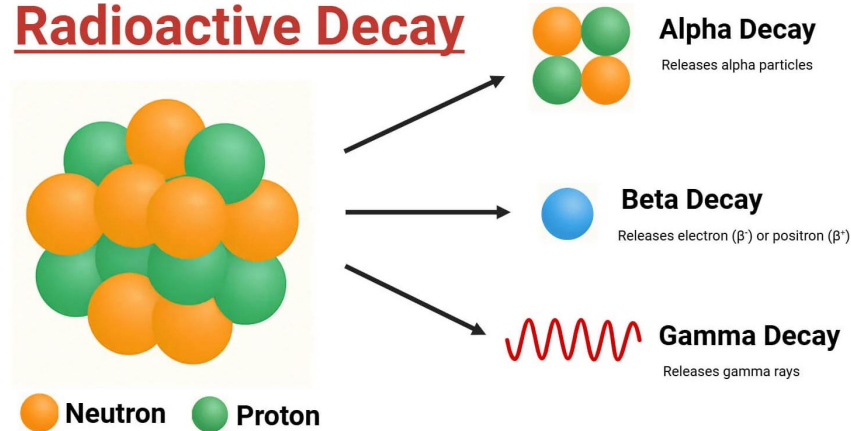


Lindblad Equation — On the Generators of Quantum Dynamical Semigroups

- Motivation

- Closed System : One-Parameter Group Unitary Transformation in Hilbert Space
- Irreversible Processed? (Decay of Unstable Particles, Approach to Thermodynamic Equilibrium, Measurement Processes)

Radioactive Decay



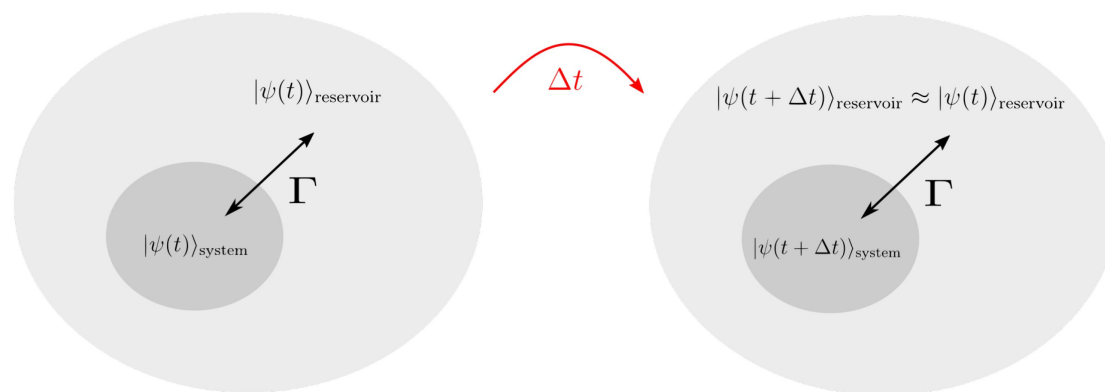
Lindblad Equation — On the Generators of Quantum Dynamical Semigroups

- To Avoid Unitary Evolution $\Rightarrow$ Non-Hamiltonian Systems

System (S) $\otimes$ Reservoir (R)

- Measure Process (System + Ancilla/Pointer)
- Stochastic External Force in Brownian Motion (Particle + Fluid)

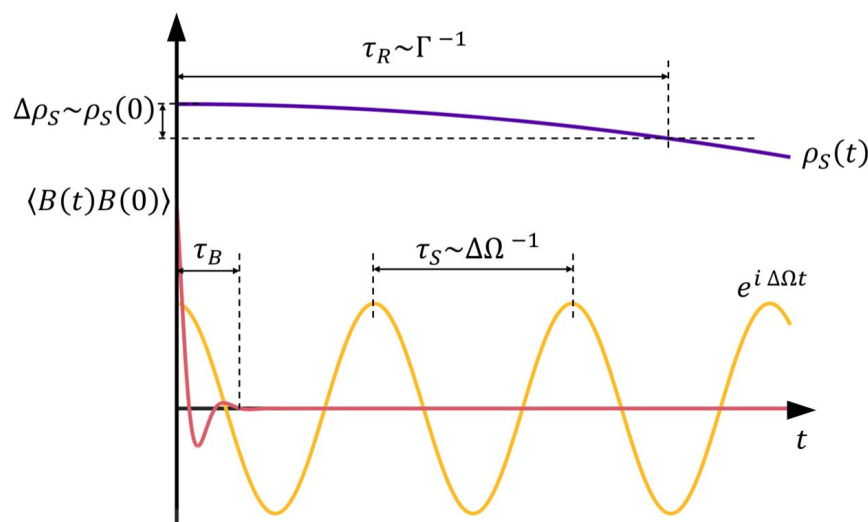
D.o.F of S (**Macroscopic**) $\ll$ D.o.F of R (**Microscopic**)



Lindblad Equation — On the Generators of Quantum Dynamical Semigroups

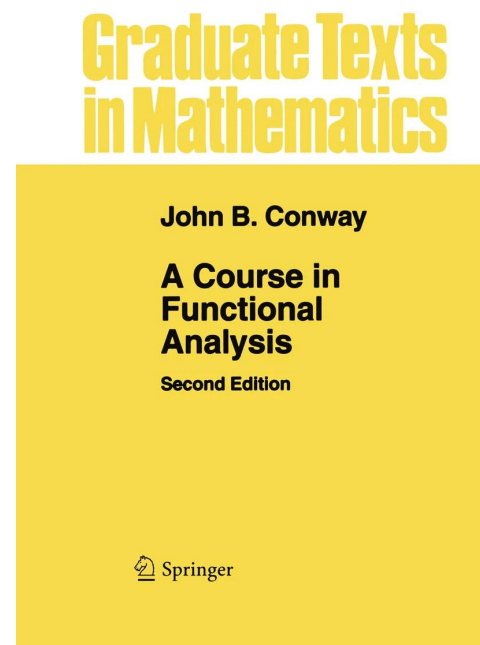
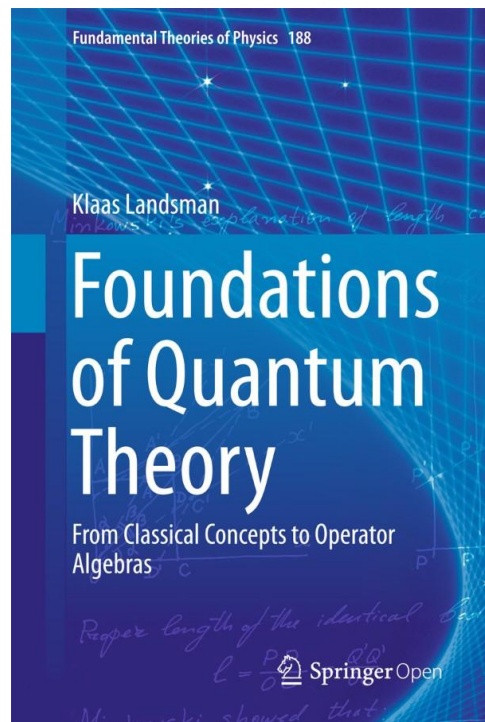
- Is Lindblad for me? (arXiv:2506.22436)
- Hierarchy of Timescales

Relaxation Time of System $\tau_R \gg$ Time of System Internal Dynamics $\tau_S \gg$ Relaxation Time of Reservoir τ_R



Lindblad Equation — An Introduction to Operator Algebra

- Landsman, K. (2017). Foundations of Quantum Theory: From Classical Concepts to Operator Algebras. Springer.
- Conway J B. A course in functional analysis[M]. Springer, 2019.
- Yoichi Kazama, von Neumann Algebra and Quantum Gravity,
<https://www2.yukawa.kyoto-u.ac.jp/~qft.web/2024/slides/kazama-yoichi.pdf>



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Basics of Analysis

- Normed Spaces and Banach Spaces

A *normed space* $(X, \|\cdot\|)$ over $\mathbb{C}$ is a vector space with a norm satisfying positivity, homogeneity, and the triangle inequality. A Cauchy sequence (x_n) is one such that $|x_n - x_m| \rightarrow 0$ as $n, m \rightarrow \infty$. If every Cauchy sequence converges in X , then X is *complete* and is called a *Banach space*.

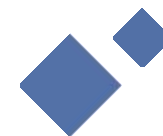
- Bounded Linear Maps and Operator Norm

A linear map $T : X \rightarrow Y$ between normed spaces is *bounded* if $\sup_{\|x\| \leq 1} \|Tx\| < \infty$; its norm is

$$\|T\| := \sup_{\|x\| = 1} \|Tx\|.$$

Boundedness is equivalent to continuity for linear maps.

$$\|T\| := \sup_{\langle x|x \rangle \leq 1} \|T|x \rangle\| = \sup_{\langle x|x \rangle \leq 1} (\langle x|T^\dagger)(T|x \rangle)$$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Basics of Analysis
 - Hilbert Spaces and Adjoints

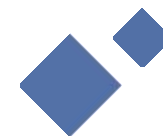
A complex *Hilbert space* $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a complete inner-product space (linear in the second slot by our convention). For every bounded operator $A \in \mathcal{B}(\mathcal{H})$ there is a unique $A^\dagger \in \mathcal{B}(\mathcal{H})$ such that $\langle x, Ay \rangle = \langle A^\dagger x, y \rangle$ for all x, y ; $A^\dagger$ is the *adjoint*.

- Spectral Notions

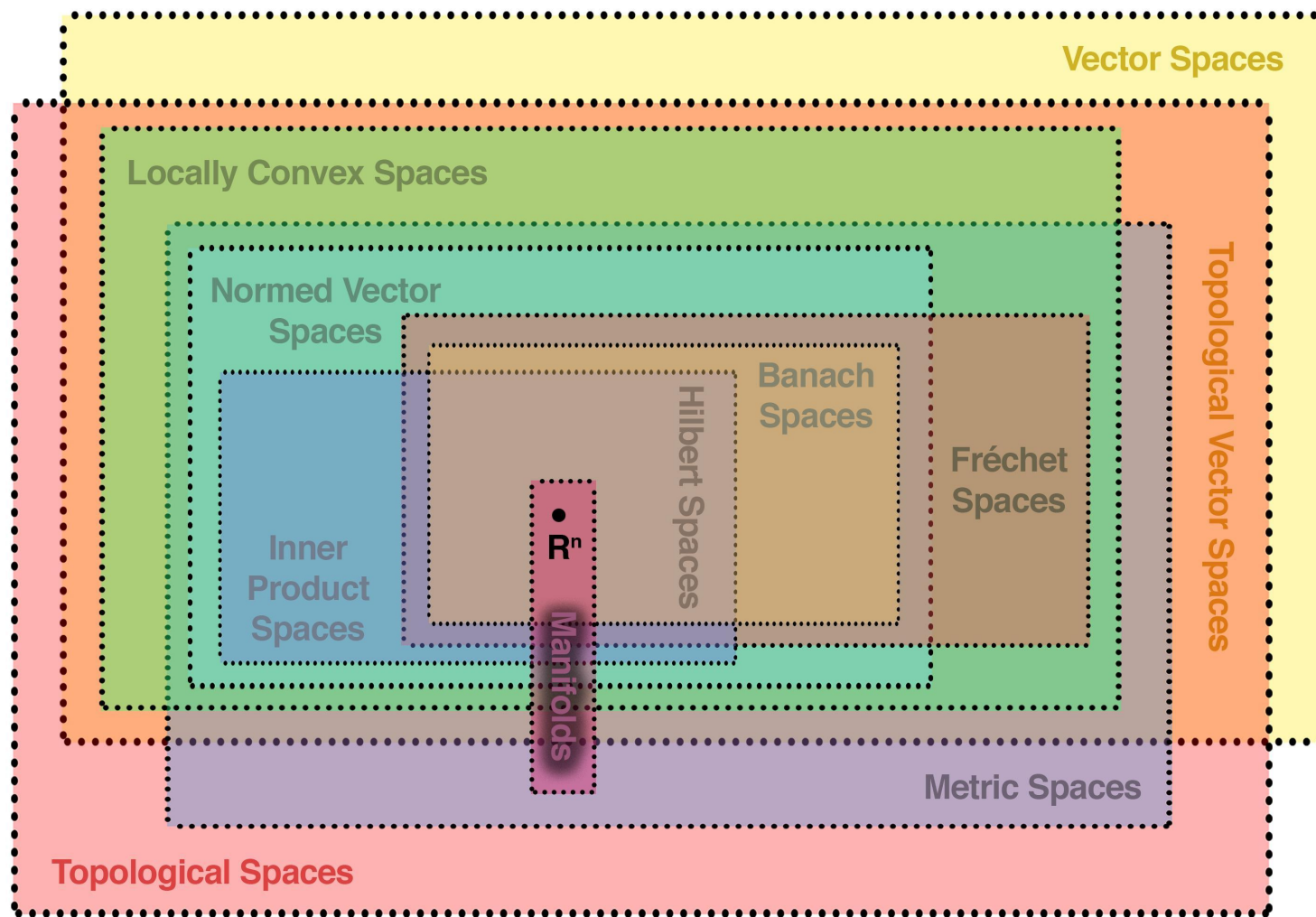
For a in a unital Banach algebra, the *spectrum* $\sigma(a)$ is the set of $\lambda \in \mathbb{C}$ such that $a - \lambda \mathbb{1}$ is not invertible. Its radius is $r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}$. A fundamental fact is the *spectral radius formula*

$$r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}$$

In C^* -algebras, spectra of norm elements encode geometry via the functional calculus (see below).



Lindblad Equation — An Introduction to Operator Algebra



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Basics of Analysis
 - Finite-Dimensional Sanity Check

In finite dimensions all norms are equivalent, every linear map is bounded, and adjoints are the usual conjugate-transpose with respect to the standard inner product. This lets you test general definitions on $M_d(\mathbb{C})$ before returning to the infinite dimensional setting.

- C^* -Algebra in Functional Analysis
 - $*$ -Algebra

A complex $*$ -algebra $(\mathcal{A}, \cdot, *)$ is a complex algebra with an involution $a \mapsto a^*$ such that

$$(ab)^* = b^*a^*, \quad (\lambda a)^* = \overline{\lambda}a^*, \quad a^{**} = a, \quad \forall a, b \in \mathcal{A}, \lambda \in \mathbb{C}$$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Basics of Analysis
 - Banach $*$ -Algebra

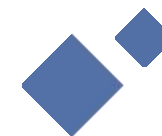
A *Banach $*$ -algebra* is a $*$ -algebra equipped with a complete submultiplicative norm $\|\cdot\|$ (i.e. $\|ab\| \leq \|a\|\|b\|$) such that $\|a^*\| = \|a\|$ need not hold a priori.

- C^* -Algebra

A C^* -algebra is a Banach $*$ -algebra $(\mathcal{A}, \cdot, *)$ satisfying the C^* -identity

$$\|a^*a\| = \|a\|^2$$

If $\mathcal{A}$ has a multiplicative identity $\mathbb{1}$, it is called *unital*.



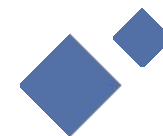
Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Basics of Analysis
 - Immediate Consequences of C^* -Identity

For a C^* -algebra $\mathcal{A}$:

- ▶ The norm is uniquely determined by the $*$ -algebraic structure: if another norm makes $\mathcal{A}$ a C^* -algebra, it must agree with $\|\cdot\|$.
- ▶ $\|a\| = r(a)$ for every normal element ($a^*a = aa^*$), in particular for self-adjoints ($a = a^*$) and unitaries ($a^*a = aa^* = \mathbb{1}$).
- ▶ If $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a $*$ -homomorphism, then it is automatically contractive: $\|\pi(a)\| \leq \|a\|$; if injective, it is isometric.



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Basics of Analysis

- Positivity

An element $x \in \mathcal{A}$ is **positive** (written as $x \geq 0$) if $x = x^*$ and $\sigma(x) \subset [0, \infty)$. Equivalently, $x = b^*b$ for some $b \in \mathcal{A}$. We define $a \leq b$ iff $b - a \geq 0$; this turns the self-adjoints into an ordered vector space.

- Functional Calculus (for Normal Elements)

If $a \in \mathcal{A}$ is normal, then for every continuous f on $\sigma(a)$ there is a unique $f(a) \in \mathcal{A}$ such that $f \mapsto f(a)$ is a $*$ -homomorphism $C(\sigma(a)) \rightarrow C^*(a, \mathbb{1})$, the C^* -subalgebra generated by a and $\mathbb{1}$. This generalizes “apply a function to a diagonalizable matrix”

$$\sigma(a) \leftrightarrow \{E_n\} \text{ (spectrum of eigenvalues)}$$

$$f(a) \leftrightarrow f(\hat{A}) = \sum_n f(E_n) |n\rangle \langle n| \text{ (spectral decomposition)}$$

$$\text{Classical Functions: } C(\sigma(a)) \leftrightarrow \text{span}\{\delta_{E_k} : E_k \in \sigma(a)\}$$

$$\text{Quantum Operators: } C^*(a, \mathbb{1}) \leftrightarrow \text{span}\{\hat{A}^k, (\hat{A}^*)^j : k, j \geq 0\}$$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

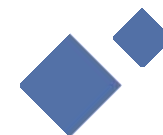
- Canonical Examples
 - Bounded Operators on a Hilbert Space

Let $\mathcal{H}$ be a complex Hilbert space. $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators $X : \mathcal{H} \rightarrow \mathcal{H}$ with operator norm

$$\|X\| = \sup_{\|\psi\|=1} \|X\psi\|$$

Involution is the adjoint $X \mapsto X^\dagger$. The C^* -identity holds:

$$\|X^\dagger X\| = \|X\|^2$$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Canonical Examples
 - Bounded Operators on a Hilbert Space

Proof:

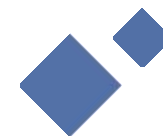
► Submultiplicativity $\|X^\dagger X\| \leq \|X^\dagger\| \|X\| = \|X\|^2$

► $\forall |\psi_n\rangle, \langle\psi_n|\psi_n\rangle = 1, \text{ s.t. } \lim_{n \rightarrow \infty} \|X|\psi_n\rangle\| = \|X\|$

$$\langle\psi_n|X^\dagger X|\psi_n\rangle = \|X|\psi_n\rangle\|^2 \rightarrow \|X\|^2$$

$$\Rightarrow \|X^\dagger X\| := \sup_{\langle\psi_n|\psi_n\rangle=1} \langle\psi_n|X^\dagger X|\psi_n\rangle \geq \lim_{n \rightarrow \infty} \langle\psi_n|X^\dagger X|\psi_n\rangle = \|X\|^2$$

$$\Rightarrow \|X^\dagger X\| = \|X\|^2$$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Canonical Examples

- Matrix Algebras

$M_d(\mathbb{C}) \cong \mathcal{B}(\mathbb{C}^d)$ with involution $A \mapsto A^\dagger = \overline{A}^\top$ and the operator norm induced by the Euclidean norm on $\mathbb{C}^d$. All finite-dimensional C^* -algebras are $*$ -isomorphic to finite direct sums $\oplus_k M_{d_k}(\mathbb{C})$.

- ▶ For rank-one $A = |u\rangle\langle v|$, $\|A\| = \|u\|\|v\|$; $A^\dagger A = \|u\|^2 |v\rangle\langle v| = \|u\|^2 \|v\|^2 \left| \frac{v}{\|v\|} \right\rangle \left\langle \frac{v}{\|v\|} \right|$ so $\|A^\dagger A\| = \|u\|^2 \|v\|^2 = \|A\|^2$ verifies the C^* -identity by hand.
- ▶ Positivity: $A \geq 0$ iff $A = A^\dagger$ and all eigenvalues are ≥ 0 ; equivalently $A = B^\dagger B$.



Lindblad Equation — An Introduction to Operator Algebra

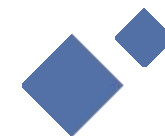
C^* -Algebras: the Minimal Package

- Canonical Examples

- Commutative example $C_0(X)$

Let X be locally compact Hausdorff. $C_0(X)$ is the algebra of continuous $f : X \rightarrow \mathbb{C}$ that **vanish at infinity**, meaning for every $\varepsilon > 0$ there exists a compact $K \subset X$ with $|f(x)| < \varepsilon$ for all $x \notin K$. Operations are pointwise; involution is complex conjugation $f^* = \bar{f}$; norm is the sup-norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$.

- ▶ $C_0(X)$ is complete in $\|\cdot\|_\infty$ (uniform limit of continuous functions is continuous, and vanishing at infinity is closed under uniform limits).
- ▶ C^* -identity holds: $\|f^* f\|_\infty = \|\bar{f} f\|_\infty = \| |f|^2 \|_\infty = \|f\|_\infty^2$.
- ▶ For X compact, $C_0(X) = C(X)$ is unital with $\mathbb{1} = 1_X$. For noncompact X , one can adjoin a unit (one-point compactification).



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Canonical Examples

- Remark (Gelfand–Naimark for Commutative C^* -Algebras)

Every commutative unital C^* -algebra $\mathcal{A}$ is $*$ -isomorphic to $C(\hat{X})$ for a compact Hausdorff space $\hat{X}$, called the *Gelfand spectrum* of $\mathcal{A}$. This space is defined as

$$\hat{X} := \{\omega : \mathcal{A} \rightarrow \mathbb{C} \mid \omega \text{ is a nonzero multiplicative } * \text{-linear functional}\}.$$

The algebra $C(\hat{X})$ consists of all continuous functions $f : \hat{X} \rightarrow \mathbb{C}$ with pointwise operations and the sup norm. Thus, Example above is not just a model; it characterizes the entire commutative world.

- States and Expectations

- Positive Linear Functional and State

A linear functional $\omega : \mathcal{A} \rightarrow \mathbb{C}$ is *positive* if $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}$. It is a *state* if, additionally, $\|\omega\| = 1$ (equivalently, for unital $\mathcal{A}$, $\omega(1) = 1$).



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- States and Expectations
 - Cauchy–Schwarz for Positive Functionals

If ω is positive, then for all $a, b \in \mathcal{A}$

$$|\omega(b^*a)|^2 \leq \omega(a^*a)\omega(b^*b).$$

Hence $|\omega(a)| \leq \omega(\mathbb{1})^{\frac{1}{2}}\omega(a^*a)^{\frac{1}{2}}$ when $\mathcal{A}$ is unital.

- Remark (Order Unit and Expectations)

On self-adjoints, $a \leq b \Rightarrow \omega(a) \leq \omega(b)$. Thus a state is precisely an order-preserving normalized functional, playing the role of an *expectation value* in physics.

$$\hat{A} \leq \hat{B} \quad \Rightarrow \quad \omega(\hat{A}) = \langle \psi | \hat{A} | \psi \rangle \leq \omega(\hat{B}) = \langle \psi | \hat{B} | \psi \rangle$$



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C^* -Algebras: the Minimal Package

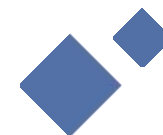
- States and Expectations
 - Two Canonical Representation Theorems Linking Analysis and States.
 - Theorem (Riesz–Markov–Kakutani for $C_0(X)$)

Every positive linear functional ω on $C_0(X)$ is integration against a unique regular Borel measure μ_ω :

$$\omega(f) = \int_X f \, d\mu_\omega, \quad f \in C_0(X).$$

States correspond to probability measures ($\mu_\omega(X) = 1$). This identifies the classical notion “state = probability distribution”.

Statistical	$f = A$	$\omega(f) = \int A \left(e^{-\beta H(p,q)} d\Gamma \right)$
Quantum	$f = \hat{A}$	$\omega(\hat{A}) = \text{Tr}(\hat{\rho}\hat{A})$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- States and Expectations

- Theorem (Normal states on $\mathcal{B}(\mathcal{H})$)

*Let $\mathcal{T}(\mathcal{H})$ be the trace-class operators on $\mathcal{H}$ (those T with $\|T\|_1 := \text{Tr } \sqrt{T^\dagger T} < \infty$). For every $T \in \mathcal{T}(\mathcal{H})$, the map $\omega_T : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$, $\omega_{T(A)} = \text{Tr}(TA)$ is a bounded linear functional. The **normal** linear functionals on $\mathcal{B}(\mathcal{H})$ are exactly those of this form. States correspond to density operators $\rho \in \mathcal{T}(\mathcal{H})$ with $\rho \geq 0$ and $\text{Tr } \rho = 1$:*

$$\omega_{\rho(A)} = \text{Tr}(\rho A).$$

- Example (Classical expectation as a $C_0(X)$ -state)

Let $X = \mathbb{R}$ with Borel probability measure μ . Then $\omega(f) = \int f \, d\mu$ is a state on $C_0(\mathbb{R})$. For $f(x) = \frac{x^2}{1+x^4}$ (which vanishes at infinity), $\omega(f)$ is the bounded second-moment surrogate $\int \frac{x^2}{1+x^4} \, d\mu$.



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- States and Expectations
 - Two Canonical Representation Theorems Linking Analysis and States.
 - Example (Quantum expectations on $M_d(\mathbb{C})$)

Let $\rho \in M_d(\mathbb{C})$ be positive with $\text{Tr } \rho = 1$. Then for $A \in M_d(\mathbb{C})$, $\omega_\rho(A) = \text{Tr}(\rho A)$ is a state. If $\rho = |\psi\rangle\langle\psi|$ is pure, $\omega_\rho(A) = \langle\psi|A|\psi\rangle$. If $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, then ω_ρ is the convex combination $\sum_i p_i \langle\psi_i|A|\psi_i\rangle$.

- Example (Projective measurement as expectation)

For a projection $P = P^2 = P^\dagger$ in $M_d(\mathbb{C})$ and a state ρ , the number $\text{Tr}(\rho P) \in [0, 1]$ is the Born probability of obtaining the “yes” outcome for the yes/no observable P .



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- States and Expectations
 - Pure v.s. Mixed States and the Krein-Milman Picture
 - Definition (Pure and Mixed States)

A state ω on a C^* -algebra $\mathcal{A}$ is called *pure* if it cannot be written as a nontrivial convex combination of other states. That is, ω is pure if whenever $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$ with $\lambda \in (0, 1)$ and ω_1, ω_2 states, then $\omega_1 = \omega_2 = \omega$. A state that is not pure is called *mixed*. Equivalently, ω is pure if and only if it is an extreme point of the convex set $S(\mathcal{A})$ of all states on $\mathcal{A}$.



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- States and Expectations
 - Pure v.s. Mixed States and the Krein-Milman Picture
 - Physical Interpretation
 - **Pure states** represent maximal knowledge about a quantum system - they correspond to preparation procedures that cannot be decomposed into probabilistic mixtures of other preparations.
 - **Mixed states** arise from statistical uncertainty or incomplete knowledge about the system's preparation, and can be written as convex combinations $\omega = \sum_i p_i \omega_i$ where ω_i are pure states and $p_i > 0$, $\sum_i p_i = 1$.



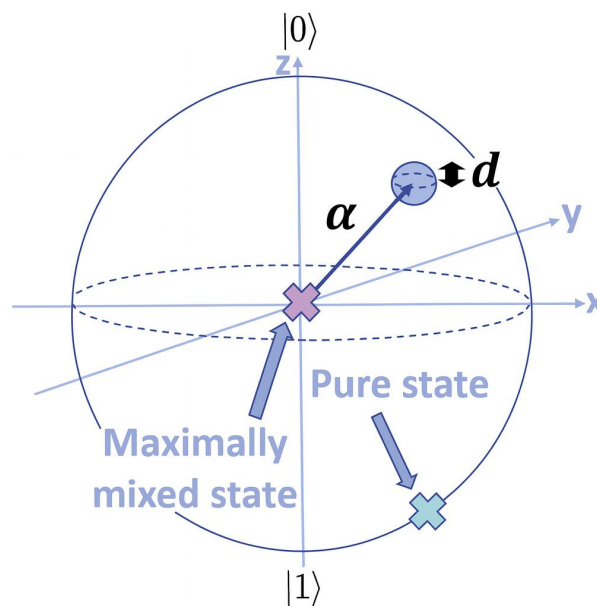
Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- States and Expectations
- The Krein–Milman Framework

The state space $S(\mathcal{A})$ is convex and weak* compact. By the Krein–Milman theorem:

- ▶ The extreme points of $S(\mathcal{A})$ are precisely the **pure** states
- ▶ Every state is a weak* limit of convex combinations of pure states
- ▶ Pure states form the “building blocks” from which all other states can be constructed



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- States and Expectations
 - Pure v.s. Mixed States and the Krein-Milman Picture
 - Concrete Realizations
 - On $\mathcal{B}(\mathcal{H})$: Pure normal states are exactly vector states $\omega_\psi(A) = \langle \psi | A | \psi \rangle$ with $\|\psi\| = 1$
 - On $M_d(\mathbb{C})$: Pure states correspond to rank-1 density matrices $\rho = |\psi\rangle\langle\psi|$, while mixed states correspond to density matrices $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with at least two nonzero p_i
 - On $C(X)$ (commutative case): Pure states are point evaluations $\omega_x(f) = f(x)$, while mixed states correspond to probability measures with support on more than one point



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Why C^* -algebras for physics? A structured checklist

- **Unification.**

Classical observables form $C_0(X)$; quantum finite-level observables form $M_d(\mathbb{C})$; both are C^* -algebras with a common positivity and order formalism.

- **Intrinsic norm and spectrum.**

The C^* -identity ties the norm to spectral data for normal elements, enabling a robust functional calculus (e.g. $f(H)$ for Hamiltonians $H = H^\dagger$).



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Why C^* -algebras for physics? A structured checklist

- **States as expectations.**

The abstract “state” matches probability measures in the classical case and density matrices in the quantum case

- **Gateway to dynamics.**

Once observables live in a C^* -algebra, completely positive (CP) maps and semigroups are the natural dynamics; the GKSL theorem characterizes their generators.



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- A First Encounter with Functional Calculus (Hands-on)

Let $H = H^\dagger$ on $\mathcal{H}$. For any bounded continuous $f : \sigma(H) \rightarrow \mathbb{R}$, $f(H)$ is defined in $\mathcal{B}(\mathcal{H})$ such that

$$\|f(H)\| = \sup_{\lambda \in \sigma(H)} |f(\lambda)|, \quad g \circ f \mapsto g(f(H)) \text{ and } \overline{f} \mapsto f(H)^\dagger.$$

In finite dimension, if $H = U \operatorname{diag}(\lambda_1, \dots, \lambda_d) U^\dagger$, then

$$f(H) = U \operatorname{diag}(f(\lambda_1), \dots, f(\lambda_d)) U^\dagger.$$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- A First Encounter with Functional Calculus (Hands-on)

Examples:

1. $f(x) = e^{-itx}$ yields the unitary e^{-itH} :

$$e^{-itH} = \int_{\sigma(H)} e^{-it\lambda} P(d\lambda),$$

where $P(d\lambda)$ is the spectral measure associated with H . This operator represents the time evolution operator in quantum mechanics. The integral sums over the eigenvalues λ of H , with each term weighted by the function $e^{-it\lambda}$.



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- A First Encounter with Functional Calculus (Hands-on)

Examples:

2. $f(x) = \mathbf{1}_{\Delta(x)}$ yields the spectral projection onto the energy window Δ :

$$P_{\Delta} = \int_{\sigma(H)} \mathbf{1}_{\Delta(\lambda)} P(d\lambda) = P(\Delta),$$

where $P(\Delta)$ is the spectral projection operator corresponding to the energy window Δ . This projection operator projects the state onto the subspace of $\mathcal{H}$ associated with the eigenvalues of H that lie within the set Δ .



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Worked Micro-Exercises
 - Show $C_0(X)$ is a C^* -algebra.

Step 1: Algebraic Structure

The set $C_0(X)$ is clearly a complex vector space under pointwise addition and scalar multiplication. Furthermore, it is closed under pointwise multiplication since the product of two functions vanishing at infinity also vanishes at infinity. The associative, distributive, and commutative properties follow from the corresponding properties of complex numbers.

Step 2: Involution

Define the involution $*$: $C_0(X) \rightarrow C_0(X)$ by $f^*(x) = \overline{f(x)}$ (complex conjugation). This involution satisfies: $(f + g)^* = f^* + g^*$, $(\lambda f)^* = \overline{\lambda} f^*$ for $\lambda \in \mathbb{C}$, $(fg)^* = g^* f^*$, $(f^*)^* = f$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Worked Micro-Exercises
 - Show $C_0(X)$ is a C^* -algebra.

Step 3: Norm and Completeness

Equip $C_0(X)$ with the supremum norm: $\|f\|_\infty = \sup_{x \in X} |f(x)|$. This norm makes $C_0(X)$ a Banach space. The norm satisfies: $\|f^* f\|_\infty = \|f\|_\infty^2$ (C^* -identity). To verify the C^* -identity, note that: $\|f^* f\|_\infty = \sup_{x \in X} |\overline{f(x)} f(x)| = \sup_{x \in X} |f(x)|^2 = \left(\sup_{x \in X} |f(x)| \right)^2 = \|f\|_\infty^2$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Worked Micro-Exercises

- **Rank-One Operator Norm.** For $|u\rangle, |v\rangle \in \mathcal{H}$, $A = |u\rangle\langle v|$. Show $\|A\| = \|u\| \|v\|$.

Step 1: Upper bound $\|A\| \leq \|u\| \|v\|$

For any $|\psi\rangle \in \mathcal{H}$ with $\|\psi\| = 1$, we have: $A|\psi\rangle = |u\rangle\langle v|\psi\rangle = \langle v|\psi\rangle |u\rangle$. Therefore, $\|A|\psi\rangle\| = |\langle v|\psi\rangle| \cdot \|u\| \leq \|v\| \cdot \|\psi\| \cdot \|u\| = \|u\| \|v\|$ where we used the Cauchy-Schwarz inequality $|\langle v|\psi\rangle| \leq \|v\| \cdot \|\psi\|$. Since this holds for all unit vectors $|\psi\rangle$, we conclude: $\|A\| = \sup_{\|\psi\|=1} \|A|\psi\rangle\| \leq \|u\| \|v\|$

Step 2: Lower bound $\|A\| \geq \|u\| \|v\|$

To show the reverse inequality, consider the vector $|\varphi\rangle = \frac{|v\rangle}{\|v\|}$ (if $v \neq 0$). Then $\|\varphi\| = 1$, and: $A|\varphi\rangle = |u\rangle\langle v|\varphi\rangle = \|v\| |u\rangle$. Thus, $\|A|\varphi\rangle\| = \|v\| \cdot \|u\|$. Therefore, $\|A\| = \sup_{\|\psi\|=1} \|A|\psi\rangle\| \geq \|A|\varphi\rangle\| = \|u\| \|v\|$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Worked Micro-Exercises
 - Cauchy–Schwarz for States.

For positive ω and $a, b \in \mathcal{A}$, consider $\omega((a + \lambda b)^*(a + \lambda b)) \geq 0$ for all $\lambda \in \mathbb{C}$; optimize over λ to get the inequality.

Proof

$$\omega((a + \lambda b)^*(a + \lambda b)) = \omega(a^*a) + \|\lambda\|^2\omega(b^*b) + \lambda\omega(a^*b) + \lambda^*\omega(b^*a)$$

$$\frac{d\omega}{d\lambda} = 0 \Rightarrow \lambda^* = -\frac{\omega(a^*b)}{\omega(b^*b)} \Rightarrow \lambda = -\frac{\omega(b^*a)}{\omega(b^*b)}$$

Insert the λ back to the first expression and require the minimum to be no less than 0:

$$\omega(a^*b)\omega(b^*a) \leq \omega(a^*a)\omega(b^*b)$$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Worked Micro-Exercises
 - Cauchy–Schwarz for States.

For positive ω and $a, b \in \mathcal{A}$, consider $\omega((a + \lambda b)^*(a + \lambda b)) \geq 0$ for all $\lambda \in \mathbb{C}$; optimize over λ to get the inequality.

Proof

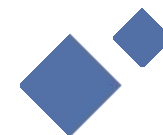
$$\omega(a^*b)\omega(b^*a) \leq \omega(a^*a)\omega(b^*b)$$

In particular,

$$\langle \psi_0 | \hat{A}^\dagger \hat{B} | \psi_0 \rangle \langle \psi_0 | \hat{B}^\dagger \hat{A} | \psi_0 \rangle \leq \langle \psi_0 | \hat{A}^\dagger \hat{A} | \psi_0 \rangle \langle \psi_0 | \hat{B}^\dagger \hat{B} | \psi_0 \rangle$$

. Let $\hat{A}|\psi_0\rangle = |\psi\rangle$, $\hat{B}|\psi_0\rangle = |\phi\rangle$ and we derive the Cauchy-Schwarz inequality in QM:

$$\|\langle \psi | \phi \rangle\|^2 \leq \langle \psi | \psi \rangle \langle \phi | \phi \rangle$$



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Worked Micro-Exercises
 - Spectral radius formula in $M_d(\mathbb{C})$.

Verify $r(A) = \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}$ by diagonalizing A in Jordan form and bounding $\|A^n\|$ above and below by the largest modulus eigenvalue.

Proof

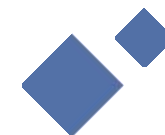
Rewrite A in Jordan form (J is the block-diagonal matrix)

$$A = PJP^{-1} \Rightarrow A^n = PJ^nP^{-1} \quad J^n = P^{-1}A^nP$$

The corresponding norm satisfies

$$A^n = PJ^nP^{-1} \Rightarrow \|A^n\| \leq \|P\| \|J^n\| \|P^{-1}\| \Rightarrow \|A^n\| \leq c_1 \|J^n\|$$

$$J^n = P^{-1}A^nP \Rightarrow \|J^n\| \leq \|P^{-1}\| \|A^n\| \|P\| \Rightarrow \|A^n\| \geq c_2 \|J^n\|$$



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C^* -Algebras: the Minimal Package

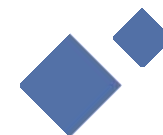
- Worked Micro-Exercises
 - Spectral radius formula in $M_d(\mathbb{C})$.

Verify $r(A) = \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}$ by diagonalizing A in Jordan form and bounding $\|A^n\|$ above and below by the largest modulus eigenvalue.

Proof

With $\lim_{n \rightarrow \infty} c^{\frac{1}{n}} = 1$, we have

$$(c_2 \|J^n\|)^{\frac{1}{n}} \leq \|A^n\|^{\frac{1}{n}} \leq (c_1 \|J^n\|)^{\frac{1}{n}} \Rightarrow r(A) = \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}} = \sup\{|\lambda|, \lambda \in \sigma(A)\}$$



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C^* -Algebras: the Minimal Package

- Common Pitfalls Clarified
 - “Positive” $\neq$ “Nonnegative Entries”

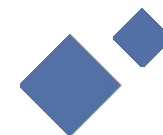
In $M_d(\mathbb{C})$, $A \geq 0$ means $x^\dagger A x \geq 0$ for all x , not that each entry is ≥ 0 .

- Trace-Class vs Hilbert–Schmidt.

On infinite-dimensional $\mathcal{H}$, $\mathcal{T}(\mathcal{H}) \subset \text{HS}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$; only trace-class defines normal states via $\rho \mapsto \text{Tr}(\rho \cdot)$.

- Vanishing at Infinity.

In $C_0(X)$, functions need not be compactly supported; they merely become small outside a compact set. This is crucial for completeness and duality with measures.



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- GNS Construction: Definition, Meaning, and Examples

- What does “GNS” stand for?

GNS stands for *Gel’fand–Naimark–Segal*. The GNS construction is a fundamental tool in operator algebras that takes a state ω on a C^* -algebra $\mathcal{A}$ and produces a Hilbert space $\mathcal{H}_\omega$ and a representation $\pi_\omega : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\omega)$ such that ω is realized as a vector state.

- Definition (Cyclic Vector)

A vector $|\Omega_\omega\rangle \in \mathcal{H}_\omega$ is called *cyclic* for the representation π_ω if the set of vectors

$$\{\pi_\omega(a)|\Omega_\omega\rangle \mid a \in \mathcal{A}\}$$

is dense in $\mathcal{H}_\omega$. This means that by acting on $|\Omega_\omega\rangle$ with all elements of the algebra (via π_ω), we can approximate any vector in $\mathcal{H}_\omega$. The cyclicity ensures that the representation is “large enough” to capture the full algebra $\mathcal{A}$.



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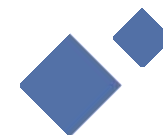
C^* -Algebras: the Minimal Package

- GNS Construction: Definition, Meaning, and Examples
 - Unification of States

The GNS construction unifies the treatment of pure states (described by vectors in a Hilbert space) and mixed states (described by density operators) by representing both as vector states in the GNS Hilbert space:

$$\omega(a) = \langle \Omega_\omega | \pi_\omega(a) | \Omega_\omega \rangle \quad \forall a \in \mathcal{A}.$$

Pure states: If ω is a pure state, the GNS representation is irreducible, and $|\Omega_\omega\rangle$ is the unique (up to phase) vector representing ω .



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- GNS Construction: Definition, Meaning, and Examples
 - Unification of States

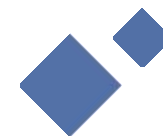
Mixed states: If ω is a mixed state with density operator $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ on a Hilbert space $\mathcal{H}$, the GNS space $\mathcal{H}_\omega$ is isomorphic to $\mathcal{H} \otimes \mathcal{H}$, and the cyclic vector becomes a purification of ρ :

$$|\Omega_\omega\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle \otimes |\psi_i\rangle \in \mathcal{H} \otimes \mathcal{H}.$$

The representation acts as $\pi_\omega(a) = a \otimes \mathbb{1}$, and the expectation value reproduces the mixed-state trace:

$$\langle\Omega_\omega|(a \otimes \mathbb{1})|\Omega_\omega\rangle = \sum_i p_i \langle\psi_i|a|\psi_i\rangle = \text{Tr}(\rho a).$$

Thus, the GNS framework treats all states uniformly as vector states, with cyclicity ensuring the algebra is fully represented.



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C^* -Algebras: the Minimal Package

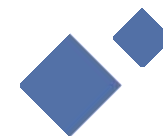
- GNS Construction: Definition, Meaning, and Examples

- Definition (State)

A **state** on a C^* -algebra $\mathcal{A}$ is a linear functional $\omega : \mathcal{A} \rightarrow \mathbb{C}$ such that $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}$ and $\|\omega\| = 1$ (equivalently, $\omega(\mathbb{1}) = 1$ when $\mathcal{A}$ is unital). For $\mathcal{A} = \mathcal{B}(\mathcal{H})$, normal states are precisely $\omega(a) = \text{Tr}(\rho a)$ with $\rho \geq 0$, $\text{Tr } \rho = 1$.

- Definition (GNS construction (Gel'fand–Naimark–Segal))

Given a state ω on a C^* -algebra $\mathcal{A}$, the **GNS triple** $(\pi_\omega, \mathcal{H}_\omega, |\Omega_\omega\rangle)$ is obtained as follows.



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- GNS Construction: Definition, Meaning, and Examples
 - **Definition (GNS construction (Gel'fand–Naimark–Segal))**
 - Step 1 (pre-Hilbert space).

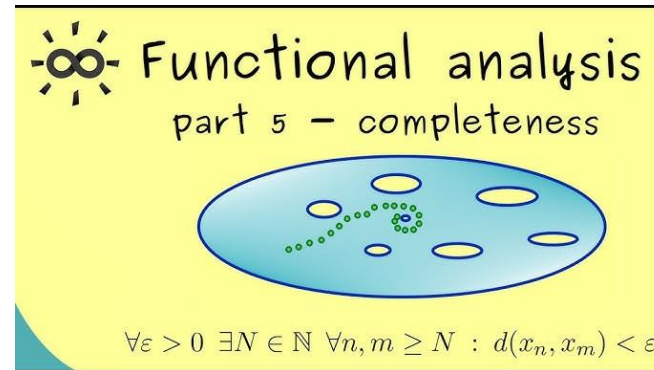
Define a sesquilinear form on $\mathcal{A}$ by $\langle a, b \rangle_\omega := \omega(a^*b)$, $a, b \in \mathcal{A}$. Let the null space (left ideal)

$$\mathcal{N}_\omega := \{a \in \mathcal{A} : \omega(a^*a) = 0\}.$$

Form the quotient vector space $\frac{\mathcal{A}}{\mathcal{N}_\omega}$ and denote the class of a by $[a]$. The form $\langle \cdot, \cdot \rangle_\omega$ descends to an inner product

$$\langle [a], [b] \rangle_\omega := \omega(a^*b),$$

making $\frac{\mathcal{A}}{\mathcal{N}_\omega}$ a pre-Hilbert space.



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C^* -Algebras: the Minimal Package

- GNS Construction: Definition, Meaning, and Examples
 - **Definition (GNS construction (Gel'fand–Naimark–Segal))**

► Step 2 (completion).

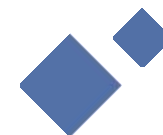
Let $\mathcal{H}_\omega$ be the Hilbert-space completion of $\frac{\mathcal{A}}{\mathcal{N}_\omega}$ in the norm $\|[a]\|_\omega := \sqrt{\omega(a^*a)}$.

► Step 3 (representation).

Define $\pi_\omega : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\omega)$ by left multiplication:

$$\pi_\omega(x)[a] := [xa], \quad x, a \in \mathcal{A},$$

which is well-defined, bounded, and a $*$ -representation.



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- GNS Construction: Definition, Meaning, and Examples
 - **Definition (GNS construction (Gel'fand–Naimark–Segal))**

► Step 4 (cyclic vector).

Let $|\Omega_\omega\rangle := [1] \in \mathcal{H}_\omega$. Then

$$\omega(a) = \langle \Omega_\omega, \pi_\omega(a) \Omega_\omega \rangle_\omega, \quad \text{and} \quad \overline{\text{span}}\{\pi_\omega(a) |\Omega_\omega\rangle : a \in \mathcal{A}\} = \mathcal{H}_\omega.$$

- Theorem (Uniqueness up to unitary equivalence)

If $(\pi, \mathcal{H}, |\Omega\rangle)$ is any cyclic representation such that $\omega(a) = \langle \Omega | \pi(a) | \Omega \rangle$ for all $a \in \mathcal{A}$, then there exists a unique unitary $U : \mathcal{H}_\omega \rightarrow \mathcal{H}$ with $U |\Omega_\omega\rangle = |\Omega\rangle$ and $U \pi_\omega(a) = \pi(a) U$ for all $a \in \mathcal{A}$.



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- Key Properties and Physical Meaning.

- Vector-State realization.

Every abstract state is realized as a concrete vector state on some Hilbert space, with observables represented by bounded operators via π_ω .

- Purity vs Irreducibility.

ω is *pure* $\Leftrightarrow \pi_\omega$ is *irreducible* (has no nontrivial closed invariant subspaces). This is the rigorous correspondence between pure states and rays.

- Faithfulness.

ω is *faithful* ($\omega(a^*a) = 0 \Rightarrow a = 0$) $\Rightarrow \pi_\omega$ is faithful (injective), hence identifies $\mathcal{A}$ with a concrete operator algebra.



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C^* -Algebras: the Minimal Package

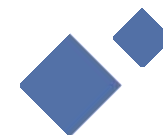
- Key Properties and Physical Meaning.

- From Abstract to Concrete.

In physics, we often start from $\mathcal{A}$ and a physically prepared state ω (e.g. a thermal/KMS state). GNS provides the “right” Hilbert space $\mathcal{H}_\omega$ and the observable representation tailored to ω .

- Bridge to Stinespring.

Stinespring’s dilation for CP maps uses a GNS-like construction for the positive functional ($a \mapsto \langle \psi, \Phi(a)\psi \rangle$); technically, GNS underlies many dilation theorems in operator theory.



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C^* -Algebras: the Minimal Package

- GNS Construction: Definition, Meaning, and Examples

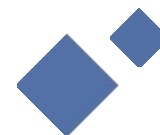
- Finite-dimensional examples (QI notation).

- **Example (Matrix algebra with a density operator)**

Let $\mathcal{A} = M_d(\mathbb{C})$ and $\omega(a) = \text{Tr}(\rho a)$ with $\rho \geq 0$, $\text{Tr } \rho = 1$. Define inner product on $\mathcal{A}$ by $\langle X, Y \rangle_\omega := \text{Tr}(\rho X^\dagger Y)$. Then

$$\mathcal{H}_\omega \cong (M_d(\mathbb{C}), \langle \cdot, \cdot \rangle_\omega), \quad \pi_\omega(a) : X \mapsto aX, \quad |\Omega_\omega\rangle \hat{=} \text{Id}.$$

One checks $\text{Tr}(\rho a) = \langle \mathbb{1}, a\mathbb{1} \rangle_\omega$, and π_ω is faithful iff ρ is full rank. If ρ has support projection $s(\rho)$ of rank r , then π_ω is equivalent to the standard representation on $\mathbb{C}^r$.



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- GNS Construction: Definition, Meaning, and Examples
 - Finite-dimensional examples (QI notation).

► Example (Pure state (vector state))

Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\omega(a) = \langle \psi | a | \psi \rangle$ with $\|\psi\| = 1$. Then $\mathcal{N}_\omega = \{a : a | \psi \rangle = 0\}$, $\mathcal{H}_\omega$ is isomorphic to the closure of $\{a | \psi \rangle : a \in \mathcal{B}(\mathcal{H})\}$ (which is all of $\mathcal{H}$ if $\mathcal{H}$ is separable and ψ is cyclic), π_ω is the identity representation (up to unitary equivalence), and $|\Omega_\omega\rangle$ corresponds to $|\psi\rangle$. Thus GNS reproduces the usual Dirac formalism for pure states.



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- GNS Construction: Definition, Meaning, and Examples
 - Finite-dimensional examples (QI notation).

► Example (Tracial state and L^2 -picture)

On $M_d(\mathbb{C})$ with the tracial state $\tau(a) = \frac{1}{d} \operatorname{Tr}(a)$, the GNS Hilbert space is the Hilbert–Schmidt space $(M_d(\mathbb{C}), \langle X, Y \rangle_\tau = \frac{1}{d} \operatorname{Tr}(X^\dagger Y))$, with π_ω the left-regular representation $a : X \mapsto aX$ and $|\Omega_\omega\rangle = \operatorname{Id}$. This is the finite-dimensional prototype of the L^2 -space $L^2(\mathcal{A}, \tau)$ used in noncommutative integration.



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C^* -Algebras: the Minimal Package

- GNS Construction: Definition, Meaning, and Examples

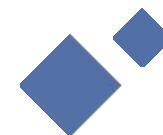
- Finite-dimensional examples (QI notation).

- **Commutative case and the Riesz–Markov theorem.**

If $\mathcal{A} = C_0(X)$ (continuous functions vanishing at infinity on a locally compact Hausdorff space X), any state ω comes from a unique probability measure μ by the Riesz–Markov theorem: $\omega(f) = \int_X f \, d\mu$. The GNS space is $\mathcal{H}_\omega = L^2(X, \mu)$, with

$$\pi_\omega(f) \text{ acting as multiplication by } f, \quad \Omega_\omega = \mathbf{1} \in L^2(X, \mu).$$

Thus, in the classical case GNS reduces to the standard L^2 representation of observables as multiplication operators and the state as an L^2 -unit vector.



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C^* -Algebras: the Minimal Package

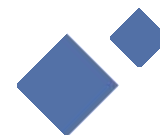
- GNS Construction: Definition, Meaning, and Examples
 - Useful facts (at a glance).
 - ▶ **Cyclicity:** $|\Omega_\omega\rangle$ is cyclic by construction; $\pi_\omega(\mathcal{A}) |\Omega_\omega\rangle$ is dense in $\mathcal{H}_\omega$.
 - ▶ **Separating vector:** If ω is faithful, then $|\Omega_\omega\rangle$ is separating for $\pi_\omega(\mathcal{A})$ (i.e. $\pi_\omega(a) |\Omega_\omega\rangle = 0 \Rightarrow a = 0$).
 - ▶ **Normal states on $\mathcal{B}(\mathcal{H})$:** For $\omega(a) = \text{Tr}(\rho a)$, the GNS triple is unitarily equivalent to the standard form built on the support of ρ ; if ρ is pure, GNS collapses to the usual Hilbert-space picture.
 - ▶ **Purity test:** ω pure $\Leftrightarrow$ the commutant $\pi_\omega(\mathcal{A})'$ contains only scalars (Schur's lemma).



Lindblad Equation — An Introduction to Operator Algebra

C^* -Algebras: the Minimal Package

- GNS Construction: Definition, Meaning, and Examples
 - Why GNS matters for these notes.
 - ▶ It justifies representing abstract observables in $\mathcal{A}$ by concrete operators on some $\mathcal{H}_\omega$, with expectations given by vector states. This is the backbone for moving between abstract C^* -algebras and $B(\mathcal{H})$.
 - ▶ It clarifies the role of *normal* states and ultraweak continuity in the W^* (von Neumann) setting: normal states correspond to trace-class objects via the predual, and their GNS representations mesh well with partial traces and dilations.
 - ▶ It connects directly to Stinespring dilations of CP maps (next section), which can be viewed as a GNS construction for a positive semidefinite kernel induced by the map.



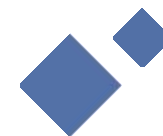
Lindblad Equation — An Introduction to Operator Algebra

von Neumann (W^*) Algebras and the Right Topologies

This section explains what von Neumann (W^*) algebras are, why the *weak/strong operator topologies* are the correct ones for infinite systems and open dynamics, and how *normal* states/maps arise from the unique *predual*. Examples are chosen with quantum mechanics, quantum optics, and (briefly) QFT in mind.

- Operator Topologies on $\mathcal{B}(\mathcal{H})$ (physicist-friendly refresher)

Let $\mathcal{B}(\mathcal{H})$ denote bounded operators on a Hilbert space $\mathcal{H}$. Besides the norm topology $\|X\|$, we use weaker topologies defined via matrix elements.



Lindblad Equation — An Introduction to Operator Algebra

von Neumann (W^*) Algebras and the Right Topologies

- **Definition (Weak and Strong Operator Topologies on Bounded Sets)**

For a net $(X_\alpha) \subset \mathcal{B}(\mathcal{H})$ and $X \in \mathcal{B}(\mathcal{H})$:

- ▶ **Weak operator topology (WOT):** $X_\alpha \rightarrow X$ iff $\langle \varphi, X_\alpha \psi \rangle \rightarrow \langle \varphi, X \psi \rangle$ for all $\varphi, \psi \in \mathcal{H}$.
- ▶ **Strong operator topology (SOT):** $X_\alpha \rightarrow X$ iff $\|X_\alpha \psi - X \psi\| \rightarrow 0$ for all $\psi \in \mathcal{H}$.

On *bounded* sets (i.e., $\sup_\alpha \|X_\alpha\| < \infty$), these are metrizable if $\mathcal{H}$ is separable. One always has

$$\text{norm} \Rightarrow \text{SOT} \Rightarrow \text{WOT}.$$



Lindblad Equation — An Introduction to Operator Algebra

von Neumann (W^*) Algebras and the Right Topologies

- **Definition (Ultraweak and Ultrastrong Topologies)**

Using the trace-class $\mathcal{T}(\mathcal{H})$ as the predual of $\mathcal{B}(\mathcal{H})$ (Theorem below):

- ▶ **Ultraweak (UW/ $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{T}(\mathcal{H}))$):** $X_\alpha \rightarrow X$ iff $\text{Tr}(TX_\alpha) \rightarrow \text{Tr}(TX)$ for all $T \in \mathcal{T}(\mathcal{H})$.
- ▶ **Ultrastrong:** $X_\alpha \rightarrow X$ iff $\|(X_\alpha - X)\rho^{\frac{1}{2}}\|_2 \rightarrow 0$ for all $\rho \in \mathcal{T}(\mathcal{H})_+$, where $\|\cdot\|_2$ is Hilbert–Schmidt norm.

On bounded sets, ultraweak = WOT and ultrastrong = SOT. The “ultra” language is convenient beyond $\mathcal{B}(\mathcal{H})$ in general von Neumann algebras.

Physically: WOT/UW track convergence of expectation values $\langle \varphi | X | \psi \rangle$ or $\text{Tr}(\rho X)$; SOT/US track convergence of action on state vectors or density-operator square roots.



Lindblad Equation — An Introduction to Operator Algebra

von Neumann (W^*) Algebras and the Right Topologies

- What is a von Neumann algebra?
 - **Definition (Commutant and Double Commutant)**

For $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$, the *commutant* is

$$\mathcal{S}' := \{X \in \mathcal{B}(\mathcal{H}) : [X, S] = 0 \ \forall S \in \mathcal{S}\}.$$

It is a unital $*$ -subalgebra. The *double commutant* is $\mathcal{S}'' := (\mathcal{S}')'$.



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von Neumann (W^*) Algebras and the Right Topologies

- What is a von Neumann algebra?
 - **Theorem (von Neumann Double Commutant Theorem)**

If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a unital $*$ -subalgebra, then the following are equivalent:

- $\mathcal{A}$ is WOT-closed (equivalently, SOT-closed) in $\mathcal{B}(\mathcal{H})$;
- $\mathcal{A} = \mathcal{A}''$ (it equals its double commutant).

A unital $*$ -subalgebra satisfying either condition is called a **von Neumann algebra** (or W^* -algebra).



Lindblad Equation — An Introduction to Operator Algebra

von Neumann (W^*) Algebras and the Right Topologies

- What is a von Neumann algebra?
 - Remark (Closing under weak/strong limits matters)

In experiments and theory, we approximate observables by sequences/nets (finite-resolution detectors, finite-volume/energy cutoffs, Trotter products, etc.). Demanding closure under WOT/SOT ensures limits of physically meaningful approximations remain in the algebra.

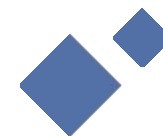


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von Neumann (W^*) Algebras and the Right Topologies

- Examples a Theoretical Physicist Knows
 - **Example (All bounded observables)**

$\mathcal{B}(\mathcal{H})$ itself is a von Neumann algebra: it is already WOT/SOT closed and equals its double commutant. Type classification: $\mathcal{B}(\mathcal{H})$ is a type I factor (center = $\mathbb{C}1$).



Lindblad Equation — An Introduction to Operator Algebra

von Neumann (W^*) Algebras and the Right Topologies

- Examples a Theoretical Physicist Knows
 - **Example (Multiplication operators: classical probability)**

Let (X, μ) be a σ -finite measure space and $\mathcal{H} = L^2(X, \mu)$. Define

$$L^\infty(X, \mu) := \{M_f : \psi \mapsto f\psi \mid f \in L^\infty(X, \mu)\} \subset \mathcal{B}(\mathcal{H}).$$

This is a commutative von Neumann algebra (WOT-closed). Its commutant is itself: $L^\infty(X, \mu)' = L^\infty(X, \mu)$.



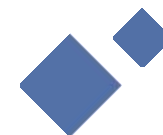
Lindblad Equation — An Introduction to Operator Algebra

von Neumann (W^*) Algebras and the Right Topologies

- Examples a Theoretical Physicist Knows

- Example (Finite-level quantum systems)

For finite d , $M_d(\mathbb{C}) = \mathcal{B}(\mathbb{C}^d)$ is a von Neumann algebra (and a C^* -algebra). Any finite direct sum $\oplus_k M_{d_k}(\mathbb{C})$ acting on $\oplus_k \mathbb{C}^{d_k}$ is also a von Neumann algebra. These model superselection sectors or block-diagonal Hamiltonians.



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von Neumann (W^*) Algebras and the Right Topologies

- Examples a Theoretical Physicist Knows
 - Example (Quantum optics and CCR via Weyl operators)

Let $\mathcal{H}$ be the Fock space of a bosonic mode. The *Weyl operators* $W(\xi) = \exp(i(\hat{q}\xi_q + \hat{p}\xi_p))$ are unitary and bounded. The von Neumann algebra $\mathcal{M}$ generated by $\{W(\xi) : \xi \in \mathbb{R}^2\}$ in a fixed regular representation is WOT-closed and contains all bounded functions of the number operator via the spectral calculus. This provides a rigorous bounded-operator setting for continuous-variable systems.

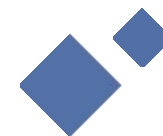


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von Neumann (W^*) Algebras and the Right Topologies

- Examples a Theoretical Physicist Knows
 - Example (Local algebras in algebraic QFT (AQFT))

For a spacetime region $\mathcal{O}$, the local bounded observables generate a von Neumann algebra $\mathcal{M}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$. In relativistic QFT, these are typically *type III* factors (no trace, no density matrices as normal faithful tracial states), explaining the subtleties of entropy and partial trace in QFT. For our open-systems notes we mainly use type I, but the W^* framework scales to QFT.



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von Neumann (W^*) Algebras and the Right Topologies

- Predual and normality

- Theorem (Unique Predual)

Every von Neumann algebra $\mathcal{M}$ is the dual of a uniquely determined Banach space $\mathcal{M}_*$ (its *predual*): $\mathcal{M} = (\mathcal{M}_*)^*$. The $\sigma(\mathcal{M}, \mathcal{M}_*)$ topology is the *ultraweak* topology.

- Example (Predual of $\mathcal{B}(\mathcal{H})$)

If ω is positive, then for all $a, b \in \mathcal{A}$

$$|\omega(b^*a)|^2 \leq \omega(a^*a)\omega(b^*b).$$

Hence $|\omega(a)| \leq \omega(\mathbb{1})^{\frac{1}{2}}\omega(a^*a)^{\frac{1}{2}}$ when $\mathcal{A}$ is unital.



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von Neumann (W^*) Algebras and the Right Topologies

- Predual and normality

- Remark (Order Unit and Expectations)

The predual is the trace-class $\mathcal{T}(\mathcal{H})$; the dual pairing is $\langle T, X \rangle = \text{Tr}(TX)$, $T \in \mathcal{T}(\mathcal{H})$, $X \in \mathcal{B}(\mathcal{H})$. Thus ultraweak convergence is exactly convergence of all density-matrix expectations $\text{Tr}(\rho X_\alpha)$.

- Proposition (Normal States on $\mathcal{B}(\mathcal{H})$)

Normal states are exactly $\omega_\rho(X) = \text{Tr}(\rho X)$ with $\rho \in \mathcal{T}(\mathcal{H})$, $\rho \geq 0$, $\text{Tr } \rho = 1$. Non-normal states exist (singular states), but they are unphysical for open-systems dynamics driven by partial traces and limits of finite-rank approximations.



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von Neumann (W^*) Algebras and the Right Topologies

- Predual and normality
 - Remark (Why Normality for Dynamics)

In open quantum systems we repeatedly use:

- ▶ **Partial trace** $\text{Tr}_E : \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_E) \rightarrow \mathcal{B}(\mathcal{H}_S)$ is a *normal*, completely positive (CP), unital map in Heisenberg picture (CPTP in Schrödinger picture).
- ▶ **Limits of Trotter products** and weak-coupling limits produce UW/SOT limits; normality ensures these limits commute with expectations $\text{Tr}(\rho \cdot)$.
- ▶ **Semigroups** $\{\Phi_t\}_{t \geq 0}$ used in GKSL are assumed normal so that $\rho \mapsto \text{Tr}(\rho, \Phi_t(X))$ varies continuously with t for all trace-class ρ .



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von Neumann (W^*) Algebras and the Right Topologies

- Projections, support, and conditional expectations

- **Definition (Projections and Support)**

A projection $P \in \mathcal{M}$ satisfies $P = P^\dagger = P^2$. For a positive normal functional (state) ω on $\mathcal{M}$, its **support projection** $s(\omega)$ is the smallest $P \in \mathcal{M}$ with $\omega(\mathbb{1} - P) = 0$. For a positive $A \in \mathcal{M}$, $s(A)$ is the projection onto $\overline{\text{Ran}(A)}$ in $\mathcal{B}(\mathcal{H})$.

- **Proposition (Lattice of projections)**

In a von Neumann algebra, projections form a complete lattice: any family has a least upper bound (supremum) and greatest lower bound (infimum), given by strong/ultraweak limits of increasing/decreasing nets. This fails in general C^* -algebras.



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von Neumann (W^*) Algebras and the Right Topologies

- Projections, support, and conditional expectations
 - **Definition (Conditional expectation)**

A (faithful, normal) conditional expectation $E : \mathcal{M} \rightarrow \mathcal{N}$ onto a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ is a unital CP idempotent map $E^2 = E$, normal and $\mathcal{N}$ -bimodular: $E(N_1 X N_2) = N_1 E(X) N_2$. Example: partial trace $E = \text{Tr}_E$ onto $\mathcal{B}(\mathcal{H}_S) \otimes \mathbb{1}_E$.



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von Neumann (W^*) Algebras and the Right Topologies

- Cyclic and Separating Vectors; Standard Form (Physics View)

- Definition (Cyclic and Separating)

A vector $\Omega \in \mathcal{H}$ is *cyclic* for $\mathcal{M}$ if $\overline{\mathcal{M}\Omega} = \mathcal{H}$. It is *separating* if $X\Omega = 0$ implies $X = 0$ for $X \in \mathcal{M}$.

- Example (Thermal/Gibbs States)

For $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\rho_\beta = \frac{e^{-\beta H}}{Z}$ full-rank, the vector $\Omega_\beta = \rho_\beta^{1/2}$ in the GNS (or standard) Hilbert space L^2 -picture is cyclic and separating. This underpins the KMS condition and, more generally, Tomita–Takesaki modular theory (not needed in detail here, but it explains why type III algebras naturally encode thermal/time evolution).



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von Neumann (W^*) Algebras and the Right Topologies

- Type Classification at 10,000 ft (Why You might Care)

A von Neumann algebra $\mathcal{M}$ has a **center** $Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$; if $Z(\mathcal{M}) = \mathbb{C}1$, $\mathcal{M}$ is a **factor**.

Factors are classified as:

- **Type I:** contains minimal projections; includes $\mathcal{B}(\mathcal{H})$. Standard quantum mechanics/open systems mostly live here.
- **Type II:** has a faithful trace but no minimal projections (e.g. hyperfinite II_1 factor); appears in statistical mechanics and subfactor theory.
- **Type III:** no trace at all; arises in relativistic QFT local algebras. Entropy and partial trace require modular theory, not density matrices.

Even if your system is type I, environments or scaling limits can generate non-type-I behavior; W^* language keeps the formalism robust.



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von Neumann (W^*) Algebras and the Right Topologies

- Worked micro-exercises (with solutions)
 - **WOT closure test.**

Let (P_n) be spectral projections of a Hamiltonian H onto energy windows $[-n, n]$. Show $P_n \downarrow P$ in SOT for some projection P and that the set of all spectral projections of H is SOT-closed.

Solution: Spectral theorem gives $P_n \psi \rightarrow P \psi$ for all ψ ; projections form a complete lattice in a von Neumann algebra.



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von Neumann (W^*) Algebras and the Right Topologies

- Worked micro-exercises (with solutions)
 - **Commutant of multiplication operators.**

Prove that on $\mathcal{H} = L^2(X, \mu)$, the commutant of $L^\infty(X, \mu)$ is itself: $L^\infty(X, \mu)' = L^\infty(X, \mu)$.

Solution: If T commutes with all M_f , it must preserve the ranges of M_{1_E} for measurable E ; from this one deduces $T = M_g$ for some $g \in L^\infty$.



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von Neumann (W^*) Algebras and the Right Topologies

- Worked micro-exercises (with solutions)

- **Normality of partial trace.**

For finite-rank $X = \sum_i A_i \otimes B_i$, check UW-continuity of Tr_E against any $\rho_S \in \mathcal{T}(\mathcal{H}_S)$ using $\text{Tr}[\rho_S \text{Tr}_E(X)] = \text{Tr}[(\rho_S \otimes \mathbb{1})X]$, then extend by density and UW continuity.

- **Identify a factor.**

Show $\mathcal{B}(\mathcal{H})$ has trivial center $\mathbb{C}\mathbb{1}$; hence it is a factor (type I).

Solution: If Z commutes with all rank-one operators $|\varphi\rangle\langle\psi|$, then Z is a scalar multiple of $\mathbb{1}$.



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von Neumann (W^*) Algebras and the Right Topologies

- Takeaways for open quantum systems and GKSL
 - The *right topology* for infinite systems and limits is ultraweak/weak operator; von Neumann algebras are exactly the UW/SOT-closed $*$ -algebras.
 - *Normal* states/maps are those continuous for these topologies; they coincide with trace-class generated expectations and CPTP dynamics relevant in experiments.
 - Partial traces, conditional expectations, and Stinespring dilations are *normal* CP maps between von Neumann algebras; this is the natural stage for GKSL semigroups.



Lindblad Equation — An Introduction to Operator Algebra

Positivity, Complete Positivity, and Dilations

- Positive vs Completely Positive (CP)

- Definition (CP maps)

A linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is *positive* if $\Phi(X) \geq 0$ whenever $X \geq 0$. It is ***completely positive*** (CP) if, for every $n \in \mathbb{N}$, the amplification

$$\Phi_n \equiv \Phi \otimes \mathbb{1}_n : \mathcal{A} \otimes M_n(\mathbb{C}) \rightarrow \mathcal{B} \otimes M_n(\mathbb{C})$$

is positive. We identify $\mathcal{A} \otimes M_n(\mathbb{C}) \cong M_n(\mathcal{A})$.

Physically, CP means the map remains positive on the system even when we extend by an arbitrary idle n -dimensional ancilla, possibly entangled with the system.



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Positivity, Complete Positivity, and Dilations

- Positive vs Completely Positive (CP)
 - Choi–Jamiołkowski isomorphism (finite dimension)

For $\mathcal{A} = M_d(\mathbb{C})$, define the (unnormalized) maximally entangled vector $|\Omega\rangle = \sum_{i=1}^d |i\rangle \otimes |i\rangle$ and the *Choi matrix*

$$J_\Phi := (\Phi \otimes \mathbb{1}_d)(|\Omega\rangle\langle\Omega|).$$

Then Φ is CP iff $J_\Phi \geq 0$. The rank of J_Φ is the minimal number of Kraus operators.



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Positivity, Complete Positivity, and Dilations

- Stinespring Dilation and Kraus Form

- Theorem (Stinespring)

Let $\mathcal{A}$ be a C^* -algebra and $\Phi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ be CP (and normal if $\mathcal{A}$ is W^*). Then there exists a Hilbert space $\mathcal{H}_E$, a $*$ -representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_E)$, and a bounded $V : \mathcal{H} \rightarrow \mathcal{H}_E$ such that

$$\Phi(a) = V^\dagger \pi(a) V \quad (\forall a \in \mathcal{A}).$$

For channels on $\mathcal{B}(\mathcal{H}_S)$ (Schrödinger picture), there exist $\mathcal{H}_E$, a unitary U on $\mathcal{H}_S \otimes \mathcal{H}_E$, and a fixed environment state σ_E such that

$$\Phi_*(\rho) = \text{Tr}_E [U(\rho \otimes \sigma_E)U^\dagger].$$



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Positivity, Complete Positivity, and Dilations

- Stinespring Dilation and Kraus Form
 - **Proposition (Kraus representation)**

In finite dimension (or for normal CP on $\mathcal{B}(\mathcal{H})$), there exist operators $\{K_k\}$ such that

$$\Phi_*(\rho) = \sum_k K_k \rho K_k^\dagger, \quad \sum_k K_k^\dagger K_k \leq \mathbb{1}.$$

If Φ_* is trace preserving (CPTP), then $\sum_k K_k^\dagger K_k = \mathbb{1}$. Heisenberg dual: $\Phi(X) = \sum_k K_k^\dagger X K_k$ (unital iff $\sum_k K_k K_k^\dagger = \mathbb{1}$).

- **Why CP (and not just positive)?**

A merely positive map can map a valid bipartite state ρ_{SA} into a non-positive operator when applied as $\Phi_S \otimes \mathbb{1}_A$ if ρ_{SA} is entangled (counterexample: matrix transpose). CP guarantees physicality in the presence of arbitrary idle ancillas.



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Positivity, Complete Positivity, and Dilations

- Stinespring Dilation and Kraus Form
 - Kadison–Schwarz Inequality

If Φ is CP and unital ($\Phi(\mathbb{1}) = \mathbb{1}$), then

$$\Phi(X^\dagger X) \geq \Phi(X^\dagger)\Phi(X).$$

This inequality seeds generator constraints by differentiation.



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Quantum Dynamical Semigroups and Generators

- Definition (QDS)

A *quantum dynamical semigroup* (QDS) $\{\Phi_t\}_{t \geq 0}$ on a W^* -algebra $\mathcal{M}$ is a family of CP (usually unital in Heisenberg, trace preserving in Schrödinger) normal maps with

$$\Phi_0 = \mathbb{1}, \quad \Phi_{t+s} = \Phi_t \circ \Phi_s, \quad \lim_{t \rightarrow 0^+} \|\Phi_t - \mathbb{1}\| = 0 \quad (\text{norm continuity}).$$

Lindblad assumes norm continuity so the generator is bounded.



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Quantum Dynamical Semigroups and Generators

- Generator (Liouvillian)

There exists a bounded linear map $L : \mathcal{M} \rightarrow \mathcal{M}$ such that

$$\Phi_t = e^{tL} \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} L^n \quad (\text{norm-convergent}).$$

Differentially: in Heisenberg picture, $\frac{d}{dt}X_t = L(X_t)$ with $X_t := \Phi_t(X)$; in Schrödinger picture, $\frac{d}{dt}\rho_t = L_*(\rho_t)$ where $\text{Tr}[L_*(\rho)X] = \text{Tr}[\rho L(X)]$.



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Quantum Dynamical Semigroups and Generators

- From CP to generator constraints: matrix amplification

Define the n -level amplification $L_n := L \otimes \mathbb{1}_n$ acting on $M_n(\mathcal{M}) \cong \mathcal{M} \otimes M_n(\mathbb{C})$. Complete positivity of each Φ_t means $\Phi_{t,n} := \Phi_t \otimes \mathbb{1}_n$ is positive for all n . Differentiating the Kadison–Schwarz inequality for $\Phi_{t,n}$ at $t = 0$ yields **complete dissipativity**:

$$D(L_n; X, X) \equiv L_n(X^\dagger X) - L_n(X^\dagger)X - X^\dagger L_n(X) \geq 0 \quad \forall X \in M_n(\mathcal{M}), \forall n \in \mathbb{N}.$$

This is exactly where the amplification L_n is essential: it transfers CP at the semigroup level to a quadratic inequality at the generator level.



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Quantum Dynamical Semigroups and Generators

- Characterizing reversibility

$D(L; \cdot, \cdot) \equiv 0$ iff L is a derivation $L(X) = i[H, X]$ with $H = H^\dagger$. Then $\{\Phi_t\}$ is a group of $*$ -automorphisms (unitary Heisenberg evolution). Otherwise, the semigroup is genuinely irreversible.



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The Lindblad/GKSL normal form

- Heisenberg Picture

- Theorem (GKSL/Lindblad form on $\mathcal{B}(\mathcal{H})$)

Let $\{\Phi_t\}$ be a norm-continuous QDS of normal unital CP maps on $\mathcal{B}(\mathcal{H})$ with generator L . Then there exist a self-adjoint $H = H^\dagger$ and (possibly countably many) Lindblad operators $\{V_j\} \subset \mathcal{B}(\mathcal{H})$ such that

$$L(X) = \sum_j \left(V_j^\dagger X V_j - \frac{1}{2} \{V_j^\dagger V_j, X\} \right) + i[H, X].$$



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Quantum Dynamical Semigroups and Generators

- Schrödinger Dual

For density operators ρ ,

$$L_*(\rho) = -i[H, \rho] + \sum_j \left(V_j \rho V_j^\dagger - \frac{1}{2} \{V_j^\dagger V_j, \rho\} \right),$$

and the master equation reads $\dot{\rho}_t = L_*(\rho_t)$.



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Quantum Dynamical Semigroups and Generators

- Non-Uniqueness (Gauge)
 - **Unitary mixing of jumps:** $V_j \mapsto \sum_k u_{jk} V_k$ for a unitary $U = (u_{jk})$ on the index space leaves L_* invariant.
 - **Hamiltonian shift:** certain shifts $V_j \mapsto V_j + c_j \mathbb{1}$ can be absorbed into $H \mapsto H + H_{\text{LS}}$ (Lamb-shift-like), leaving L_* invariant.
 - **Minimal number of jumps:** equals rank of the Choi matrix of L_* (finite dimension).



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Quantum Dynamical Semigroups and Generators

- Complete dissipativity made manifest

With equation

$$L(X) = \sum_j \left(V_j^\dagger X V_j - \frac{1}{2} \{ V_j^\dagger V_j, X \} \right) + i[H, X],$$

one computes

$$D(L; X, X) = \sum_j [X, V_j]^\dagger [X, V_j] \geq 0,$$

and the same holds at all matrix levels by amplification, confirming equation

$$D(L_n; X, X) \equiv L_n(X^\dagger X) - L_n(X^\dagger)X - X^\dagger L_n(X) \geq 0 \quad \forall X \in M_n(\mathcal{M}), \forall n \in \mathbb{N}.$$



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Physical Modeling, Ancillas, and Micro-to-Master

- Why $\Phi \otimes \mathbb{1}_n$ must be positive for all n

Any real device can be part of a larger system (registers, reference frames, probe pointers). The system may start entangled with an external idle ancilla. A physically valid evolution must never produce negative eigenvalues on such inputs. CP is exactly this ancilla-robustness.



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Physical Modeling, Ancillas, and Micro-to-Master

- From microscopic unitary to GKSL (big picture)

Start with $H_{\text{tot}} = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + \lambda V_{SE}$ on $\mathcal{H}_S \otimes \mathcal{H}_E$. Reduced dynamics:

$$\rho_S(t) = \text{Tr}_E \left[U_t(\rho_S(0) \otimes \sigma_E) U_t^\dagger \right].$$

Under weak coupling, fast-decaying bath correlations, Markov and secular approximations (e.g. Davies weak-coupling limit), one obtains a CPTP semigroup with a GKSL generator. Thermal baths impose detailed balance for the rates and produce a Gibbs fixed point (up to Lamb shifts).



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Physical Modeling, Ancillas, and Micro-to-Master

- Reversibility vs Irreversibility
 - Purely Hamiltonian $L(X) = i[H, X]$
 $\Rightarrow$ automorphism group (reversible).
 - Any nonzero dissipator
 $\Rightarrow$ proper CP semigroup (irreversible), contractive in trace distance in Schrödinger picture; arrow of time emerges.



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Physical Modeling, Ancillas, and Micro-to-Master

- Worked examples

- Example (Qubit pure dephasing)

Let $V = \sqrt{\gamma}Z$ and $H = 0$. Then

$$L_*(\rho) = \gamma(Z\rho Z - \rho).$$

Bloch vector $(x, y, z) \mapsto (e^{-2\gamma t}x, e^{-2\gamma t}y, z)$.



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Physical Modeling, Ancillas, and Micro-to-Master

- Worked examples

- ▶ Example (Qubit amplitude damping ($T = 0$))

Let $V = \sqrt{\gamma} |0\rangle\langle 1| = \sqrt{\gamma}\sigma_-$. Then

$$L_*(\rho) = \gamma \left(\sigma_- \rho \sigma_+ - \frac{1}{2} \{ \sigma_+ \sigma_-, \rho \} \right).$$

Excited population decays as $e^{-\gamma t}$; ground state $|0\rangle$ is fixed.



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Physical Modeling, Ancillas, and Micro-to-Master

- Worked examples

- **Example (Thermalization (Davies generator) for $H_S = \frac{\omega}{2}Z$)**

Take $V_{\downarrow} = \sqrt{\gamma_{\downarrow}}\sigma_{-}$, $V_{\uparrow} = \sqrt{\gamma_{\uparrow}}\sigma_{+}$ with $\frac{\gamma_{\uparrow}}{\gamma_{\downarrow}} = e^{-\beta\omega}$ and add dephasing $V_{\varphi} = \sqrt{\gamma_{\varphi}}Z$. Then

$$\begin{aligned} L_*(\rho) = & -i[H_S + H_{LS}, \rho] + \gamma_{\downarrow} \left(\sigma_{-}\rho\sigma_{+} - \frac{1}{2}\{\sigma_{+}\sigma_{-}, \rho\} \right) \\ & + \gamma_{\uparrow} \left(\sigma_{+}\rho\sigma_{-} - \frac{1}{2}\{\sigma_{-}\sigma_{+}, \rho\} \right) + \gamma_{\varphi}(Z\rho Z - \rho), \end{aligned}$$

with Gibbs state $\frac{e^{-\beta H_S}}{Z}$ as unique fixed point.



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Physical Modeling, Ancillas, and Micro-to-Master

- Worked examples

- Example (Depolarizing semigroup in d)

Let $\{F_j\}_{j=1}^{d^2-1}$ be an orthonormal operator basis with $\text{Tr}(F_j^\dagger F_k) = \delta_{jk}$, and set $V_j = \sqrt{\gamma} F_j$. Then

$$L_*(\rho) = -\gamma \left(\rho - \frac{\mathbb{1}}{d} \right),$$

i.e. exponential contraction to the maximally mixed state.



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Deepening the Operator-Algebra Background

- Positive Cones and Order

In any C^* -algebra, an element a is said to be positive, denoted $a \geq 0$, if there exists an element b such that $a = b^\dagger b$. This can also be understood in terms of the spectrum of a , where $a \geq 0$ if and only if the spectrum $\sigma(a)$ is contained in the non-negative real line, i.e., $\sigma(a) \subset [0, \infty)$.

For operators in $\mathcal{B}(\mathcal{H})$, the set of bounded operators on a Hilbert space $\mathcal{H}$, the condition $a \geq 0$ can be described using the expectation value: an operator a is positive if and only if for all $|\psi\rangle \in \mathcal{H}$, $\langle\psi| a |\psi\rangle \geq 0$.



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Deepening the Operator-Algebra Background

- Projections and Commutants

Projections are idempotent operators, meaning $p = p^2$, and they are also self-adjoint, i.e., $p^\dagger = p$. The set of all projections in a W^* -algebra forms a complete lattice, which essentially encodes the structure of yes/no events or dichotomic outcomes in quantum mechanics.

The commutant $\mathcal{M}'$ of an algebra $\mathcal{M}$ is defined as the set of all operators that commute with every element of $\mathcal{M}$. That is, $\mathcal{M}' = \{a \in \mathcal{B}(\mathcal{H}) \mid [a, b] = 0, \forall b \in \mathcal{M}\}$. The double commutant $\mathcal{M}''$ is the von Neumann closure of $\mathcal{M}$, which is the smallest weak-* closed algebra containing $\mathcal{M}$.



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Deepening the Operator-Algebra Background

- Normal Maps

A linear map $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ between two von Neumann algebras is said to be *normal* if it satisfies the following properties:

- **Continuity:** The map Φ is continuous with respect to the σ -topologies on $\mathcal{M}$ and $\mathcal{N}$, i.e., Φ is $\sigma(\mathcal{M}, \mathcal{M}_*)$ - $\sigma(\mathcal{N}, \mathcal{N}_*)$ continuous on bounded sets.
- **Preservation of suprema:** The map Φ preserves the suprema of bounded increasing nets of positive elements in $\mathcal{M}$.
- **Duality under trace-class duality:** For $\mathcal{B}(\mathcal{H})$, Φ is continuous under the trace-class duality. This means that for any $T \in \mathcal{T}(\mathcal{H})$, the map Φ satisfies $\text{Tr}[T\Phi(X)]$ depends ultraweakly on X for each fixed T .



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Deepening the Operator-Algebra Background

- Semigroups and Bounded Generators

A semigroup of maps $\{\Phi_t\}_{t \geq 0}$ is said to be *norm-continuous* if the map $t \mapsto \Phi_t(X)$ is continuous in operator norm for each fixed X .

The generator L of a semigroup $\{\Phi_t\}$ is defined as the limit:

$$L = \lim_{t \rightarrow 0^+} \frac{\Phi_t - \text{id}}{t}.$$

If Φ_t is norm-continuous, then the generator L is bounded. Conversely, if L is bounded, then the semigroup e^{tL} is norm-continuous for all $t \geq 0$.



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Deepening the Operator-Algebra Background

- Complete Dissipativity (Precise Form)

For a bounded $*$ -map $L : \mathcal{M} \rightarrow \mathcal{M}$, the quantity $D(L; X, Y)$ is defined as:

$$D(L; X, Y) := L(X^\dagger Y) - L(X^\dagger)Y - X^\dagger L(Y).$$

The map L is said to be *completely dissipative* if it satisfies the following conditions:

- **Unitality:** $L(\mathbb{I}) = 0$.
- **$*$ -preservation:** L is $*$ -preserving, i.e., $L(a^\dagger) = (L(a))^\dagger$ for all $a \in \mathcal{M}$.
- **Dissipativity:** $D(L_n; X, X) \geq 0$ for all $X \in M_n(\mathcal{M})$ and all n .

The Lindblad equation shows that a map Φ_t is unital and completely positive (CP) for all $t \geq 0$ if and only if the generator L is completely dissipative.



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Deepening the Operator-Algebra Background

- Heisenberg–Schrödinger Duality

In $\mathcal{B}(\mathcal{H})$, the dual generator L_* acts on the space of trace-class operators $\mathcal{T}(\mathcal{H})$ and satisfies the relation:

$$\mathrm{Tr}[L_*(\rho)X] = \mathrm{Tr}[\rho L(X)],$$

for any $\rho \in \mathcal{T}(\mathcal{H})$ and $X \in \mathcal{B}(\mathcal{H})$. If Φ_t is a unital CP map, its predual $\Phi_{t,*}$ is a CPTP (Completely Positive and Trace-Preserving) map. The equations in the Heisenberg and Schrödinger pictures are dual to each other.



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Practical Derivations and Checkpoints

- Positive but not CP: Matrix Transpose

Consider the transpose map $T : M_d \rightarrow M_d$ defined by $T(X) = X^\top$, where X is a $d \times d$ matrix. The map T is positive, meaning that for any $X \geq 0$, we have $T(X) \geq 0$. However, $(T \otimes \text{id}_2)(|\Omega\rangle\langle\Omega|)$ has a negative eigenvalue, where $|\Omega\rangle\langle\Omega|$ is a maximally entangled state, violating the Peres–Horodecki criterion. Therefore, T is not completely positive.



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Practical Derivations and Checkpoints

- From Stinespring to Kraus (Finite Dimension)

In the finite-dimensional case, consider an orthonormal basis $\{|e_k\rangle\}$ of the environment Hilbert space $\mathcal{H}_E$. The Stinespring dilation theorem states that a completely positive map Φ can be represented as:

$$\Phi(X) = V^\dagger (X \otimes \mathbb{I}) V,$$

where V is a unitary operator. Expanding this using the basis $\{|e_k\rangle\}$, we obtain the Kraus representation:

$$\Phi_*(\rho) = \sum_k K_k \rho K_k^\dagger,$$

where $K_k = \langle e_k | V | \psi \rangle$ are the Kraus operators.



Lindblad Equation — An Introduction to Operator Algebra

Practical Derivations and Checkpoints

- Deriving GKSL from complete dissipativity (sketch)

From the condition $D(L_n; X, X) \geq 0$ at all matrix levels and applying a hyperfinite approximation as in Lindblad's work, we can construct a normal CP map Θ such that:

$$L(X) = \Theta(X) - \frac{1}{2}\{\Theta(\mathbb{I}), X\} + i[H, X].$$

By Kraus-decomposing Θ , we can recover the Lindblad equation in both the Heisenberg and Schrödinger pictures, as in the equations:

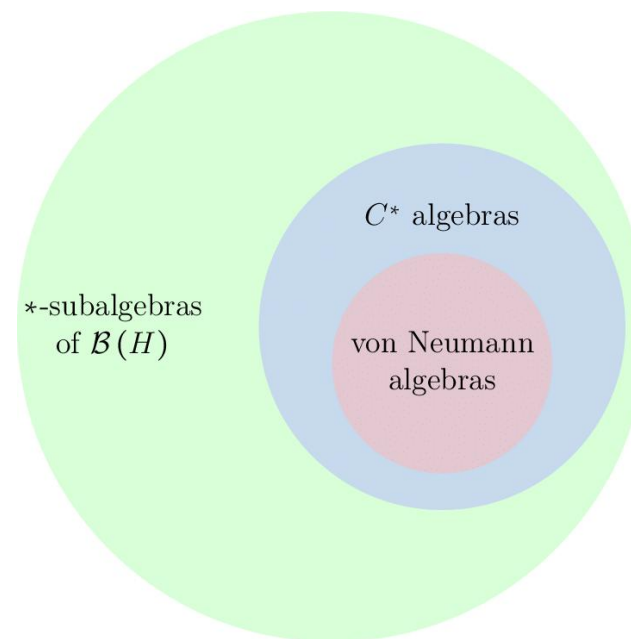
$$\frac{d\rho}{dt} = L(\rho),$$

where the dissipator is given by the sum of jump operators and a Hamiltonian term.



Lindblad Equation — On the Generators of Quantum Dynamical Semigroups

- Unitary Evolution : Group
- Non-Unitary Evolution : Semigroup (Preferred Direction in Time)
- Axioms for a Dynamical Semigroup (Ingarden and Kossakowski)
 - Let $\mathcal{A}$ be a W^* -algebra. A dynamical semigroup is a one-parameter family Φ_t of maps of $\mathcal{A}$ into itself satisfying
 - a) Φ_t is positive
 - b) $\Phi_t(I) = I$
 - c) $\Phi_s \cdot \Phi_t = \Phi_{s+t}$
 - d) $\Phi_t(X) \rightarrow X$ ultraweakly, $t \rightarrow 0$
 - e) Φ is normal (ultraweakly continuous)

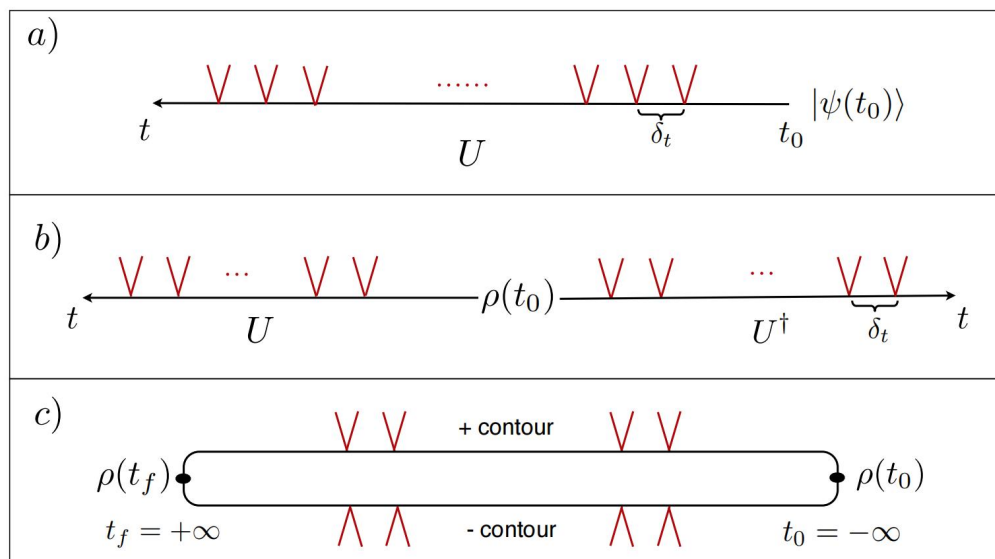


From Keldysh to Lindblad — Coherent State Representation for Lindbladian Evolution

- Formal Solution of the Master Equation

$$\partial_t \rho = \hat{\mathcal{L}}[\rho] \quad \Rightarrow \quad \rho(t) = \mathbb{T} e^{\int_{t'}^t dt \hat{\mathcal{L}}(t)} \rho(t')$$

- Infinitesimal Steps $\rho_{j+1} = e^{\delta_t \hat{\mathcal{L}}_j} \rho_j \approx (1 + \delta_t \hat{\mathcal{L}}_j) \rho_j \quad \delta_t = \int_{t_j}^{t_{j+1}} dt$
- Bosonic Case & Two Sets of Coherent State Over-Complete Basis



- Instantaneous Density Matrix Operator

$$\begin{aligned}\hat{\rho}_j &= \hat{1} \hat{\rho}_j \hat{1} = \left(\int d[\bar{\phi}_j^+, \phi_j^+] e^{-|\phi_j^+|^2} |\phi_j^+\rangle \langle \phi_j^+| \right) \hat{\rho}_j \left(\int d[\bar{\phi}_j^-, \phi_j^-] e^{-|\phi_j^-|^2} |\phi_j^-\rangle \langle \phi_j^-| \right) \\ &= \iint d[\bar{\phi}_j^+, \phi_j^+] d[\bar{\phi}_j^-, \phi_j^-] e^{-|\phi_j^+|^2 - |\phi_j^-|^2} |\phi_j^+\rangle \langle \phi_j^+| \hat{\rho}_j |\phi_j^-\rangle \langle \phi_j^-|\end{aligned}$$

- Express Matrix Elements of Reduced Density Matrix $\hat{\rho}_{j+1}$ with $\hat{\rho}_j$

$$\begin{aligned}\langle \phi_{j+1}^+ | \hat{\rho}_{j+1} | \phi_{j+1}^- \rangle &= \langle \phi_{j+1}^+ | e^{\delta_t \hat{\mathcal{L}}_j} \hat{\rho}_j | \phi_{j+1}^- \rangle = \langle \phi_{j+1}^+ | e^{\delta_t \hat{\mathcal{L}}_j} \left[\iint d[\bar{\phi}_j^+, \phi_j^+] d[\bar{\phi}_j^-, \phi_j^-] e^{-|\phi_j^+|^2 - |\phi_j^-|^2} |\phi_j^+\rangle \langle \phi_j^+| \hat{\rho}_j |\phi_j^-\rangle \langle \phi_j^-| \right] | \phi_{j+1}^- \rangle \\ &= \iint d[\bar{\phi}_j^+, \phi_j^+, \bar{\phi}_j^-, \phi_j^-] e^{-|\phi_j^+|^2 - |\phi_j^-|^2} \langle \phi_j^+ | \hat{\rho}_j | \phi_j^- \rangle \langle \phi_{j+1}^+ | e^{\delta_t \hat{\mathcal{L}}_j} [|\phi_j^+\rangle \langle \phi_j^-|] | \phi_{j+1}^- \rangle \\ &= \iint d[\bar{\phi}_j^+, \phi_j^+, \bar{\phi}_j^-, \phi_j^-] e^{-|\phi_j^+|^2 - |\phi_j^-|^2} \langle \phi_j^+ | \hat{\rho}_j | \phi_j^- \rangle \langle \phi_{j+1}^+ | \phi_j^+ \rangle \langle \phi_j^- | \phi_{j+1}^- \rangle e^{\delta_t \mathcal{L}_j(\bar{\phi}_{j+1}^+, \bar{\phi}_j^-, \phi_j^+, \phi_{j+1}^-)} \\ &= \iint d[\bar{\phi}_j^+, \phi_j^+, \bar{\phi}_j^-, \phi_j^-] \exp \left\{ \delta_t \left[\frac{\bar{\phi}_{j+1}^+ - \bar{\phi}_j^+}{\delta_t} \phi_j^+ + \bar{\phi}_j^- \frac{\phi_{j+1}^- - \phi_j^-}{\delta_t} + \mathcal{L}_j(\bar{\phi}_{j+1}^+, \bar{\phi}_j^-, \phi_j^+, \phi_{j+1}^-) \right] \right\} \langle \phi_j^+ | \hat{\rho}_j | \phi_j^- \rangle \\ &= \iint d[\bar{\phi}_j^+, \phi_j^+, \bar{\phi}_j^-, \phi_j^-] e^{\delta_t [\partial_t \bar{\phi}_j^+ \phi_j^+ + \bar{\phi}_j^- \partial_t \phi_j^- + \mathcal{L}_j(\bar{\phi}_{j+1}^+, \bar{\phi}_j^-, \phi_j^+, \phi_{j+1}^-)]} \langle \phi_j^+ | \hat{\rho}_j | \phi_j^- \rangle\end{aligned}$$



From Keldysh to Lindblad — Coherent State Representation for Lindbladian Evolution

- Instantaneous Density Matrix Operator

$$\begin{aligned}\hat{\rho}_j &= \hat{1} \hat{\rho}_j \hat{1} = \left(\int [\bar{\phi}_j^+, \phi_j^+] e^{-|\phi_j^+|^2} |\phi_j^+\rangle \langle \phi_j^+| \right) \hat{\rho}_j \left(\int [\bar{\phi}_j^-, \phi_j^-] e^{-|\phi_j^-|^2} |\phi_j^-\rangle \langle \phi_j^-| \right) \\ &= \iint d[\bar{\phi}_j^+, \phi_j^+] d[\bar{\phi}_j^-, \phi_j^-] e^{-|\phi_j^+|^2 - |\phi_j^-|^2} |\phi_j^+\rangle \langle \phi_j^+| \hat{\rho}_j |\phi_j^-\rangle \langle \phi_j^-|\end{aligned}$$

- Express Matrix Elements of Reduced Density Matrix $\hat{\rho}_{j+1}$ with $\hat{\rho}_j$
- Iterate to the Final Step

$$\begin{aligned}\langle \phi_N^+ | \hat{\rho}_N | \phi_N^- \rangle &= \iint d[\bar{\phi}_{N-1}^+, \phi_{N-1}^+, \bar{\phi}_{N-1}^-, \phi_{N-1}^-] \exp \left\{ i\delta_t \left[-i \frac{\bar{\phi}_N^+ - \bar{\phi}_{N-1}^+}{\delta_t} \phi_{N-1}^+ - i \bar{\phi}_{N-1}^- \frac{\phi_N^- - \phi_{N-1}^-}{\delta_t} + \mathcal{L}_{N-1}(\bar{\phi}_N^-, \bar{\phi}_{N-1}^-, \phi_{N-1}^+, \phi_N^-) \right] \right\} \langle \phi_{N-1}^+ | \hat{\rho}_{N-1} | \phi_{N-1}^- \rangle \\ &= \iint \left(\prod_{j=N-2}^{N-1} d[\bar{\phi}_j^+, \phi_j^+, \bar{\phi}_j^-, \phi_j^-] \right) \exp \left\{ i\delta_t \sum_{j=N-2}^{N-1} \left[-i \frac{\bar{\phi}_{j+1}^+ - \bar{\phi}_j^+}{\delta_t} \phi_j^+ - i \bar{\phi}_j^- \frac{\phi_{j+1}^- - \phi_j^-}{\delta_t} + \mathcal{L}_j \right] \right\} \langle \phi_{N-2}^+ | \hat{\rho}_{N-2} | \phi_{N-2}^- \rangle \\ &\quad \dots \\ &= \iint \left(\prod_{j=0}^{N-1} d[\bar{\phi}_j^+, \phi_j^+, \bar{\phi}_j^-, \phi_j^-] \right) \exp \left\{ i\delta_t \sum_{j=0}^{N-1} \left[-i \frac{\bar{\phi}_{j+1}^+ - \bar{\phi}_j^+}{\delta_t} \phi_j^+ - i \bar{\phi}_j^- \frac{\phi_{j+1}^- - \phi_j^-}{\delta_t} + \mathcal{L}_j \right] \right\} \langle \phi_0^+ | \hat{\rho}_0 | \phi_0^- \rangle\end{aligned}$$



From Keldysh to Lindblad — Coherent State Representation for Lindbladian Evolution

- Normalization of the “Partition Function”

$$Z = \text{Tr } \hat{\rho}(t) = 1$$

↓

$$\begin{aligned} \text{Tr } \hat{\rho}(t) &= \sum_{n=0}^{\infty} \langle n | \hat{\rho}(t) | n \rangle = \iint d[\bar{\phi}_N^+, \phi_N^+, \bar{\phi}_N^-, \phi_N^-] e^{-|\phi_N^+| - |\phi_N^-|} \langle n | \phi_N^+ \rangle \langle \phi_N^+ | \hat{\rho}_N(t) | \phi_N^- \rangle \langle \phi_N^- | n \rangle \\ &= \iint d[\bar{\phi}_N^+, \phi_N^+, \bar{\phi}_N^-, \phi_N^-] e^{-|\phi_N^+| - |\phi_N^-|} \langle \phi_N^+ | \hat{\rho}_N(t) | \phi_N^- \rangle \left(\sum_{n=0}^{\infty} \langle \phi_N^- | n \rangle \langle n | \phi_N^+ \rangle \right) \\ &= \iint d[\bar{\phi}_N^+, \phi_N^+, \bar{\phi}_N^-, \phi_N^-] e^{-|\phi_N^+| - |\phi_N^-|} \langle \phi_N^+ | \hat{\rho}_N(t) | \phi_N^- \rangle e^{\bar{\phi}_N^- \phi_N^+} \end{aligned}$$

↓

$$\begin{aligned} \text{Tr } \hat{\rho}(t) &= \iint d[\bar{\phi}_N^+, \phi_N^+, \bar{\phi}_N^-, \phi_N^-] e^{-|\phi_N^+| - |\phi_N^-|} \left\{ \iint \left(\prod_{j=0}^{N-1} d[\bar{\phi}_j^+, \phi_j^+, \bar{\phi}_j^-, \phi_j^-] \right) \exp \left\{ i \delta_t \sum_{j=0}^{N-1} \left[-i \frac{\bar{\phi}_{j+1}^+ - \bar{\phi}_j^+}{\delta_t} \phi_j^+ - i \bar{\phi}_j^- \frac{\phi_{j+1}^- - \phi_j^-}{\delta_t} + \mathcal{L}_j \right] \right\} \langle \phi_0^+ | \hat{\rho}_0 | \phi_0^- \rangle \right\} e^{\bar{\phi}_N^- \phi_N^+} \\ &= \iint \left(\prod_{j=0}^N d[\bar{\phi}_j^+, \phi_j^+, \bar{\phi}_j^-, \phi_j^-] \right) \exp \left\{ i \left[i \left(|\phi_N^+|^2 - |\phi_N^-|^2 - \bar{\phi}_N^- \phi_N^+ \right) + \delta_t \sum_{j=0}^{N-1} \left(-i \frac{\bar{\phi}_{j+1}^+ - \bar{\phi}_j^+}{\delta_t} \phi_j^+ - i \bar{\phi}_j^- \frac{\phi_{j+1}^- - \phi_j^-}{\delta_t} + \mathcal{L}_j \right) \right] \right\} \langle \phi_0^+ | \hat{\rho}_0 | \phi_0^- \rangle \end{aligned}$$



From Keldysh to Lindblad — Coherent State Representation for Lindbladian Evolution

- Action in the Continuum Limit / Effective Keldysh Action

$$\begin{aligned}
 S &= i \left(|\phi_N^+|^2 + |\phi_N^-|^2 - \bar{\phi}_N^- \phi_N^+ \right) \\
 &+ \left\{ -i \left[|\phi_N^+|^2 + \bar{\phi}_{N-1}^+ \left(\phi_{N-1}^+ - \phi_{N-2}^+ + |\phi_0^+|^2 \right) \right] + i\delta_t \left(\sum_{j=0}^{N-1} \bar{\phi}_{j+1}^+ \frac{\phi_{j+1}^+ - \phi_j^+}{\delta_t} + \bar{\phi}_N^+ \frac{\phi_N^+ - \phi_{N-1}^+}{\delta_t} \right) \right\} \\
 &+ \left\{ i\bar{\phi}_N^- (\phi_N^- - \phi_{N-1}^-) - i\delta_t \left(\sum_{j=0}^{N-1} \bar{\phi}_j^- \frac{\phi_{j+1}^- - \phi_j^-}{\delta_t} + \bar{\phi}_N^- \frac{\phi_N^- - \phi_{N-1}^-}{\delta_t} \right) \right\} + \mathcal{L}_j \\
 &\stackrel{\phi_N^+ = \phi_N^-}{\underset{N \rightarrow \infty}{\approx}} -i|\phi_0^+|^2 + i \int dt \left(\bar{\phi}^+ i\partial_t \phi^+ - \bar{\phi}^- i\partial_t \phi^- + \mathcal{L}_j \right) \\
 &\Downarrow \\
 S[\bar{\phi}, \phi] &= \int dt \left[\bar{\phi}^+ i\partial_t \phi^+ - \bar{\phi}^- i\partial_t \phi^- - H_p'^+ + H_p'^- - i \sum_a \gamma_a \left(L_a^+ \bar{L}_a^- - \frac{1}{2} \bar{L}_a^+ L_a^+ - \frac{1}{2} \bar{L}_a^- L_a^- \right) \right]
 \end{aligned}$$



From Keldysh to Lindblad — Coherent State Representation for Lindbladian Evolution

- Effective Keldysh Action

$$S[\bar{\phi}, \phi] = \int dt \left[\bar{\phi}^+ i \partial_t \phi^+ - \bar{\phi}^- i \partial_t \phi^- - H_p'^+ + H_p'^- - i \sum_a \gamma_a \left(L_a^+ \bar{L}_a^- - \frac{1}{2} \bar{L}_a^+ L_a^+ - \frac{1}{2} \bar{L}_a^- L_a^- \right) \right]$$

- No Dissipative $\Rightarrow$ Factorised Action (Unbroken Time Reversal Symmetry)

$$\gamma_a = 0 \quad \Rightarrow \quad S = S[\phi^+] - S[\phi^-]$$

- Mixing Terms (broken Time Reversal Symmetry)

$$\hat{L}_a \hat{\rho} \hat{L}_a^\dagger \quad \Rightarrow \quad L_a^+ \bar{L}_a^-$$



From Keldysh to Lindblad — Coherent State Representation for Lindbladian Evolution

- Effective Keldysh Action

$$S[\bar{\phi}, \phi] = \int dt \left[\bar{\phi}^+ i_t \phi^+ - \bar{\phi}^- i \partial_t \phi^- - H_p'^+ + H_p'^- - i \sum_a \gamma_a \left(L_a^+ \bar{L}_a^- - \frac{1}{2} \bar{L}_a^+ L_a^+ - \frac{1}{2} \bar{L}_a^- L_a^- \right) \right]$$

- Quantum Jump Operators Correspond to an Approximately Time-Local Effective Keldysh Action (Non-Markovian)
- Simplest Jump Operators

$$\hat{L}_1 = \hat{b} \quad \hat{L}_2 = \hat{b}^\dagger \quad \Rightarrow \quad \hat{b} \rightarrow \phi^\pm \quad \hat{b}^\dagger \rightarrow \bar{\phi}^\pm$$

$\Downarrow$

$$\begin{aligned} -i\gamma_1 \left(\phi^+ \bar{\phi}^- - \frac{1}{2} \bar{\phi}^+ \phi^+ - \frac{1}{2} \bar{\phi}^- \phi^- \right) &= \frac{1}{2} (\bar{\phi}^{\text{cl}}, \bar{\phi}^{\text{q}}) \begin{pmatrix} 0 & -i\gamma_1 \\ i\gamma_1 & 2i\gamma_1 \end{pmatrix} \begin{pmatrix} \phi^{\text{cl}} \\ \phi^{\text{q}} \end{pmatrix} \\ -i\gamma_2 \left(\bar{\phi}^+ \phi^- - \frac{1}{2} \bar{\phi}^+ \phi^+ - \frac{1}{2} \bar{\phi}^- \phi^- \right) &= \frac{1}{2} (\bar{\phi}^{\text{cl}}, \bar{\phi}^{\text{q}}) \begin{pmatrix} 0 & i\gamma_2 \\ -i\gamma_2 & 2i\gamma_2 \end{pmatrix} \begin{pmatrix} \phi^{\text{cl}} \\ \phi^{\text{q}} \end{pmatrix}. \end{aligned}$$

$\Downarrow$

$$\kappa = (\gamma_1 - \gamma_2)/2 \quad \kappa_1 = (\gamma_1 + \gamma_2)/2$$



From Keldysh to Lindblad — Coherent State Representation for Lindbladian Evolution

- General Interaction

$$\hat{H}_{\text{int}} = \sum_s \frac{g_s}{\sqrt{2\omega_s}} \left(\hat{a}_s^\dagger \hat{L} + \hat{L}^\dagger \hat{a}_s \right) \quad \hat{L} = \hat{b}^n$$

- Two Competing Jump Operators

$$\text{Down Jumps} \quad \hat{L}_1 = \hat{L} \quad \text{Up Jumps} \quad \hat{L}_2 = \hat{L}^\dagger$$

- Ratio of the Corresponding Down to Up Rates

$$\frac{\gamma_1}{\gamma_2} = \frac{n_B + 1}{n_B} \quad n_B = \frac{1}{e^{n\omega_0/T} - 1}$$

- Independent of Details of the Coupling Mechanism (Rotating-Wave FDT)



Linear Systems

- Recall : Keldysh Rotation

$$\phi^{\text{cl}} \rightarrow \phi^{\text{cl}} - \phi_\eta \quad \rightarrow \quad \begin{array}{l} \phi^+ \rightarrow \phi^+ - \frac{\phi_\eta}{\sqrt{2}} \\ \phi^- \rightarrow \phi^- - \frac{\phi_\eta}{\sqrt{2}} \end{array} \quad \rightarrow \quad \begin{array}{l} \hat{b} \rightarrow \hat{b} - \frac{\phi_\eta}{\sqrt{2}} \\ \hat{b}^\dagger \rightarrow \hat{b}^\dagger - \frac{\bar{\phi}_\eta}{\sqrt{2}} \end{array}$$

- Consider Unitary Transformation

$$\hat{\mathcal{D}}(\phi_\eta) = \exp \left\{ \frac{1}{\sqrt{2}} \left(\phi_\eta \hat{b}^\dagger - \bar{\phi}_\eta \hat{b} \right) \right\}$$

- Check

$$\hat{\mathcal{D}}(\phi_\eta) \hat{b} \hat{\mathcal{D}}^\dagger(\phi_\eta) = e^{\phi_\eta \hat{b}^\dagger / \sqrt{2}} \hat{b} e^{-\phi_\eta \hat{b}^\dagger / \sqrt{2}} = \hat{b} + \left[\phi_\eta \hat{b}^\dagger / \sqrt{2}, \hat{b} \right] = \hat{b} - \frac{\phi_\eta}{\sqrt{2}}$$

$$\hat{\mathcal{D}}(\phi_\eta) \hat{b}^\dagger \hat{\mathcal{D}}^\dagger(\phi_\eta) = e^{-\phi_\eta \hat{b} / \sqrt{2}} \hat{b}^\dagger e^{\phi_\eta \hat{b} / \sqrt{2}} = \hat{b}^\dagger + \left[-\phi_\eta \hat{b} / \sqrt{2}, \hat{b}^\dagger \right] = \hat{b}^\dagger - \frac{\bar{\phi}_\eta}{\sqrt{2}}$$



Linear Systems

- Corresponding Lindblad Equation

$$\begin{aligned}
 \hat{\mathcal{D}}(\partial_t \hat{\rho}) \hat{\mathcal{D}}^\dagger &= \partial_t (\hat{\mathcal{D}} \hat{\rho} \hat{\mathcal{D}}^\dagger) \equiv \partial_t (\hat{\rho}') \\
 &= -i [\hat{\mathcal{D}} H'_p \hat{\mathcal{D}}^\dagger, \hat{\rho}'] + \sum_a \gamma_a \left[(\hat{\mathcal{D}} \hat{L}_a \hat{\mathcal{D}}^\dagger) \hat{\rho}' (\hat{\mathcal{D}} \hat{L}_a^\dagger \hat{\mathcal{D}}^\dagger) - \frac{1}{2} \{ \hat{\mathcal{D}} \hat{L}_a^\dagger \hat{L}_a \hat{\mathcal{D}}^\dagger, \hat{\rho}' \} \right] \\
 &= -i [\hat{H}'_p, \hat{\rho}'] - i \left[\left(i\eta - \frac{\Delta}{\sqrt{2}} \phi_\eta \right) \hat{b}^\dagger - \left(i\eta + \frac{\Delta}{\sqrt{2}} \bar{\phi}_\eta \right) \hat{b} + \frac{\Delta}{2} |\phi_\eta|^2 + \frac{i\eta}{\sqrt{2}} (\phi_\eta - \bar{\phi}_\eta), \hat{\rho}' \right] \\
 &+ \left\{ \gamma_1 \left[\hat{b} \hat{\rho}' \hat{b}^\dagger - \frac{1}{2} \{ \hat{b}^\dagger \hat{b}, \hat{\rho}' \} - \frac{1}{\sqrt{2}} (\phi_\eta \hat{\rho}' \hat{b}^\dagger + \bar{\phi}_\eta \hat{b} \hat{\rho}') \right] + \gamma_2 \left[\hat{b}^\dagger \hat{\rho}' \hat{b} - \frac{1}{2} \{ \hat{b} \hat{b}^\dagger, \hat{\rho}' \} - \frac{1}{\sqrt{2}} (\phi_\eta \hat{b}^\dagger \hat{\rho}' + \bar{\phi}_\eta \hat{\rho}' \hat{b}) \right] \right\} + \frac{\gamma_1 + \gamma_2}{2\sqrt{2}} \{ \phi_\eta \hat{b}^\dagger + \bar{\phi}_\eta \hat{b}, \hat{\rho}' \} \\
 &= -i [\hat{H}'_p, \hat{\rho}'] + \gamma_1 \left(\hat{b} \hat{\rho}' \hat{b}^\dagger - \frac{1}{2} \{ \hat{b}^\dagger \hat{b}, \hat{\rho}' \} \right) + \gamma_2 \left(\hat{b}^\dagger \hat{\rho}' \hat{b} - \frac{1}{2} \{ \hat{b} \hat{b}^\dagger, \hat{\rho}' \} \right) \\
 &+ \left[\left(\eta + \frac{i}{\sqrt{2}} \Delta \phi_\eta + \frac{\gamma_1 - \gamma_2}{2\sqrt{2}} \phi_\eta \right) \hat{b}^\dagger \hat{\rho}' - \left(\eta + \frac{i}{\sqrt{2}} \Delta \phi_\eta + \frac{\gamma_1 - \gamma_2}{2\sqrt{2}} \phi_\eta \right) \hat{\rho}' \hat{b}^\dagger + \left(\eta - \frac{i}{\sqrt{2}} \Delta \bar{\phi}_\eta + \frac{\gamma_1 - \gamma_2}{2\sqrt{2}} \bar{\phi}_\eta \right) \hat{\rho}' \hat{b} - \left(\eta - \frac{i}{\sqrt{2}} \Delta \bar{\phi}_\eta + \frac{\gamma_1 - \gamma_2}{2\sqrt{2}} \bar{\phi}_\eta \right) \hat{b} \hat{\rho}' \right]
 \end{aligned}$$

- Coefficients

$$\frac{\gamma_1}{\gamma_2} = \frac{\kappa_1 + \kappa}{\kappa_1 - \kappa} = 1 + \frac{1}{n_B} \quad \Rightarrow \quad \begin{aligned} \gamma_1 &= 2(1 + n_B)\kappa \\ \gamma_2 &= 2n_B\kappa \end{aligned}$$



Linear Systems

- Corresponding Lindblad Equation

$$\partial_t \hat{\rho}' = -i [\hat{H}'_p, \hat{\rho}'] + \left\{ \left[\eta + \frac{i\phi_\eta}{\sqrt{2}} (\Delta - i\kappa) \right] [\hat{b}^\dagger, \hat{\rho}'] + \left[\eta - \frac{i\bar{\phi}_\eta}{\sqrt{2}} (\Delta + i\kappa) \right] [\hat{\rho}', \hat{b}] \right\} \\ \gamma_1 \left(\hat{b} \hat{\rho}' \hat{b}^\dagger - \frac{1}{2} \{ \hat{b}^\dagger \hat{b}, \hat{\rho}' \} \right) + \gamma_2 \left(\hat{b}^\dagger \hat{\rho}' \hat{b} - \frac{1}{2} \{ \hat{b} \hat{b}^\dagger, \hat{\rho}' \} \right)$$

- Shift Equation

$$(\Delta - i\kappa)\phi_\eta + \frac{g}{2} |\phi_\eta|^2 \phi_\eta = \sqrt{2} i \eta$$

$$(\Delta + i\kappa)\bar{\phi}_\eta + \frac{g}{2} |\phi_\eta|^2 \bar{\phi}_\eta = -\sqrt{2} i \eta$$

- Lindblad Equation for Linear Systems $g = 0$ $\gamma_1 = 2(1 + n_B)\kappa$
 $\gamma_2 = 2n_B\kappa$

$$\partial_t \hat{\rho} = -i [\hat{H}'_p, \hat{\rho}] + 2\kappa (n_B + 1) \left(\hat{b} \hat{\rho} \hat{b}^\dagger - \frac{1}{2} \{ \hat{b}^\dagger \hat{b}, \hat{\rho} \} \right) + 2\kappa n_B \left(\hat{b}^\dagger \hat{\rho} \hat{b} - \frac{1}{2} \{ \hat{b} \hat{b}^\dagger, \hat{\rho} \} \right)$$



Linear Systems

- Glauber-Sudarshan Coherent State Representation

$$\hat{\rho}(t) = \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \mathcal{P}_{\text{GS}}(\bar{\phi}, \phi, t) |\phi\rangle\langle\phi|$$

- Recall Properties of Coherent States

$$|\phi\rangle = e^{\phi\hat{b}^\dagger}|0\rangle \quad \langle\phi| = \langle 0|e^{\bar{\phi}\hat{b}}$$

$$\Downarrow$$

$$\hat{b}|\phi\rangle = \phi|\phi\rangle \quad \hat{b}^\dagger|\phi\rangle = \partial_\phi|\phi\rangle \quad \langle\phi|\hat{b}^\dagger = \bar{\phi}\langle\phi| \quad \langle\phi|\hat{b} = \partial_{\bar{\phi}}\langle\phi|$$

- Lindblad Equation in GS Representation

$$\partial_t \hat{\rho} = \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \partial_t \mathcal{P}_{\text{GS}} |\phi\rangle\langle\phi|$$

$$\begin{aligned} -i [\hat{H}'_p, \hat{\rho}] &= \int d[\bar{\phi}, \phi] e^{-|\phi|^2} (-i\mathcal{P}_{\text{GS}}) \left(\hat{H}'_p |\phi\rangle\langle\phi| - |\phi\rangle\langle\phi| \hat{H}'_p \right) \\ &= \int d[\bar{\phi}, \phi] e^{-|\phi|^2} i\Delta \mathcal{P}_{\text{GS}} \left(|\phi\rangle\langle\phi| \hat{b}^\dagger \hat{b} - \hat{b}^\dagger \hat{b} |\phi\rangle\langle\phi| \right) = \int d[\bar{\phi}, \phi] e^{-|\phi|^2} i\Delta \mathcal{P}_{\text{GS}} \left[\bar{\phi} |\phi\rangle \left(\partial_{\bar{\phi}} \langle\phi| \right) - \phi \left(\partial_\phi |\phi\rangle \right) \langle\phi| \right] \\ &= \int d[\bar{\phi}, \phi] i\Delta \left[-\partial_{\bar{\phi}} \left(e^{-|\phi|^2} \mathcal{P}_{\text{GS}} \bar{\phi} \right) + \partial_\phi \left(e^{-|\phi|^2} \mathcal{P}_{\text{GS}} \phi \right) \right] |\phi\rangle\langle\phi| = \int d[\bar{\phi}, \phi] i\Delta \left(\phi \partial_\phi \mathcal{P}_{\text{GS}} - \bar{\phi} \partial_{\bar{\phi}} \mathcal{P}_{\text{GS}} \right) e^{-|\phi|^2} |\phi\rangle\langle\phi| \end{aligned}$$



Linear Systems

- Lindblad Equation in GS Representation

$$\begin{aligned}
 \hat{b}\hat{\rho}\hat{b}^\dagger - \frac{1}{2}\{\hat{b}^\dagger\hat{b}, \hat{\rho}\} &= \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \mathcal{P}_{\text{GS}} \left[\hat{b}|\phi\rangle\langle\phi|\hat{b}^\dagger - \frac{1}{2}(\hat{b}^\dagger\hat{b}|\phi\rangle\langle\phi| + |\phi\rangle\langle\phi|\hat{b}^\dagger\hat{b}) \right] \\
 &= \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \mathcal{P}_{\text{GS}} \left\{ |\phi|^2 |\phi\rangle\langle\phi| - \frac{1}{2} \left[\phi (\partial_\phi |\phi\rangle) \langle\phi| + |\phi\rangle (\partial_{\bar{\phi}} \langle\phi|) \bar{\phi} \right] \right\} \\
 &= \int d[\bar{\phi}, \phi] \left\{ e^{-|\phi|^2} \mathcal{P}_{\text{GS}} |\phi|^2 + \frac{1}{2} \left[\partial_\phi (e^{-|\phi|^2} \mathcal{P}_{\text{GS}} \phi) + \partial_{\bar{\phi}} (e^{-|\phi|^2} \mathcal{P}_{\text{GS}} \bar{\phi}) \right] \right\} |\phi\rangle\langle\phi| \\
 &= \int d[\bar{\phi}, \phi] \left[\mathcal{P}_{\text{GS}} + \frac{1}{2} (\phi \partial_\phi \mathcal{P}_{\text{GS}} + \bar{\phi} \partial_{\bar{\phi}} \mathcal{P}_{\text{GS}}) \right] e^{-|\phi|^2} |\phi\rangle\langle\phi|
 \end{aligned}$$

$$\begin{aligned}
 \hat{b}^\dagger \hat{\rho} \hat{b} - \frac{1}{2} \{ \hat{b} \hat{b}^\dagger, \hat{\rho} \} &= \int d[\hat{\phi}, \phi] e^{-|\phi|^2} \mathcal{P}_{\text{GS}} \left[\hat{b}^\dagger |\phi\rangle\langle\phi| \hat{b} - \frac{1}{2} (\hat{b} \hat{b}^\dagger |\phi\rangle\langle\phi| + |\phi\rangle\langle\phi| \hat{b} \hat{b}^\dagger) \right] \\
 &= \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \mathcal{P}_{\text{GS}} \left\{ (\partial_\phi |\phi\rangle) (\partial_\phi \langle\phi|) - \frac{1}{2} \left[(\hat{b}^\dagger \hat{b} + 1) |\phi\rangle\langle\phi| + |\phi\rangle\langle\phi| (\hat{b}^\dagger \hat{b} + 1) \right] \right\} \\
 &= \int d[\bar{\phi}, \phi] \left\{ \partial_{\bar{\phi}\phi}^2 (e^{-|\phi|^2} \mathcal{P}_{\text{GS}}) + \frac{1}{2} \left[\partial_\phi (e^{-|\phi|^2} \mathcal{P}_{\text{GS}} \phi) + \partial_{\bar{\phi}} (e^{-|\phi|^2} \mathcal{P}_{\text{GS}} \bar{\phi}) - 2e^{-|\phi|^2} \mathcal{P}_{\text{GS}} \right] \right\} |\phi\rangle\langle\phi| \\
 &= \int d[\bar{\phi}, \phi] \left[-\mathcal{P}_{\text{GS}} - \frac{1}{2} (\phi \partial_\phi \mathcal{P}_{\text{GS}} + \bar{\phi} \partial_{\bar{\phi}} \mathcal{P}_{\text{GS}}) + \partial_{\bar{\phi}\phi}^2 \mathcal{P}_{\text{GS}} \right] e^{-|\phi|^2} |\phi\rangle\langle\phi|
 \end{aligned}$$



Linear Systems

- Lindblad Equation in GS Representation

$$\Rightarrow \int d[\bar{\phi}, \phi] \left\{ \partial_t \mathcal{P}_{\text{GS}} - i\Delta(\phi \partial_{\phi} \mathcal{P}_{\text{GS}} - \bar{\phi} \partial_{\bar{\phi}} \mathcal{P}_{\text{GS}}) - 2\kappa(n_B + 1) \left[\mathcal{P}_{\text{GS}} + \frac{1}{2}(\phi \partial_{\phi} \mathcal{P}_{\text{GS}} + \bar{\phi} \partial_{\bar{\phi}} \mathcal{P}_{\text{GS}}) \right] \right. \\ \left. - 2\kappa n_B \left[-\mathcal{P}_{\text{GS}} - \frac{1}{2}(\phi \partial_{\phi} \mathcal{P}_{\text{GS}} + \bar{\phi} \partial_{\bar{\phi}} \mathcal{P}_{\text{GS}}) + \partial_{\bar{\phi}\phi}^2 \mathcal{P}_{\text{GS}} \right] \right\} e^{-|\phi|^2} |\phi\rangle\langle\phi| \\ = \int d[\bar{\phi}, \phi] \left\{ \partial_t \mathcal{P}_{\text{GS}} - (i\Delta + \kappa) [\partial_{\phi}(\phi \mathcal{P}_{\text{GS}}) - \mathcal{P}_{\text{GS}}] - (\kappa - i\Delta) [\partial_{\bar{\phi}}(\bar{\phi} \mathcal{P}_{\text{GS}}) - \mathcal{P}_{\text{GS}}] - 2\kappa \mathcal{P}_{\text{GS}} - 2\kappa n_B \partial_{\bar{\phi}\phi}^2 \mathcal{P}_{\text{GS}} \right\} e^{-|\phi|^2} |\phi\rangle\langle\phi| = 0$$

$\Downarrow$

$$\partial_t \mathcal{P}_{\text{GS}} = \left[(\kappa + i\Delta) \partial_{\phi} (\phi \mathcal{P}_{\text{GS}}) + (\kappa - i\Delta) \partial_{\bar{\phi}} (\bar{\phi} \mathcal{P}_{\text{GS}}) \right] + 2\kappa n_B \partial_{\bar{\phi}\phi}^2 \mathcal{P}_{\text{GS}}$$

- Fokker-Planck Equation for the Probability Distribution in the Phase Space
- Dissipative Terms $\propto K$

$$\kappa \left[\partial_{\phi} V_{\bar{\phi}} + \partial_{\bar{\phi}} V_{\phi} \right] \mathcal{P} \quad V(\phi, \bar{\phi}) = \bar{\phi}\phi.$$

- Dynamical Terms : Generate Rotation in the Phase Space $\sim i\Delta$
- Boltzmann-Like Distribution



Linear Systems

- Lindblad Equation in GS Representation

$$\partial_t \mathcal{P}_{\text{GS}} = \left[(\kappa + i\Delta) \partial_\phi (\phi \mathcal{P}_{\text{GS}}) + (\kappa - i\Delta) \partial_{\bar{\phi}} (\bar{\phi} \mathcal{P}_{\text{GS}}) \right] + 2\kappa n_B \partial_{\bar{\phi}\phi}^2 \mathcal{P}_{\text{GS}}$$

- Boltzmann-Like Distribution (Stationary)

$$\mathcal{P}_{\text{GS}}(\bar{\phi}, \phi) = \frac{1}{n_B} \exp \left\{ -\frac{|\phi|^2}{n_B} \right\} \quad \text{Tr } \hat{\rho} = \int d[\bar{\phi}, \phi] \mathcal{P}_{\text{GS}}(\bar{\phi}, \phi) = 1$$

$$\hat{\rho} = \frac{1}{n_B} \int d[\bar{\phi}, \phi] \exp \left\{ -e^{\omega_0/T} |\phi|^2 \right\} |\phi\rangle \langle \phi|$$

- In Number State Basis $|\phi\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{\sqrt{n!}} |n\rangle$

$$\begin{aligned} \hat{\rho} &= \frac{1}{n_B} \sum_{m,n=0}^{\infty} \frac{|m\rangle \langle n|}{\sqrt{m!n!}} \int d[\bar{\phi}, \phi] \phi^m \bar{\phi}^n \exp \{ -e^{\omega_0/T} |\phi|^2 \} \\ &= \frac{1}{n_B} \sum_{n=0}^{\infty} \frac{|n\rangle \langle n|}{n!} \int d[\bar{\phi}, \phi] |\phi|^{2n} \exp(-e^{\omega_0/T} |\phi|^2) = \frac{1}{n_B} \sum_{n=0}^{\infty} \frac{|n\rangle \langle n|}{n!} e^{-(n+1)\omega_0/T} \int d[\bar{\phi}, \phi] |\phi|^{2n} e^{-|\phi|^2} \\ &= \frac{1}{n_B} \sum_{n=0}^{\infty} |n\rangle \langle n| e^{-(n+1)\omega_0/T} = \left(\frac{e^{-\omega_0/T}}{1 - e^{-\omega_0/T}} \right)^{-1} \sum_n |n\rangle \langle n| e^{-(n+1)\omega_0/T} = \frac{1}{n_B + 1} \sum_{n=0}^{\infty} |n\rangle \langle n| e^{-n\omega_0/T} \end{aligned}$$



Linear Systems

- Lindblad Equation in GS Representation

$$\partial_t \mathcal{P}_{\text{GS}} = \left[(\kappa + i\Delta) \partial_\phi (\phi \mathcal{P}_{\text{GS}}) + (\kappa - i\Delta) \partial_{\bar{\phi}} (\bar{\phi} \mathcal{P}_{\text{GS}}) \right] + 2\kappa n_B \partial_{\bar{\phi}\phi}^2 \mathcal{P}_{\text{GS}}$$

- Equilibrium Thermal Density Matrix

$$\hat{\rho} = \sum_{n=0}^{\infty} \mathcal{P}_n |n\rangle \langle n| \quad \mathcal{P}_n = \frac{e^{-n\omega_0/T}}{n_B + 1}$$

- Fluctuations around the Driven Path in the Phase Space
- Linear Drive : No Nonequilibrium Distribution
- Parametric Drive

- GS Stationary Function in the Unshifted Frame

$$\mathcal{P}_{\text{GS}} \propto \exp \left\{ |\phi - \phi_\eta|^2 / n_B \right\}$$

- Zero Temperature $\Rightarrow$ Pure Coherent State Density Matrix

$$\mathcal{P}_{\text{GS}} \propto \exp \left\{ |\phi - \phi_\eta|^2 / n_B \right\} \longrightarrow \mathcal{P}_{\text{GS}} = \delta(\phi - \phi_\eta) \delta(\bar{\phi} - \bar{\phi}_\eta)$$

$\Downarrow$

$$\hat{\rho} = e^{-|\phi_\eta|^2} |\phi_\eta\rangle \langle \phi_\eta|$$



Linear Systems

- Hubbard-Stratonovich Transformation of $e^{-2\kappa(2n_B+1) \int dt |\phi^q|^2}$
 - Integration over Quantum Component Gives Langevin Equation

$$\partial_t \phi^{\text{cl}} = -(\kappa + i\Delta)\phi^{\text{cl}} - i\xi(t) \quad \partial_t \bar{\phi}^{\text{cl}} = -(\kappa - i\Delta)\bar{\phi}^{\text{cl}} + i\bar{\xi}(t)$$

$$\langle \xi(t) \bar{\xi}(t') \rangle = 2\kappa(2n_B + 1) \delta(t - t')$$

- Corresponding Fokker-Plank Equation

$$\partial_t \mathcal{P} = \left[(\kappa + i\Delta) \partial_{\phi^{\text{cl}}} \phi^{\text{cl}} + (\kappa - i\Delta) \partial_{\bar{\phi}^{\text{cl}}} \bar{\phi}^{\text{cl}} \right] \mathcal{P} + 2\kappa(2n_B + 1) \partial_{\bar{\phi}^{\text{cl}}}^2 \mathcal{P}$$

- Normalized Stationary Solution

$$\mathcal{P}(\bar{\phi}^{\text{cl}}, \phi^{\text{cl}}) = \frac{1}{2n_B + 1} \exp \left\{ -\frac{|\phi^{\text{cl}}|^2}{2n_B + 1} \right\} \Rightarrow \langle \phi^{\text{cl}}(t) \bar{\phi}^{\text{cl}}(t) \rangle = 2n_B + 1$$





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Thanks for Your Listening

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